CHAIN UNION CLOSURES AND CONVEXITY

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Abstract. We study spherical completeness of ball spaces and its stability under expansions. We give some criteria for ball spaces that guarantee that spherical completeness is preserved when the ball space is closed under unions of chains. This applies in particular to the spaces of closed ultrametric balls in ultrametric spaces with linearly ordered value sets, or with countable narrow value sets. We show that in general, chain union closures of ultrametric spaces with partially ordered value sets do not preserve spherical completeness. Further, we introduce and study the notions of chain union stability and of chain union rank, which measure how often the process of closing a ball space under all unions of chains has to be iterated until a ball space is obtained that is closed under unions of chains. In suitable ball spaces we introduce the notion of precise balls and a notion of convex subsets and discuss the question how many iterations are needed to generate all convex sets from the precise balls. We apply this to partially ordered sets and to ultrametric spaces, using the advantage of ball spaces, as they cover both cases simultaneously.

Contents

1. Introduction

In [12, 13, 16, 2, 4], a *theory of ball spaces* is developed in order to provide a general framework for fixed point theorems that in some way or the other work with

Date: May 9, 2026.
2020 *Mathematics Subject Classification.* 06A06, 54A05.
Key words and phrases. Ball space, ultrametric space, spherical completeness, narrow poset.
The second author was partially supported by Opus grant 2017/25/B/ST1/01815 from the National Science Centre of Poland.

contractive functions. A *ball space* $(X, \mathcal{B})$ is a nonempty set X together with any nonempty collection of nonempty subsets of X. The completeness property necessary for the proof of fixed point theorems is then encoded as follows. A *chain of balls* (also called a *nest*) in $(X, \mathcal{B})$ is a nonempty subset of $\mathcal{B}$ which is linearly ordered by inclusion. A ball space $(X, \mathcal{B})$ is called *spherically complete* if every chain of balls has a nonempty intersection.

The notion "ball space" indicates that a generalization of ultrametric spaces is developed. In valuation theory, ultrametric balls play a main role and can be used to define topologies. However, even more than the properties of such topologies, the notion of spherical completeness is of central importance. A crucial class of valued fields consists of the *maximal* ones, which do not admit any nontrivial extensions with the same value group and residue field. These feature prominently in Kaplansky's seminal paper [9]. It has been shown by Prieß-Crampe and others that a valued field is maximal if and only if its underlying ultrametric space, the ball space consisting of all so-called closed ultrametric balls, is spherically complete. This has been used to prove ultrametric fixed point theorems (see for example the articles of Prieß-Crampe and Ribenboim cited in [4]), and to prove various versions of Hensel's Lemma for maximal valued fields (see [18, 11]).

The definition of "spherically complete ball space" is a direct adaptation of the definition of "spherically complete ultrametric space". By dropping any assumptions what the balls in our ball spaces actually are, the important notion of spherical completeness becomes flexible enough to be applied to different sets of balls. For example, if we use the larger set of all nonempty intersections of chains of closed ultrametric balls instead of the set of all closed balls, is a spherically complete ultrametric space still spherically complete with respect to this larger set? A positive answer is given in [10] by studying, more generally, the closure of ball spaces under nonempty intersections of chains of balls.

Likewise, if in a spherically complete ultrametric space we add all *generalized ultrametric balls*, i.e., unions of chains of the already existing ultrametric balls, will the larger ball space still be spherically complete? This question will be answered in the present paper.

The flexibility of the notion of ball spaces allows us to apply the concept of spherical completeness also in settings where no ultrametric is around. This can be used to interpret and study completeness or compactness notions in such settings (see [4]). Let us give two examples for successful applications of the ball space approach in settings which are not ultrametric.

In [14], a new proof of the Caristi–Kirk Fixed Point Theorem, which holds in complete metric spaces, is given. To this end, balls (that are different from the usual metric balls) are defined using the condition of the theorem. Then it is shown that the so obtained ball space is spherically complete if the metric space is complete. In [3], the same approach is used to prove, for complete metric spaces, the Oettli–Théra Fixed Point Theorem as well as related results such as Ekeland's variational principle, Takahashi's theorem and the flower petal theorem.

In [19], Saharon Shelah introduces symmetrically complete ordered fields, which means that for every Dedekind cut in them the cofinality of the lower cut set is different from the coinitially of the upper cut set. He proves that every ordered field can be embedded in a symmetrically complete ordered field. In [15], his definition is extended to ordered sets and ordered abelian groups. Taking the balls to be the nonempty closed bounded intervals, it is then shown that the corresonding ball space is spherically complete if and only if symmetrical completeness holds. This is used to generalize Banach's Fixed Point Theorem to symmetrically complete ordered fields, replacing the usual metric of the reals by a distance function that is derived from the ordering. Further, a characterization and construction of symmetrically complete ordered fields, ordered abelian groups and ordered sets is given. The ball spaces of all closed bounded intervals in symmetrically complete ordered fields (see Example 4.7) can not only serve as interesting examples, but also as a motivation for our present investigations. For instance, it is an interesting question how often the process of taking unions over increasing chains of already generated convex subsets has to be iterated to generate *all* convex subsets (see Section 6).

After having studied intersections of chains of balls in [10], in the present paper we will investigate the unions of such chains. Let us describe a main application we have in mind. The flexibility of the notion of ball spaces allows to treat ultrametric spaces and partially ordered sets simultaneously. In both of them, there is a natural notion of *convex subset*. We will generalize this to suitable ball spaces in which we can also introduce the notion of *precise ball*. Then the convex subsets can be generated by recursively closing the set of precise balls under unions of chains, and the question arises how often this procedure has to be repeated. This leads to the notion of the *chain union rank* of a ball space, which we will study in Section 6. Let us now introduce the basic notions we will need, and some of our main results.

We say that a ball space $(X,\mathcal{B})$ is *chain union closed* if the union of every chain in $\mathcal{B}$ is a member of $\mathcal{B}$. We define $\mathrm{cu}(\mathcal{B})$ to be the family of all sets of the form $\bigcup\mathcal{C}$, where $\mathcal{C}\subseteq\mathcal{B}$ is a chain (recall that, by default, chains of sets are supposed to be nonempty). More formally,

$$\mathrm{cu}(\mathcal{B})=\left\{\bigcup\mathcal{C}\mid\emptyset\neq\mathcal{C}\subseteq\mathcal{B},\ \mathcal{C}\text{ is a chain}\right\}.$$

Hence a ball space $(X,\mathcal{B})$ is chain union closed if and only if $\mathrm{cu}(\mathcal{B})=\mathcal{B}$. In the present paper, we study the process of obtaining a chain union closed ball space from a given ball space and the question under which conditions the spherical completeness of $(X,\mathcal{B})$ implies the spherical completeness of $(X,\mathrm{cu}(\mathcal{B}))$.

Given two ball spaces $(X,\mathcal{B})$ and $(X,\mathcal{B}')$ on the same set X, we call $(X,\mathcal{B}')$ an *expansion* of $(X,\mathcal{B})$ if $\mathcal{B}\subseteq\mathcal{B}'$. Hence $(X,\mathrm{cu}(\mathcal{B}))$ is an expansion of $(X,\mathcal{B})$. The reader should note that if $\mathcal{B}\subsetneq\mathcal{B}'$, then spherical completeness of $(X,\mathcal{B})$ does not always imply spherical completeness of $(X,\mathcal{B}')$ since there are simply more chains in the latter; see Proposition 3.1. On the other hand, the reverse implication always holds. We note that if $(X,\mathrm{cu}(\mathcal{B}))$ is not spherically complete, then no chain union closed expansion of $\mathcal{B}$ will be spherically complete as it must contain $\mathrm{cu}(\mathcal{B})$.

A ball space $(X,\mathcal{B})$ is said to be *chain union stable* if $(X,\mathrm{cu}(\mathcal{B}))$ is chain union closed, or in other words, for every chain $\mathcal{D}\subseteq \mathrm{cu}(\mathcal{B})$ there exists a chain $\mathcal{C}\subseteq\mathcal{B}$ with $\bigcup\mathcal{C}=\bigcup\mathcal{D}$. Clearly, every chain union closed space is chain union stable. Furthermore, $(X,\mathcal{B})$ is chain union stable if and only if $(X,\mathrm{cu}(\mathcal{B}))$ is chain union closed.

Example 1.1. Let $\mathcal{B}$ be the family of all finite nonempty subsets of a fixed set $X\neq\emptyset$. Then $\mathrm{cu}(\mathcal{B})$ is the family of all nonempty countable subsets of X. Note that $(X,\mathcal{B})$ is chain union stable if and only if X is countable. $\diamondsuit$

The main inspiration for these definitions and questions is taken from the theory of ultrametric spaces and their ultrametric balls. An *ultrametric* u on a set X is a function from $X\times X$ to a partially ordered set Γ with smallest element $\perp$, such that for all $x,y,z\in X$ and all $\gamma\in\Gamma$,

(U1) $u(x,y)=\perp$ if and only if $x=y$,
(U2) if $u(x,y)\leq\gamma$ and $u(y,z)\leq\gamma$, then $u(x,z)\leq\gamma$,
(U3) $u(x,y)=u(y,x)$ (symmetry).

Condition (U2) is the ultrametric triangle law; if Γ is linearly ordered, it can be replaced by

(UT) $u(x,z)\leq\max\{u(x,y),u(y,z)\}$.

