# VALUATIONS ON RATIONAL FUNCTION FIELDS THAT ARE INVARIANT UNDER PERMUTATION OF THE VARIABLES

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ABSTRACT. We study and characterize the class of valuations on rational functions fields that are invariant under permutation of the variables and can be extended to valuations with the same property whenever a finite number of new variables is adjoined. The Gauß valuation is in this class, which constitutes a natural generalization of the concept of Gauß valuation. Further, we apply our characterization to show that the most common ad hoc generalization of the Gauß valuation is also in this class.

## 1. Introduction

In this paper, we will work with (Krull) valuations $v$ and write them in the classical additive way, that is, the value group of $v$ on a field $K$, denoted by $vK$, is an additively written ordered abelian group, and the ultrametric triangle law reads as $v(a+b) \geq \min\{va, vb\}$. We denote by $Kv$ the residue field of $v$ on $K$, by $va$ the value of an element $a \in K$, and by $av$ its residue.

Take any field $K$ and a rational function field $K(X_1, \ldots, X_m)$. In [6], the possible extensions of a valuation on $K$ to $K(X_1, \ldots, X_m)$ have been studied. While a quite good description of all possible value groups and residue fields of the extensions has been achieved, a detailed description and characterization of the extensions themselves seems rather out of reach. In this situation, restricting the problem to a subset of extensions with additional properties may well be of help.

In situations where at least one extension of a nontrivial valuation of $K$ has to be constructed on $K(X_1, \ldots, X_m)$, the **Gauß valuation** is often the canonical choice. Once we have a valuation on

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a polynomial ring, it extends in a unique way to the quotient field. On $K[X_1,\ldots,X_m]$, the Gauß valuation is defined as follows. Given $f\in K[X_1,\ldots,X_m]$, write

$$f \;=\; \sum_{\underline{i}} d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} \tag{1}$$

where the sum runs over multi-indices $\underline{i}=(i_1,\ldots,i_m)$ and each $d_{\underline{i}}$ is an element of $K$. Then define

$$vf \;:=\; \min_{\underline{i}} vd_{\underline{i}}\,. \tag{2}$$

This indeed defines a valuation on $K[X_1,\ldots,X_m]$; see Corollary 2.2 below.

We have occasionally witnessed ad hoc attempts to generalize the concept of Gauß valuation, but these attempts did not consider its particular properties in any systematic way. So let us take a closer look at some of them.

First, we notice that definition (2) is invariant under permutation of the variables. Second, the same definition can be used to extend the valuation to $K[X_1,\ldots,X_n]$ for each $n>m$, preserving the property of being invariant. Third, it is well known (and follows from Lemma 2.4 below) that for the Gauß valuation $v$ on $K[X_1,\ldots,X_n]$, the residues $X_1v,\ldots,X_mv$ are algebraically independent over the residue field $Kv$. This implies that $v$ is an Abhyankar valuation on the function field $K(X_1,\ldots,X_n)|K$; the definition of this notion will be given after Theorem 1.3.

The property that the residues $X_1v,\ldots,X_mv$ are algebraically independent over $Kv$ singles out the Gauß valuation. In order to obtain a generalization of the notion of Gauß valuation, we have to drop this property. Our goal therefore is to characterize the valuations that have the first two properties. Thereafter we will clarify their relation to the property of being Abhyankar valuations.

For each permutation $\pi\in S_m$ we denote by $\tau_\pi$ the automorphism of $K(X_1,\ldots,X_m)$ over $K$ induced by the corresponding permutation

$$(X_1,\ldots,X_m)\;\mapsto\;(X_{\pi(1)},\ldots,X_{\pi(m)})\,.$$

Its restriction to the polynomial ring $K[X_1,\ldots,X_m]$ is an automorphism as well. A valuation $v$ on $K(X_1,\ldots,X_m)$ or $K[X_1,\ldots,X_m]$ will be called **symmetric** if $v\tau_\pi=v$ for all $\pi\in S_m$, that is, if it is invariant under permutation of the variables. Several classes of symmetric valuations have been studied in [9, 10, 11, 12].

The present paper revisits the content of [10] and presents it in a concise form, with shorter and more accessible proofs. A symmetric valuation $v$ on $K(X_1,\ldots,X_m)$ (or $K[X_1,\ldots,X_m]$) will be called **openly-symmetric** if it can be extended to a symmetric valuation on $K(X_1,\ldots,X_n)$ (or $K[X_1,\ldots,X_n]$, respectively) for every $n > m$. Note that such valuations were called "symmetrically-open" in [10], but this name appears to indicate a topological meaning, which is not intended.

As we have seen above, Gauß valuations are openly-symmetric.

A valuation is symmetric on $K(X_1,\ldots,X_m)$ if and only if it is symmetric on $K[X_1,\ldots,X_m]$, and the same holds for "openly-symmetric". Further, every valuation on the polynomial ring $K[X_1,\ldots,X_m]$ defines a valuation on the rational function field $K(X_1,\ldots,X_m)$. Therefore, as was done in [10], we will often work with the polynomial ring in place of the rational function field.

In Section 3 we will prove the following useful result. By $K^{\mathrm{ac}}$ we will denote the algebraic closure of $K$.

**Proposition 1.1.** *If $v$ is an openly-symmetric valuation on the polynomial ring $K[X_1,\ldots,X_m]$, then each of its extensions to $K^{\mathrm{ac}}[X_1,\ldots,X_m]$ is again openly-symmetric.*

**Remark 1.2.** Take a partition $I\dot\cup J = \{1,\ldots,m\}$. If $v$ is symmetric on $K[X_1,\ldots,X_m]$, then it is also symmetric on the polynomial ring

$$K(X_i \mid i \in I)[X_j \mid j \in J]$$

(with respect to permutations of the variables $X_j$, $j \in J$). If $v$ is openly-symmetric on $K[X_1,\ldots,X_m]$, then it is also openly-symmetric on $K(X_i \mid i \in I)[X_j \mid j \in J]$. From Proposition 1.1, it follows that each extension of $v$ to the polynomial ring $K(X_i \mid i \in I)^{\mathrm{ac}}[X_j \mid j \in J]$ is again openly-symmetric.

In Section 5 we will give an example which shows that the openness condition in Proposition 1.1 is necessary. There we will construct a valuation on $\mathbb{Q}^{\mathrm{ac}}(X,Y)$ which is not symmetric, but has a symmetric restriction to $\mathbb{Q}(X,Y)$. (This example shows that Proposition 3.5 of [9] is in error; its attempted proof contains a gap that cannot be filled in general. But it can be filled in the case of openly-symmetric valuations, as Proposition 1.1 shows.)

