TOPICS IN HIGHER RAMIFICATION THEORY I: RAMIFICATION IDEALS

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Abstract. We introduce and study the notion of ramification ideals in higher ramification theory. After general results on their computation for finite extensions, we discuss their connection with the possibly nontrivial defect of the extensions. We compute them for Artin-Schreier extensions and Kummer extensions of prime degree equal to the residue characteristic, which may or may not have nontrivial defect. We present an example that shows that nontrivial defect in an extension of degree p^2, p a prime, may not imply the existence of a nonprincipal ramification ideal.

Dedicated to Dr. Sudesh K. Khanduja
on the occasion of her 75th birthday

1. Introduction

Higher ramification theory is the theory of valued field extensions $\mathcal{E} = (L|K, v)$ where (K, v) has positive residue characteristic p and is its own **absolute ramification field** (see Section 2.3). The latter means that (K, v) is henselian, its **value group** vK is divisible by all primes different from p, and its **residue field** Kv is separable-algebraically closed. The **absolute Galois group** $\mathrm{Gal}\, K := \mathrm{Gal}\, K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K, is then a pro-p-group ([9, (20.18), p. 167]. This implies that every finite Galois extension of K is a tower of Galois extensions of degree p. In **equal characteristic**, i.e., if $\mathrm{char}\, K = \mathrm{char}\, Kv = p$, the latter are **Artin-Schreier extensions**, and in **mixed characteristic**, i.e., if $\mathrm{char}\, K = 0$ and $\mathrm{char}\, Kv = p$, they are **Kummer extensions** because K contains all p-th roots of unity (see Section 3.3).

Our interest in higher ramification theory originated from the following well known deep open valuation theoretical problems:

1) local uniformization, the local form of resolution of singularities in arbitrary dimension and positive characteristic,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$ with q elements, and of its perfect hull, where q is a power of a prime p.

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Both problems are connected with the structure theory of valued function fields of positive characteristic p. The main obstruction here is the phenomenon of the **defect**, which we define in Section 2.2. For background on the defect and its impact on the above problems, see [18] for generalities on the defect and a large collection of examples for defect extensions, [15, 13, 14] for partial solutions of the local uniformization problem and the connection with the defect, [20, 25] for model theoretic results on tame and separably tame fields, and [16, 21] for elimination of ramification (and hence of the influence of the defect) in valued function fields, which provides the crucial tools for the proofs of the main results in [13, 14, 20, 25].

An important approach in the study of the defect in extensions of arbitrary finite degree is what Cossart and Piltant use in [4, §8] and call "Abhyankar's strategy"; it was introduced by Abhyankar in [1]. Using the above mentioned property of the absolute ramification field, it reduces the investigation of a given finite extension of valued fields to that of purely inseparable extensions and of Galois extensions of degree $p = \mathrm{char}\, Kv > 0$; if the original extension has nontrivial defect, then the same will be true for some of the latter extensions (cf. [6, Section 2.1]). This principle has been crucially used in [16, 21].

Defects of Galois extensions $\mathcal{E} = (L|K, v)$ of prime degree have been classified ("dependent" vs. "independent" defect) first in [17] for the equal characteristic case and then in [27] in general. Theorem 1.4 of [27] presents various criteria for independent defect. One of them uses the ramification ideal $I_{\mathcal{E}}$ of the extension $\mathcal{E}$, which we define in Section 2.5.

Section 3 is then devoted to the computation of ramification ideals. Starting in Section 3.1 with a first approach described by Ribenboim in [36] we develop more elaborate computations. Of particular interest is the case of extensions that have valuation bases; for this notion, see Section 2.6. Based on this, Proposition 3.7 treats towers of two Galois extensions where the upper one has a valuation basis, which we will apply in Section 3.5. We do not know an analogue of Proposition 3.7 where the lower extension has a valuation basis.

In Section 3.2 we discuss the relation between defect and the existence of non-principal ramification ideals. While it is true that a finite Galois extension without defect has only principal ramification ideals (see Proposition 3.1), the converse does not hold. We give an example for this phenomenon in Section 3.5 for the equal characteristic case. We do not know whether there is a similar example in mixed characteristic. The example consists of a tower of two Galois extensions of prime degree where the lower has defect but the upper does not. We do not know of an example where this is reversed.

In Section 3.3 we first compute the unique ramification ideals $I_{\mathcal{E}}$ for Artin-Schreier and Kummer extensions $\mathcal{E} = (L|K, v)$ of degree $p = \mathrm{char}\, Kv$ without defect; the results are applied in [7]. We then take a closer look at the ramification ideals $I_{\mathcal{E}}$ in the case when $\mathcal{E}$ is a defect extension, which are computed in [6].

In Section 3.4 we determine the cases where $I_{\mathcal{E}}$ is equal to the maximal ideal $\mathcal{M}_L$ of the valuation ring on L. We discuss the significance of this equality and give examples of several cases where it holds. Already in Proposition 3.16 a dichotomy can be observed when it comes to the nature of generators for Kummer extensions without defect; it appears again in the computation of their ramification ideals

(see Theorem 3.17). The examples we present for the cases of Kummer extensions illuminate the dichotomy.

The following theorem shows the particular importance of extensions of prime degree of a valued field (K,v). For the definition of the notions of **deeply ramified field** and of **module of relative differentials**, see [23]. Denote the valuation ring of (K,v) by $\mathcal{O}_K$.

Theorem 1.1. [7, Theorem 1.3] *Let (K,v) be a valued field of residue characteristic $p>0$. If K has characteristic 0, then assume in addition that it contains all p-th roots of unity. Then (K,v) is a deeply ramified field if and only if the module $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ of relative differentials equals zero for all unibranched Galois extensions $(L|K,v)$ of degree p.*

This theorem as well as Abhyankar's strategy justify our particular focus on Artin-Schreier extensions and Kummer extensions of prime degree in the present paper.

Several of the results of this paper are used in [23], where we present $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ using $\mathcal{O}_L$-ideals that are first order definable in the language of valued fields with a predicate for membership in K. These results are:
1) Proposition 2.6 which establishes an inclusion preserving bijection from the set of all higher ramification groups of a valued Galois extension to the set of all higher ramification ideals;
2) Proposition 2.7 which shows that the definition of ramification ideals used in [23] is equivalent to the definition given in [27] by use of ramification jumps;
3) Proposition 3.1 which shows that every ramification ideal of a unibranched Galois extension without defect is principal;
4) Theorem 3.17 which presents the ramification ideals of unibranched Galois extensions of prime degree without defect;
5) part 1) of Corollary 3.20 which gives a criterion for a valued Artin-Schreier extension to have nontrivial defect without the assumption that the extension is unibranched (we have not found such a criterion in the literature).

Let us mention that some approaches and results of the present paper are generalized in [32]. In particular, Theorem 3.17 and Proposition 3.21 can be directly deduced from Proposition 3.16 by using [32, Theorem 3.1].

2. Preliminaries

For basic facts from valuation theory, see [9, 10, 34, 39, 40].

Take a valued field (K,v). We denote its value group by vK, its residue field by Kv, its valuation ring by $\mathcal{O}_K$, with its maximal ideal $\mathcal{M}_K$. For $a\in K$, we write va for its value and av for its residue. *We will tacitly assume all valuations appearing in this paper to be nontrivial.*

By $(L|K,v)$ we denote an extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto.

By $\tilde{K}$ we denote the algebraic closure of K. Every valuation on $K^{\rm sep}$ has a unique extension to $\tilde{K}$; this allows us to identify the absolute Galois group $\mathrm{Gal}\,K=\mathrm{Gal}\,K^{\rm sep}|K$ with the automorphism group $\mathrm{Aut}\,\tilde{K}|K$.

2.1. **Final segments and ideals.** A subset S of a totally ordered set Γ is an **initial segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \leq \alpha$, it follows that $\gamma \in S$. Symmetrically, S is a **final segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \geq \alpha$, it follows that $\gamma \in S$. *Throughout, we will assume initial and final segments to be nonempty and not equal to* Γ.

Take a valued field (L, v). A subset $I \subset L$ is a **fractional $\mathcal{O}_L$-ideal** if there is some $a \in L$ such that aI is an $\mathcal{O}_L$-ideal (contained in $\mathcal{O}_L$). In particular, every fractional $\mathcal{O}_L$-ideal is a proper subset of L.

The function

$$v:\ I \mapsto \Sigma_I := \{vb \mid 0 \neq b \in I\} \tag{1}$$

is a bijection from the set of all nonzero fractional $\mathcal{O}_L$-ideals onto the set of all final segments of vL. Both sets are linearly ordered by inclusion, and the function (1) is order preserving: $J \subseteq I$ holds if and only if $\Sigma_J \subseteq \Sigma_I$ holds. The inverse of the above function is the order preserving function

$$\Sigma \mapsto I_\Sigma := \{a \in L \mid va \in \Sigma\} \cup \{0\} = (a \in L \mid va \in \Sigma) \tag{2}$$

from the set of all final segments of vL onto the set of all nonzero fractional $\mathcal{O}_L$-ideals.

2.2. **The defect.**

A valued field extension $(L|K, v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation. A valued field (K, v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, if all of its algebraic extensions are unibranched. A **henselization** of (K, v) is an algebraic extension of (K, v) which admits a valuation preserving embedding in every other henselian extension of (K, v). Henselizations always exist and are unique up to valuation preserving isomorphism over K; therefore we will talk of *the* henselization of (K, v) and denote it by $(K, v)^h = (K^h, v)$. The henselization of (K, v) is an immediate separable-algebraic extension. The valued field (K, v) is henselian if and only if it is equal to its henselization.

If $(L|K, v)$ is a finite unibranched extension, then by the Lemma of Ostrowski ([40, Corollary to Theorem 25, Section G, p. 78]),

$$[L : K] = \tilde{p}^\nu \cdot (vL : vK)[Lv : Kv]\,, \tag{3}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K, v) := \tilde{p}^\nu$ is the **defect** of the extension $(L|K, v)$. We call $(L|K, v)$ a **defect extension** if $d(L|K, v) > 1$, and a **defectless extension** if $d(L|K, v) = 1$. Throughout this paper, when we talk of a **defect extension** $(L|K, v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. If the extension is of prime degee, then it follows from (3) that $[L : K] = \operatorname{char} Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$, that is, $(L|K, v)$ is an immediate extension.

Nontrivial defect only appears when $\operatorname{char} Kv = p > 0$, in which case $\tilde{p} = p$. A henselian field (K, v) is called a **defectless field** if all of its finite extensions are defectless.

The following lemma shows that the defect is multiplicative. This is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. We leave the straightforward proof to the reader.

Lemma 2.1. *Take a valued field (K,v). If $L|K$ and $M|L$ are finite extensions and the extension of v from K to M is unique, then*

$$\mathrm{d}(M|K,v) = \mathrm{d}(M|L,v)\cdot \mathrm{d}(L|K,v) \tag{4}$$

In particular, $(M|K,v)$ is defectless if and only if $(M|L,v)$ and $(L|K,v)$ are defectless.

For every extension $(L|K,v)$ of valued fields and $a\in L$ we define

$$v(a-K) := \{v(a-c)\,|\,c\in K\}\,.$$

The set $v(a-K)\cap vK$ is an initial segment of vK. For more information on its properties, see [22].

Lemma 2.2. *Take a unibranched algebraic extension $(K(a)|K,v)$ and an extension of v from $K(a)$ to $\tilde{K}$. Denote by (K^h,v) the henselization of (K,v) in $(\tilde{K},v)$. Then:*
a) $K(a)|K$ is linearly disjoint from $K^h|K$,
b) $(K^h(a)|K^h,v)$ is a defect extension if and only if $(K(a)|K,v)$ is, and
c) $v(a-K^h)=v(a-K)$.

Proof. Our first assertion follows from [3, Lemma 2.1]. For the proof of the second assertion, recall that henselizations are immediate extensions, so we have $vK^h = vK$ and $K^hv = Kv$. Further, we have $K^h(a) = K(a)^h$ since on the one hand, $K^h(a)$ is henselian, being an algebraic extension of K^h, and on the other hand, it contains $K(a)$. Hence, $vK^h(a) = vK(a)$ and $K^h(a)v = K(a)v$. Since $K(a)|K$ is linearly disjoint from $K^h|K$, we also have $[K^h(a):K^h] = [K(a):K]$. As an algebraic extension of a henselian field, $(K^h(a)|K^h,v)$ is unibranched. It follows that

$$\begin{aligned} d(K^h(a)|K^h,v) &= [K^h(a):K^h]/(vK^h(a):vK^h)[K^h(a)v:K^hv] \\ &= [K(a):K]/(vK(a):vK)[K(a)v:Kv] \\ &= d(K(a)|K,v)\,. \end{aligned}$$

This proves our second assertion.