When dealing with such ultrametric spaces, we can say that a set A has *diameter* $\leq\gamma$ if $u(x,y)\leq\gamma$ for every $x,y\in A$. On the other hand, the diameter of A may not be defined, unless the value set Γ is a complete meet semilattice. An *ultrametric ball* is a set $B_\alpha(x):=\{y\in X\mid u(x,y)\leq\alpha\}$, where $x\in X$ and $\alpha\in\Gamma$. The problem with general ultrametric spaces is that closed ultrametric balls $B_\alpha(x)$ are not necessarily precise, that is, there may not be any $y\in X$ such that $u(x,y)=\alpha$. Therefore, we prefer to work only with *precise ultrametric balls*

$$B(x,y):=\{z\in X\mid u(x,z)\leq u(x,y)\},$$

where $x,y\in X$. Note that a precise ultrametric ball $B(x,y)$ has diameter precisely $u(x,y)$. We leave it to the reader to prove:

Lemma 1.2. *Take an ultrametric space (X,u) and $x,y,x',y'\in X$. If $x,y\in B(x',y')$, then $u(x,y)\leq u(x',y')$ and $B(x,y)\subseteq B(x',y')$.*

We obtain the *ultrametric ball space* $(X,\mathcal{B}_u)$ from (X,u) by taking $\mathcal{B}_u$ to be the set of all precise ultrametric balls $B(x,y)$. Specifically, $\mathcal{B}_u:=\{B(x,y)\mid x,y\in X\}$.

More generally, an *ultrametric ball* (also called *generalized ultrametric ball*) is a set

$$B_S(x):=\{y\in X\mid u(x,y)\in S\},$$

where $x\in X$ and S is an initial segment of Γ. We call X together with the collection of all ultrametric balls the *full ultrametric ball space* of (X,u). In particular, this now also contains the *open ultrametric balls* $B^\circ_\alpha(x):=\{y\in X\mid u(x,y)<\alpha\}$, where

$x \in X$ and $\alpha \in \Gamma$ is not its smallest element. Every ultrametric ball can be written as the union over a chain of precise ultrametric balls:

$$B_S(x) \;=\; \bigcup\{B(x,y) \mid u(x,y) \in S\}\;.$$

Hence the full ultrametric ball space is just $(X, \mathrm{cu}(\mathcal{B}_u))$.

Typically, ultrametric spaces are considered with a linearly ordered value set, in which case the ball structure is a tree, in the sense that given a ball B, the set $\{C \supseteq B \mid C \text{ is a ball }\}$ is a chain. This is not true when the distance set is partially ordered, however we still have the following easy and well-known weaker fact.

Proposition 1.3. *Let (X,u) be an ultrametric space and let B_0, B_1 be ultrametric balls with nonempty intersection and the same diameter. Then $B_0 = B_1$.*

We will exploit the fact that classical ultrametric spaces are tree-like. A ball space $(X,\mathcal{B})$ is called *tree-like* if for every $B_1, B_2 \in \mathcal{B}$ the following implication holds.

$$\text{(I)} \qquad B_1 \cap B_2 \neq \emptyset \implies B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1.$$

See [10] for some remarks on tree-like ball spaces.

We now formulate some main results.

Theorem 1.4. *Let $(X,\mathcal{B})$ be a tree-like ball space. Then $\mathrm{cu}(\mathcal{B})$ is tree-like and the following assertions hold:*

(1) *The ball space $(X,\mathcal{B})$ is chain union stable.*
(2) *If $(X,\mathcal{B})$ is spherically complete, then so is $(X,\mathrm{cu}(\mathcal{B}))$.*

We will prove Theorem 1.4 in Section 7.

Corollary 1.5. *The assertions of Theorem 1.4 hold for the ball space $(X,\mathcal{B}_u)$ of every ultrametric space (X,u) with linearly ordered value set.*

This result is a consequence of Theorem 1.4 and the fact that ultrametric spaces with linearly ordered value set are tree-like.

The next result will be proven by a simple example in the next section. It shows that in general, we cannot expect the existence of chain union closed expansions which preserve spherical completeness:

Theorem 1.6. *There exists a countable spherically complete ultrametric space with a countable partially ordered value set, whose ultrametric ball space does not admit any expansion that is chain union closed and spherically complete.*

So we cannot hope for extending Theorem 1.5(2) to partially ordered value sets. However, it turns out that part (1) of this theorem holds in a more general setting. We say that a partially ordered set is *ω_1-free* if it does not contain any uncountable strictly increasing sequence.

Theorem 1.7. *Assume (X,u) is an ultrametric space with an ω_1-free partially ordered value set. Then its ball space $(X,\mathcal{B}_u)$ is chain union stable.*

We shall deduce this result in Section 5 from a more abstract statement involving the concept of chain regularity, inspired by a similar notion in category theory. Example 4.3 will show that the condition "countable" is necessary.

2. Precise balls and convexity

Let us generalize the notion of *precise ultrametric balls* and introduce a corresponding notion of convexity. If $(X, \mathcal{B})$ is a ball space, then we call a ball $B \in \mathcal{B}$ *precise* if there are $x, y \in B$ such that $B \subseteq B'$ for every $B' \in \mathcal{B}$ for which $x, y \in B'$. In this case, we write $B = B(x, y)$. Note that for every $x, y \in X$ there is at most one precise ball containing x and y in $\mathcal{B}$. The precise ultrametric balls we have already defined are precise in every full ultrametric ball space.

Take a partially ordered set $(X, \leq)$. Then the closed bounded intervals $[x, y] = \{z \in X \mid a \leq z \leq y\}$ where $x, y \in X$ with $x \leq y$ are exactly all precise balls in the ball space consisting of all convex subsets of X. We can write $B(x, y) = B(y, x) = [x, y]$ if $x \leq y$, and $B(x, y) = B(y, x) = [y, x]$ otherwise. We call $(X, \mathcal{B}_{\leq})$, where

$$\mathcal{B}_{\leq} := \{[x, y] \mid x, y \in X \text{ with } x \leq y\} ,$$

the *interval ball space* of $(X, \leq)$. Note that in this case, $B(x, y) = B(x', y')$ only holds if $x = x'$ and $y = y'$. This is in general not true in the case of ultrametric balls.

The ball spaces $(X, \mathcal{B}_{\leq})$ are in general not tree-like, but they have the following property: if two balls in $\mathcal{B}_{\leq}$ have nonempty intersection, then their intersection as well as their union are again elements of $\mathcal{B}_{\leq}$.

Which ball spaces $(X, \mathcal{B})$ contain precise balls $B(x, y)$ for all $x, y \in X$? Here is a criterion.

Proposition 2.1. *Assume that for $x, y \in X$ the ball space $(X, \mathcal{B})$ contains at least one ball that contains both x and y, and that the intersection over all such balls is again a ball. Then $\mathcal{B}$ contains a precise ball $B(x, y)$.*

The condition about the intersections is always satisfied if $\mathcal{B}$ is closed under intersections of centered systems of balls in $\mathcal{B}$ (that is, $(X, \mathcal{B})$ is an $\mathbf{S}_5^d$ ball space in the sense of [4]). A *centered system of balls* is a nonempty collection of balls such that the intersection of any finite number of balls in the collection is nonempty.

In a ball space $(X, \mathcal{B})$ that contains precise balls $B(x, y)$ for all $x, y \in X$ we can define a notion of convexity as follows. A subset $S \subset X$ is $\mathcal{B}$-convex if $\mathcal{B}(x, y) \subset S$ for all $x, y \in S$. In particular, all precise balls are $\mathcal{B}$-convex.

If $(X, \leq)$ is a partially ordered set, then a subset X is convex in the usual sense if and only if it is $\mathcal{B}_{\leq}$-convex.

Now the question arises: if the ball space $(X, \mathcal{B})$ contains precise balls $B(x, y)$ for all $x, y \in X$, how can we generate an expansion that consists of all $\mathcal{B}$-convex subsets of X? Every union over an increasing chain of $\mathcal{B}$-convex sets is again a $\mathcal{B}$-convex set. However, as we will see later, closing under such unions just once may not produce all $\mathcal{B}$-convex sets from the precise balls. Then we have to iterate the process to arrive at what we call the *chain union closure*, i.e., a smallest expansion $\mathcal{B}'$ of $\mathcal{B}$ such that $\mathcal{B}'$ is chain union closed. This process and the number of iterations needed to arrive at the chain union closure shall be studied in detail below for arbitrary ball spaces. For cardinality reasons, chain union closures exist for all ball spaces, and they are uniquely determined. However, if we start with a ball space consisting of precise balls, it is not a priori clear whether all convex sets lie in the chain union closure.

Theorem 2.2. *Take a tree-like ball space $(X,\mathcal{B})$ consisting of precise balls $B(x,y)$ for all $x,y\in X$. Then its chain union closure consists exactly of all $\mathcal{B}$-convex sets in X.*

Proof. We know already that all sets in the chain union closure are $\mathcal{B}$-convex. Take a $\mathcal{B}$-convex subset S of X and choose some $x\in S$. Then $B(x,y)\subseteq S$ for all $y\in S$, hence

$$S \;=\; \bigcup_{y\in S}\{y\} \;\subseteq\; \bigcup_{y\in S} B(x,y) \;\subseteq\; S\,,$$

so equality holds everywhere. Since $(X,\mathcal{B})$ is tree-like, $\{B(x,y)\mid y\in S\}$ is a chain. Consequently, its union S lies in the chain union closure. □

Theorem 2.3. *Take a totally ordered set X. Then $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))$ is the chain union closure of $\mathcal{B}_\le$, and it consists exactly of all $\mathcal{B}_\le$-convex sets in X.*

Proof. Again, we know that all sets in the chain union closure are $\mathcal{B}_\le$-convex. Take a $\mathcal{B}_\le$-convex subset S of X and choose some $x\in S$. Then $[x,y]=B(x,y)\subseteq S$ for all $y\in S$ with $x\le y$, hence $\{[x,y]\mid y\in S, x\le y\}$ is a chain. Therefore,

$$S_x \;:=\; \{y\in S\mid x\le y\} \;=\; \bigcup_{y\in S,\,x\le y} [x,y] \;\in\; \mathrm{cu}(\mathcal{B}_\le)\,.$$

Observe that $\{S_x\mid x\in S\}$ is a chain in $\mathrm{cu}(\mathcal{B}_\le)$. Consequently,

$$S \;=\; \bigcup_{x\in S} S_x \;\in\; \mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))\,.$$

The union over each chain in $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))$ is again convex, so it already lies in $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))$. This shows that $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))$ is chain union closed. □

Note that it can happen that $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\le))=\mathrm{cu}(\mathcal{B}_\le)$, for instance, if $X=\mathbb{N}$.