As a consequence of Proposition 1.1, every openly-symmetric valuation on $K[X_1,\ldots,X_m]$ is the restriction of an openly-symmetric valuation on $K^{\mathrm{ac}}[X_1,\ldots,X_m]$. This fact is a very useful tool in the description of the variety of openly-symmetric valuations on $K[X_1,\ldots,X_m]$.

The following theorem summarizes our main results; it will be proved in Section 4. Note that $K[X_1,\ldots,X_m]=K[X_1,X_2-X_1\ldots,X_m-X_1]$ and that every polynomial $f\in K[X_1,\ldots,X_m]$ can be written in the form

$$f \;=\; \sum_{\underline{i}} g_{\underline{i}}(X_1)(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_m-X_1)^{i_m} \tag{3}$$

where the sum runs over multi-indices $\underline{i}=(i_2,\ldots,i_m)$ and each $g_{\underline{i}}(X_1)$ is a polynomial in $K[X_1]$,

**Theorem 1.3.**
*Take an arbitrary field $K$ and a valuation $v$ on the rational function field $K^{\mathrm{ac}}(X_1,\ldots,X_m)$.*
*a) The valuation $v$ is openly-symmetric on $K[X_1,\ldots,X_m]$ if and only if for each polynomial $f\in K^{\mathrm{ac}}[X_1,\ldots,X_m]$ written in the form (3), we have that*

$$vf \;=\; \min_{\underline{i}}\left(vg_{\underline{i}}(X_1)+(i_2+\ldots+i_m)\delta\right), \tag{4}$$

*where $\delta$ is any value in some ordered abelian group extension of the value group $vK(X_1)$ that satisfies*

$$\delta \;\geq\; v(X_1-a)\;\text{ for all } a\in K^{\mathrm{ac}}\,. \tag{5}$$

*b) If $v$ is openly-symmetric on $K[X_1,\ldots,X_m]$ and $m\geq 3$, then the elements*

$$\frac{X_3-X_1}{X_2-X_1},\ldots,\;\frac{X_m-X_1}{X_2-X_1}$$

*are units in the valuation ring and their residues are algebraically independent over the residue field $K(X_1,X_2)v$.*

*c) With*

$$r\;:=\;\dim_{\mathbb{Q}} vK(X_1,\ldots,X_m)/vK\quad\text{and } t\;:=\;\mathrm{trdeg}\,K(X_1,\ldots,X_m)v|Kv,$$

*exactly the following cases are possible:*
*(a) $r=0$ and $t=m$,*
*(b) $r=0$ and $t=m-1$,*
*(c) $r=1$ and $t=m-1$,*
*(d) $r=1$ and $t=m-2$,*
*(e) $r=2$ and $t=m-2$.*
*Therefore, we always have that*

$$m-1\;\leq\;r+t\;\leq\;m\,.$$

*Cases (a), (c) and (e) always appear for arbitrary valuations on $K$, and cases (b) and (d) always appear when the valuation $v$ on $K^{\mathrm{ac}}$ admits* *an extension to* $K^{\mathrm{ac}}(X_1)$ *that does not enlarge value group and residue field.*

Note that once a valuation $v$ on $K^{\mathrm{ac}}(X_1)$ is given, the formula (4) defines a valuation on $K^{\mathrm{ac}}[X_1,\dots,X_m]$ and thus also on $K^{\mathrm{ac}}(X_1,\dots,X_m)$, for any $\delta$ (see Corollary 2.2 below). As a consequence of our theorem, arbitrary valuations on $K(X_1)$ can be extended to openly-symmetric valuations on $K(X_1,\dots,X_m)$. In the paper [1] (see also [6] and [2]), conditions are discussed under which for a given valued field $K$ its algebraic closure $K^{\mathrm{ac}}$ admits extensions of the valuation to the rational function field $K^{\mathrm{ac}}(X_1)$ that do not enlarge value group and residue field.

In the cases (a), (c) and (e) we have that

$$r+t = \operatorname{trdeg} K(X_1,\dots,X_m)|K\ ,$$

that is, $v$ is an **Abhyankar valuation** of the rational function field $K(X_1,\dots,X_m)|K$. But the remaining cases show that an openly-symmetric valuation of a rational function field is not necessarily an Abhyankar valuation, so it may not share this particular property with the Gauß valuation, which belongs to case (a). However, openly-symmetric valuations can fail to have this property by a transcendence degree of at most one. Our theorem shows that if $v$ is openly-symmetric, then for each $i\in\{1,\dots,m\}$, $v$ is an Abhyankar valuation of $K(X_1,\dots,X_m)|K(X_i)$. The importance of Abhyankar valuations can be seen from the results of the papers [5, 7], and they now play a considerable role in algebraic geometry.

Finally, let us make a few more remarks about the characterization of openly-symmetric valuations. If we are working over a field $K$ that is not algebraically closed and we only know that condition (4) is satisfied for all polynomials $f\in K[X_1,\dots,X_m]$, can we still conclude that $v$ is openly-symmetric on $K[X_1,\dots,X_m]$? The following corollary to the proof of our main theorem shows that we can, if we strengthen condition (5).

**Corollary 1.4.** *Take* $K$ *and* $v$ *as in Theorem 1.3. Then* $v$ *is openly-symmetric on* $K[X_1,\dots,X_m]$ *if and only if (4) holds for each polynomial* $f\in K[X_1,\dots,X_m]$*, where* $\delta$ *is any value in some ordered abelian group extension of the value group* $vK(X_1)$ *that satisfies*

$$\delta \;\geq\; v(X_i-a) \;\; \textit{for all } a\in K^{\mathrm{ac}} \textit{ and } 1\leq i\leq m\,. \tag{6}$$

The Gauß valuation is an example of a valuation that satisfies conditions (4) and (6). However, when we construct other openly-symmetric valuations, checking condition (6) could be more work than we are ready to invest. In this case, the following result is helpful. Again, it follows readily from the proof of Theorem 1.3.

**Corollary 1.5.** *Take $K$ and $v$ as in Theorem 1.3. If (4) holds for each polynomial $f \in K[X_1,\dots,X_m]$, where $\delta$ is any value in some ordered abelian group extension of the value group $vK(X_1)$ that satisfies*

$$\delta \;>\; v(X_1-a) \;\; \textit{for all } a\in K^{\mathrm{ac}}\,, \tag{7}$$

*then $v$ is openly-symmetric on $K[X_1,\dots,X_m]$.*

It is not true that a symmetric valuation on $K[X_1,\dots,X_m]$ is openly-symmetric already when it can be extended to a symmetric valuation on $K(X_1,\dots,X_{m+1})$. But the latter turns out to be sufficient when it is combined with additional conditions in the spirit of Remark 1.2.