Suppose that $v(a-K^h)\neq v(a-K)$. Since $v(a-K)$ is an initial segment of $vK = vK^h$, this means that there must be some $z\in K^h$ such that $v(a-z) > v(a-K)$. However, as $K(a)|K$ is linearly disjoint from $K^h|K$, we know from that this cannot be true. This proves our third assertion. □

2.3. **The ramification field.**

The **ramification field** of a Galois extension $(L|K,v)$ with Galois group $G = \mathrm{Gal}\,(L|K)$ is the fixed field in L of the **ramification group**

$$G^r := \left\{\sigma\in G \;\middle|\; \frac{\sigma b-b}{b}\in\mathcal{M}_L \text{ for all } b\in L^\times\right\}\,. \tag{5}$$

In order to reduce the study of arbitrary finite defect extensions to purely inseparable extensions and Galois extensions of degree $p = \mathrm{char}\,Kv > 0$, we fix an

extension of v from K to $\tilde{K}$. The **absolute ramification field of** (K,v) (with respect to the chosen extension of v), denoted by (K^r,v), is the ramification field of the Galois extension $(K^{\rm sep}|K,v)$.

Take a finite defect extension $(L|K,v)$. Then $(L.K^r|K^r,v)$ is a defect extension with the same defect (see [27, Proposition 2.12]). On the other hand, $K^{\rm sep}|K^r$ is a p-extension (see [17, Lemma 2.7]), so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p. Note that $(K,v)=(K^r,v)$ if and only if (K,v) is henselian, vK is divisible by all primes different from $\mathrm{char}\,Kv$, and Kv is separable-algebraically closed; this follows from [10, Corollary 5.3.8].

2.4. **Immediate extensions.**

Let us give more details about immediate extensions.

Lemma 2.3. *Take an arbitrary extension $(L|K,v)$ and $b\in L$. Then there is $c\in K$ such that $v(b-c)>vb$ if and only if $vb\in vK$ and $c'bv\in Kv$ for every $c'\in K$ such that $vc'b=0$.*

Proof. Assume first that $v(b-c)>vb$. Then $vb=vc\in vK$ and for any $c'\in K$ such that $vc'b=0$ we have $v(c'b-c'c)>0$ so that $c'bv=c'cv\in Kv$. Now assume that $vb\in vK$ and $c'bv\in Kv$ for every $c'\in K$ such that $vc'b=0$. Take $c_1\in K$ such that $vc_1=vb$ and set $c'=c_1^{-1}$. Then $vc'b=0$, hence by assumption, $c'bv\in Kv$. Take $c_2\in K$ such that $c'bv=c_2v$, so that $v(c'b-c_2)>0$. Multiplying with c_1 we obtain $v(b-c_1c_2)>vc_1=vb$. □

It follows that an extension $(L|K,v)$ is immediate if and only if for all $b\in L$ there is $c\in K$ such that $v(b-c)>vb$. This lays the basis for the proof of the following theorem; see [11, Theorem 1] and [22, Lemma 2.29].

Theorem 2.4. *If $(L|K,v)$ is an immediate extension of valued fields, then for every element $a\in L\setminus K$ the set $v(a-K)$ is an initial segment of vK without maximal element.*

The following partial converse of this theorem also holds, as shown in [17, Lemma 2.21]:

Lemma 2.5. *Assume that $(K(a)|K,v)$ is a unibranched extension of prime degree such that $v(a-K)$ has no maximal element. Then the extension $(K(a)|K,v)$ is immediate and hence a defect extension.*

2.5. **Higher ramification groups and ramification ideals.**

Take a valued field extension $\mathcal{E}=(L|K,v)$. Assume that $L|K$ is a Galois extension, and let $G=\mathrm{Gal}\,L|K$ denote its Galois group. We define the **series of upper ramification groups**

$$G_I := \left\{\sigma\in G \;\middle|\; \frac{\sigma b-b}{b}\in I \;\text{ for all } b\in L^\times\right\}, \tag{6}$$

where I runs through all $\mathcal{O}_L$-ideals (cf. [40, §12]). Every G_I is a normal subgroup of G ([40, (d) on p.79]). Note that $G_{\mathcal{O}_L}$ is the decomposition group of $\mathcal{E}$; this easily follows from its definition as $\{\sigma\in G \mid v\sigma=v \text{ on } L\}$ (see [15, §5 (16)]). This also implies that $\mathcal{E}$ is unibranched if and only if $G=G_{\mathcal{O}_L}$ (cf. [15, Theorem 5.4 b)]).

Further, $G_{\mathcal{M}_L}$ is the ramification group of $\mathcal{E}$ (see [15, §5 (18)]). We call G_I a **higher ramification group** if it is a nontrivial subgroup of $G_{\mathcal{M}_L}$. Every G_I is a higher ramification group if and only if $G = G_{\mathcal{M}_L}$. If this is the case, then the extension $\mathcal{E}$ is **purely wild** in the sense of [26], that is, $(vL : vK)$ is a p-power, where $p = \operatorname{char} Kv$, and $Lv|Kv$ is purely inseparable. Indeed, observe first that $G = G_{\mathcal{M}_L}$ means that K is the ramification field of $(L|K, v)$, which by [9, (20.15), p. 166] is equal to $K^r \cap L$. Now $K^r \cap L = K$ implies that $L|K$ and $K^r|K$ are linearly disjoint, which by [26, Lemma 4.2] is equivalent to $(L|K, v)$ being purely wild. Note that every purely wild extension is unibranched since K^r contains the henselization of (K, v) (under the chosen extension of v to the algebraic closure), and by [19, Theorem 2], L being linearly disjoint from the henselization over K implies that $(L|K, v)$ is unibranched.

The function

$$\varphi : \ I \ \mapsto \ G_I \tag{7}$$

from the set of $\mathcal{O}_L$-ideals to the set of upper ramification groups preserves $\subseteq$, that is, if $I \subseteq J$, then $G_I \subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the series of upper ramification groups is linearly ordered by inclusion. Note that in general, φ will neither be injective nor surjective as a function to the set of normal subgroups of G. This gives rise to the task to determine the smallest ideal that is sent by φ to a group G_I in its image. To this end, for each subgroup H of G we define the (possibly fractional) $\mathcal{O}_L$-ideal

$$I_H \ := \ \left(\frac{\sigma b - b}{b} \ \Big| \ \sigma \in H \, , \, b \in L^\times \right) \ = \ \left(\frac{\sigma b}{b} - 1 \ \Big| \ \sigma \in H \, , \, b \in L^\times \right) \tag{8}$$

and consider the function

$$\psi : \ H \ \mapsto \ I_H \tag{9}$$

from the set of all subgroups H of $G_{\mathcal{O}_L}$ to the set of all $\mathcal{O}_L$-ideals. Also ψ preserves $\subseteq$ and is in general neither injective nor surjective. However, it is easy to see that $G_{(0)} = \{\mathrm{id}\}$ and $I_{\{\mathrm{id}\}} = (0)$. If I_H is nonzero, then it is called it a **ramification ideal**; if in addition it is contained in $\mathcal{M}_L$, then it is called it a **higher ramification ideal**. Observe that the only ramification ideal that is not a higher ramification ideal is $\mathcal{O}_L$ and therefore, every ramification ideal of a purely wild extension is a higher ramification ideal; so when we deal with such extensions, we will drop the word "higher" for simplicity.

We note:

Proposition 2.6. *1) For every $\mathcal{O}_L$-ideal J, the group G_J is the largest of all subgroups H' of G such that $I_{H'} \subseteq J$.*

2) For every subgroup H of G, the ideal I_H is the smallest of all $\mathcal{O}_L$-ideals J' such that $H \subseteq G_{J'}$.

3) For every subgroup H of G and every $\mathcal{O}_L$-ideal J,

$$I_{G_{I_H}} \ = \ I_H \ \text{ and } \ G_{I_{G_J}} \ = \ G_J \, .$$

hence φ is an inclusion preserving bijection from the set of all ramification ideals together with (0) onto the set of all upper ramification groups, with ψ its inverse.

4) The function φ induces an inclusion preserving bijection from the set of all higher ramification ideals onto the set of all higher ramification groups, with its inverse induced by ψ.

5) A subgroup H of G is an upper ramification group if and only if it is a subgroup of $G_{\mathcal{O}_L}$ and for every subgroup H' of $G_{\mathcal{O}_L}$ we have $H \subsetneq H' \Rightarrow I_H \subsetneq I_{H'}$.

6) An $\mathcal{O}_L$-ideal J is a ramification ideal if and only if it is nonzero and for every $\mathcal{O}_L$-ideal J' we have $J' \subsetneq J \Rightarrow G_{J'} \subsetneq G_J$.

7) If $\mathcal{E} = (L|K, v)$ is a nontrivial unibranched Galois extension, then I_G is its largest ramification ideal. If in addition $\mathcal{E}$ is of prime degree, then I_G is its unique ramification ideal.

Proof. 1) and 2) follow directly from the definitions of G_J and I_H.

3): If $J = I_H$, then it follows from part 1) that $H \subseteq G_J$. Thus $J = I_H \subseteq I_{G_J} \subseteq J$, so $I_{G_J} = J$. If $H = G_J$, then it follows from part 2) that $I_H \subseteq J$. Thus $H \subseteq G_{I_H} \subseteq G_J = H$, so $G_{I_H} = H$.

4): If I_H is a higher ramification ideal, then I_H is nonzero and contained in $\mathcal{M}_L$, whence $G_{I_H} \subseteq G_{\mathcal{M}_L}$, so G_{I_H} is a higher ramification group. This shows that φ sends higher ramification ideals to higher ramification groups.

If G_J is a higher ramification group, then $G_J \subseteq G_{\mathcal{M}_L}$, hence $I_{G_J} \subseteq I_{G_{\mathcal{M}_L}} \subseteq \mathcal{M}_L$, where the second inequality follows from part 1). Moreover, since G_J is nontrivial, I_{G_J} is nonzero. This shows that ψ sends higher ramification groups to higher ramification ideals.

Now the assertion of part 4) follows from part 3).

5): Assume first that H is an upper ramification group. Then it is a subgroup of $G_{\mathcal{O}_L}$ and there is an $\mathcal{O}_L$-ideal J such that $H = G_J$. Take a subgroup H' of $G_{\mathcal{O}_L}$ which properly contains G_J. Then by part 1), $I_H = I_{G_J} \subsetneq I_{H'}$.

For the converse, assume that H is a subgroup of $G_{\mathcal{O}_L}$ such that for every subgroup H' of $G_{\mathcal{O}_L}$ we have $H \subsetneq H' \Rightarrow I_H \subsetneq I_{H'}$. By part 1), G_{I_H} is the largest of all subgroups H' of $G_{\mathcal{O}_L}$ such that $I_{H'} \subseteq I_H$. Therefore $G_{I_H} = H$, which shows that H is an upper ramification group.

6): Assume first that J is a ramification ideal. Then it is nonzero and by part 3) there is an upper ramification group H such that $J = I_H$. If J' is an $\mathcal{O}_L$-ideal properly contained in I_H, then by parts 2) and 3), $G_{J'} \subsetneq H = G_{I_H} = G_J$.

For the converse, assume that J is a nonzero $\mathcal{O}_L$-ideal such that for every $\mathcal{O}_L$-ideal J' we have $J' \subsetneq J \Rightarrow G_{J'} \subsetneq G_J$. By part 2), I_{G_J} is the smallest of all $\mathcal{O}_L$-ideals J' such that $G_J \subseteq G_{J'}$. Therefore $J = I_{G_J}$, which shows that J is a ramification ideal.

7): Since $\mathcal{E}$ is a nontrivial unibranched extension, also $G = G_{\mathcal{O}_L}$ is nontrivial, which by definition of I_G implies that I_G is nonzero. Since $G = G_{\mathcal{O}_L}$, we have $I_G \subseteq \mathcal{O}_L$. Thus I_G is a ramification ideal. As ψ preserves inclusion, I_G is the largest ramification ideal of $\mathcal{E}$.

If in addition $\mathcal{E}$ is of prime degree, then the only subgroups of G are G and $\{\mathrm{id}\}$. Since $I_{\{\mathrm{id}\}} = (0)$ is not a ramification ideal, I_G is then the unique ramification ideal of $\mathcal{E}$. □

By means of the bijection defined in (1) and (2), the upper ramification groups can be represented as

$$G_\Sigma := G_{I_\Sigma} = \left\{ \sigma \in G \;\middle|\; v\,\frac{\sigma b - b}{b} \in \Sigma \cup \{\infty\} \text{ for all } b \in L^\times \right\},$$

where Σ runs through all final segments of $(vL)^{\geq 0}$ and $\emptyset$.