3. Proof of Theorem 1.6

Proposition 3.1. *Take a partially ordered set $(X,\le)$ and assume that X contains a nonempty linearly ordered subset S without a largest element. Then $(X,\mathrm{cu}(\mathcal{B}_\le))$ is not spherically complete. This is in particular the case if X is an ordered field or a nontrivial ordered abelian group.*

Proof. For each $x\in S$, define

$$S_x := \bigcup\{[x,z]\mid z\in S \text{ with } x\le z\}\in \mathrm{cu}(\mathcal{B}_\le)\,.$$

If $x,y\in S$ with $x\le y$, then $S_y\subseteq S_x$, so $\{S_x\mid x\in S\}$ is a chain in $\mathrm{cu}(\mathcal{B}_\le)$. Since

$$\bigcap_{x\in S} S_x=\emptyset\,,$$

we see that $(X,\mathrm{cu}(\mathcal{B}_\le))$ is not spherically complete. □

Proposition 3.2. *Take an ultrametric space* (X, u) *or a partial order* $(X, \leq)$*, and consider the precise balls* $B(x, y)$ *in either of them. Then* $\Gamma := \{B(x, y) \mid x, y \in X\} \cup \{\emptyset\}$ *with set inclusion is a partially ordered set with smallest element* $\emptyset$*, and the function* $u^* : X \times X \mapsto \Gamma$ *defined by*

$$u^*(x, y) := B(x, y) \text{ if } x \neq y, \text{ and } u^*(x, x) := \emptyset \text{ for } x, y \in X$$

is an ultrametric on X *with value set* Γ*. For all* $x, y \in X$*, the precise ball* $B_{u^*}(x, y)$ *with respect to* u^* *is equal to* $B(x, y)$*.*

Proof. Properties (U1) and (U3) are clearly satisfied. To prove (U2), take $x, y, z \in X$ and $\gamma \in \Gamma$. The case of $\gamma = \emptyset$ is straightforward.

In order to treat the case of $\gamma \neq \emptyset$, let us assume first that we are dealing with an ultrametric space (X, u). Then we can write $\gamma = B(x', y')$ with $x', y' \in X$, and we assume that $u^*(x, y) \leq \gamma$ and $u^*(y, z) \leq \gamma$. That is, $B(x, y) \subseteq B(x', y')$ and $B(y, z) \subseteq B(x', y')$, which implies that $x, y, z \in B(x', y')$. By Lemma 1.2, it follows that $B(x, z) \subseteq B(x', y')$, which means that $u^*(x, z) \leq \gamma$. Further,

$$\begin{aligned} B_{u^*}(x, y) &= \{z \in X \mid u^*(x, z) \leq u^*(x, y)\} = \{z \in X \mid B(x, z) \subseteq B(x, y)\} \\ &= \{z \in X \mid u(x, z) \leq u(x, y)\} = B(x, y) \, . \end{aligned}$$

For the third equality note that $u(x, z) \leq u(x, y)$ implies that $z \in B(x, y)$ and then $B(x, z) \subseteq B(x, y)$ by Lemma 1.2.

Now assume that we are dealing with a partial order $(X, \leq)$, so we can write $\gamma = [x', y']$ with $x', y' \in X$ and $x' \leq y'$. We assume that $u^*(x, y) \leq \gamma$ and $u^*(y, z) \leq \gamma$, and in view of (U3) we may assume without loss of generality that $x \leq y$.

Let us first assume that $y \leq z$. Then we have that $[x, y] \subseteq [x', y']$ and $[y, z] \subseteq [x', y']$, whence $[x, z] \subseteq [x', y']$. This again shows that $u^*(x, z) \leq \gamma$.

Now we assume that $z \leq y$. Then we have that $[x, y] \subseteq [x', y']$ and $[z, y] \subseteq [x', y']$, whence $[x, z] \subseteq [x', y']$ if $x \leq z$, and $[z, x] \subseteq [x', y']$ if $z \leq x$. This again shows that $u^*(x, z) \leq \gamma$.

Further,

$$\begin{aligned} B_{u^*}(x, y) &= \{z \in X \mid u^*(x, z) \leq u^*(x, y)\} \\ &= \{z \in X \mid [x, z] \subseteq [x, y] \vee [z, x] \subseteq [x, y]\} = [x, y] \, . \end{aligned}$$

In both cases it follows directly from the definition of u^* that its value set is Γ. □

Using these results, we give an example that proves Theorem 1.6.

Example 3.3. Take $\mathcal{B}_\leq$ to be the interval ball space of $(\mathbb{N}, \leq)$. By Proposition 3.1, $(\mathbb{N}, \mathrm{cu}(\mathcal{B}_\leq))$ is not sperically complete. Any chain union closed expansion of $\mathcal{B}_\leq$ must contain $\mathrm{cu}(\mathcal{B}_\leq))$ and thus is also not sperically complete.

Using Proposition 3.2, we can turn $(\mathbb{N}, \leq)$ into an ultrametric ball space $(\mathbb{N}, \mathcal{B}_{u^*})$ whose balls are exactly the balls in $\mathcal{B}_\leq$. Hence from what we have already proved, the ultrametric ball space $(\mathbb{N}, \mathcal{B}_{u^*})$ does not admit any expansion that is chain union closed and spherically complete. ◇

4. Some examples concerning chain union stability

In the following we shall exhibit problems posed by uncountable cardinalities. We start with a counterpart to Proposition 3.1.

Proposition 4.1. *Take a partially ordered set $(X, \leq)$ and assume that X contains a nonempty linearly ordered subset S whose initiality and cofinality are distinct infinite cardinals. Then $\mathrm{cu}(\mathcal{B}_{\leq})$ is not chain union closed.*

Proof. Let κ be the initiality of S and λ its cofinality. Without loss of generality assume that $\lambda > \kappa$; the proof for the case of $\lambda > \kappa$ is similar. Take a chain $\mathcal{B}_0 \subset \mathcal{B}_{\leq}$. If the cardinality of $\mathcal{B}_0$ is smaller than λ, then the union over $\mathcal{B}_0$ will not be cofinal in S. If the cardinality of $\mathcal{B}_0$ is at least λ, then the union over $\mathcal{B}_0$ will not be coinitial in S. This shows that $(X, \mathcal{B}_{\leq})$ is not chain union closed. □

A similar problem is presented in the following examples.

Example 4.2. Let $X = \omega \times \omega_1$ and let $\mathcal{B}_1$ consist of all rectangles $[0, n) \times [0, \alpha)$ where both $n \in \omega$, $\alpha \in \omega_1$ are positive. Note that $\mathrm{cu}(\mathcal{B}_1)$ consists of all rectangles of the form $[0, \xi) \times [0, \beta)$, where either $\xi < \omega$, $\beta < \omega_1$ or $\xi = \omega$ and $\beta < \omega_1$ or $\xi < \omega$ and $\beta = \omega_1$. In particular, $X \in \mathrm{cu}(\mathrm{cu}(\mathcal{B}_1)) \setminus \mathrm{cu}(\mathcal{B}_1)$. It follows that $(X, \mathcal{B}_1)$ is not chain union stable. ♢

This example can be modified as follows.

Example 4.3. Take again $X = \omega \times \omega_1$ and now let $\mathcal{B}_2$ consist of all rectangles $[0, n] \times [0, \beta]$ with $n \in \omega$ and $\beta \in \omega_1$. Then $\mathrm{cu}(\mathcal{B}_2)$ consists of these rectangles together with all rectangles of the form $[0, n] \times [0, \beta)$ with $n \in \mathbb{N}$ and $\beta \leq \omega_1$, or $[0, \omega) \times [0, \beta)$ and $[0, \omega) \times [0, \beta]$ with $\beta < \omega_1$. Again, $X \in \mathrm{cu}(\mathrm{cu}(\mathcal{B}_2)) \setminus \mathrm{cu}(\mathcal{B}_2)$ and it follows that $(X, \mathcal{B}_2)$ is not chain union stable.

Further, $\mathcal{B}_{\leq}$ consists of all rectangles $[m, n] \times [\alpha, \beta]$ with $m, n \in \omega$, $m \leq n$, and $\alpha, \beta \in \omega_1$, $\alpha \leq \beta$. As before, it is shown that $(X, \mathcal{B}_{\leq})$ is not chain union stable. ♢

When X is an upper semilattice, we can define a natural ultrametric on X.

Proposition 4.4. *Take $(X, \leq)$ to be an upper semilattice with smallest element $\perp$. Then the function $u : X \mapsto \Gamma$ defined by*

$$u(x, y) := \sup(x, y) \ \text{ if } x \neq y, \text{ and } \ u(x, x) := \perp \ \text{ for } \ x, y \in X \tag{1}$$

is an ultrametric on X with value set Γ. For all $x, y \in X$ with $x \leq y$, the precise ball $B(x, y)$ with respect to u is eqal to $[x, y]$.