**Corollary 1.6.** *Take a valuation $v$ on $K(X_1,\dots,X_{m+1})^{\mathrm{ac}}$ and assume that for $1\leq i\leq m$, $v$ is symmetric on $K(X_1,\dots,X_{i-1})^{\mathrm{ac}}[X_i,\dots,X_{m+1}]$. Then $v$ is openly-symmetric on $K[X_1,\dots,X_m]$.*

To conclude with, we will show how Theorem 1.3 can be applied to analyze the most common ad hoc generalization of the Gauß valuation. Take a rational function field $K(X_1,\dots,X_m)$ over the valued field $(K,v)$ and an element $\delta$ in some ordered abelian group containing $vK$. For every $f\in K[X_1,\dots,X_m]$ written in the form (1), set

$$v_\delta f \;:=\; \min_{\underline{i}}\,(vd_{\underline{i}} + (i_1+\ldots+i_m)\delta)\,. \tag{8}$$

By Corollary 2.2 below, this defines a valuation $v_\delta$ on $K(X_1,\dots,X_m)$ which extends $v$.

**Corollary 1.7.** *The valuation $v_\delta$ on $K(X_1,\dots,X_m)$ is openly-symmetric.*

## 2. Some preliminaries

For the easy proof of the following lemma, see [3], Chapter VI, §10.3, Theorem 1.

**Lemma 2.1.** *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i,y_j\in L$, $i\in I$, $j\in J$, such that the values $vx_i$, $i\in I$, are rationally independent over $vK$, and the residues $y_jv$, $j\in J$, are algebraically independent over $Kv$. Then the elements $x_i,y_j$, $i\in I$, $j\in J$, are algebraically independent over $K$.*

*Moreover, write*

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i,y_j \mid i\in I, j\in J] \tag{9}$$

*in such a way that whenever $k\neq \ell$, then there is some $i$ s.t. $\mu_{k,i}\neq\mu_{\ell,i}$ or some $j$ s.t. $\nu_{k,j}\neq\nu_{\ell,j}$ . Then*

$$vf \;=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k vc_k \;+\; \sum_{i\in I}\mu_{k,i} vx_i\ . \tag{10}$$

*That is, the value of the polynomial $f$ is equal to the least of the values of its monomials. In particular, this implies:*

$$\begin{aligned} vK(x_i,y_j \mid i\in I, j\in J) \;&=\; vK\oplus\bigoplus_{i\in I}\mathbb{Z}vx_i \\ K(x_i,y_j \mid i\in I, j\in J)v \;&=\; Kv\,(y_jv \mid j\in J)\ . \end{aligned}$$

*The valuation $v$ on $K(x_i,y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to $K$, the values $vx_i$ and the fact that the residues $y_jv$, $j\in J$, are algebraically independent over $Kv$.*

*The residue map on $K(x_i,y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to $K$, the residues $y_jv$, and the fact that values $vx_i$ , $i\in I$, are rationally independent over $vK$.*

*Conversely, if $\delta_i$ , $i\in I$, are elements of some ordered abelian group containing $vK$ and are rationally independent over $vK$, then*

$$vf \;:=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k vc_k \;+\; \sum_{i\in I}\mu_{k,i}\delta_i \tag{11}$$

*defines a valuation that extends the valuation $v$ from $K$ to the rational function field $K(x_i,y_j \mid i\in I, j\in J)$.*

**Corollary 2.2.** *Take a rational function field $K(X_1,\ldots,X_m)$ over the valued field $(K,v)$ and an element $\delta$ in some ordered abelian group containing $vK$. Then the function $v_\delta$ defined in (8) induces a valuation on $K(X_1,\ldots,X_m)$ which extends $v$.*

*Proof.* We distinguish two cases.

First, suppose that $\delta$ is a torsion element over $vK$. Take any extension of $v$ to $K^{\mathrm{ac}}$. Then $\delta = vb$ for some $b\in K^{\mathrm{ac}}$. In this case we set $y_i := b^{-1}X_i$ for $1\leq i\leq m$. By the second part of Lemma 2.1, setting

$$w\sum_{\underline{i}} c_{\underline{i}} y_1^{i_1}\cdot\ldots\cdot y_m^{i_m} \;:=\; \min_{\underline{i}} vc_{\underline{i}}$$

induces a valuation on $K^{\mathrm{ac}}(y_1,\ldots,y_m) = K^{\mathrm{ac}}(X_1,\ldots,X_m)$ that extends $v$. For $f$ written in the form (1) we have that

$$\begin{aligned} wf &= w\sum_{\underline{i}} d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} \;=\; w\sum_{\underline{i}} d_{\underline{i}} b^{i_1+\ldots+i_m} y_1^{i_1}\cdot\ldots\cdot y_m^{i_m} \\ &= \min_{\underline{i}}\,(vd_{\underline{i}} + (i_1+\ldots+i_m)\delta) \;=\; v_\delta f\ , \end{aligned}$$

which proves that $v_\delta$ is a valuation.

Now suppose that $\delta$ is rationally independent over $vK$. In this case we set $x := X_1$ and $y_i := x^{-1}X_i$ for $2\le i\le m$. By the second part of Lemma 2.1, setting

$$w\sum_{\underline{i}} c_{\underline{i}} x^{i_1} y_2^{i_2}\cdot\ldots\cdot y_m^{i_m} \;:=\; \min_{\underline{i}}\,(vc_{\underline{i}} + i_1\delta)$$

induces a valuation on $K(x,y_2,\ldots,y_m) = K(X_1,\ldots,X_m)$ that extends $v$. For $f$ written in the form (1) we have that

$$\begin{aligned} wf &= w\sum_{\underline{i}} d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} \;=\; w\sum_{\underline{i}} d_{\underline{i}} x^{i_1+\ldots+i_m} y_2^{i_2}\cdot\ldots\cdot y_m^{i_m} \\ &= \min_{\underline{i}}\,(vd_{\underline{i}} + (i_1+\ldots+i_m)\delta) \;=\; v_\delta f\ , \end{aligned}$$

which again proves that $v_\delta$ is a valuation. □

Let $(L|K,v)$ be an extension of valued fields of finite transcendence degree. Then the following well known form of the “Abhyankar inequality” is a consequence of Lemma 2.1:

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{rr}\, vL/vK \;+\; \mathrm{trdeg}\, Lv|Kv\ , \tag{12}$$

where $\mathrm{rr}\, vL/vK := \dim_{\mathbb{Q}}(vL/vK)\otimes\mathbb{Q}$ is the **rational rank** of the abelian group $vL/vK$, i.e., the maximal number of rationally independent elements in $vL/vK$.

The straightforward proof of the following lemma is left to the reader.