Like the function (7), also the function $\Sigma \mapsto G_\Sigma$ is in general neither injective nor surjective. We call a final segment Σ of $(vL)^{\geq 0}$ a **ramification jump** if and only if

$$\Sigma' \subsetneq \Sigma \;\Rightarrow\; G_{\Sigma'} \subsetneq G_\Sigma$$

for every final segment Σ' of $(vL)^{\geq 0}$.

In [27] and [6], ramification ideals are defined by use of ramification jumps. The following result shows the equivalence with our above definition.

Proposition 2.7. *Take a nontrivial unibranched Galois extension $\mathcal{E} = (L|K, v)$. Then a nonempty final segment Σ of $(vL)^{\geq 0}$ is a ramification jump if and only if I_Σ is a ramification ideal.*

Proof. First note that for every nonempty final segment Σ of $(vL)^{\geq 0}$ the ideal I_Σ is nonzero and contained in $\mathcal{M}_L$. Now a nonempty final segment Σ of $(vL)^{\geq 0}$ is a ramification jump if and only if for every nonempty final segment Σ' of $(vL)^{\geq 0}$ we have $\Sigma' \subsetneq \Sigma \Rightarrow G_{I_{\Sigma'}} \subsetneq G_{I_\Sigma}$. This holds if and only if for every nonzero $\mathcal{O}_L$-ideal J' we have $J' \subsetneq I_\Sigma \Rightarrow G_{J'} \subsetneq G_{I_\Sigma}$. By part 6) of Proposition 2.6, this in turn holds if and only if I_Σ is a ramification ideal. □

By Propositions 2.6 and 2.7, the number of ramification ideals and ramification jumps in a unibranched Galois extension is bounded from above by the number of nontrivial subgroups of its Galois group. It may not always be equal to this number, as an example given in Section 3.5 below will show. For computations of the number of ramification ideals in certain Galois extensions, see [8].

Remark 2.8. In this paper we are particularly interested in the case where $\mathcal{E} = (L|K, v)$ is a purely wild, or at least a unibranched, Galois extension of prime degree p. Then by part 7) of Proposition 2.6, $\mathcal{E}$ has the unique ramification ideal I_G, and we denote it by $I_\mathcal{E}$. Hence $\Sigma_\mathcal{E} := \Sigma_{I_\mathcal{E}}$ is the unique ramification jump of $\mathcal{E}$. As we will show in what follows, ramification jump and ramification ideal carry important information about the extension $(L|K, v)$. #

Remark 2.9. In [27, Section 2.1] we also included zero ideals and empty segments in the definitions of the functions (1) and (2). However, classically ramification jumps have always been defined as integers in the case of discrete valuations, and as real numbers in the case of valuations of rank one, i.e., their value group is archimedean ordered. This means that it can be embedded in the ordered additive group of the reals. However, the intended meaning of "jump" does not fit well with the value $v0 = \infty$. #

Further, we want to quickly discuss the **series of lower ramification groups**

$$G_I^l := \{\sigma \in G \mid \sigma b - b \in I \text{ for all } b \in \mathcal{O}_L\}, \tag{10}$$

where I runs through all $\mathcal{O}_L$-ideals (see [40, §12]). Note that $G^l_{\mathcal{M}_L}$ is the inertia group of $(L|K,v)$. Again, for every $\mathcal{O}_L$-ideal I, G^l_I is a normal subgroup of G (see [40, (d) on p.79]), and $G_I \subseteq G^l_I$ (see [40, (a) on p.78]). But in the case of an immediate extension $(L|K,v)$, the two groups coincide, as follows from the next, more general, result:

Lemma 2.10. *If $vL = vK$, then $G_I = G^l_I$ for all nonzero ideals I of $\mathcal{O}_L$ contained in $\mathcal{M}_L$.*

Proof. It suffices to show that $G^l_I \subseteq G_I$. Take $\sigma \in G^l_I$ and $b \in L^\times$. Since $vL = vK$, we can pick some $c \in K$ such that $vcb = 0$. As $\sigma \in G^l_I$, we have that $\sigma(cb) - cb \in I$. Since $vcb = 0$, it follows that

$$\frac{\sigma b - b}{b} \;=\; \frac{\sigma(cb) - cb}{cb} \;\in\; I\,.$$

This shows that $\sigma \in G_I$. □

2.6. **Valuation bases.**

Take an extension $(L|K,v)$. Elements $b_1, \ldots, b_n \in L$ are called **valuation independent** (over K) if for all choices of $c_1, \ldots, c_n \in K$,

$$v\sum_{i=1}^{n} c_i b_i \;=\; \min_i vc_i b_i\,.$$

If in addition these elements form a basis of $L|K$, then they are called a **valuation basis** of $(L|K,v)$. If the valuation basis contains 1, we will speak of a **valuation basis with 1**.

Recall that a unibranched extension $(L|K,v)$ is defectless if it satisfies the fundamental equality $[L:K] = \mathrm{e}\cdot\mathrm{f}$, where $\mathrm{e} = (vL:vK)$ is the ramification index and $\mathrm{f} = [Lv:Kv]$ is the inertia degree. In this case, $(L|K,v)$ admits a **standard valuation basis**, which we construct as follows: we take $y_1, \ldots, y_\mathrm{e} \in L$ such that $vy_1 + vK, \ldots, vy_\mathrm{e} + vK$ are the cosets of vK in vL, and $z_1, \ldots, z_\mathrm{f} \in L$ such that $z_1 v, \ldots, z_\mathrm{f} v$ are a basis of $Lv|Kv$. Then the products $y_i z_j$, $1 \le i \le \mathrm{e}$, $1 \le j \le \mathrm{f}$, form a valuation basis of $(L|K,v)$ (see [10, Lemma 3.2.2]). Note that we can always choose $y_1 = z_1 = 1$ so that $y_1 z_1 = 1$. We will then speak of a **standard valuation basis with 1**.

The next result has been shown in the proof of [17, Lemma 2.1].

Lemma 2.11. *Take a unibranched extension $(L|K,v)$ of prime degree p. If for $b \in L^\times$, either $vb \notin vK$ or there is some $c \in K$ such that $vcb = 0$ and $cbv \notin Kv$, then $1, b, \ldots, b^{p-1}$ forms a standard valuation basis with 1 of $(L|K,v)$.*

For the following, cf. [5, Proposition 3.4].

Lemma 2.12. *Take a finite unibranched extension $(L|K,v)$. Then the following are equivalent:*

a) $(L|K,v)$ is defectless,
b) $(L|K,v)$ admits a valuation basis,
c) $(L|K,v)$ admits a standard valuation basis,
d) $(L|K,v)$ admits a standard valuation basis with 1.

Proof. Implication a)⇒d) has just been shown above. Implications d)⇒c) and c)⇒b) are trivial. For the implication b)⇒a), see the proof of [5, Proposition 3.4]. □

In particular, for a finite unibranched defectless extension there is always a valuation basis with 1.

Lemma 2.13. *Take a finite unibranched defectless extension $(L|K,v)$ and $a \in L$. Then the set $\{v(a-c) \mid c \in K\}$ has a maximum. More precisely, if we choose a valuation basis $b_1,\dots,b_n$ for $(L|K,v)$ with $b_1 = 1$ and write*

$$a \;=\; \sum_{i=1}^{n} c_i b_i \,,$$

then $v(a-c_1)$ is the maximum of $\{v(a-c) \mid c \in K\}$.

Proof. For every $c \in K$,

$$\begin{aligned} v(a-c_1) \;&=\; v\sum_{i=2}^{n} c_i b_i \;=\; \min_{2\le i\le n} vc_ib_i \;\ge\; \min\{v(c_1-c)\,,\, vc_ib_i \mid 2\le i\le n\} \\ &=\; v\left(c_1 - c + \sum_{i=2}^{n} c_i b_i\right) \;=\; v(a-c)\,. \end{aligned}$$

□

Corollary 2.14. *Take a unibranched defectless extension of prime degree p and $a_0 \in L \setminus K$. Then there is some $c \in K$ such that for $a = a_0 - c$, the elements $1, a, \dots, a^{p-1}$ form a valuation basis.*

Proof. By Lemma 2.13 there is some $c \in K$ such that $v(a_0-c) = \max\{v(a_0-c) \mid c \in K\}$. By Lemma 2.3 this can only happen if either $v(a_0-c) \notin vK$ or there is some $d \in K$ such that $vd(a_0-c) = 0$ and $d(a_0-c)v \notin Kv$. We set $a = a_0 - c$; then in both cases, the elements $1, a, \dots, a^{p-1}$ form a valuation basis by Lemma 2.11. □

For a more general setting, see Lemma 2.10 and Corollary 2.11 of [5].

3. Computation of ramification ideals

3.1. **Basic computations.**

We are going to prove:

Proposition 3.1. *Take a finite unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ with Galois group G. Then every ramification ideal is principal.*

Take a nontrivial subgroup H of G. We will prove the proposition by giving an algorithm for the computation of an element $b_{\min}$ such that for some $\sigma \in H$,

$$v\left(\frac{\sigma b_{\min}}{b_{\min}} - 1\right) \;=\; \min\left\{ v\left(\frac{\sigma b}{b} - 1\right) \,\middle|\, \sigma \in H\,,\, b \in L^\times \right\}\,, \tag{11}$$

which means that $\frac{\sigma b_{\min}}{b_{\min}} - 1$ generates the ramification ideal (8).

Remark 3.2. This proposition was proven in 1970 by P. Ribenboim in [36] for rank one valuations. He uses that after embedding vL in $\mathbb{R}$, every subset of vL bounded from below has an infimum in $\mathbb{R}$. Our computations presented below are inspired by his. As they will show, the assumption of rank one is not necessary. The use of infima can be replaced by the use of final segments of vL. In the same way, results proven by Ribenboim in [33] for rank one have been generalized to arbitrary rank in [24].

A different version of the computation was presented by M. Marshall in [29]. He does not assume that (L, v) has rank one, but that (K, v) is maximally complete and that the extension $Lv|Kv$ is separable. The assumption that (K, v) is maximally complete means that it has no nontrivial immediate extensions, and this implies that (K, v) is defectless and henselian.

In [35] Ribenboim attempts to prove Proposition 3.1 for all unibranched Galois extensions with non-discrete rank one valuations without the "defectless" assumption, but this is false. Indeed, take Example 3.12 of a defect extension of prime degree from [18] (originally, Abhyankar gave this example for a different purpose without noticing the defect). By part 3) of Corollary 3.20 below, its unique ramification ideal is a nonprincipal $\mathcal{O}_L$-ideal.

Ribenboim's mistake was noticed by J. L. Chabert. In [36] Ribenboim then gives a correct proof of Proposition 3.1 for all finite defectless unibranched Galois extensions in the case of rank one valuations. Curiously, his false proposition from [35] is later repeated in [37] with literally the same incorrect proof. #

We shall now present computations that will not only prove the above proposition, but will also be used later for more advanced results. Let us start with some useful basic principles.

Lemma 3.3. *Let K be any field and take $\sigma \in \operatorname{Gal} K$.*

1) For all $a, b \in \tilde{K}^\times$ and $c \in K$,

$$\frac{\sigma cab}{cab} - 1 \;=\; \frac{\sigma ab}{ab} - 1 \;=\; \left(\frac{\sigma a}{a} - 1\right)\left(\frac{\sigma b}{b} - 1\right) \;+\; \left(\frac{\sigma a}{a} - 1\right) \;+\; \left(\frac{\sigma b}{b} - 1\right) . \tag{12}$$

2) Assume that v is a valuation on $\tilde{K}$ and that $a \in \tilde{K}$ is such that $v\left(\frac{\sigma a}{a} - 1\right) > 0$. Take $i \in \mathbb{N}$ and assume that $i < \operatorname{char} Kv$ if $\operatorname{char} Kv > 0$. Then

$$v\left(\frac{\sigma a^i}{a^i} - 1\right) \;=\; v\left(\frac{\sigma a}{a} - 1\right) . \tag{13}$$

Proof. 1): We leave the straightforward proof to the reader.