Proof. Properties (U1) and (U3) are clearly satisfied. To prove (U2), take $x, y, z \in X$ and $\gamma \in X$. The case of $\gamma = \perp$ is straightforward, hence we assume that $\gamma \neq \perp$ as well as $\sup(x, y) = u(x, y) \leq \gamma$ and $\sup(y, z) = u(y, z) \leq \gamma$. It follows that $u(x, z) = \sup(x, z) \leq \gamma$, which proves that (U2) holds.

We observe that for every $x \in X$, $x = u(\perp, x) \in u(X \times X)$, so the value set of U is X. □

Example 4.5. Let X and $\mathcal{B}_{\leq}$ be as in the previous example. By Proposition 4.4, the function u defined in (1) is an ultrametric on X with precise balls $B(x, y) = [x, y]$

for $x \leq y$. Hence it follows from Example 4.3 that this ultrametric ball space is not chain union stable. ♢

Let us build again on Example 4.3 by constructing a more advanced ultrametric space from it. Take an abelian group G. A *group valuation* v on G is a mapping from G onto a partially ordered value set Γ with smallest element $\perp$ such that for all $x, y, z \in G$ and all $\gamma \in \Gamma$,

(V1) $v(x) = \perp$ if and only if $x = 0$,
(V2) $v(x) = v(-x)$,
(V3) if $v(x) \leq \gamma$ and $v(y) \leq \gamma$, then $v(x - y) \leq \gamma$.

From this we obtain an ultrametric with value set Γ on G by setting

$$u(x, y) \ := \ v(x - y)\,. \tag{2}$$

Now take an upper semilattice L with smallest element $\perp$ and G to be the free abelian group generated by L. We write the nonzero elements of G in the form $z_1\ell_1 + \ldots + z_n\ell_n$ with $n \in \mathbb{N}$, $z_i \in \mathbb{Z}$ and $\ell_i \in L$. Then

$$v(z_1\ell_1 + \ldots + z_n\ell_n) \ := \ \sup(\ell_1, \ldots, \ell_n), \ \ v(0) \ = \ \perp\,,$$

is a group valuation on G with value set L. Note that the restriction to L of the ultrametric u defined by (2) coincides with the ultrametric defined in (1).

Example 4.6. Take L to be the lattice X from the above examples, and G to be the free abelian group generated by L. We leave it to the reader to show that the function u defined by (2) is not only an ultrametric on L, but also on G with value set L. For its precise ultrametric balls $B(g, h)$, where $g, h \in G$ with $g \neq h$, we have:

$$B(g, h) \ = \ \{h' \in G \mid u(g, h') \leq u(g, h)\}\,,$$

so $B(g, h)$ is the set of all $h' \in G$ for which $u(g, h')$ lies in $\{x \in X \mid x \leq u(g, h)\} \in \mathcal{B}_2$. Now g and h lie in the union of a chain of precise balls all containing g if and only if $u(g, h)$ lies in the union of a chain of balls in $\mathcal{B}_2$, that is, in $\mathrm{cu}(\mathcal{B}_2)$. Hence again it follows from Example 4.3 that this ultrametric ball space is not chain union stable. ♢

Let us give an interesting example using symmetrically complete ordered fields. By definition, an ordered field $(K, <)$ is *symmetrically complete* if for every Dedekind cut in K, the cofinality of the left cut set is different from the coinitiality of the right cut set.

Example 4.7. Take K to be a symmetrically complete ordered field with its spherically complete ball space $\mathcal{B}_K$ consisting of all closed bounded intervals in K. Let us compute $\mathrm{cu}(\mathcal{B}_K)$. The interval (∞, ∞), all intervals $[a, b)$ with $a \in K$, $b \in K \cup \{\infty\}$, and all intervals $(a, b]$ with $a \in K \cup \{\infty\}$, $b \in K$ are unions of chains of closed bounded intervals in K. From the fact that the cofinality of $\{c \in K \mid c < 0\}$ equals the coinitiality of $\{c \in K \mid c > 0\}$ as well as the coinitiality and cofinality of K, it follows that also all intervals (a, b) with $a \in K \cup \{\infty\}$ are unions of chains of closed bounded intervals in K. So $\mathrm{cu}\,\mathcal{B}_K$ contains all of these intervals together with the intervals in $\mathcal{B}$. Proposition 3.1 shows that $(K, \mathrm{cu}(\mathcal{B}_K))$ is not spherically complete.

All coinitialities and cofinalities in subfields K of $\mathbb{R}$ are equal to 1 or $\aleph_0$, and it follows that $\mathrm{cu}(\mathcal{B}_K)$ is precisely the set of all intervals in K. Hence for such K we have that $(K,\mathcal{B}_K)$ is chain union stable.

However, according to [16, 19] there are symmetrically complete ordered fields K that properly contain $\mathbb{R}$, and for them the situation is very different. Since $\mathbb{R}$ is the only cut complete ordered field, they must contain Dedekind cuts for which cofinality and coinitiality are distinct infinite cardinals. Take such a cut (Λ^L,Λ^R) where Λ^L is the left cut set and Λ^R is the right cut set. It gives rise to convex subsets $[a,b]\cap\Lambda^R$, where $a\in\Lambda^L$ and $b\in\Lambda^R$, $[a,b)\cap\Lambda^R$, where $a\in\Lambda^L$ and $b\in\Lambda^R\cup\{\infty\}$, $[a,b]\cap\Lambda^L$, where $a\in\Lambda^L$ and $b\in\Lambda^R$, and $(a,b]\cap\Lambda^L$, where $a\in\Lambda^L\cup\{\infty\}$ and $b\in\Lambda^R$. Further, each second Dedekind cut $(\Lambda_1^L,\Lambda_1^R)$ with $\Lambda_1^L\subsetneq\Lambda^L$ gives rise to the convex subset $\Lambda_1^R\cap\Lambda^L$. Among these convex subsets, those that have one endpoint and those for which their coinitiality equals their cofinality lie in $\mathrm{cu}(\mathcal{B}_K)$, all others do not. The latter can be shown to always exist in symmetrically complete ordered fields $K\neq\mathbb{R}$. Indeed, take $(\Lambda_1^L,\Lambda_1^R)$ to be the cut (Λ^L,Λ^R) shifted by 1 to the left. Since the cofinality of Λ^L is not equal to the coinitiality of Λ^R and the shift preserves cofinality and coinitiality, the coinitiality of $\Lambda_1^R\cap\Lambda^L$ is not equal to its cofinality. Since (Λ^L,Λ^R) and hence also $(\Lambda_1^L,\Lambda_1^R)$ are not realized, neither the cofinality nor the coinitiality of $\Lambda_1^R\cap\Lambda^L$ is equal to 1. Hence it follows from Proposition 4.1 that for those K, $\mathrm{cu}(\mathcal{B}_K)$ is not chain union closed. However, the next lemma will show that $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_K))$ is chain union closed.

We note that while $(K,\mathcal{B}_K)$ is spherically complete, Proposition 3.1 implies that $(K,\mathrm{cu}(\mathcal{B}_K))$ is not. If $\mathcal{B}$ is the ball space of all convex subsets of K, then it contains $\mathrm{cu}(\mathcal{B}_K)$ and hence it is also not spherically complete. $\diamondsuit$

Lemma 4.8. *Take a linearly ordered set X and consider the interval ball space $(X,\mathcal{B}_\leq)$. Then $\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\leq))$ is chain union closed.*

Proof. As we have remarked already, the union over every chain in $\mathrm{cu}(\mathcal{B}_\leq)$ is a $\mathcal{B}_\leq$-convex subset S of X. If $\alpha\in S$, then $S_\alpha:=\{\beta\in S\mid\beta\geq\alpha\}=\bigcup_{\beta>\alpha}[\alpha,\beta]\in\mathrm{cu}(\mathcal{B}_\leq)$. Further, $S=\bigcup_{\alpha\in S}S_\alpha\in\mathrm{cu}(\mathrm{cu}(\mathcal{B}_\leq))$. $\square$

Remark. While symmetrically complete fields have the extra property that they are spherically complete, other ordered fields with Dedekind cuts having given cofinalities are much easier to construct. Indeed, take distinct regular cardinals $\kappa_1,\dots,\kappa_n,\lambda_1,\dots,\lambda_n$ and set $I:=\kappa_1+-\lambda_1+\kappa_2+-\lambda_2+\ldots+\kappa_n+-\lambda_n$, where $-\lambda_i$ denotes the set λ_i with reverse ordering. Then take G to be the lexicographic product over copies of $\mathbb{R}$ with index set I; this is an ordered abelian group and has a group valuation v_G with value set I. Then take K to be the power series field with coefficients in $\mathbb{R}$ and exponents in G. This is a real closed field with a compatible valuation v having value group G. The preimages $v_g^{-1}(\kappa_i)$ and $v^{-1}(v_g^{-1}(\kappa_i))$ again have cofinality κ_i, and the preimages $v_g^{-1}(-\lambda_i)$ and $v^{-1}(v_g^{-1}(-\lambda_i))$ again have coinitiality λ_i. This shows that for $1\leq i\leq n$ the Dedekind cut in I which is the upper edge of κ_i lifts up to a Dedekind cut in K with lower cut set of cofinality κ_i and upper cut set of coinitiality λ_i.

For background on the construction methods we have used here, see [15].