**Lemma 2.3.** *The value group of an algebraically closed field is divisible, and its residue field is algebraically closed.*

We will also need the following easy result:

**Lemma 2.4.** *Take a valued field $(K(x),v)$. If $a\in K$ is such that*

$$v(x-a) \;=\; \max\{v(x-c)\mid c\in K\}\ , \tag{13}$$

*then either $v(x-a)\notin vK$, or for every $d\in K$ with $vd = v(x-a)$, we have that $\frac{x-a}{d}v\notin Kv$.*

*If in addition $K$ is algebraically closed, then in the first case, $v(x-a)$ is rationally independent over $vK$, and in the second case, $\frac{x-a}{d}v$ is transcendental over $Kv$.*

*Proof.* Assume that $v(x-a) \in vK$ and take $d \in K$ with $vd = v(x-a)$. Assume further that $\frac{x-a}{d}v \in Kv$. Then take $c \in K$ such that $\frac{x-a}{d}v = cv$, so that

$$v\left(\frac{x-a}{d} - c\right) > 0\,.$$

Then $v(x-a-cd) > vd = v(x-a)$, which contradicts (13) since $a + cd \in K$. This proves the first assertion of the lemma.

The second assertion of the lemma follows by Lemma 2.3 from what we have already shown. □

## 3. Proof of Proposition 1.1

We will need the following auxiliary result.

**Lemma 3.1.** *Let $m, n, k$ be natural numbers such that $n = m + 2k$. Assume that $v$ is a valuation on $K[X_1, \ldots, X_n]$ whose restriction to $K[X_1, \ldots, X_m]$ is not symmetric. Then the set*

$$\{v\tau_\pi \mid \pi \in S_n\}$$

*has at least $k + 1$ many elements.*

*Proof.* Since $v$ is not symmetric on $K[X_1, \ldots, X_m]$, there is $\pi \in S_m$ such that $v\tau_\pi \neq v$ on $K[X_1, \ldots, X_m]$. Since $S_m$ is generated by transpositions, $v\tau_\pi \neq v$ must already hold for some transposition $\pi$. W.l.o.g. we may assume that $\pi = (1\,2)$. Pick some $f \in K[X_1, \ldots, X_m]$ such that $v\tau_\pi f \neq vf$. Now using $(1\,2)$ as an element of $S_n$, we fix the automorphism $\tau = \tau_{(1\,2)}$ of $K[X_1, \ldots, X_n]$. Then we have that $v\tau f \neq vf$.

Pick $i, j$ such that $m < i < j \leq n$. Let $\tau_{ij} = \tau_{(1\,i)(2\,j)}$ be the automorphism of $K[X_1, \ldots, X_n]$ that exchanges $X_1$ with $X_i$ and $X_2$ with $X_j$ and leaves the other variables fixed. We have that

$$f_{ij} := \tau_{ij} f = f(X_i, X_j, X_3, \ldots, X_m) \in K[X_i, X_j, X_3, \ldots, X_m]$$

and that

$$v\tau_{ij} f_{ij} = vf \neq v\tau f = v\tau\tau_{ij} f_{ij}\,.$$

It follows that the restrictions of $v\tau_{ij}$ and $v\tau\tau_{ij}$ to the polynomial ring $K[X_i, X_j, X_3, \ldots, X_m]$ are distinct, so at least one of them is distinct from the restriction of $v$ to $K[X_i, X_j, X_3, \ldots, X_m]$. If it is $v\tau_{ij}$, then we set $v_{ij} := v\tau_{ij}$; otherwise, we set $v_{ij} := v\tau\tau_{ij}$.

Take $i', j'$ such that $m < i' < j' \leq n$ and $\{i, j\} \cap \{i', j'\} = \emptyset$. Then the restrictions of $\tau_{i\,j}$ and $\tau$ to $K[X_{i'}, X_{j'}, X_3, \ldots, X_m]$ are the identity and therefore, $v$ and $v_{i\,j}$ coincide on $K[X_{i'}, X_{j'}, X_3, \ldots, X_m]$, which shows that $v_{i\,j} \neq v_{i'\,j'}$.

By what we have proved, it follows that the $k+1$ many valuations

$$v,\ v_{m+1\,m+2}\ ,\ v_{m+3\,m+4}\ ,\ \ldots\ ,\ v_{n-1\,n}$$

are distinct on $K[X_1, \ldots, X_n]$. □

Now we are ready for the **Proof of Proposition 1.1:**

Take a valuation $v$ on $K[X_1, \ldots, X_m]$ which is openly-symmetric. Further, choose some extension to $K^{\mathrm{ac}}[X_1, \ldots, X_m]$ and call it again $v$. Suppose that $v$ is not symmetric on $K^{\mathrm{ac}}[X_1, \ldots, X_m]$. Then there is already a finite normal extension $L|K$ so that the restriction $v_0$ of $v$ to $L[X_1, \ldots, X_m]$ is not symmetric. Set $k = [L : K]$ and $n = m + 2k$. Then choose a symmetric extension $w$ of $v$ from $K[X_1, \ldots, X_m]$ to $K[X_1, \ldots, X_n]$.

We show that there is an extension of $v_0$ to $L[X_1, \ldots, X_n]$ which also extends $w$. Choose any extension $w'$ of $w$ to $L[X_1, \ldots, X_n]$. The restriction $w_0$ of $w'$ to $L[X_1, \ldots, X_m]$ is like $v_0$ an extension of $v$ from $K[X_1, \ldots, X_m]$ to $L[X_1, \ldots, X_m]$. As the extension

$$L[X_1, \ldots, X_m] | K[X_1, \ldots, X_m]$$

is algebraic and normal, $v_0$ and $w_0$ are conjugate over $K[X_1, \ldots, X_m]$ (cf. [4, Theorem 3.2.15]). Choose an automorphism $\sigma_0$ of $L[X_1, \ldots, X_m]$ over $K[X_1, \ldots, X_m]$ such that $v_0 = w_0\sigma_0$.

The purely transcendental extension

$$K[X_1, \ldots, X_n] | K[X_1, \ldots, X_m]$$

is linearly disjoint from the algebraic extension

$$L[X_1, \ldots, X_m] | K[X_1, \ldots, X_m]$$

(cf. [8, Chapter X, \$5, Prop. 3]). Therefore, $\sigma_0$ can be extended to an automorphism $\sigma$ of $L[X_1, \ldots, X_n]$ over $K[X_1, \ldots, X_n]$. The restriction of $w'\sigma$ to $L[X_1, \ldots, X_m]$ is $w_0\sigma_0 = v_0$. Since $\sigma$ is trivial on $K[X_1, \ldots, X_n]$, the restriction of $w'\sigma$ to $K[X_1, \ldots, X_n]$ coincides with that of $w'$. Hence the valuation $w\sigma$ on $L[X_1, \ldots, X_n]$ extends both $v_0$ and $w$; to simplify notation, we call it again $v$.