2): By our assumption on i, we have $vi = 0$. Using this together with equation (12), one proves equation (13) by induction on i. □

Further, we will need the following facts. We note that if $(L|K, v)$ is a unibranched Galois extension, then for every $\sigma \in \operatorname{Gal} L|K$ and $b \in L^\times$,

$$\frac{\sigma b}{b} - 1 \;\in\; \mathcal{O}_L . \tag{14}$$

Lemma 3.4. *Assume that $(L|K, v)$ is a purely wild Galois extension. Then for every $\sigma \in \operatorname{Gal} L|K$ and all $a, b \in L^\times$,*

$$\frac{\sigma b}{b} - 1 \ \in \ \mathcal{M}_L \tag{15}$$

and

$$v\left(\frac{\sigma cab}{cab} - 1\right) \ \geq \ \min\left\{v\left(\frac{\sigma a}{a} - 1\right), v\left(\frac{\sigma b}{b} - 1\right)\right\}, \tag{16}$$

with equality holding if $v\left(\frac{\sigma a}{a} - 1\right) \neq v\left(\frac{\sigma b}{b} - 1\right)$.

Proof. Equation (15) holds since by the definition of "purely wild extension", $\operatorname{Gal} L|K = G_{\mathcal{M}_L}$. Equation (16) follows from equations (12) and (15). □

Lemma 3.5. *Take a normal unibranched extension $(L|K, v)$ and an intermediate field L_0. Assume that the elements $b_1, \ldots, b_n \in L$ are valuation independent over L_0 and write*

$$0 \ \neq \ b \ = \ \sum_{i=1}^{n} a_i b_i \tag{17}$$

with elements $a_1, \ldots, a_n \in L_0$. Then for each automorphism σ of L over K, not necessarily trivial on L_0,

$$v\left(\frac{\sigma b}{b} - 1\right) \ \geq \ \min_i v\left(\frac{\sigma a_i b_i}{a_i b_i} - 1\right) \tag{18}$$

where the minimum runs over all i for which $a_i \neq 0$. If in addition σ is trivial on all b_i, then $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_i}{a_i} - 1$. On the other hand, if σ is trivial on all a_i, then $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma b_i}{b_i} - 1$.

Proof. We have

$$\frac{\sigma b}{b} - 1 \ = \ \sum_i \left(\frac{\sigma a_i b_i}{a_i b_i} - 1\right) \cdot \frac{a_i b_i}{b} \tag{19}$$

where the sum runs over all i for which $a_i \neq 0$. Since $vb \leq va_ib_i = v\sigma a_i b_i$ for $1 \leq i \leq n$, this implies (18) and that $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_L$-ideal generated by the elements $\frac{\sigma a_i b_i}{a_i b_i} - 1$. If in addition $\sigma b_i = b_i$ for all i, then $\frac{\sigma a_i b_i}{a_i b_i} - 1 = \frac{\sigma a_i}{a_i} - 1$. On the other hand, if $\sigma a_i = a_i$ for all i, then $\frac{\sigma a_i b_i}{a_i b_i} - 1 = \frac{\sigma b_i}{b_i} - 1$. □

Corollary 3.6. *Assume that $\mathcal{E} = (L|K, v)$ is a finite defectless unibranched Galois extension with Galois group G.*

1) Choose a valuation basis b_i, $1 \leq i \leq n$. Then for all $b \in L^\times$,

$$v\left(\frac{\sigma b}{b} - 1\right) \ \geq \ \gamma_{\mathcal{E}} \ := \ \min\left\{v\left(\frac{\sigma b_i}{b_i} - 1\right) \,\middle|\, \sigma \in G, \, 1 \leq i \leq n\right\} \ \geq \ 0\,. \tag{20}$$

Hence I_G is principal with generator $b_{\min}$ which can be chosen to be b_i for suitable i. Then $vb_{\min} = \gamma_{\mathcal{E}}$.

2) Assume in addition that $\mathcal{E}$ is purely wild and choose a standard valuation basis y_iz_j, $1 \le i \le e$, $1 \le j \le f$ of $(L|K,v)$ as described in Section 2.6. Then for all $b \in L^\times$,

$$v\left(\frac{\sigma b}{b}-1\right) \ge \min\left\{ v\left(\frac{\sigma y_i}{y_i}-1\right), v\left(\frac{\sigma z_j}{z_j}-1\right) \,\middle|\, \sigma \in G\,,\ 1 \le i \le \mathrm{e}\,,\ 1 \le j \le \mathrm{f} \right\}.$$

Proof. 1): It follows from (14) that no values in (20) are less than 0. Since $\sigma \in G$ is trivial on all $a \in L_0$, our statement follows from Lemma 3.5, where we take $L_0 = K$.

2): Take $b \in L$ and write it in the form $b = \sum_{i,j} c_{i\,j} y_i z_j$. Then our assertion follows from part 1) of our corollary together with (16). □

Proposition 3.7. *Assume that $(L|K,v)$ is a finite purely wild Galois extension with Galois group G. Assume further that L_0 is an intermediate field of $L|K$ such that $(L|L_0,v)$ is defectless and that b_i, $1 \le i \le n$ is a valuation basis of $(L|L_0,v)$. Define*

$$\gamma := \min\left\{ v\left(\frac{\sigma b_i}{b_i}-1\right) \,\middle|\, \sigma \in G\,,\ 1 \le i \le n \right\} > 0\,. \tag{21}$$

Assume further that there is $\gamma_0 \in vL$ such that

$$v\left(\frac{\sigma a}{a}-1\right) \ge \gamma_0 \quad \text{for all } \sigma \in G \text{ and } a \in L_0^\times\,. \tag{22}$$

Then

$$v\left(\frac{\sigma b}{b}-1\right) \ge \min\{\gamma_0\,,\gamma\} \quad \text{for all } \sigma \in G \text{ and } b \in L^\times\,. \tag{23}$$

If $\gamma \le \gamma_0$, then

$$\min\left\{ v\left(\frac{\sigma b}{b}-1\right) \,\middle|\, \sigma \in G\,,\ b \in L^\times \right\} = \gamma\,, \tag{24}$$

hence I_G is principal and $b_{\min}$ can be chosen to be b_i for suitable i.

Proof. Take $b \in L^\times$ and write it in the form (17) with $a_i \in L_0$. Using (18) together with (16), we obtain:

$$\begin{aligned} v\left(\frac{\sigma b}{b}-1\right) &\ge \min\left\{ v\left(\frac{\sigma a_i b_i}{a_i b_i}-1\right) \,\middle|\, \sigma \in G\,,\ a_i \in L_0^\times\,,\ 1 \le i \le n \right\} \\ &\ge \min\left\{ v\left(\frac{\sigma a_i}{a_i}-1\right), v\left(\frac{\sigma b_i}{b_i}-1\right) \,\middle|\, \sigma \in G\,,\ a_i \in L_0^\times\,,\ 1 \le i \le n \right\} \\ &= \min\{\gamma_0\,,\gamma\}\,, \end{aligned}$$

which proves (23). The last assertion is obvious. □

Proof of Proposition 3.1:
Take any nontrivial subgroup H of $G_{\mathcal{M}_L}$ and denote its fixed field in L by K_H. Then also $\mathcal{E}_H := (L|K_H,v)$ is a finite unibranched Galois extension, by Lemma 2.1 it is again defectless, and its Galois group is H. Applying part 1) of Corollary 3.6 with $\mathcal{E}_H$ in place of $\mathcal{E}$, equation (20) yields that I_H is principal. This proves our proposition. □

Finally, we prove a generalization of a fact that has been used in [38, Section 7.1]. A unibranched algebraic extension $(L|K, v)$ is called **tame** if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by $\operatorname{char} Kv$,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ is defectless.

Every subextension of a tame extension is tame. For detailed information on tame extensions and their connection with purely wild extensions, see [20, 26]. We will need:

Lemma 3.8. *Take a henselian field (K, v), a purely wild Galois extension $(L|K, v)$ and a tame Galois extension $(K'|K, v)$. Let L' be the field compositum $L.K'$ of L and K', endowed with the unique extension of v from K. Then $L|K$ is linearly disjoint from $K'|K$, $(L'|K', v)$ is a purely wild Galois extension of degree $[L : K]$, and $(L'|L, v)$ is a tame extension of degree $[K' : K]$.*

Proof. By [26, Proposition 4.1], an extension $(K'|K, v)$ is tame if and only if it is a subextension of $(K^r|K, v)$, where (K^r, v) is the absolute ramification field of (K, v). By It follows that $L|K$ and $K'|K$ are linearly disjoint, $[L' : K'] = [L : K]$ and $[L' : L] = [K' : K]$.

From [9, (20.15), p. 166] we know that $(L^r, v) = (L.K^r, v)$ and $((K')^r, v) = (K'.K^r, v)$. Since $(K'|K, v)$ is tame, we have $K' \subseteq K^r$ and so, $L' = L.K' \subseteq L.K^r = L^r$, showing that $(L'|L, v)$ is tame. Since $(L|K, v)$ is purely wild, $L|K$ is linearly disjoint from $K^r|K$. It follows that $L' = L.K'$ is linearly disjoint from $K'.K^r = (K')^r$ over K', showing that $(L'|K', v)$ is purely wild. □

Proposition 3.9. *Take a henselian field (K, v), a finite purely wild Galois extension $(L|K, v)$ and a tame Galois extension $(K'|K, v)$. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K', v)$ is a purely wild Galois extension of degree $[L : K]$, and*

$$I \;\mapsto\; I\mathcal{O}_{L'} \tag{25}$$

is a bijection between the ramification ideals of $(L|K, v)$ and those of $(L'|K', v)$. Its inverse is the function

$$I' \;\mapsto\; I' \cap \mathcal{O}_L \tag{26}$$

from the ramification ideals of $(L'|K', v)$ to those of $(L|K, v)$.

Proof. By Lemma 3.8, $L|K$ and $K'|K$ are linearly disjoint. Therefore, $L'|K'$ is a Galois extension with its Galois group G isomorphic to $\operatorname{Gal} L|K$ via the restriction

$$\operatorname{Gal} L'|K' \ni \sigma \;\mapsto\; \sigma|_L \in \operatorname{Gal} L|K\ .$$

For a subgroup $H' \subseteq \operatorname{Gal} L'|K'$, denote its image under the restriction by H. If we start with a subgroup $H \subseteq \operatorname{Gal} L|K$, then the mentioned isomorphism yields $H' \in \operatorname{Gal} L'|K'$.

Take a finite Galois subextension $(K_0'|K, v)$ of $(K'|K, v)$. It is again tame, hence by Lemma 3.8, $(L_0'|L, v)$ is a finite tame Galois extension, where $L_0' = L.K_0'$. In particular, every valuation basis $b_1, \ldots, b_n$ of $(K_0'|K, v)$ is also a valuation basis of $(L_0'|L, v)$.

Each element $b \in L'$ already lies in such a compositum $L'_0 = L.K'_0$, so it can be written as $b = \sum_{1 \le i \le n} a_i b_i$ with $b_1, \ldots, b_n$ a valuation basis of $(K'_0|K, v)$ and suitable elements $a_i \in L$.

