From Zygmund's ideas to HK_r and P_r integrals

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1. Introduction
   - Motivation I: History of recovering a function from its derivative
   - Motivation II: Suitable method of integration

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2. The Denjoy, Henstock and Perron Integrals
   - Denjoy-integral
   - Henstock-Kurzweil-integral
   - Perron-integral
   - Variational-integral

An open problem to solve

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Function with derivative not L-integrable

Define $F : [0,1] \to \mathbf{R}$ in the following way:

$$F(x) := \begin{cases} x^2 \sin(\frac{\pi}{x^2}), & \text{if } x \in (0,1], \\ 0, & \text{if } x = 0 \end{cases}$$

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The function has a derivative at each point on $[0,1]$, but F is not absolutely continuous on $[0,1]$.

Absolute Continuity and Absolute Continuity in restricted sense

The oscillation of the function F on an interval $[c, d]$ is

$$\omega(F, [c, d]) = \sup\{|F(y) - F(x)| : c \leq x < y \leq d\}$$

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AC- and AC^*-function

F is AC-function (or AC^*-function) on $E \subset [a, b]$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |F(d_i) - F(c_i)| < \varepsilon$, (or $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \varepsilon$) whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

Generalized Absolute Continuity and Generalized Absolute Continuity in restricted sense

The function F is Generalized Absolutely Continuous on $E \subset [a, b]$, in short F is an ACG-function on E (F is Generalized Absolutely Continuous in the restricted sense on $E \subset [a, b]$, in short F is an ACG^*-function on E) as the following definition:

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To simplify our notation we shall consider $E = [a, b]$ and we shall say F is an ACG or ACG^* function.

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This integral recovers a function from its derivative, i.e., supposing F is continuous on $[a, b]$, and differentiable nearly everywhere on $[a, b]$, then F' is integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a, b]$.

Perron approach, minor and major functions

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Let $f : [a, b] \to \mathbb{R}$. A function $U : [a, b] \to \mathbb{R}$ is major function of f on $[a, b]$ if $\underline{D}U(x) > -\infty$ and $\underline{D}U(x) \geq f(x)$ for all $x \in [a, b]$; a function $V : [a, b] \to \mathbb{R}$ is minor function of f on $[a, b]$ if $\overline{D}V(x) < +\infty$ and $\overline{D}V(x) \leq f(x)$ for all $x \in [a, b]$.

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Supposing that U is a major function and V minor function of f on $[a, b]$ it can be proved that

$$-\infty < \sup\{V(b) - V(a)\} \leq \inf\{U(b) - U(a)\} < +\infty$$

where the supremum is taken over all minor function of f and the infimum is taken over all major function of f.

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The same value is obtained if we consider the family of continuous major and minor functions getting the so-called Perron continuous integral that is equivalent to the above Perron integral. Moreover, another equivalent Perron integral is obtained if in the definition of derivates of major and minor function we suppose that $\underline{U}(x) > -\infty$ and $\overline{V}(x) < +\infty$ nearly everywhere on $[a,b]$ and that $\underline{U}(x) \geq f(x)$ and $\overline{V}(x) \leq f(x)$ almost everywhere.

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A measurable function $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable on $[a, b]$ if and only if for each $\varepsilon > 0$ there exist an absolutely continuous major function U and an absolutely continuous minor functions V of f on $[a, b]$ such that $[U(b) - U(a)] - [V(b) - V(a)] < \varepsilon$.

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Riemann integral

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a value L with the following property: for each $\varepsilon > 0$ there exists a positive number δ such that $|\sum_{i=1}^{n} f(x_i)(d_i - c_i) - L| < \varepsilon$ for any partition $P = \{[c_i, d_i] : 1 \le i \le n\}$ such that $x_i \in [c_i, d_i]$ and $\max |d_i - c_i| < \delta$, $1 \le i \le n$.

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Let δ a positive function defined on $[a, b]$, a tagged partition of $[a, b]$ subordinate to δ is a finite collection $\mathbb{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ of pairs point-interval such that $\cup_{i=1}^{n} [c_i, d_i] = [a, b]$, $x_i \in [c_i, d_i]$, $[c_i, d_i] \subset [x_i - \delta(x_i), x_i - \delta(x_i)]$ for each $i = 1, ..., n$.

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It is easy to understand that, by the Henstock-Kurzweil-integral definition, the main "work" is to find a suitable positive function δ in order to verify the integrability of the function f.
There are some "tricks" to force the "tags" x_i to have some special properties: for example in the points where the function is not continuous.