By assumption, $v$ is not symmetric on $L[X_1, \ldots, X_m]$. From Lemma 3.1 it follows that the set $\{v\tau_\pi \mid \pi \in S_n\}$ of valuations on $L[X_1, \ldots, X_n]$ has at least $k + 1 = [L : K] + 1$ many elements. But their restrictions to $K[X_1,\dots,X_n]$ are $w\tau_\pi$, where $\tau_\pi$ is now understood to be an automorphism of $K[X_1,\dots,X_n]$. As $w$ was chosen to be symmetric on $K[X_1,\dots,X_n]$, all of these restrictions coincide with $w$. Therefore, we have at least $[L:K]+1$ many extensions of $w$ to $L[X_1,\dots,X_n]$. On the other hand, as the extension $L[X_1,\dots,X_n]|K[X_1,\dots,X_n]$ is algebraic of degree $[L:K]$, all of the extensions must be conjugate over $K[X_1,\dots,X_n]$ and there can be no more than $[L:K]$ many. This contradiction shows that the valuation $v$ must be symmetric on $K^{\mathrm{ac}}[X_1,\dots,X_m]$. □

## 4. Proof of Theorem 1.3 and its corollaries

Take an arbitrary field $K$ and a valuation $v$ on $K(X_1,\dots,X_m)^{\mathrm{ac}}$. For $m=1$ there is nothing to show, so we assume that $m\geq 2$.

Let us first show that if for every $f\in K^{\mathrm{ac}}[X_1,\dots,X_m]$ conditions (4) and (5) hold, then $v$ is openly-symmetric. The proof will not depend on $m$; so if we just show that $v$ is symmetric on $K[X_1,\dots,X_m]$ it will also follow that an extension of $v$ which satisfies conditions (4) and (5) on $K[X_1,\dots,X_n]$ (with $n$ in place of $m$), where $n>m$, is also symmetric. Taking condition (4) for the definition of such an extension, we then have proved that $v$ is even openly-symmetric. Thus it actually suffices to show that $v$ is symmetric.

What we need for the proof of the symmetry are the following equations:

$$v(X_j-X_i)=\delta \quad \text{for all distinct } i,j\in\{1,\dots,m\}, \tag{14}$$

$$v(X_j-a)=v(X_1-a) \quad \text{for all } a\in K^{\mathrm{ac}} \text{ and } 2\leq j\leq m\,. \tag{15}$$

In order to obtain (14), we only need that condition (4) holds for polynomials in $K[X_1,\dots,X_m]$ because then, for every choice of distinct $i,j\in\{2,\dots,m\}$,

$$\begin{aligned} v(X_j-X_i) &= v(X_j-X_1+X_1-X_i)\\ &= \min\{v(X_j-X_1),v(X_1-X_i)\} \;=\; \delta\,. \end{aligned}$$

There are various conditions that ensure the validity of (15). In the case of the assumptions of our theorem, it works as follows: for $2\leq j\leq m$ and every $a\in K^{\mathrm{ac}}$,

$$\begin{aligned} v(X_j-a) &= v(X_j-X_1+X_1-a) \;=\; \min\{v(X_j-X_1),v(X_1-a)\}\\ &= v(X_1-a) \end{aligned}$$

where the second equality follows from (4) and the third equality follows from (5).

If we assume condition (4) only for polynomials in $K[X_1, \dots, X_m]$, as is the case in Corollaries 1.4 and 1.5, then we can proceed as follows. Suppose that condition (6) holds, but $v(X_j - a) \neq v(X_1 - a)$ for some $a \in K^{\mathrm{ac}}$. If $v(X_j - a) < v(X_1 - a)$, then

$$\delta = v(X_j - X_1) = \min\{v(X_j - a), v(X_1 - a)\} = v(X_j - a) < v(X_1 - a) \leq \delta,$$

where the first equality holds by (14). This is a contradiction. If on the other hand $v(X_j - a) > v(X_1 - a)$, then the same argument with 1 and $j$ interchanged also leads to a contradiction. This proves that (15) holds in this case.

Now suppose that condition (7) holds. Then we obtain (15) because $v(X_j - X_1) = \delta > v(X_1 - a)$ implies that

$$v(X_j - a) \;=\; \min\{v(X_j - X_1), v(X_1 - a)\} \;=\; v(X_1 - a)\,.$$

Now we proceed to the proof of the symmetry. Take a polynomial $g \in K[X]$ and write it as

$$g = c \prod_{k=1}^{\deg g} (X - a_k)$$

with $c, a_k \in K^{\mathrm{ac}}$. Then for $2 \leq j \leq m$, using (15),

$$\begin{aligned} vg(X_1) &= vc + \sum_{k=1}^{\deg g} v(X_1 - a_k) \\ &= vc + \sum_{k=1}^{\deg g} v(X_j - a_k) \;=\; vg(X_j)\,. \end{aligned}$$

Now take any polynomial $f \in K[X_1, \dots, X_m]$ and any $\pi \in S_m$. Write $f$ in the form (3). Using the equalities we have computed above together with (14), we find:

$$\begin{aligned} vf &= \min_{\underline{i}} \; vg_{\underline{i}}(X_1) + (i_2 + \ldots + i_m)\delta \\ &= \min_{\underline{i}} \; vg_{\underline{i}}(X_{\pi(1)}) + i_2 v(X_{\pi(2)} - X_{\pi(1)}) + \ldots + i_m v(X_{\pi(m)} - X_{\pi(1)}) \\ &= \min_{\underline{i}} \; vg_{\underline{i}}(X_{\pi(1)})(X_{\pi(2)} - X_{\pi(1)})^{i_2} \cdot \ldots \cdot (X_{\pi(m)} - X_{\pi(1)})^{i_m} \\ &\leq v \sum_{\underline{i}} g_{\underline{i}}(X_{\pi(1)})(X_{\pi(2)} - X_{\pi(1)})^{i_2} \cdot \ldots \cdot (X_{\pi(m)} - X_{\pi(1)})^{i_m} \\ &= vf(X_{\pi(1)}, \dots, X_{\pi(m)}) \;=\; v\tau_\pi f\,. \end{aligned}$$

Since $f$ and $\pi$ were arbitrary, we also have that

$$v\tau_\pi f \;\leq\; v\tau_{\pi^{-1}}(\tau_\pi f) \;=\; v(\tau_{\pi^{-1}}\tau_\pi f) \;=\; vf\,.$$

Altogether, this shows that $vf = v\tau_\pi f$. We have thus shown that $v$ is symmetric. This proves the corresponding implications in part a) of Theorem 1.3 and in Corollary 1.4, as well as Corollary 1.5.

Now we prove that if $v$ is openly-symmetric on $K[X_1,\ldots,X_m]$, then $v$ must be as described in Theorem 1.3 and Corollary 1.4. From Proposition 1.1 we know that $v$ is also openly-symmetric on $K^{\mathrm{ac}}[X_1,\ldots,X_m]$.