Now take a nontrivial subgroup H of $\mathrm{Gal}\, L|K$ and consider the ramification ideal $I_{H'}$ of $(L'|K', v)$. Take $\sigma \in H'$ and $0 \neq b \in L'_0$ such that $\frac{\sigma b}{b} - 1 \in I_{H'}$ and write b in the form as indicated above. Since σ is trivial on K', Lemma 3.5 with L'_0 in place of L and L in place of K shows that $\frac{\sigma b}{b} - 1$ lies in the $\mathcal{O}_{L'_0}$-ideal generated by the elements

$$\frac{\sigma a_i}{a_i} - 1 \;=\; \frac{\sigma|_L\, a_i}{a_i} - 1 \;\in\; I_H\,.$$

Therefore,

$$\frac{\sigma b}{b} - 1 \;\in\; I_H \mathcal{O}_{L'_0} \;\subseteq\; I_H \mathcal{O}_{L'}\,, \tag{27}$$

which shows that the ramification ideal $I_{H'}$ of $(L'|K', v)$ is a subset of $I_H \mathcal{O}_{L'}$. To prove the reverse inclusion, take an element $\frac{\tau a}{a} - 1 \in I_H$, where $\tau \in H$ and $a \in L^\times$. We write $\tau = \sigma|_L$ with $\sigma \in H'$. Then

$$\frac{\tau a}{a} - 1 \;=\; \frac{\sigma a}{a} - 1 \;\in\; I_{H'}\,.$$

Thus $I_H \subseteq I_{H'}$, and we obtain

$$I_H \mathcal{O}_{L'} \;=\; I_{H'}\,.$$

This proves that the function (25) sends ramification ideals of $(L|K, v)$ to ramification ideals of $(L'|K', v)$. It also shows that $I_{H'}$ is the collection of all elements in L' whose value is not less than the value of some element in I_H. This implies that $I_{H'} \cap \mathcal{O}_L$ is the collection of all elements in L whose value is not less than the value of some element in I_H. In other words, $I_{H'} \cap \mathcal{O}_L = I_H \mathcal{O}_L = I_H$. Hence, $I_H \mathcal{O}_{L'} \cap \mathcal{O}_L = I_H$, which proves that the function (25) is a bijection, with the function (26) its inverse. □

Remark 3.10. In [38, Section 7.1] only the special case is considered where (K, v) is a henselian field of mixed characteristic, $L|K$ has prime degree p and $K' = K(\zeta_p)$ where ζ_p is a p-th root of unity. The latter implies that $(K'|K, v)$ is a tame extension. This case is of interest when $L|K$, though being Galois, is not a Kummer extension, since $L'|K'$ will be a Kummer extension. #

With a proof adapted from the one of the previous proposition, the following can be shown:

Proposition 3.11. *Take a henselian field (K, v), a finite immediate Galois extension $(L|K, v)$ and a Galois extension $(K'|K, v)$ for which every finite subextension is defectless. Then with the unique extension of v to the compositum $L' = L.K'$, also $(L'|K', v)$ is an immediate Galois extension of degree $[L : K]$, and (25) is again a bijection between the ramification ideals of $(L|K, v)$ and those of $(L'|K', v)$.* □

3.2. **Ramification ideals and defect.**

We start with the following easy observation. If $(K(a)|K,v)$ is any algebraic extension of valued fields and $\sigma \in \mathrm{Gal}\,(K)$ is such that $\sigma a \neq a$, then

$$\left\{ v\,\frac{\sigma(a-c)-(a-c)}{a-c} \;\middle|\; c\in K \right\} = \left\{ v\,\frac{\sigma a-a}{a-c} \;\middle|\; c\in K \right\} = -v(a-K)+v(\sigma a-a)\,.$$

Lemma 3.12. *Take an extension $(K(a)|K,v)$ of prime degree and any $b\in K(a)^\times$. Assume that $v(a-K)$ has no maximal element. Then for all $\sigma\in\mathrm{Gal}\,(K)$ such that $\sigma a\neq a$ and $v(a-K)<v(\sigma a-a)$, there is some $c\in K$ such that*

$$v\,\frac{\sigma b-b}{b} \;>\; -v(a-c)\,+\,v(\sigma a-a)\;. \tag{28}$$

Proof. Since we are already assuming that $v(a-K)$ has no maximal element and that $v(a-K)<v(\sigma a-a)$, the proof can be taken over from the proof of [27, Lemma 3.3], using [27, Lemma 2.11]. □

Now take a Galois extension $\mathcal{E}=(L|K,v)$ of prime degree p with Galois group G. For $\sigma\in G\setminus\{\mathrm{id}\}$ we set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma b-b}{b}\right) \;\middle|\; b\in L^\times \right\}\;. \tag{29}$$

Theorem 3.13. *Assume that $a\in L$ is a generator of $\mathcal{E}$ such that $v(a-K)$ has no maximal element. Then for every $\sigma\in G\setminus\{\mathrm{id}\}$ such that $v(a-K)<v(\sigma a-a)$,*

$$\Sigma_\sigma \;=\; -v(a-K)+v(\sigma a-a)\,, \tag{30}$$

which is a final segment of vL.

Proof. Since $v(a-K)$ is an initial segment of vK by Theorem 2.4, $-v(a-K)$ is a final segment of vK, hence $-v(a-K)+v(\sigma a-a)$ is a final segment of vL. Therefore, Equation (30) follows from Lemma 3.12 together with our observation at the start of the section. □

We set

$$\Sigma_{\mathcal{E}} \;:=\; \bigcup_{\mathrm{id}\neq\sigma\in G} \Sigma_\sigma \;=\; \left\{ v\left(\frac{\sigma b-b}{b}\right) \;\middle|\; \mathrm{id}\neq\sigma\in G\,,\, b\in L^\times \right\}\;. \tag{31}$$

As a union of final segments of vK, $\Sigma_{\mathcal{E}}$ is itself a final segment of vK. Thus we can set

$$I_{\mathcal{E}} \;:=\; I_{\Sigma_{\mathcal{E}}}\;. \tag{32}$$

If all Σ_σ are bouded from below, then so is $\Sigma_{\mathcal{E}}$ and $I_{\mathcal{E}}$ is a (possibly fractional) $\mathcal{O}_L$-ideal.

Proposition 3.14. *Assume that $a\in L$ is a generator of $\mathcal{E}$ such that $v(a-K)$ has no maximal element.*

1) If $\mathcal{E}$ is unibranched, then $v(\sigma a-a)$ and Σ_σ are independent of the choice of $\sigma\in G\setminus\{\mathrm{id}\}$.

2) With $\Sigma_{\mathcal{E}}$ and $I_{\mathcal{E}}$ as in (31) and (32), the following assertions are equivalent:

a) for all $\sigma\in G\setminus\{\mathrm{id}\}$, $v(a-K)$ is bounded from above by $v(\sigma a-a)$,

b) $\Sigma_{\mathcal{E}} \subseteq vL^{>0}$,

c) $I_{\mathcal{E}}$ *is a higher ramification ideal,*

d) $\mathcal{E}$ *is purely wild,*

e) $\mathcal{E}$ *is an immediate defect extension,*

f) $\mathcal{E}$ *is unibranched.*

Proof. 1): Take any $\sigma \in G \setminus \{\mathrm{id}\}$. Since $\mathcal{E}$ is unibranched, we have $v = v\sigma$ on L. Thus, $v(\sigma^2 a - \sigma a) = v\sigma(\sigma a - a) = v(\sigma a - a)$ and therefore,

$$v(\sigma^2 a - a) \;\geq\; \min\{v(\sigma^2 a - \sigma a), v(\sigma a - a)\} \;=\; v(\sigma a - a)\,.$$

Assume that $k < p-1$ and we have already shown that $v(\sigma^k a - a) \geq v(\sigma a - a)$. Then $v(\sigma^{k+1} a - \sigma a) = v\sigma(\sigma^k a - a) = v(\sigma^k a - a) \geq v(\sigma a - a)$ and therefore,

$$v(\sigma^{k+1} a - a) \;\geq\; \min\{v(\sigma^{k+1} a - \sigma a), v(\sigma a - a)\} \;\geq\; v(\sigma a - a)\,.$$

By induction, it follows that $v(\sigma^i a - a) \geq v(\sigma a - a)$ for all $i \leq p-1$. Since both σ and σ^i for $i \leq p-1$ are generators of G and we had chosen an arbitrary generator σ, it follows that the value $v(\sigma a - a)$ and hence also Σ_σ is independent of our choice of σ.

2): The proof of the equivalence of assertions a), b), c) and d) is straightforward.

d)⇒e): The purely wild extension $\mathcal{E}$ cannot be defectless, because then Lemma 2.13 shows that $v(a-K)$ would have a maximal element for every $a \in L \setminus K$. Hence $\mathcal{E}$ is a defect extension, and since it has prime degree, it is immediate.

e)⇒f): By the definition of defect extensions.

f)⇒e): By Lemma 2.5, a unibranched extension $(K(a)|K, v)$ such that $v(a-K)$ has no maximal element is an immediate defect extension.

e)⇒d): Every defect extension $(L|K, v)$ of prime degree p is immediate, hence $(vL : vK)$ is a p-power and $Lv|Kv$ is purely inseparable, which shows that it is purely wild. □

In Remark 2.8, for a unibranched Galois extension $\mathcal{E} = (L|K, v)$ of prime degree, $I_{\mathcal{E}}$ was defined to be I_G, and we set $\Sigma_{\mathcal{E}} := \Sigma_{I_{\mathcal{E}}}$. If $\mathcal{E}$ has a generator $a \in L$ such that $v(a-K)$ has no maximal element, then by implication f)⇒c) of the proposition, the ideal $I_{\mathcal{E}}$ as defined in (32) is a higher ramification ideal, and as $\mathcal{E}$ has a unique ramification ideal, it must coincide with the ideal defined in Remark 2.8.

Corollary 3.15. *Assume that $a \in L$ is a generator of $\mathcal{E}$ such that $v(a-K)$ has no maximal element, and that $\Sigma_{\mathcal{E}} \subseteq vL^{>0}$. Then $\mathcal{E}$ is unibranched and an immediate defect extension, and its unique ramification ideal is the nonprincipal ideal*

$$I_{\mathcal{E}} \;=\; \left(\frac{\sigma_0\, a - a}{a - c} \;\mid\; c \in K\right) \;=\; \left(\frac{\sigma_0(a-c)}{a-c} - 1 \;\mid\; c \in K\right)\,, \tag{33}$$

where σ_0 is any generator of G.

Proof. Under the assumptions of the corollary, implications b)⇒f) and b)⇒e) of the previous proposition show that $\mathcal{E}$ is unibranched and an immediate defect extension. Implication b)⇒a) shows that $\Sigma_{\mathcal{E}} = \Sigma_\sigma$ for each $\sigma \in G \setminus \{\mathrm{id}\}$; this proves (33). Since $\Sigma_{\mathcal{E}}$ has no smallest element, $I_{\mathcal{E}}$ is nonprincipal. □

3.3. **Unibranched Galois extensions of prime degree.**

A Galois extension of degree p of a field K of characteristic $p > 0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** $X^p - X - c$ with $c \in K$. A Galois extension of degree p of a field K of characteristic 0 which contains all p-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^p \in K$. For these facts, see [28, Chapter VIII, §8].

If $(L|K, v)$ is a Galois defect extension of degree p of fields of characteristic 0, then a Kummer generator of $L|K$ can be chosen to be a 1-unit. Indeed, choose any Kummer generator η. Since $(L|K, v)$ is immediate, we have that $v\eta \in vK(\eta) = vK$, so there is $c \in K$ such that $vc = -v\eta$. Then $v\eta c = 0$, and since $\eta cv \in K(\eta)v = Kv$, there is $d \in K$ such that $dv = (\eta cv)^{-1}$. Then $v(\eta cd) = 0$ and $(\eta cd)v = 1$. Hence ηcd is a 1-unit. Furthermore, $K(\eta cd) = K(\eta)$ and $(\eta cd)^p = \eta^p c^p d^p \in K$. Thus we can replace η by ηcd and assume from the start that η is a 1-unit. It follows that also $\eta^p \in K$ is a 1-unit.

Throughout this article, whenever we speak of "Artin-Schreier extension" we refer to fields of positive characteristic, and with "Kummer extension" we refer to fields of characteristic 0.

3.3.1. *The defectless case.*

The following proposition is taken from [6]. For the convenience of the reader, and as an illustration of the usefulness of Lemma 2.13, we include its proof here.

Proposition 3.16. *1) Take a valued field (K, v) of equal positive characteristic p and a unibranched defectless Artin-Schreier extension $(L|K, v)$.*

If $\mathrm{f}(L|K, v) = p$, then the extension has an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $Lv = Kv(\tilde{c}\vartheta v)$ for every $\tilde{c} \in K$ with $v\tilde{c}\vartheta = 0$; the extension $Lv|Kv$ is separable if and only if $v\vartheta = 0$.

If $\mathrm{e}(L|K, v) = p$, then the extension has an Artin-Schreier generator ϑ such that $vL = vK + \mathbb{Z}v\vartheta$. Every such ϑ satisfies $v\vartheta < 0$.

2) Take a valued field (K, v) of mixed characteristic and a unibranched defectless Kummer extension $(L|K, v)$ of degree p not necessarily equal to $\mathrm{char}\, Kv$*. Then the extension has a Kummer generator η such that:*

a) if $\mathrm{f}(L|K, v) = p$, then either ηv generates the residue field extension, in which case it is inseparable if and only if $p =$ $\mathrm{char}\, Kv$*, or η is a 1-unit and for some $\tilde{c} \in K$, $\tilde{c}(\eta - 1)v$ generates the residue field extension;*

b) if $\mathrm{e}(L|K, v) = p$, then either $v\eta$ generates the value group extension, or η is a 1-unit and $v(\eta - 1)$ generates the value group extension.

Proof. 1): Take any Artin-Schreier generator y of $(L|K, v)$. Then by Lemma 2.13 there is $c \in K$ such that either $v(y - c) \notin vK$, or for every $\tilde{c} \in K$ such that $v\tilde{c}(y - c) = 0$ we have $\tilde{c}(y - c)v \notin Kv$. Since p is prime, in the first case it follows that $\mathrm{e}(L|K, v) = p$ and that $v(y - c)$ generates the value group extension. In the second case it follows that $\mathrm{f}(L|K, v) = p$ and that $\tilde{c}(y - c)v$ generates the residue field extension. In both cases, $\vartheta = y - c$ is an Artin-Schreier generator. Let $\vartheta^p - \vartheta = b \in K$.