5. Chain regularity

We say a ball space $(X, \mathcal{B})$ is *chain regular* if for every chain $\mathcal{C} \subseteq \mathcal{B}$, for every ball $B \subseteq \bigcup \mathcal{C}$ there exists $C \in \mathcal{C}$ with $B \subseteq C$. This notion is well justified by the following:

Proposition 5.1. *Every ultrametric ball space (with a partially ordered value set) is chain regular.*

Proof. Assume $B \subseteq \bigcup \mathcal{C}$, where $\mathcal{C}$ is a chain of precise ultrametric balls. Take $a, b \in B$ such that $B = B(a, b)$ and choose $C \in \mathcal{C}$ such that $a, b \in C$. Then the diameter of $u(a, b)$ is greater than or equal to the diameter of B, therefore $B \subseteq C$. □

Lemma 5.2. *Assume $(X, \mathcal{B})$ to be a chain regular ball space such that all chains in $\mathrm{cu}(\mathcal{B})$ have countable cofinality. Then $(X, \mathcal{B})$ is chain union stable.*

Proof. Let $\mathcal{C} = \{C_n\}_{n\in\omega}$ be a chain in $\mathrm{cu}(\mathcal{B})$. For each $n \in \omega$ choose a chain $\mathcal{A}_n \subseteq \mathcal{B}$ whose union is C_n. By our assumptions, each $\mathcal{A}_n$ has countable cofinality.

Using standard induction together with chain regularity, construct a matrix of balls $\{B_{i,j}\}_{i,j<\omega}$ such that

(1) $B_{i,j} \in \mathcal{A}_i$ and $\bigcup_{n\in\omega} B_{i,n} = C_i$,
(2) $B_{i,j} \subseteq B_{i+1,j}$,

for every $i, j < \omega$. Finally, the union of the diagonal $\{B_{n,n}\}_{n\in\omega}$ is $\bigcup_{n\in\omega} C_n$. □

We shall need the following simple result concerning partially ordered sets.

Lemma 5.3. *Let $(P, \leq)$ be a partially ordered set, $Q \subseteq P$ and C a chain in P such that each element of C is the supremum of a chain in Q. Let κ be an uncountable regular cardinal such that $|Q| < \kappa$. Then C does not contain a copy of $(\kappa, \in)$.*

Proof. Assume $\varphi\colon \kappa \to C$ preserves $\leq$. We are going to show that φ is eventually constant. Refining[1] Q, we may assume that for each $q \in Q$ there is $\alpha_q < \kappa$ with $q \leq \varphi(\alpha_q)$. Let $\delta = \sup\{\alpha_q \mid q \in Q\}$. Then $\delta < \kappa$, because κ is regular. Furthermore $\varphi(\delta)$ is an upper bound of the whole set Q, therefore $\varphi(\beta) = \varphi(\delta)$ for every $\beta > \delta$. □

Proof of Theorem 1.7. We can apply Lemma 5.2, as long as we prove that all chains in $\mathrm{cu}(\mathcal{B}_u)$ have countable cofinality.

Assume $\mathcal{C} = \{C_\alpha\}_{\alpha<\omega_1}$ is an ω_1-chain in $\mathrm{cu}(\mathcal{B}_u)$, i.e., $C_\alpha \subseteq C_\beta$ whenever $\alpha < \beta$. Choose $z \in C_0$. Each C_α is the union of a chain of balls $\{B_{\alpha,n}\}_{n\in\omega} \subseteq \mathcal{B}_u$ and we may assume that $z \in B_{\alpha,0}$ for every $\alpha < \omega_1$. We claim that the family $\mathcal{F} = \{B_{\alpha,n} \mid \alpha < \omega_1, n \in \omega\}$ is countable. Then Lemma 5.3 gives $C_\beta = C_\delta$ for all big enough $\beta, \delta < \omega_1$.

Given $\alpha, \alpha' < \omega_1$, $n, n' < \omega$, note that $B_{\alpha,n} = B_{\alpha',n'}$ whenever $\mathrm{diam}(B_{\alpha,n}) = \mathrm{diam}(B_{\alpha',n'})$ (Proposition 1.3). Since the value set of (X, u) is countable, this shows that $\mathcal{F}$ is countable, which completes the proof. □

[1] By *refining* a set we mean removing "unnecessary" or "irrelevant" elements.

6. Chain union and chain intersection closures

Let $\mathcal{B}$ be a nonempty family of nonempty sets. Using transfinite recursion, we define $\mathrm{cu}_\alpha(\mathcal{B})$ and $\mathrm{ci}_\alpha(\mathcal{B})$ for each ordinal α, as follows:

$$\mathrm{cu}_0(\mathcal{B}) \;=\; \mathcal{B}, \qquad \mathrm{cu}_\alpha(\mathcal{B}) \;=\; \mathrm{cu}\left(\bigcup_{\xi<\alpha} \mathrm{cu}_\xi(\mathcal{B})\right) \text{ for } \alpha > 0\,,$$

$$\mathrm{ci}_0(\mathcal{B}) \;=\; \mathcal{B}, \qquad \mathrm{ci}_\alpha(\mathcal{B}) = \mathrm{ci}\left(\bigcup_{\xi<\alpha} \mathrm{ci}_\xi(\mathcal{B})\right) \text{ for } \alpha > 0\,,$$

where $\mathrm{ci}(\mathcal{B})$ is defined in [10] as

$$\mathrm{ci}(\mathcal{B}) = \left\{\bigcap \mathcal{C} \mid \emptyset \neq \mathcal{C} \subseteq \mathcal{B},\ \mathcal{C} \text{ is a chain}\right\} \setminus \{\emptyset\}\,.$$

We have $\mathrm{cu}(\mathcal{B}) = \mathrm{cu}_1(\mathcal{B})$ and $\mathrm{ci}(\mathcal{B}) = \mathrm{ci}_1(\mathcal{B})$. We observe that

(3) $$\mathcal{B} \subseteq \mathcal{B}' \;\Rightarrow\; \mathrm{ci}_\alpha(\mathcal{B}) \subseteq \mathrm{ci}_\alpha(\mathcal{B}') \text{ and } \mathrm{cu}_\alpha(\mathcal{B}) \subseteq \mathrm{cu}_\alpha(\mathcal{B}') \text{ for all } \alpha\,.$$

We define the *chain union rank* of $\mathcal{B}$, denoted by $\mathrm{cur}(\mathcal{B})$, to be the smallest ordinal α such that $\mathrm{cu}_{\alpha+1}(\mathcal{B}) = \mathrm{cu}_\alpha(\mathcal{B})$. Thus, $\mathrm{cur}(\mathcal{B}) = 0$ if and only if $\mathcal{B}$ is chain union closed, while $\mathrm{cur}(\mathcal{B}) \leq 1$ means that in order to make $\mathcal{B}$ chain union closed, it suffices to expand it by adding all unions of chains. Note that $(X, \mathrm{cu}_\alpha(\mathcal{B}))$, with $\alpha = \mathrm{cur}(\mathcal{B})$, is the chain union closure of $(X, \mathcal{B})$. It could also be described as a ball space $(X, \mathcal{B}')$, where $\mathcal{B}' \supseteq \mathcal{B}$ is minimal such that $\mathcal{B}'$ is stable under unions of chains.

Recall from the introduction that a ball space $(X, \mathcal{B})$ is *chain union stable* if $(X, \mathrm{cu}(\mathcal{B}))$ is chain union closed. Obviously, this is equivalent to $\mathrm{cur}(X, \mathcal{B}) \leq 1$. In [10], the notions of *chain intersection rank*, denoted by $\mathrm{cir}(\mathcal{B})$, of *chain intersection closure* and of *chain intersection stable* were defined analogously, just with "cu" replaced by "ci". Note that both cur and cir are well-defined, because they are bonded by the cardinality of the powerset of the set X.

If $(X, \mathcal{B})$ is a ball space, then also $(X, \mathrm{cpl}\mathcal{B})$ with

$$\mathrm{cpl}\mathcal{B} \;:=\; \{X \setminus B \mid B \in \mathcal{B}, B \neq X\}$$

is a ball space; we will call it the *complement ball space* of $(X, \mathcal{B})$. Note that $\mathrm{cpl}\mathcal{B} = \mathrm{cpl}(\mathcal{B} \setminus \{X\})$. We collect a number of useful properties of chain unions and chain intersections.

Lemma 6.1. *Take a ball space* $(X, \mathcal{B})$*.*
1) If $\mathcal{S} \subseteq \mathcal{B}$ *is a finite set and* $\mathcal{B} \setminus \mathcal{S} \neq \emptyset$*, then*
i) $\mathrm{ci}_\alpha(\mathcal{B}) = \mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S}$ *and* $\mathrm{cu}_\alpha(\mathcal{B}) = \mathrm{cu}_\alpha(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S}$ *for all* α*,*
ii) $\mathrm{cir}(\mathcal{B}) \leq \mathrm{cir}(\mathcal{B} \setminus \mathcal{S})$ *and* $\mathrm{cur}(\mathcal{B}) \leq \mathrm{cur}(\mathcal{B} \setminus \mathcal{S})$*.*

2) If $X \in \mathcal{B}$ *and* $\mathcal{B} \setminus \{X\} \neq \emptyset$*, then* $\mathrm{ci}_\alpha(\mathcal{B}) \setminus \mathrm{ci}_\alpha(\mathcal{B} \setminus \{X\}) = \{X\}$ *for all* α *and* $\mathrm{cir}(\mathcal{B}) = \mathrm{cir}(\mathcal{B} \setminus \{X\})$*.*

3) For all α*,* $(X, \mathrm{ci}_\alpha(\mathrm{cpl}\mathcal{B}))$ *is the complement ball space of* $(X, \mathrm{cu}_\alpha(\mathcal{B}))$*.*

4) We have that

$$\mathrm{cur}(\mathcal{B}) \;\geq\; \mathrm{cir}(\mathrm{cpl}\mathcal{B})\,.$$

If $\mathrm{cir}(\mathrm{cpl}\mathcal{B}) = \beta$ *and* $X \in \mathrm{cu}_\beta(\mathcal{B})$*, then* $\mathrm{cur}(\mathcal{B}) = \mathrm{cir}(\mathrm{cpl}\mathcal{B})$*.*

Proof. 1): We treat the case of chain intersections; the case of chain unions is analogous. We have that $\mathrm{ci}(\mathcal{B}) = \mathrm{ci}(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S}$ as all members of $\mathcal{S}$ can be removed from any infinite chain without changing the intersection. This implies assertion i) for $\alpha = 1$ in the case of chain intersections.