First of all, because $\delta := v(X_2 - X_1)$ is invariant under permutations of the variables, all values $v(X_j - X_i)$ are equal to $\delta$, provided that $i \neq j$. Suppose that $\delta < v(X_1 - a)$ for some $a \in K^{\mathrm{ac}}$. Then

$$v(X_2 - a) = \min\{v(X_2 - X_1), v(X_1 - a)\} = \delta < v(X_1 - a)\ ,$$

contradicting the symmetry. Hence, $\delta \geq v(X_1 - a)$ for all $a \in K^{\mathrm{ac}}$, showing that (5) holds. Since $v(X_i - a) = v(X_1 - a)$ by symmetry, also (6) holds.

Choose any $d \in K(X_1, X_2)^{\mathrm{ac}}$ with $vd = \delta$. For instance, $d = X_2 - X_1$ is a possible choice, but later on we will also want to consider other choices. Then all elements $\frac{X_j - X_1}{d}$, $2 \leq j \leq m$, are units in the valuation ring.

Using that $v$ is openly-symmetric on $K^{\mathrm{ac}}[X_1,\ldots,X_m]$, extend it to an openly-symmetric valuation on $K^{\mathrm{ac}}[X_1,\ldots,X_{m+1}]$. Extend $v$ further to a valuation on $K(X_1,\ldots,X_{m+1})^{\mathrm{ac}}$. Remark 1.2 shows that $v$ is symmetric on $K(X_1,\ldots,X_{i-1})^{\mathrm{ac}}[X_i,\ldots,X_{m+1}]$ for $2 \leq i \leq m$. Note that we are now in the situation where the assumptions of Corollary 1.6 are satisfied.

The same arguments as before show that

$$(16) \quad v(X_i - X_1) \;=\; \delta \;\geq\; v(X_i - a) \;\text{ for all } a \in K(X_1,\ldots,X_{i-1})^{\mathrm{ac}}.$$

For $3 \leq i \leq m$ we have that $v\frac{X_i - X_1}{d} = 0$, hence Lemma 2.4, (16) implies that the residue of $\frac{X_i - X_1}{d}$ is transcendental over the algebraically closed residue field

$$K(X_1,\ldots,X_{i-1})^{\mathrm{ac}}v \;=\; (K(X_1,\ldots,X_{i-1})v)^{\mathrm{ac}}\,.$$

This shows that the residues of the elements $\frac{X_3 - X_1}{d},\ldots,\frac{X_m - X_1}{d}$ are algebraically independent over $K(X_1, X_2)^{\mathrm{ac}}v$, hence also over $K(X_1, X_2)v$. Taking $d = X_2 - X_1$, this proves part b) of the theorem. We continue with the proof of part a).

If $\delta \in vK(X_1)^{\mathrm{ac}}$, then we can take the above arguments one step further. In this case we can take $d \in K(X_1)^{\mathrm{ac}}$. Similarly as before, we obtain that $v\frac{X_2 - X_1}{d} = 0$ and that $\frac{X_2 - X_1}{d}v$ is transcendental over $K(X_1)^{\mathrm{ac}}v$, hence also over $K(X_1)v$. We now have that the residues of the elements $\frac{X_2-X_1}{d},\ldots,\frac{X_m-X_1}{d}$ are algebraically independent over $K(X_1)v$.

Take any $f\in K^{\mathrm{ac}}[X_1,\ldots,X_m]$. If $\delta\in vK(X_1)^{\mathrm{ac}}$, then we take $d\in K(X_1)^{\mathrm{ac}}$ and rewrite the representation (3) as follows:

$$\begin{aligned} f &= \\ &= \sum_{\underline{i}} g_{\underline{i}}(X_1)\cdot(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_m-X_1)^{i_m} \\ &= \sum_{\underline{i}} g_{\underline{i}}(X_1)\cdot d^{i_2+\ldots+i_m}\cdot\left(\frac{X_2-X_1}{d}\right)^{i_2}\cdot\ldots\cdot\left(\frac{X_m-X_1}{d}\right)^{i_m}. \end{aligned}$$

Using Lemma 2.1, taking $K(X_1)^{\mathrm{ac}}$ in place of $K$, $I=\emptyset$ and $J=\{2,\ldots,n\}$ with $y_j=\frac{X_j-X_1}{d}$, we obtain that

$$\begin{aligned} vf &= \\ &= \min_{\underline{i}}\, vg_{\underline{i}}(X_1)\cdot d^{i_2+\ldots+i_m}\cdot\left(\frac{X_2-X_1}{d}\right)^{i_2}\cdot\ldots\cdot\left(\frac{X_m-X_1}{d}\right)^{i_m} \\ &= \min_{\underline{i}}\left(vg_{\underline{i}}(X_1)+(i_2+\ldots+i_m)\delta\right), \end{aligned}$$

which proves (4) in this case.

If $\delta\notin vK(X_1)^{\mathrm{ac}}$, then we take $d=X_2-X_1$ and rewrite the representation (3) as follows:

$$\begin{aligned} f &= \\ &= \sum_{\underline{i}} g_{\underline{i}}(X_1)(X_2-X_1)^{i_2}\cdots(X_m-X_1)^{i_m} \\ &= \sum_{\underline{i}} g_{\underline{i}}(X_1)(X_2-X_1)^{i_2+\ldots+i_m}\left(\frac{X_3-X_1}{X_2-X_1}\right)^{i_3}\cdots\left(\frac{X_m-X_1}{X_2-X_1}\right)^{i_m}. \end{aligned}$$

Using Lemma 2.1, taking $K(X_1)^{\mathrm{ac}}$ in place of $K$, $I=\{2\}$ with $x_2=X_2-X_1$, and $J=\{3,\ldots,m\}$ with $y_j=\frac{X_j-X_1}{X_2-X_1}v$, we obtain that

$$\begin{aligned} vf &= \\ &= \min_{\underline{i}}\, vg_{\underline{i}}(X_1)(X_2-X_1)^{i_2+\ldots+i_m}\left(\frac{X_3-X_1}{X_2-X_1}\right)^{i_3}\cdots\left(\frac{X_m-X_1}{X_2-X_1}\right)^{i_m} \\ &= \min_{\underline{i}}\left(vg_{\underline{i}}(X_1)+(i_2+\ldots+i_m)\delta\right), \end{aligned}$$

which proves (4) in this second case. This completes the proof of part a) of Theorem 1.3, and of Corollary 1.4 and Corollary 1.6.