Assume that $\mathrm{f}(L|K,v) = p$. If $v\vartheta < 0$, then $v(\vartheta^p - b) = v\vartheta > pv\vartheta = v\vartheta^p$, whence $v((\tilde{c}\vartheta)^p - \tilde{c}^p b) = v\tilde{c}^p\vartheta > v(\tilde{c}\vartheta)^p$ for $\tilde{c} \in K$ with $v\tilde{c}\vartheta = 0$ and therefore, $(\tilde{c}\vartheta)^p v = \tilde{c}^p bv \in Kv$. In this case, the residue field extension is inseparable. Now assume that $v\vartheta \geq 0$ and hence also $vb \geq 0$. The reduction of $X^p - X - b$ to $Kv[X]$ is a separable polynomial, so $Lv|Kv$ is separable. The polynomial $X^p - X - bv$ cannot have a zero in Kv, since otherwise the p distinct roots of this polynomial give rise to p distinct extensions of v from K to L, contradicting our assumption that $(L|K,v)$ is unibranched. Consequently, $bv \neq 0$, whence $vb = 0$ and $v\vartheta = 0$.

Assume that $\mathrm{e}(L|K,v) = p$. If $v\vartheta \geq 0$, then $vb \geq 0$ and ϑv is a root of $X^p - X - bv$. If this polynomial does not have a zero in Kv, then ϑv generates a nontrivial residue field extension, contradicting our assumption that $\mathrm{e}(L|K,v) = p$. If the polynomial has a zero in Kv, then similarly as before one deduces that $(L|K,v)$ is not unibranched, contradiction. Hence $v\vartheta < 0$.

2): Take any Kummer generator y of $(L|K,v)$. If there is a Kummer generator η such that $v\eta \notin vK$, then it follows as before that $\mathrm{e}(L|K,v) = p$ and that $v\eta$ generates the value group extension. Now assume that there is no such η.

If there is a Kummer generator y and some $\tilde{c} \in K$ such that $v\tilde{c}y = 0$ and $\tilde{c}yv \notin Kv$, then it follows as before that $\mathrm{f}(L|K,v) = p$ and that $\tilde{c}yv$ generates the residue field extension. We set $\eta = \tilde{c}y$ and observe that also η is a Kummer generator. Since $(\eta v)^p \in Kv$, in this case $Lv|Kv$ is purely inseparable if and only if $p = \mathrm{char}\, Kv$.

Now assume that the above cases do not appear, and choose an arbitrary Kummer generator y of $(L|K,v)$. Consequently, we have that $vy \in vK$ and $\tilde{c}yv \in Kv$ for all $\tilde{c} \in K$ with $v\tilde{c}y = 0$. Then as described at the start of this section, there are $c_1, c_2 \in K$ such that c_2c_1y is a Kummer generator of $(L|K,v)$ which is a 1-unit. We replace y by c_2c_1y.

By Lemma 2.13 there is $c \in K$ such that $v(y-c)$ is maximal in $v(y-K)$ and either $v(y-c) \notin vK$ or there is some $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since y is a 1-unit, we know that $v(y-1) > 0$, hence also $v(y-c) > 0 = vy$, showing that also c is a 1-unit. Then $\eta := c^{-1}y$ is again a Kummer generator of $(L|K,v)$ which is a 1-unit. Since $vc = 0$, we know that $v(\eta - 1) = vc(\eta - 1) = v(y - c)$. Hence if $v(y-c) \notin vK$, then $v(\eta - 1)$ generates the value group extension.

Now assume that there is $\tilde{c} \in K$ such that $v\tilde{c}(y-c) = 0$ and $\tilde{c}(y-c)v \notin Kv$. Since c is a 1-unit, it follows that $v\tilde{c}(\eta - 1) = v\tilde{c}c(\eta - 1) = v\tilde{c}(y - c) = 0$ and $\tilde{c}(\eta - 1)v = \tilde{c}c(\eta - 1)v = \tilde{c}(y - c)v$. We find that $\tilde{c}(\eta - 1)v$ generates the residue field extension. □

From this proposition we deduce:

Theorem 3.17. *Take a unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ of prime degree p. Let $\gamma_{\mathcal{E}}$ be as defined in (20).*

1) If $\mathcal{E}$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis for $(L|K,v)$. The element $b_{\min}$ as in (11) can be chosen to be ϑ, so that

$$I_{\mathcal{E}} = \left(\frac{1}{\vartheta}\right) \tag{34}$$

and $\gamma_{\mathcal{E}} = -v\vartheta$. *We have* $I_{\mathcal{E}} = \mathcal{O}_L$ *if and only if* $v\vartheta = 0$, *and this holds if and only if* $Lv|Kv$ *is separable of degree* p.

2) Let $\mathcal{E}$ *be a Kummer extension, and fix a* p*-th root of unity* ζ_p. *Then there are two cases:*

a) $\mathcal{E}$ *admits a Kummer generator* η *such that* $v\eta \geq 0$ *and* $1, \eta, \ldots, \eta^{p-1}$ *form a valuation basis for* $\mathcal{E}$. *In this case,* $b_{\min}$ *can be chosen to be* η *and we have* $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$ *and*

$$I_{\mathcal{E}} \;=\; (\zeta_p - 1)\,. \tag{35}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ *if and only if* $p \neq \operatorname{char} Kv$.

b) $\mathcal{E}$ *admits a Kummer generator* η *which is a* 1*-unit with* $v(\eta - 1) \leq v(\zeta_p - 1)$, $p = \operatorname{char} Kv$, *and* $1, \eta - 1, \ldots, (\eta - 1)^{p-1}$ *a valuation basis for* $\mathcal{E}$. *In this case,* $b_{\min}$ *can be chosen to be* $\eta - 1$ *and we have* $\gamma_{\mathcal{E}} = v(\zeta_p - 1) - v(\eta - 1)$ *and*

$$I_{\mathcal{E}} \;=\; \left(\frac{\zeta_p - 1}{\eta - 1}\right)\,. \tag{36}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ *if and only if* $v(\eta - 1) = v(\zeta_p - 1)$, *and this holds if and only if* $Lv|Kv$ *is separable of degree* p.

Proof. Throughout the proof we use part 1) of Corollary 3.6.

1): By part 1) of Proposition 3.16 there exists an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $v\vartheta$ generates the value group extension, or $v\tilde{c}\vartheta = 0$ and $Lv = Kv(\tilde{c}\vartheta v)$ for some $\tilde{c} \in K$. By Lemma 2.11, it follows that $1, \vartheta, \ldots, \vartheta^{p-1}$ is a valuation basis for $\mathcal{E}$.

If $v\vartheta < 0$, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) \;=\; v\left(\frac{\sigma\vartheta - \vartheta}{\vartheta}\right) \;=\; -v\vartheta \;=\; v\left(\frac{1}{\vartheta}\right) \;>\; 0 \tag{37}$$

for every $\sigma \in \operatorname{Gal} L|K \setminus \{\mathrm{id}\}$ since then $\sigma\vartheta - \vartheta \in \mathbb{F}_p \setminus \{0\}$. Hence by Lemma 3.3, for $1 \leq j \leq p-1$ we have

$$v\left(\frac{\sigma\vartheta^j}{\vartheta^j} - 1\right) \;=\; v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) \;=\; v\left(\frac{1}{\vartheta}\right)\,.$$

This proves that $b_{\min}$ can be chosen to be ϑ in this case.

If $v\vartheta = 0$, which by part 1) of Proposition 3.16 holds if and only if $Lv|Kv$ is separable of degree p, then

$$v\left(\frac{\sigma\vartheta}{\vartheta} - 1\right) \;=\; v\left(\frac{1}{\vartheta}\right) \;=\; 0\,,$$

and as the value $\gamma_{\mathcal{E}}$ is non-negative, this is equivalent to $I_{\mathcal{E}} = \mathcal{O}_L$.

2): By part 2) of Proposition 3.16 there exists a Kummer generator η such that either

a) $v\eta$ generates the value group extension, or ηv generates the residue field extension, or

b) η is a 1-unit and $v(\eta - 1)$ generates the value group extension or for some $\tilde{c} \in K$, $\tilde{c}(\eta - 1)v$ generates the residue field extension.

We first consider case a). By Lemma 2.11, it follows that $1, \eta, \ldots, \eta^{p-1}$ is a valuation basis for $(L|K, v)$. If $v\eta$ generates the value group extension, then so does $v\eta^{-1}$. Therefore, we can assume that $v\eta \geq 0$. For $1 \leq j \leq p-1$ and every $\sigma \in \operatorname{Gal} L|K \setminus \{\mathrm{id}\}$,

$$v\left(\frac{\sigma\eta^j}{\eta^j} - 1\right) \;=\; v\left(\frac{\sigma\eta^j - \eta^j}{\eta^j}\right) \;=\; v\left(\frac{\zeta_p^k\eta^j - \eta^j}{\eta^j}\right) \;=\; v(\zeta_p^k - 1) \;=\; v(\zeta_p - 1)$$

for some $k \in \mathbb{N}$; the last equation holds since $v(\zeta - 1) = vp/(p-1)$ for every primitive p-th root of unity ζ (cf. [6, Lemma 2.5]). This proves that in case a), $b_{\min}$ can be chosen to be η and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1)$. Finally, $v(\zeta_p - 1) = 0$ if and only if $p \neq \operatorname{char} Kv$.

Now we consider case b), where η is a 1-unit. Again by Lemma 2.11, $1, \eta - 1, \ldots, (\eta-1)^{p-1}$ is a valuation basis for $(L|K, v)$. Since $v\eta = 0$, we have for every $\sigma \in \operatorname{Gal} L|K \setminus \{\mathrm{id}\}$:

$$v\left(\frac{\sigma(\eta-1)}{\eta-1} - 1\right) \;=\; v\left(\frac{\sigma\eta - \eta}{\eta - 1}\right) \;=\; v(\zeta_p - 1) - v(\eta - 1)\,.$$

This value must be non-negative since it is not less than $\gamma_{\mathcal{E}}$. If it is equal to 0, then it must be equal to $\gamma_{\mathcal{E}}$. If it is positive, then we can apply Lemma 3.3, obtaining that for $1 \leq j \leq p-1$,

$$v\left(\frac{\sigma(\eta-1)^j}{(\eta-1)^j} - 1\right) \;=\; v\left(\frac{\sigma(\eta-1)}{\eta-1} - 1\right)$$

and consequently, this value is again equal to $\gamma_{\mathcal{E}}$. Hence in case b), $b_{\min}$ can be chosen to be $\eta - 1$ and we have $\gamma_{\mathcal{E}} = v(\zeta_p - 1) - v(\eta - 1)$. Since $v(\eta - 1) > 0$ and $\gamma_{\mathcal{E}} \geq 0$, we must have $p = \operatorname{char} Kv$ because otherwise, $v(\zeta_p - 1) = 0$.

Since $I_{\mathcal{E}}$ is the unique ramification ideal of $\mathcal{E}$, the equality $I_{\mathcal{E}} = \mathcal{O}_L$ is equivalent to L being the ramification field of $(L|K, v)$. This in turn holds if and only if $(L|K, v)$ is a tame extension ([26, Proposition 4.1]). Since $(L|K, v)$ is assumed to be unibranched and defectless of degree p, this is equivalent to $Lv|Kv$ being separable of degree p. □

Remark 3.18. Equation (36) also holds in case 2 a) of the previous theorem since in this case, $v(\eta - 1) = 0$. Indeed, in that case we have $v\eta \geq 0$, and $1, \eta, \ldots, \eta^{p-1}$ form a valuation basis for $(L|K, v)$. If $v\eta > 0$, then $v(\eta - 1) = 0$. If $v\eta = 0$, then $1, \eta v, \ldots, (\eta v)^{p-1}$ form a basis of $Lv|Kv$, so $\eta v \neq 1$, whence $v(\eta - 1) = 0$ again. #

3.3.2. *The defect case.*

Proposition 3.19. *If $\mathcal{E} = (L|K, v)$ is an Artin-Schreier extension with Artin-Schreier generator ϑ such that $v(\vartheta - K)$ has no maximal element, then $v(\sigma\vartheta - \vartheta) = 0$ for all $\sigma \in G \setminus \{\mathrm{id}\}$, and*

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta - K)\,. \tag{38}$$

If $\mathcal{E}$ is a Kummer extension with Kummer generator η of value 0 such that $v(\eta - K)$ has no maximal element, then $v(\sigma\eta - \eta) = v(\zeta_p - 1)$ for all $\sigma \in G \setminus \{\mathrm{id}\}$,

where ζ_p is an arbitrary primitive p-th root of unity, and

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p - 1) \;-\; v(\eta - K) \;=\; \frac{vp}{p-1} \;-\; v(\eta - K)\,. \tag{39}$$

Proof. Every Artin-Schreier generator ϑ satisfies $\sigma\vartheta - \vartheta \in \mathbb{F}_p$ and hence $v(\sigma\vartheta - \vartheta) = 0$ for all $\sigma \in G \setminus \{\mathrm{id}\}$. Every Kummer generator η satisfies $\sigma\eta - \eta = (\zeta_p - 1)\eta$ for some primitive p-th root of unity, and as $v(\zeta_p - 1)$ is the same for all primitive p-th roots of unity ζ_p, we have $v(\sigma\eta - \eta) = v(\zeta_p - 1) + v\eta = v(\zeta_p - 1)$ for all $\sigma \in G \setminus \{\mathrm{id}\}$, for ζ_p an arbitrary primitive p-th root of unity.