Now we proceed by induction on α. Assume that assertion i) holds for ci and ci_α. Then

$$\begin{aligned}\mathrm{ci}_{\alpha+1}(\mathcal{B}) &= \mathrm{ci}(\mathrm{ci}_\alpha(\mathcal{B})) = \mathrm{ci}(\mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S}) = \mathrm{ci}((\mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S}) \setminus \mathcal{S}) \cup \mathcal{S} \\ &= \mathrm{ci}(\mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S})) \cup \mathcal{S} = \mathrm{ci}_{\alpha+1}(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S},\end{aligned}$$

where we have used our assertion for ci_α for the second equality, and our assertion for ci for the third equality, with $\mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S})$ in place of $\mathcal{B}$. This proves the successor case of the induction. The limit case is straightforward.

In order to prove assertion ii), assume that $\mathrm{cir}(\mathcal{B} \setminus \mathcal{S}) = \alpha$, that is, $\mathrm{ci}_{\alpha+1}(\mathcal{B} \setminus \mathcal{S})) = \mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S})$. Then by assertion i),

$$\mathrm{ci}_{\alpha+1}(\mathcal{B}) = \mathrm{ci}_{\alpha+1}(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S} = \mathrm{ci}_\alpha(\mathcal{B} \setminus \mathcal{S}) \cup \mathcal{S} = \mathrm{ci}_\alpha(\mathcal{B}) .$$

This proves that $\mathrm{cir}(\mathcal{B}) \leq \alpha = \mathrm{cir}(\mathcal{B} \setminus \mathcal{S})$.

2): Since the only chain that has X as its intersection is $\{X\}$, no ball space $\mathcal{B}$ satisfies $X \in \mathrm{ci}(\mathcal{B} \setminus \{X\})$. By induction, $X \notin \mathrm{ci}_\alpha(\mathcal{B} \setminus \{X\})$ for all α. Hence if $X \in \mathcal{B}$ and $\mathcal{B} \setminus \{X\} \neq \emptyset$, then $X \in \mathrm{ci}_\alpha(\mathcal{B}) \setminus \mathrm{ci}_\alpha(\mathcal{B} \setminus \{X\})$, and it follows from assertion i) of part 1) that $\mathrm{ci}_\alpha(\mathcal{B}) \setminus \mathrm{ci}_\alpha(\mathcal{B} \setminus \{X\}) = \{X\}$.

From assertion ii) of part 1) we know that $\mathrm{cir}(\mathcal{B}) \leq \mathrm{cir}(\mathcal{B} \setminus \{X\})$; we have to show that also "$\geq$" holds. Assume that $\mathrm{ci}_{\alpha+1}(\mathcal{B}) = \mathrm{ci}_\alpha(\mathcal{B})$. Then by what we have just proved,

$$\mathrm{ci}_{\alpha+1}(\mathcal{B} \setminus \{X\}) = \mathrm{ci}_{\alpha+1}(\mathcal{B}) \setminus \{X\} = \mathrm{ci}_\alpha(\mathcal{B}) \setminus \{X\} = \mathrm{ci}_\alpha(\mathcal{B} \setminus \{X\}) .$$

This proves the desired inequality and thus the second assertion of part 2).

3): The assertion is proven by induction on α using the fact that the complement of the union of a chain $\{B_i\}_{i\in I}$ is the intersection of the chain $\{X \setminus B_i\}_{i\in I}$.

4): Let $\mathrm{cur}(\mathcal{B}) = \alpha$, that is, $\mathrm{cu}(\mathrm{cu}_\alpha(\mathcal{B})) = \mathrm{cu}_\alpha(\mathcal{B})$. Pick a chain $\mathcal{C}$ in $\mathrm{ci}_\alpha(\mathrm{cpl}\mathcal{B})$ such that $\bigcap \mathcal{C} \neq \emptyset$. By part 3), $\{X \setminus B \mid B \in \mathcal{C}\}$ is a subset of $\mathrm{cu}_\alpha(\mathcal{B})$, and it is also a chain. By assumption, $B' := \bigcup\{X \setminus B \mid B \in \mathcal{C}\} \in \mathrm{cu}_\alpha(\mathcal{B})$. Since $\bigcap \mathcal{C} \neq \emptyset$, we have that $B' \neq X$. Using part 3) again, $\bigcap \mathcal{C} = X \setminus B' \in \mathrm{ci}_\alpha(\mathrm{cpl}\mathcal{B})$. We have proved that $\mathrm{ci}_\alpha(\mathrm{cpl}\mathcal{B})$ is chain intersection closed, which shows that $\mathrm{cir}(\mathrm{cpl}\mathcal{B}) \leq \alpha$. Hence our first assertion holds.

Now assume that $\mathrm{cir}(\mathrm{cpl}\mathcal{B}) = \beta$ and that $X \in \mathrm{cu}_\beta(\mathcal{B})$. By what we have proved before, it suffices to show that $\mathrm{cur}(\mathcal{B}) \leq \beta$. Pick a chain $\mathcal{C}$ in $\mathrm{cu}_\beta(\mathcal{B})$; we wish to show that $\bigcup \mathcal{C} \in \mathrm{cu}_\beta(\mathcal{B})$. As $X \in \mathrm{cu}_\beta(\mathcal{B})$, we may assume that $\bigcup \mathcal{C} \neq X$, so that $B' := \bigcap\{X \setminus B \mid B \in \mathcal{C}\} \neq \emptyset$. By part 3), $\{X \setminus B \mid B \in \mathcal{C}\}$ is a subset of $\mathrm{ci}_\beta(\mathrm{cpl}\mathcal{B})$, and it is also a chain. Since $\mathrm{cir}(\mathrm{cpl}\mathcal{B}) = \beta$, we find that $B' \in \mathrm{ci}_\beta(\mathrm{cpl}\mathcal{B})$. Using part 3) again, $\bigcup \mathcal{C} = X \setminus B' \in \mathrm{cu}_\beta(\mathcal{B})$. This shows that $\mathrm{cur}(\mathcal{B}) \leq \beta = \mathrm{cir}(\mathrm{cpl}\mathcal{B})$, as desired. □

Note that it can happen that

$$\mathrm{cir}(\mathrm{cpl}\mathcal{B}) \;=\; \mathrm{cur}(\mathcal{B}) \;<\; \mathrm{cur}(\mathcal{B}\setminus\{X\})\,.$$

For example, take $X=\mathbb{N}$ and $\mathcal{B}$ to be the collection of all initial segments of $\mathbb{N}$. Then $(X,\mathcal{B})$ is chain union closed, while $\mathrm{cur}(\mathcal{B}\setminus\{X\})=1$. Further, we see that $\mathrm{cpl}(\mathcal{B})$ is the collection of all final segments of $\mathbb{N}$. It is chain intersection closed, i.e., $\mathrm{cir}(\mathrm{cpl}(\mathcal{B}))=\mathrm{cur}(\mathcal{B})=0$.

We will now demonstrate by an example that both the chain union rank and the chain intersection rank of a ball space can be equal to any given ordinal α. Since the ball space $(X,\mathcal{P}(X)\setminus\{\emptyset\})$ for nonempty X is both chain union and chain intersection closed, we have to show this only for the case of $\alpha\geq 1$.

Example 6.2. Take an ordinal $\alpha\geq 1$ and set $X:=\aleph_\alpha$. For β any ordinal, define $\mathcal{B}_\beta$ to be the collection of all nonempty subsets of $\aleph_\alpha$ of cardinality smaller than or equal to $\aleph_\beta$. Set $\mathcal{B}:=\mathcal{B}_0$. We note that $X\notin\mathcal{B}$ since $\alpha\geq 1$. A standard transfinite induction shows that $\mathrm{cu}_\beta(\mathcal{B})=\mathcal{B}_\beta$ for every $\beta\leq\alpha$.

Since the subsets of $\aleph_\alpha$ have cardinality at most $\aleph_\alpha$, it follows that $\mathrm{cu}_\alpha(\mathcal{B})=\mathcal{B}_\alpha=\mathcal{P}(\aleph_\alpha)\setminus\{\emptyset\}=\mathcal{P}(X)\setminus\{\emptyset\}$. Therefore, $\mathrm{cu}_\alpha(\mathcal{B})$ is chain union closed. On the other hand, $\mathrm{cu}_\beta(\mathcal{B})$ is not chain union closed for any $\beta<\alpha$. Hence, $\mathrm{cur}(\mathcal{B})=\alpha$.