We turn to the proof of part c) of Theorem 1.3. Lemma 2.1 shows that in the first of the above cases,

$$\begin{aligned}
&\operatorname{rr} vK(X_1,\ldots,X_m)/vK(X_1) \;=\\
&\quad=\;\operatorname{rr} vK(X_1)^{\mathrm{ac}}(X_2,\ldots,X_m)/vK(X_1)^{\mathrm{ac}} \;=\; 0\,,\\
&\operatorname{trdeg} K(X_1,\ldots,X_m)v|K(X_1)v \;=\\
&\quad=\;\operatorname{trdeg} K(X_1)^{\mathrm{ac}}(X_2,\ldots,X_m)v|K(X_1)^{\mathrm{ac}}v \;=\; m-1\,,
\end{aligned}$$

and in the second case,

$$\begin{aligned}
&\operatorname{rr} vK(X_1,\ldots,X_m)/vK(X_1) \;=\\
&\quad=\;\operatorname{rr} vK(X_1)^{\mathrm{ac}}(X_2,\ldots,X_m)/vK(X_1)^{\mathrm{ac}} \;=\; 1\,,\\
&\operatorname{trdeg} K(X_1,\ldots,X_m)v|K(X_1)v\\
&\quad=\;\operatorname{trdeg} K(X_1)^{\mathrm{ac}}(X_2,\ldots,X_m)v|K(X_1)^{\mathrm{ac}}v \;=\; m-2\,.
\end{aligned}$$

Both cases obviously appear as we are free to choose $\delta$ as long as it satisfies condition (5).

We have seen that the valuation on $K(X_1)$ can be arbitrary. By Lemma 2.1 there are always extensions of $v$ from $K$ to $K(X_1)$ such that $K(X_1)v|Kv = 1$, which by the Abhyankar inequality (12) forces $\operatorname{rr} vK(X_1)/vK = 0$. Combining such a valuation with the two cases above, we obtain openly-symmetric valuations on $K(X_1,\ldots,X_m)$ such that

$$\operatorname{rr} vK(X_1,\ldots,X_m)/vK = 0 \;\text{ and }\; \operatorname{trdeg} K(X_1,\ldots,X_m)v|Kv = m$$

in the first case, and

$$\operatorname{rr} vK(X_1,\ldots,X_m)/vK = 1 \;\text{ and }\; \operatorname{trdeg} K(X_1,\ldots,X_m)v|Kv = m-1$$

in the second case. This shows that cases (a) and (c) of Theorem 1.3 can appear.

Likewise, Lemma 2.1 shows that there are always extensions of $v$ from $K$ to $K(X_1)$ such that $\operatorname{rr} vK(X_1)/vK = 1$, which by the Abhyankar inequality (12) forces $K(X_1)v|Kv = 0$. Combining such a valuation with the two cases above, we obtain openly-symmetric valuations on $K(X_1,\ldots,X_m)$ such that

$$\operatorname{rr} vK(X_1,\ldots,X_m)/vK = 1 \;\text{ and }\; \operatorname{trdeg} K(X_1,\ldots,X_m)v|Kv = m-1$$

in the first case, and

$$\operatorname{rr} vK(X_1,\ldots,X_m)/vK = 2 \;\text{ and }\; \operatorname{trdeg} K(X_1,\ldots,X_m)v|Kv = m-2$$

in the second case. This shows that, again, case (c) can appear, and that case (e) can appear. Note that we have realized case (c) in two different ways.

If the valuation $v$ on $K$ admits an extension to $K(X_1)$ that does not enlarge value group and residue field, then we get $\mathrm{rr}\, vK(X_1)/vK = 0$ and $\mathrm{trdeg}\, K(X_1)v|Kv = 0$. Combining this with the two cases above, we obtain openly-symmetric valuations on $K(X_1,\dots,X_m)$ such that

$$\mathrm{rr}\, vK(X_1,\dots,X_m)/vK = 0 \quad\text{and}\quad \mathrm{trdeg}\, K(X_1,\dots,X_m)v|Kv = m-1$$

in the first case, and

$$\mathrm{rr}\, vK(X_1,\dots,X_m)/vK = 1 \quad\text{and}\quad \mathrm{trdeg}\, K(X_1,\dots,X_m)v|Kv = m-2$$

in the second case. Thus cases (b) and (d) can appear. This completes the proof of Theorem 1.3. $\square$

It remains to prove Corollary 1.7. Assume that the valuation $v_\delta$ is defined on $K(X_1,\dots,X_m)$ by definition (8). We can extend $v_\delta$ to $K^{\mathrm{ac}}(X_1,\dots,X_m)$ by applying the same definition to all polynomials $f \in K^{\mathrm{ac}}(X_1,\dots,X_m)$. Now it suffices to prove that $v_\delta$ is openly-symmetric on $K^{\mathrm{ac}}(X_1,\dots,X_m)$.

From our definition of $v_\delta$ on $K^{\mathrm{ac}}(X_1,\dots,X_m)$, we obtain that for all $a \in K^{\mathrm{ac}}$,

$$\delta \;\geq\; \min\{\delta, va\} \;=\; v(X_1-a)\,. \tag{17}$$

Furthermore, we obtain that for $2 \leq i \leq m$,

$$v_\delta(X_i - X_1) \;=\; \min\{\delta,\delta\} \;=\; \delta\,. \tag{18}$$

We consider the valuation $v$ defined by (4) on $K^{\mathrm{ac}}(X_1,\dots,X_m)$, where we take $v = v_\delta$ on $K^{\mathrm{ac}}(X_1)$. From (17) it follows that condition (5) is satisfied, so we know from Theorem 1.3 that $v$ is openly-symmetric. Therefore, we obtain that for $2 \leq i \leq m$,

$$vX_i \;=\; vX_1 \;=\; v_\delta X_1 \;=\; \delta\,.$$

We show that $v_\delta$ coincides with $v$ on $K^{\mathrm{ac}}(X_1,\dots,X_m)$. Indeed, for every $f \in K^{\mathrm{ac}}[X_1,\dots,X_m]$ written in the form (1),

$$\begin{aligned} v_\delta f &= \min_{\underline{i}}\,(vd_{\underline{i}} + (i_1+\ldots+i_m)\delta) \;=\; \min_{\underline{i}} v\, d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} \\ &\leq v\sum_{\underline{i}} d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} \;=\; vf\,, \end{aligned}$$

and if we write $f$ in the form (3), then by definition of $v$ and by (18),

$$\begin{aligned} vf &= \min_{\underline{i}}\,(vg_{\underline{i}}(X_1) + (i_2+\ldots+i_m)\delta) \\ &= \min_{\underline{i}} v_\delta\, g_{\underline{i}}(X_1)(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_m-X_1)^{i_m} \\ &\leq v_\delta \sum_{\underline{i}} g_{\underline{i}}(X_1)(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_m-X_1)^{i_m} \;=\; v_\delta f\,. \end{aligned}$$

It follows that $v_\delta = v$ on $K^{\mathrm{ac}}(X_1,\ldots,X_m)$, which proves that $v_\delta$ is openly-symmetric. $\square$

Finally, let us note that the comparison method of the above proof can be adapted to show the following fact, which is of independent interest.