Now equations (38) and (39) follow from equation (30). □

From this proposition together with Corollary 3.15, we obtain:

Corollary 3.20. *1) An Artin-Schreier extension with Artin-Schreier generator ϑ is a defect extension if and only if $v(\vartheta - K)$ has no maximal element and is contained in $vL^{<0}$.*

2) A Kummer extension with Kummer generator η of value 0 is a defect extension if and only if $v(\eta - K)$ has no maximal element and is bounded from above by $v(\zeta_p - 1)$.

3) Let $(L|K, v)$ be an Artin-Schreier extension or a Kummer extension of prime degree. Then $\mathcal{E}$ is a defect extension if and only if $\Sigma_{\mathcal{E}}$ is a subset of $vL^{>0}$ without a minimal element and therefore, $I_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_L$-ideal.

Theorem 3.21. *Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p.*
1) If $\mathcal{E}$ is an Artin-Schreier extension with Artin-Schreier generator ϑ, then

$$\begin{aligned} I_{\mathcal{E}} &= \left(\frac{1}{\vartheta - c} \mid c \in K \right) \\ &= \left(\frac{1}{b} \mid b \text{ an Artin-Schreier generator of } L|K \right). \end{aligned}$$

2) Let $\mathcal{E}$ be a Kummer extension with a Kummer generator η which is a 1-unit, and ζ_p a primitive p-th root of unity. Then

$$\begin{aligned} I_{\mathcal{E}} &= \left(\frac{\zeta_p - 1}{\eta - c} \mid c \in K \text{ a 1-unit} \right) \\ &= \left(\frac{\zeta_p - 1}{b - 1} \mid b \text{ a Kummer generator of } L|K \text{ which is a 1-unit} \right). \end{aligned}$$

Proof. 1): The first equation follows from equation (33) of Corollary 3.15, where we take σ_0 such that $\sigma_0\vartheta = \vartheta + 1$. The ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation because $\vartheta - c$ is again an Artin-Schreier generator for every $c \in K$. Further, by Corollary 3.15 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the two ideals are equal.

2): The first equation follows from equation (33) of Corollary 3.15, where we take σ_0 such that $\sigma_0\eta = \zeta_p\eta$, because then $\sigma_0(\eta - c) - (\eta - c) = (\zeta_p - 1)\eta$ and we can drop η since it is a unit. Further, we can restrict c to 1-units since if c is not a 1-unit, then $v(\eta - c) \leq 0 < v(\eta - 1)$ and $\frac{\zeta_p - 1}{\eta - c} \in \left(\frac{\zeta_p - 1}{\eta - 1} \right)$.

When c is a 1-unit, then $\eta - c = c(\frac{\eta}{c} - 1)$, the quotient $b = \frac{\eta}{c}$ is again a Kummer generator which is a 1-unit, and we can drop the unit factor c. This shows that the ideal on the right hand side of the second equation contains the ideal on the right hand side of the first equation. Further, by Corollary 3.15 the ideal on the right hand side of the second equation is contained in $I_{\mathcal{E}}$. Hence the two ideals are equal. □

3.4. When does the equality $I_{\mathcal{E}} = \mathcal{M}_L$ hold?

Throughout, we assume that $\mathcal{E} = (L|K, v)$ is a purely wild Galois extension of degree $p = \operatorname{char} Kv$. If $\operatorname{char} K = 0$, we assume in addition that K contains a primitive p-th root of unity ζ_p, so that $L|K$ is a Kummer extension. Under these assumptions, we will determine the cases where $I_{\mathcal{E}} = \mathcal{M}_L$.

Let us discuss the meaning of the equality $I_{\mathcal{E}} = \mathcal{M}_L$. If $\mathcal{E}$ is defectless, then $I_{\mathcal{E}}$ is a principal $\mathcal{O}_L$-ideal, hence so is $\mathcal{M}_L$, that is, both vL and vK have a minimal positive element. For a defect extension $\mathcal{E}$ we will show in Section 3.4.2 that the equality implies that $\mathcal{E}$ has independent defect in the sense of [27] (see also [23, Section 1.2]). However, the converse does only hold if vL is archimedean, because otherwise $\mathcal{E}$ has independent defect if and only if $I_{\mathcal{E}}$ is equal to the maximal ideal of some coarsening of $\mathcal{O}_L$ whose maximal ideal is nonprincipal. In any case, a defect extension $\mathcal{E}$ satisfying $I_{\mathcal{E}} = \mathcal{M}_L$ does not yield a definable proper coarsening of $\mathcal{O}_L$ in the sense of [23].

3.4.1. *The defectless case.*

We assume $\mathcal{E}$ to be defectless. Then we know from Proposition 3.1 that $I_{\mathcal{E}}$ is principal. Hence for it to be equal to $\mathcal{M}_L$, the latter and also $\mathcal{M}_K$ must be principal. As $\mathcal{E}$ is purely wild, the extension $Lv|Kv$ cannot be separable of degree p.

Proposition 3.22. *Let the assumptions be as described above.*

1) Assume that $\mathcal{E}$ is an Artin-Schreier extension. If $(vL : vK) = p$, then $I_{\mathcal{E}} = \mathcal{M}_L$ if and only if $\mathcal{E}$ admits an Artin-Schreier generator ϑ such that $-v\vartheta$ is the smallest positive element of vL and $-pv\vartheta$ is the smallest positive element of vK. If $[Lv : Kv] = p$, then $I_{\mathcal{E}} = \mathcal{M}_L$ if and only if $\mathcal{E}$ admits an Artin-Schreier generator ϑ such that $-v\vartheta$ is the smallest positive element of $vL = vK$.

2) Assume that $\mathcal{E}$ is a Kummer extension. Then $I_{\mathcal{E}} = \mathcal{M}_L$ holds if and only if $\mathcal{E}$ admits a Kummer generator η such that one of the following cases holds:

(a) $\mathrm{e}\,(L|K, v) = p$ and $v\eta$ generates the value group extension or $\mathrm{f}\,(L|K, v) = p$, $v\eta = 0$ and ηv generates the residue field extension, and $\mathcal{M}_L = (\zeta_p - 1)\mathcal{O}_L$ as well as $\mathcal{M}_K = (\zeta_p - 1)\mathcal{O}_K$,

(b) $\mathrm{e}\,(L|K, v) = p$, η a 1-unit, $v(\eta - 1)$ generates the value group extension or $\mathrm{f}\,(L|K, v) = p$ and $v\tilde{c}(\eta - 1)$ generates the residue field extension for some $\tilde{c} \in K$, and $\mathcal{M}_L = \frac{\zeta_p - 1}{\eta - 1}\mathcal{O}_L$.

Proof. Our statements follow from Proposition 3.16 together with Theorem 3.17. In case 2(a) note that $\mathcal{M}_L = (\zeta_p - 1)\mathcal{O}_L$ means that $v(\zeta_p - 1) \in vK$ is the smallest positive element in vL, which implies that it also is the smallest positive element in vK. □

Note that if case 2(a) holds with $\mathrm{e}\,(L|K,v) = p$, then vK cannot be archimedean, because $v(\zeta_p - 1)$ is the smallest positive element of both vL and vK.

Let us give examples for the different types of extensions appearing in the proposition.

• Artin-Schreier extension with $(vL : vK) = p$: take a valued field (K, v) of characteristic $p > 0$ such that vK has a smallest positive element vc, $c \in K$. Let ϑ be a root of $X^p - X - c^{-1}$. Then $v\vartheta = -vc/p$. Thus $v\vartheta^{-1}$ is the smallest positive element of $vK(\vartheta)$, whence $I_{\mathcal{E}} = \mathcal{M}_L$ for $L = K(\vartheta)$.

• Artin-Schreier extension with $[Lv : Kv] = p$: take (K, v) and $c \in K$ as before. Further, assume that Kv contains an element dv, $d \in \mathcal{O}_K^\times$, which does not have a p-th root in Kv. Let ϑ be a root of $X^p - X - c^{-p}d$. Then $v\vartheta = -vc$ and $v((c\vartheta)^p - d) > 0$, so that $c\vartheta v$ is a p-th root of dv. We obtain $[Lv : Kv] = p$ for $L = K(\vartheta)$, so $vL = vK$. As $v\vartheta^{-1} = vc$ is the smallest positive element of $vK = vL$, it follows that $I_{\mathcal{E}} = \mathcal{M}_L$.

• Kummer extension with Kummer generator η such that $v\eta$ generates the value group extension, with $(vL : vK) = p$: take $K = \mathbb{Q}_p(\zeta_p, t)$, where t is transcendental over $\mathbb{Q}_p(\zeta_p)$ and extend the p-adic valuation to a valuation v of K in such a way that vK is the lexicographic product $vt\,\mathbb{Z} \times v(\zeta_p - 1)\mathbb{Z}$. Let η be a root of $X^p - t$ and set $L := K(\eta)$. Then $(vL : vK) = p$, $v\eta$ generates the value group extension, and vL is the lexicographic product $\frac{vt}{p}\mathbb{Z} \times v(\zeta_p - 1)\mathbb{Z}$. Consequently, $v(\zeta_p - 1)$ is still the smallest positive element of vL, showing that $I_{\mathcal{E}} = \mathcal{M}_L$.

• Kummer extension with Kummer generator η such that $v\eta = 0$ and ηv generates the residue field extension, with $[Lv : Kv] = p$: take again $K = \mathbb{Q}_p(\zeta_p, t)$, but now extend the p-adic valuation to a valuation v of K in such a way that v is the Gauß valuation of the rational function field $K = \mathbb{Q}_p(\zeta_p)(t)$. Then tv is transcendental over $\mathbb{Q}_p(\zeta_p)v = \mathbb{F}_p$ and does not have a p-th root in Kv. Let η be a root of $X^p - t$ and set $L := K(\eta)$. Then $\eta v = (tv)^{1/p}$ and ηv generates the residue field extension. Since $[Lv : Kv] = p$ implies $vL = vK$, $v(\zeta_p - 1)$ is still the smallest positive element of vL, showing that $I_{\mathcal{E}} = \mathcal{M}_L$.

• To construct Kummer extensions described in case 2(b), take a valued field (K, v) of characteristic 0 with residue characteristic $p > 0$ and assume that $\zeta_p \in K$. Further, assume that $c \in K$ is a 1-unit such that $v(c - 1) = vp$ and that $a \notin K$ is a root of the polynomial

$$f(X) \;=\; (X+1)^p - c \;=\; X^p + \sum_{0<i<p} \binom{p}{i} X^{p-i} + 1 - c\,. \tag{40}$$

Then $\eta := a + 1$ is 1-unit and a Kummer generator of $K(a)|K$.

For $0 < i < p$, each binomial coefficient in (40) has value $vp > 0$ and we have $v\binom{p}{i}a^{p-i} \geq vp + va > vp$. Since $f(a) = 0$ and $v(c-1) = vp$, it follows that $va^p = v(c-1)$ and $v(a^p - (c-1)) \geq vp + va > vp = va^p$.

Let us give an example of an extension with $(vL : vK) = p$. Take $K = \mathbb{Q}(\zeta_p)$ with the extension v of the p-adic valuation. Then $vK = \frac{1}{p-1}\mathbb{Z}$, so vp is not p-divisible in vK. Set $c = p + 1$ so that $c - 1 = p$. From our above computations we

obtain that $va = \frac{vp}{p}$, which generates the value group extension. Further,

$$v\,\frac{\zeta_p - 1}{\eta - 1} \;=\; \frac{vp}{p-1} - \frac{vp}{p} \;=\; \frac{vp}{p(p-1)}\,, \tag{41}$$

which is the smallest positive element in vL. In view of part 2) b) of Theorem 3.17, this shows that $I_{\mathcal{E}} = \mathcal{M}_L$.