Finally, we show that $\mathrm{cir}(\mathrm{cpl}\mathcal{B})=\alpha$. By part 4) of the preceding lemma, $\beta:=\mathrm{cir}(\mathrm{cpl}\mathcal{B})\leq\mathrm{cur}(\mathcal{B})=\alpha$. Applying part 1)i) of the same lemma with $S=\{X\}$, we obtain that $\mathrm{cu}_\beta(\mathcal{B}\cup\{X\})=\mathrm{cu}_\beta(\mathcal{B})\cup\{X\}$. We have that

$$\beta \;=\; \mathrm{cir}(\mathrm{cpl}\mathcal{B}) \;=\; \mathrm{cir}(\mathrm{cpl}(\mathcal{B}\cup\{X\})) \;=\; \mathrm{cur}(\mathcal{B}\cup\{X\})\,,$$

where the last equality follows from part 4) of the previous lemma. This implies that $\mathrm{cu}_\beta(\mathcal{B})\cup\{X\}=\mathcal{B}_\beta\cup\{X\}$ is chain union closed. But as we have seen above, this can only be if $\mathcal{B}_\beta\cup\{X\}=\mathcal{B}_\alpha$. Since X is not the only subset of X of cardinality $\aleph_\alpha$, this can only be the case if $\beta=\alpha$, showing that $\mathrm{cir}(\mathrm{cpl}\mathcal{B})=\alpha$. $\diamondsuit$

For ultrametric and tree-like ball spaces, we obtain from Theorem 1.4 and Theorem 1.7:

Corollary 6.3. *Every tree-like ball space $(X,\mathcal{B})$ has chain union rank $\mathrm{cur}(\mathcal{B})\leq 1$. The ultrametric ball space $(X,\mathcal{B}_u)$ of every ultrametric space with an ω_1-free partially ordered value set has chain union rank $\mathrm{cur}(\mathcal{B})\leq 1$.*

Let us dicuss the case of ordered sets. Lemma 4.8 implies:

Corollary 6.4. *The interval ball space of a linearly ordered set has chain union rank at most 2.*

In contrast, from Proposition 4.1 we obtain:

Corollary 6.5. *Take a partially ordered set $(X,\leq)$ and assume that X contains a nonempty linearly ordered subset S whose initiality and cofinality are distinct infinite cardinals. Then $\mathrm{cur}(\mathcal{B})\geq 2$.*

Let us give an example that is derived from the previous Example 4.7.

Example 6.6. If K is any subfield of $\mathbb{R}$, then $\mathrm{cur}(\mathcal{B}_K) = 1$. If K is a symmetrically complete ordered field properly containing $\mathbb{R}$, then $\mathrm{cur}(\mathcal{B}_K) = 2$. ♢

The ball space $\mathcal{B}_\leq$ in Examples 4.2, 4.3, 4.5 and 4.6 have chain union rank 2. The idea of Example 4.3 can be genberalized as follows. Assume that $(X_j, \mathcal{B}_j)_{j\in J}$ is a family of ball spaces. We set $X = \prod_{j\in J} X_j$ and define the *box product* $(X_j, \mathcal{B}_j)^{\mathrm{bpr}}_{j\in J}$ of the family to be $(X, (\mathcal{B}_j)^{\mathrm{bpr}}_{j\in J})$, where

$$(\mathcal{B}_j)^{\mathrm{bpr}}_{j\in J} := \left\{ \prod_{j\in J} B_j \mid \forall j \in J :\ B_j \in \mathcal{B}_j \right\} ,$$

cf. [4, Section 8.1].

Proposition 6.7. *Take distinct infinite regular cardinals $\kappa_1, \ldots, \kappa_n$. Consider each κ_j as a totally ordered set with its associated interval ball space $\left(\kappa_j, \mathcal{B}^j_\leq\right)$. Then the interval ball space of any partial order that contains the box product of these interval ball spaces has chain union rank n.*

Proof. ... □

If $(K, <)$ is an ordered field admitting Dedekind cuts with the lower cut sets having cofinalities $\kappa_1, \ldots, \kappa_n$, then $\kappa_1 \times \ldots \times \kappa_n$ can be embedded in the partially ordered vector space K^n and it follows that its interval ball space has chain union rank at least n.

7. Chain union closed or stable ball spaces

Proof of Theorem 1.4. Fix $D_0, D_1 \in \mathrm{cu}(\mathcal{B})$ with $a \in D_0 \cap D_1$. Let $\mathcal{C}_0$, $\mathcal{C}_1$ be chains in $\mathcal{B}$ with $D_i = \bigcup \mathcal{C}_i$, $i = 0, 1$. We may assume $a \in \bigcap \mathcal{C}_i$, by refining the chain. Now $\mathcal{C}_0 \cup \mathcal{C}_1$ is a chain, because $\mathcal{B}$ is tree-like. Suppose $D_1 \not\subseteq D_0$ and fix $b \in D_1 \setminus D_0$. Choose $C_1 \in \mathcal{C}_1$ with $b \in C_1$. Now $C \subseteq C_1$ for every $C \in \mathcal{C}_0$, because the other inclusion is impossible. Hence $D_0 = \bigcup \mathcal{C}_0 \subseteq C_1 \subseteq \bigcup \mathcal{C}_1 = D_1$. This shows that $\mathrm{cu}(\mathcal{B})$ is tree-like.

Now, take a chain $\mathcal{D}$ in $\mathrm{cu}(\mathcal{B})$ and a point $a \in \bigcup \mathcal{D}$. After refining $\mathcal{D}$, we may assume $a \in D$ for all $D \in \mathcal{D}$. Next, for $D \in \mathcal{D}$, let $D = \bigcup \mathcal{C}_D$, where $\mathcal{C}_D$ is a chain in $\mathcal{B}$. Again, we may assume that $a \in C$ for every $C \in \mathcal{C}_D$. Since $(X, \mathcal{B})$ is tree-like, we conclude that $\bigcup_{D\in\mathcal{D}} \mathcal{C}_D$ is a chain in $\mathcal{B}$ with the same union as $\mathcal{D}$. Hence, $\mathrm{cu}(\mathcal{B})$ is chain union closed.

Finally, assume $(X, \mathcal{B})$ to be spherically complete and fix a strictly decreasing chain $\{D_\alpha\}_{\alpha<\kappa}$ in $\mathrm{cu}(\mathcal{B})$. Fix $\alpha < \kappa$. Choose $a \in D_{\alpha+1}$ and $b \in D_\alpha \setminus D_{\alpha+1}$. Since D_α is the union of a chain from $\mathcal{B}$, we can find B_α in that chain such that $a, b \in B_\alpha$. Hence $B_\alpha \subseteq D_\alpha$. We also have $D_{\alpha+1} \subseteq B_\alpha$, because $a \in D_{\alpha+1} \cap B_\alpha$ and $b \in B_\alpha \setminus D_{\alpha+1}$, so the opposite inclusion is impossible. Finally, $\{B_\alpha\}_{\alpha<\kappa}$ is a chain in $\mathcal{B}$ with $\bigcap_{\alpha<\kappa} B_\alpha = \bigcap_{\alpha<\kappa} D_\alpha$. □

Recall that a partially ordered set ("poset") is *narrow* if it contains no infinite sets of pairwise incomparable elements. Note that the poset X used in the first examples in Section 4 is not narrow.

For our next theorem, we will need two lemmas that reflect important and well known properties of narrow posets. The first one is an immediate consequence of the Erdős-Dushnik-Miller Theorem.

Lemma 7.1 (cf. [10, Lemma 3.2])**.** *Let $(\mathcal{P}, \leq)$ be a narrow poset and $\mathcal{A} \subseteq \mathcal{P}$ infinite. Then there exists a chain $\mathcal{C} \subseteq \mathcal{A}$ such that $|\mathcal{C}| = |\mathcal{A}|$.*

Recall that a subset A of a poset $(P, \leq)$ is *directed* if for every $a_0, a_1 \in A$ there is $b \in A$ with $a_0 \leq b$ and $a_1 \leq b$. The following fact can be found in [5]. We presented a proof in [10, Lemma 3.3].

Lemma 7.2. *Every narrow poset is a finite union of directed subsets.*

We need one more general combinatorial fact.

Lemma 7.3. *Assume $\mathcal{A} = \mathcal{A}_0 \cup \cdots \cup \mathcal{A}_{k-1}$ is a family of sets such that $\bigcup \mathcal{A}$ is the union of a chain of sets that are unions of chains in $\mathcal{A}$. Then there exists $j < k$ such that $\bigcup \mathcal{A} = \bigcup \mathcal{A}_j$.*

Proof. Our assumption says that $\bigcup \mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$ where $\mathcal{C}_\alpha \subseteq \mathcal{A}$ is a chain for each $\alpha < \kappa$ and $\bigcup \mathcal{C}_\alpha \subseteq \bigcup \mathcal{C}_\beta$ whenever $\alpha < \beta$. We may assume that κ is a regular cardinal. For each $\alpha < \kappa$ we choose $j(\alpha) < k$ such that $\bigcup \mathcal{C}_\alpha = \bigcup(\mathcal{C}_\alpha \cap \mathcal{A}_{j(\alpha)})$. If κ is finite then $\bigcup \mathcal{A} = \bigcup(\mathcal{C}_{(\kappa-1)} \cap \mathcal{A}_{j(\kappa-1)}) \subseteq \bigcup \mathcal{A}_{j(\kappa-1)}$. Otherwise, there is an unbounded set $S \subseteq \kappa$ such that $j(\alpha) = j$ for every $\alpha \in S$. Then

$$\bigcup \mathcal{A} = \bigcup_{\alpha \in S} \mathcal{C}_\alpha \subseteq \bigcup \mathcal{A}_j .$$

□

We will now extend Theorem 1.4 to a larger class of ball spaces $(X, \mathcal{B})$. For every $z \in X$, we set

$$\mathcal{B}(z) \;:=\; \{B \in \mathcal{B} \mid z \in B\} .$$

Theorem 7.4. *Let $(X, \mathcal{B})$ be a ball space such that for every $z \in X$, the poset $(\mathcal{B}(z), \subseteq)$ is narrow and ω_1-free. Then $(X, \mathcal{B})$ is chain union stable.*