**Corollary 4.1.** *Take a valued field $(K,v)$ and two valuations $w,w'$ extending $v$ from $K$ to a rational function field*

$$K(X_1,\ldots,X_m) \;=\; K(Y_1,\ldots,Y_m)\,,$$

*induced by the definitions*

$$\begin{aligned} w\sum_{\underline{i}} d_{\underline{i}} X_1^{i_1}\cdot\ldots\cdot X_m^{i_m} &:= \min_{\underline{i}}\,(vd_{\underline{i}} + i_1\delta_1+\ldots+i_m\delta_m)\,,\\ w'\sum_{\underline{j}} d_{\underline{j}} Y_1^{j_1}\cdot\ldots\cdot Y_m^{j_m} &:= \min_{\underline{j}}\,(vd_{\underline{j}} + j_1\delta_1+\ldots+j_m\delta_m)\,. \end{aligned}$$

*Then $w = w'$.*

## 5. An example

We construct an example of a valuation $v$ on the rational function field $\mathbb{Q}^{\mathrm{ac}}(X,Y)$ over $\mathbb{Q}^{\mathrm{ac}}$ which is not symmetric on $\mathbb{Q}^{\mathrm{ac}}(X,Y)$ but has a symmetric restriction to $\mathbb{Q}(X,Y)$. To this end, we write $i=\sqrt{-1}$ and extend the trivial valuation on $\mathbb{Q}^{\mathrm{ac}}(X)$ to a valuation $v$ on the rational function field $\mathbb{Q}^{\mathrm{ac}}(X,Y)=\mathbb{Q}^{\mathrm{ac}}(X,Y-iX)$ by definition (8), where we take $K=\mathbb{Q}^{\mathrm{ac}}(X)$, $\delta=1$, $m=1$ and $X_1=Y-iX$. So we have

$$v(Y-iX) \;=\; 1\,.$$

Note that

$$\begin{aligned} &(19) \qquad & v(X+iY) &\;=\; vi + v(Y-iX) \;=\; v(Y-iX) \;=\; 1\,,\\ &(20) \qquad & v(Y+iX) &\;=\; \min\{v(Y-iX),v(2iX)\} \;=\; 0\,,\\ &(21) \qquad & v(X^2+Y^2) &\;=\; v(Y-iX)+v(Y+iX) \;=\; 1\,. \end{aligned}$$

Equations (19) and (20) together show that $v$ is not symmetric on $\mathbb{Q}^{\mathrm{ac}}(X,Y)$, and even not on $\mathbb{Q}(i)(X,Y)$.

We show that $v$ is not only trivial on $\mathbb{Q}^{\mathrm{ac}}(X)$, but also on $\mathbb{Q}^{\mathrm{ac}}(Y)$. It suffices to show that $v$ is trivial on $\mathbb{Q}^{\mathrm{ac}}[Y]$, and since it is trivial on $\mathbb{Q}^{\mathrm{ac}}$ and every polynomial in $\mathbb{Q}^{\mathrm{ac}}[Y]$ splits into linear factors, it even suffices to show that $v(Y-a)=0$ for every $a\in\mathbb{Q}^{\mathrm{ac}}$. Using that $iX-a\neq 0$ since $iX\notin\mathbb{Q}^{\mathrm{ac}}$ and that $v$ is trivial on $\mathbb{Q}^{\mathrm{ac}}(X)$, we compute:

$$v(Y-a) \;=\; \min\{v(Y-iX),v(iX-a)\} \;=\; 0\,.$$

Now we show that the value of every nonzero $f \in \mathbb{Q}(X)[Y]$ with $\deg_Y f \leq 1$ is 0. This is clear when the degree is 0 since then $f \in \mathbb{Q}(X) \subset \mathbb{Q}^{\mathrm{ac}}(X)$, on which $v$ is trivial. For arbitrary $g_0, g_1 \in \mathbb{Q}(X)$ with $g_1 \neq 0$, we compute:

$$\begin{aligned} v(g_1 Y + g_0) &= vg_1 + v\left(Y + \frac{g_0}{g_1}\right) = v\left(Y + \frac{g_0}{g_1}\right) \\ &= \min\left\{v(Y - iX), v\left(iX + \frac{g_0}{g_1}\right)\right\} = 0\,, \end{aligned}$$

where we use that $iX + g_0/g_1 \neq 0$ since $-iX \notin \mathbb{Q}(X)$. Similarly, we show that the value of every $f \in \mathbb{Q}(Y)[X]$ with $\deg_X f \leq 1$ is 0. This is clear when the degree is 0 since then $f \in \mathbb{Q}(Y) \subset \mathbb{Q}^{\mathrm{ac}}(Y)$, on which $v$ is trivial. For arbitrary $h_0, h_1 \in \mathbb{Q}(Y)$ with $h_1 \neq 0$, we compute:

$$\begin{aligned} v(h_1 X + h_0) &= vh_1 + v\left(X + \frac{h_0}{h_1}\right) = v\left(X + \frac{h_0}{h_1}\right) \\ &= \min\left\{v(X + iY), v\left(-iY + \frac{h_0}{h_1}\right)\right\} = 0\,, \end{aligned}$$

where we use (19) and that $-iY + h_0/h_1 \neq 0$ since $iY \notin \mathbb{Q}(Y)$.

Set $F(Y) = X^2 + Y^2 \in \mathbb{Q}(X)[Y]$. Using the $F$-adic expansion in $\mathbb{Q}(X)[Y]$, every polynomial in $f \in \mathbb{Q}[X, Y]$ can be written in the form

$$f(X, Y) = \sum_i f_i(X, Y)(X^2 + Y^2)^i$$

with $f_i(X, Y) \in \mathbb{Q}(X)[Y]$ such that $\deg_Y f_i(X, Y) \leq 1$. Then also $\deg_X f_i(Y, X) \leq 1$, and by what we have shown before,

$$vf_i(X, Y) = 0 = vf_i(Y, X)$$

for all $i$ such that $f_i(X, Y) \neq 0$. In view of (21), we obtain:

$$\begin{aligned} vf(X, Y) &= v\sum_i f_i(X, Y)(X^2 + Y^2)^i = \\ &= \min\{i \mid f_i(X, Y) \neq 0\} = \min\{i \mid f_i(Y, X) \neq 0\} \\ &= v\sum_i f_i(Y, X)(Y^2 + X^2)^i = vf(Y, X)\,, \end{aligned}$$

which shows that $v$ is symmetric on $\mathbb{Q}(X, Y)$.

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