Now we give an example of an extension with $[Lv : Kv] = p$. Choose $\tilde{c}$ such that $\tilde{c}^{-p} = p$ and extend the p-adic valuation to $\mathbb{Q}(\zeta_p, \tilde{c})$. Further, take a transcendental element t, and v to be the Gauß valuation on the rational function field $K := \mathbb{Q}(\zeta_p, \tilde{c})(t)$. Then tv is transcendental over $\mathbb{Q}(\zeta_p, \tilde{c})v = \mathbb{F}_p$ and does not have a p-th root in Kv. Set $c = 1 + pt$ so that $c - 1 = pt$ and $v(c-1) = vpt = vp$. Now we obtain from our above computations that $v(a^p - \tilde{c}^{-p}t) = v(a^p - (c-1)) > va^p$. This implies

$$v(\tilde{c}^p(\eta-1)^p - t) \;=\; v(\tilde{c}^p a^p - t) \;>\; v\tilde{c}^p a^p \;=\; 0\,,$$

whence $\tilde{c}(\eta - 1)v = (tv)^{1/p}$. We set $L := K(\eta) = K(a)$ and observe that $[Lv : Kv] = p$, so that $vL = vK$. Further, (41) again holds, so that $v\frac{\zeta_p - 1}{\eta - 1}$ is the smallest positive element in $vK = vL$. Again by part 2) b) of Theorem 3.17, it follows that $I_{\mathcal{E}} = \mathcal{M}_L$.

3.4.2. *The defect case.*
We assume $\mathcal{E}$ to be an Artin-Schreier or a Kummer defect extension. Then part 3) of Corollary 3.20 shows that $vI_{\mathcal{E}} = \Sigma_{\mathcal{E}}$ has no minimal element, so $I_{\mathcal{E}}$ is nonprincipal. Hence for it to be equal to $\mathcal{M}_L$, both $\mathcal{M}_L$ and $\mathcal{M}_K$ must be nonprincipal. We note:

Lemma 3.23. *Let the assumptions be as described above. Then $I_{\mathcal{E}} = \mathcal{M}_L$ holds if and only if $\Sigma_{\mathcal{E}} = vL^{>0}$.*

For a famous example of an Artin-Schreier extension with independent defect satisfying the equation $I_{\mathcal{E}} = \mathcal{M}_L$ originally due to Shreeram Abhyankar, see [18, Example 3.12]. For other examples, see [31, Proposition 6.14]. In that paper, also Galois extensions $(L|K, v)$ with higher powers of p as their degrees are constructed whose unique ramification ideal is equal to $\mathcal{M}_L$: see Corollary 6.10 and the interesting system of Artin-Schreier extensions in Section 6.3.

Finally, we present an example of a Kummer extension with independent defect satisfying the equation $I_{\mathcal{E}} = \mathcal{M}_L$; it has been developed with the help of Konstantinos Kartas. We work over $\mathbb{Q}_p$ and assume $p \neq 2$ (for simplicity). Let K be the p-cyclotomic field which is obtained by adjoining a primitive p-th root of unity ζ_p to $\mathbb{Q}_p$ and then closing under a compatible system of p^n-th roots of ζ_p. The p-adic valuation of $\mathbb{Q}_p$ extends (uniquely) to a valuation v on K, which again is henselian.

We consider the unibranched extension $\mathcal{E} = (K(p^{1/p})|K, v)$. First, we note that $p^{1/p} \notin K$; otherwise $\mathbb{Q}_p(p^{1/p})$ would be a subfield of the abelian extension $K|\mathbb{Q}_p$ and hence would be Galois over $\mathbb{Q}_p$ (since every subgroup of an abelian group is normal), which is not the case. Therefore, $K(p^{1/p})|K$ is a proper extension and thus must be of degree p. As $\zeta_p \in K$, it is a Kummer extension. We claim that $(K(p^{1/p})|K, v)$ is immediate. Since vK is p-divisible, it suffices to show that

$K(p^{1/p})v = Kv$. But if this were not true, then $K(p^{1/p})$ would be contained in the maximal unramified extension of K, which is equal to $\mathbb{Q}_p^{ab}$. Again, this is a contradiction because this would imply that $\mathbb{Q}_p(p^{1/p})|\mathbb{Q}_p$ is Galois. Therefore $(K(p^{1/p})|K, v)$ must be immediate. As the extension is unibranched, it has defect p.

It is known that (K, v) is a deeply ramified field (this follows e.g. from computations in [12]). Hence by [27, part (1) of Theorem 1.10], $(K(p^{1/p})|K, v)$ has independent defect. We set $L := K(p^{1/p})$; as an algebraic extension of $\mathbb{Q}_p$, it has archimedean value group. The implication a)⇒b) of [6, Theorem 1.4] shows that $I_{\mathcal{E}}$ is the maximal ideal of some nontrivial coarsening of v. But as the value group is archimedean, this can only be $\mathcal{M}_L$, so $I_{\mathcal{E}} = \mathcal{M}_L$.

3.5. **Defect does not always imply nonprincipality of ramification ideals.** We are going to give an example of a Galois defect extension $(L|K, v)$ of degree p^2 which is a tower of two Galois extensions $L|L_0$ and $L_0|K$ of degree $p = \operatorname{char} K$ and rank one, the upper one defectless and the lower a defect extension, but has only one ramification ideal, this being principal. We need a criterion for $L|K$ to be Galois. We set $\wp(X) := X^p - X$. The following is Lemma 2.9 in [30]:

Lemma 3.24. *Take Artin-Schreier extensions $L|L_0$ and $L_0|K$, and an Artin-Schreier generator ϑ of $L|L_0$ with $\vartheta^p - \vartheta = b \in L_0$. Then $L|K$ is a Galois extension if and only if $\sigma_0 b - b \in \wp(L_0)$ for some generator σ_0 of $\operatorname{Gal} L_0|K$.*

In order to construct a suitable field K from which we can obtain the Artin-Schreier defect extension $(L_0|K, v)$ we need, we choose a prime q different from p and use [2, Theorem 4.7]. According to this theorem there exists a valued rational function field $K = \widetilde{\mathbb{F}_p}(x, y)$ with value group $\Gamma := \frac{1}{q^\infty}\mathbb{Q}$ such that $(K(y^{1/p})|K, v)$ is immediate with $v(y^{1/p} - K)$ having supremum $-1/p$. According to the proof we can define

$$y \;:=\; \sum_{i\in\mathbb{N}} x^{p\gamma_i} \;\in\; \widetilde{\mathbb{F}_p}((x^\Gamma))\,,$$

where the increasing sequence $\gamma_i = \frac{n_i}{q^i}$ with suitable $n_i \in \mathbb{Z}$ converges to $-1/p$. We are free to choose $\gamma_1 = -1$.

Let ϑ_0 be a root of the Artin-Schreier polynomial $X^p - X - y$ and set $L_0 := K(\vartheta_0)$. We compute:

$$v(\vartheta_0 - y^{1/p}) \;=\; \frac{1}{p}(\vartheta_0^p - y) \;=\; \frac{1}{p}v\vartheta_0 \;=\; \frac{1}{p}vy^{1/p} \;=\; -1/p\,.$$

From [22, part 4) of Lemma 2.9] it follows that $v(\vartheta_0 - K) = v(y^{1/p} - K) \subseteq vL_0^{<0}$. Hence the Artin-Schreier extension $\mathcal{E}_0 := (L_0|K, v)$ satisfies $\Sigma_{\mathcal{E}} = -v(\vartheta_0 - K) \subseteq vL_0^{>0}$ and is therefore an immediate defect extension by Corollary 3.15. We have that $I_{\mathcal{E}_0}$ is nonprincipal, with $vI_{\mathcal{E}_0}$ having infimum $\gamma := 1/p$. However, we will construct the extension $(L|K, v)$ such that $I_{\mathcal{E}_0}$ does not induce a nonprincipal ramification ideal of it.

Let ϑ be a root of the Artin-Schreier polynomial $X^p - X - \vartheta_0$, and set $L := L_0(\vartheta) = K(\vartheta_0, \vartheta)$. Since $v\vartheta_0 = -vy/p = -1 < 0$, we have $v\vartheta = -1/p \notin vK = vL_0$. Hence by Corollary 2.14, the elements $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis

of $\mathcal{E}_1 := (L|K(\vartheta_0), v)$, showing that this extension is defectless. By part 1) of Theorem 3.17,

$$I_{\mathcal{E}_1} \;=\; \left(\frac{1}{\vartheta}\right)\,,$$

so the minimum of $vI_{\mathcal{E}_1}$ is $-v\vartheta = 1/p = \gamma$.

Since $\vartheta^p - \vartheta = \vartheta_0$, we have $L = K(\vartheta)$. To show that $L|K$ is a Galois extension, take some generator σ_0 of $\operatorname{Gal} L_0|K$. Since $\sigma_0\vartheta_0$ is also a root of $X^p - X - y$, we have $\sigma_0\vartheta_0 - \vartheta_0 = i$ for some $i \in \mathbb{F}_p$. As K contains $\widetilde{\mathbb{F}_p}$, it contains the Artin-Schreier roots of i, i.e., $i \in \wp(K) \subseteq \wp(L_0)$. Now Lemma 3.24 shows that $L|K$ is a Galois extension. We set $G := \operatorname{Gal} L|K$. At this point, we can apply Proposition 3.7 to find that (24) holds.

By Corollary 2.10 of [30], G it is not cyclic, and the discussion leading up to this corollary shows the following. Take $\sigma \in G = \operatorname{Gal} L|K$ such that $\sigma\vartheta_0 - \vartheta_0 = 1$. Then $\xi := \sigma\vartheta - \vartheta$ satisfies

$$\xi^p - \xi \;=\; (\sigma\vartheta - \vartheta)^p - (\sigma\vartheta - \vartheta) \;=\; (\sigma\vartheta)^p - \vartheta^p - \sigma\vartheta + \vartheta \tag{42}$$

$$=\; \sigma(\vartheta^p - \vartheta) - (\vartheta^p - \vartheta) \;=\; \sigma\vartheta_0 - \vartheta_0 \;=\; 1 \tag{43}$$

and is therefore an element of $\widetilde{\mathbb{F}_p} \subset K_0$. Hence $\sigma\xi = \xi$, and by induction on $n \in \mathbb{N}$ one shows that $\sigma^i\vartheta = i\xi + \vartheta$ for every $i \in \mathbb{N}$.

Further, take $\tau \in \operatorname{Gal} L|L_0 \subset G$ such that $\tau\vartheta - \vartheta = 1$. Then τ is trivial on L_0 and σ and τ commute. Thus every element in G can be written as $\sigma^i\tau^j$, $0 \le i, j \le p-1$.

By induction on $j \in \mathbb{N}$ one shows that $\tau^j\vartheta = \vartheta + j$ for every $n \in \mathbb{N}$. So we have

$$\sigma^i\tau^j\vartheta - \vartheta \;=\; \sigma^i(\vartheta + j) - \vartheta \;=\; i\xi + \vartheta + j - \vartheta \;=\; i\xi + j \;\in\; \widetilde{\mathbb{F}_p}\,,$$

hence $v(\sigma^i\tau^j\vartheta - \vartheta) = 0$ and

$$v\left(\frac{\sigma^i\tau^j\vartheta}{\vartheta} - 1\right) \;=\; -v\vartheta \;=\; \gamma\,. \tag{44}$$

Take any nontrivial subgroup H of G. Then (24) implies that

$$\min\left\{ v\left(\frac{\sigma b}{b} - 1\right) \;\middle|\; \sigma \in H\,,\, b \in L^\times \right\} \;\ge\; \gamma\,.$$

On the other hand, H must contain some $\sigma^i\tau^j$ with $0 \le i, j \le p-1$, so (44) shows that (24) actually holds with H in place of G. Hence $I_H = I_G = I_{\mathcal{E}_1}$ for all nontrivial subgroups H of G. We have now proved:

Proposition 3.25. *There are Galois extensions of degree p^2 of valued fields of rank one in equal characteristic $p > 0$ that have only one ramification ideal, and this ramification ideal is principal although the extension is not defectless.*

The connection between the number of ramification ideals in a finite Galois extension and its depth is studied in [31, 32]. For the notion of depth, see [30].

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