Proof. Fix a strictly increasing chain $\mathcal{D} = \{D_\alpha\}_{\alpha<\kappa}$, where κ is an infinite regular cardinal. We may assume that for each $\alpha < \kappa$ there is some $z_\alpha \in D_\alpha \setminus \bigcup_{\xi<\alpha} D_\xi$ (otherwise, replace D_α by $D_{\alpha+1}$). Let $z := z_0$. Each D_α is the union of a chain from $\mathcal{B}(z)$, so we may restrict attention to the narrow poset $(\mathcal{B}(z), \subseteq)$ in which all chains have countable cofinality. Using transfinite induction, we construct for each $\alpha < \kappa$ a chain $\{B_{\alpha,n}\}_{n\in\omega} \subseteq \mathcal{B}(z)$ with $\bigcup_{n\in\omega} B_{\alpha,n} = D_\alpha$ and $z_\alpha \in B_{\alpha,0}$. By this way, if $B_{\alpha,n} \subseteq B_{\alpha',n'}$ then $\alpha \leq \alpha'$, because otherwise $z_\alpha \in B_{\alpha,n} \setminus B_{\alpha',n'}$. It follows that $\mathcal{A} := \{B_{\alpha,n}\}_{\alpha<\kappa, n<\omega}$ is well-founded.

By Lemma 7.1, we conclude that $\mathcal{A}$ is countable and consequently $\kappa = \aleph_0$. Finally, Lemma 7.2 says that $\mathcal{A}$ is the union of finitely many directed subfamilies. The union of one of them must be equal to $\bigcup \mathcal{D}$, according to Lemma 7.3. A countable directed family of sets obviously has a cofinal chain, therefore $\bigcup \mathcal{D}$ is the union of an ω-chain of balls. □

8. Further results on chain union stability

Given a poset Γ, define $\sigma\Gamma$ to be the set of all nonempty directed initial segments of Γ of countable cofinality. In other words, $C \in \sigma\Gamma$ if there exists a sequence $p_0 \le p_1 \le \cdots$ in Γ such that $C = \{x \in \Gamma \mid (\exists\, n)\; x \le p_n\}$. Then $\sigma\Gamma$ is a poset with inclusion that can be viewed as the "ω-completion" of Γ, namely, Γ is naturally embedded into $(\sigma\Gamma, \subseteq)$ and every countable chain in $\sigma\Gamma$ has a least upper bound (the union of that chain is directed and has a countable cofinality, by a standard diagonalization).

A poset Γ is *up-countable* if for every $p \in \Gamma$ the set $\{x \in \Gamma \mid p < x\}$ is countable. Recall that Γ is *ω_1-free* if it does not contain a strictly increasing sequence of type ω_1. Clearly, this is stronger than being up-countable: Perhaps the simplest example distinguishing these properties is the family of at most one-element sets of a fixed uncountable set, endowed with inclusion—there are no chains of length more than 2, while there are uncountably many elements above the empty set.

Given a ball space $(X, \mathcal{B})$, an *ultradiameter* is a function $\delta \colon \mathcal{B} \to \Gamma$, where Γ is a poset, satisfying

(D1) δ is increasing (i.e., $\delta(B_0) \le \delta(B_1)$ whenever $B_0 \subseteq B_1$),
(D2) if $B_0, B_1 \in \mathcal{B}$, $B_0 \cap B_1 \neq \emptyset$, and $\delta(B_0) \le \delta(B_1)$ then $B_0 \subseteq B_1$.

If $(X, \mathcal{B}_u)$ is the ball space of an ultrametric space (X, u), then it admits the ultradiameter

$$\delta(B(x, y)) \;:=\; u(x, y)$$

whose value set is the same as that of (X, u). For this and applications of ultradiameters, see [10, Section 3].

Theorem 8.1. *Assume $(X, \mathcal{B})$ is a chain regular ball space with an ultradiameter $\delta \colon \mathcal{B} \to \Gamma$ such that Γ is up-countable. Then $(X, \mathcal{B})$ is chain union stable.*

Proof. Fix a chain $\mathcal{F} = \{F_\alpha\}_{\alpha<\kappa}$ in $\mathrm{cu}(\mathcal{B})$ such that $\alpha < \beta \implies F_\alpha \subsetneq F_\beta$, where κ is an infinite cardinal. For each $\alpha < \kappa$ choose a chain $\mathcal{C}_\alpha \subseteq \mathcal{B}$ with $\bigcup \mathcal{C}_\alpha = F_\alpha$. Choose $C_0 \in \mathcal{C}_0$. We may assume C_0 is minimal in $\mathcal{C}_0$.

Using the chain regularity of $\mathcal{B}$ and refining each of the chains $\mathcal{C}_\alpha$, we may assume that $C_0 \subseteq C$ for every $C \in \mathcal{C}_\alpha$, for every $\alpha < \kappa$.

Let $\mathcal{B}(C_0) = \{B \in \mathcal{B} \mid C_0 \subseteq B\}$. So $\mathcal{C}_\alpha \subseteq \mathcal{B}(C_0)$ for every $\alpha < \kappa$. Note that δ restricted to $\mathcal{B}(C_0)$ satisfies

$$\delta(B_0) \le \delta(B_1) \iff B_0 \subseteq B_1.$$

This is thanks to property (D2). Hence $\delta \restriction \mathcal{B}(C_0)$ is an isomorphic embedding of $(\mathcal{B}(C_0), \subseteq)$ into Γ. Since Γ is up-countable, we conclude that $\mathcal{B}(C_0)$ is countable. Lemma 5.3 implies that $\kappa = \omega$. Finally, Lemma 5.2 finishes the proof. □

Lemma 8.2. *Let $(X, \mathcal{B})$ admit an ultradiameter δ with values in an ω_1-free narrow poset Γ. Then the following assertions hold for each $z \in X$.*

1) $(\mathcal{B}, \subseteq)$ *and hence also* $(\mathcal{B}(z), \subseteq)$ *is* ω_1*-free.*

2) $(\mathcal{B}(z), \subseteq)$ *is narrow.*

Proof. 1): By (D1) and (D2), if $B_0 \subseteq B_1$ and $\delta(B_0) = \delta(B_1)$ then $B_0 = B_1$, therefore δ is one-to-one on each chain in $\mathcal{B}$.

2): Take $\{B_i\}_{i<\omega} \subseteq \mathcal{B}(z)$. Then there are $k < \ell < \omega$ such that $\delta(B_k) \leq \delta(B_\ell)$ or $\delta(B_\ell) \leq \delta(B_k)$. Since the intersection of B_k and B_ℓ is nonempty as they have the element z in common, it follows from property (D2) that $B_k \subseteq B_\ell$ or $B_\ell \subseteq B_k$. This proves that $(\mathcal{B}(z), \subseteq)$ is narrow. □

From this lemma we obtain one more criterion for chain union stability.

Theorem 8.3. *Assume* $(X, \mathcal{B})$ *is a ball space admitting an ultradiameter with values in an* ω_1*-free narrow poset. Then* $(X, \mathcal{B})$ *is chain union stable.*

Question 8.4. Can "narrow" be omitted from Theorems 7.4 and 8.3? Can "ω_1-free" be omitted from Theorem 8.3?

9. An example

Example 9.1. Let $\{D_\alpha\}_{\alpha<\omega_1}$ be a collection of countable sets satisfying the following condition:

($\Cap$) $\forall\, \alpha < \omega_1, \;\; D_\alpha \setminus \bigcup_{\xi<\alpha} D_\xi$ is infinite.

Let $X = \bigcup_{\alpha<\omega_1} D_\alpha$. Then there exists a ball structure $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ on X such that $(\mathcal{B}, \subseteq)$ is well-founded, all chains in $\mathcal{B}$ are countable, and each $\mathcal{B}_\alpha$ is an ω-chain with $\bigcup \mathcal{B}_\alpha = D_\alpha$.

Note that $\{D_\alpha\}_{\alpha<\omega_1}$ could be a chain, therefore the chain union closure of a ball space in which all ω-chains of balls are countable may lead to a strictly increasing ω_1-sequence.

The required ball structure is constructed by a transfinite induction, as follows. We start with any ω=chain $\mathcal{B}_0$ consisting of infinite sets such that $\bigcup \mathcal{B}_0 = D_0$. Having defined $\mathcal{B}_\xi$ for all $\xi < \alpha$, we set

$$A := \bigcup_{\xi<\alpha} D_\xi, \qquad C := D_\alpha \setminus A.$$

By ($\Cap$), the set C is infinite, therefore we can choose a strictly increasing sequence $\{C_n\}_{n\in\omega}$ of infinite sets such that $C = \bigcup_{n\in\omega} C_n$. The set A is countable, therefore we can enumerate it, so $A = \{a_n\}_{n\in\omega}$. Define $B_n := C_n \cup \{a_i \mid i < n\}$ and $\mathcal{B}_\alpha := \{B_n\}_{n\in\omega}$. Then $\mathcal{B}_\alpha$ forms an ω-chain whose union is D_α. Furthermore, every set from $\mathcal{B}_\alpha$ has a finite intersection with all the sets from $\bigcup_{\xi<\alpha} \mathcal{B}_\xi$. It follows that every chain in $\mathcal{B} := \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$ is contained in $\mathcal{B}_\beta$ with a fixed $\beta < \omega_1$. In particular, all chains in $\mathcal{B}$ are countable.

♢

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