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Variational-integral

The Denjoy, Perron and Henstock-Kurzweil integrals are all equivalent. They are equivalent also the the following integral definition that represents a transition between Henstock-Kurweil-integral and Perron-integral. It avoids Riemann sums and derivates.

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A function $f : [a, b] \to \mathbb{R}$ is variational integrable on $[a, b]$ if there exists a function $F[a, b] \to \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a non-decresing function φ defined on $[a, b]$ and a positive function δ defined on $[a, b]$ such that $\varphi(b) - \varphi(a) < \varepsilon$ and

$$|f(x)(d-c) - (F(d) - F(c))| < \varphi(d) - \varphi(c)$$

whenever $(x, [c, d])$ is a tagged interval in $[a, b]$ subordinate to δ.

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We consider now integrals defined to deal with another type of derivative, the L^r-derivative, i.e., a derivative in the metric L^r. It was introduced by Calderon and Zygmund in 1961 in order to establish pointwise estimates for solutions of elliptic partial differential equations.

Zygmund-Calderon ideas

L. Gordon in 1968 described a Perron-type integral, the P_r-integral, that recovers a function from its L^r-derivative, and considered an application of the L^r-derivative and the P_r-integral to Fourier series. In 2004, Musial and Sagher defined the L^r-Henstock-Kurzweil integral, the HK_r-integral, that also recovers a function from its L^r-derivative, and showed that it is an extension of the P_r-integral.

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Nevertheless the HK_r-integral is equivalent to L^r-variational integral.

L^r-derivatives

We assume that $r \geq 1$ and we work on the closed interval $[a, b]$. We start with the definition of the upper right L^r-derivative

Definition: upper right L^r-derivative (L. Gordon 1968)

Let $f \in L^r[a, b]$. The upper right L^r-derivate of f at x, denoted by $D_r^+ f(x)$, is the greatest lower bound of all α such that

$$\left(\frac{1}{h}\int_0^h [f(x+t) - f(x) - \alpha t]_+^r \, dt\right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+. \quad (1)$$

If no real number α satisfies (1), we set $D_r^+ f(x) = +\infty$. If (1) holds for every real number α, we set $D_r^+ f(x) = -\infty$.

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We assume that $r \geq 1$ and we work on the closed interval $[a, b]$. We start with the definition of the upper right L^r-derivative

Definition: upper right L^r-derivative (L. Gordon 1968)

Let $f \in L^r[a, b]$. The upper right L^r-derivate of f at x, denoted by $D_r^+ f(x)$, is the greatest lower bound of all α such that

$$\left(\frac{1}{h} \int_0^h [f(x+t) - f(x) - \alpha t]_+^r \, dt \right)^{\frac{1}{r}} = o(h) \text{ as } h \to 0^+. \qquad (1)$$

If no real number α satisfies (1), we set $D_r^+ f(x) = +\infty$. If (1) holds for every real number α, we set $D_r^+ f(x) = -\infty$.

We define the lower right L^r-derivate, $D_{+,r} f(x)$, the upper left L^r-derivate, $D_r^- f(x)$, and the lower left L^r-derivate, $D_{-,r} f(x)$, in a similar manner.

L^r-derivative

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The upper (two-sided) L^r derivate is as follows:

$$\overline{D}_r f(x) = \max\left\{D_r^+ f(x), D_r^- f(x)\right\}.$$

Similarly we define the lower (two-sided) L^r-derivate as follows:

$$\underline{D}_r f(x) = \min\left\{D_{+,r} f(x), D_{-,r} f(x)\right\}.$$

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Definition: L^r-derivative (L. Gordon 1968)

Let $f \in L^r[a,b]$. If $\overline{D}_r f(x)$ and $\underline{D}_r f(x)$ are the same real number, i.e., if all four L^r-derivates are equal and finite, then we say that f is L^r-differentiable at x. The common value, denoted by $f_r'(x)$, is the L^r-derivative of f at x.

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It is clear that if a function f is differentiable at a point x then it is also L^r-differentiable at the same point and $f_r'(x) = f'(x)$.

L^r-continuity

To define L^r-major functions and L^r-minor functions, we need a notion of L^r-continuity.

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Definition: L^r-continuity

A function $F \in L^r[a,b]$ is said to be L^r-continuous at $x \in [a,b]$ if

$$\lim_{h \to 0} \frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy = 0.$$

.

L^r- major and -minor function

Definition: L^r-major function

Suppose f is a function defined on $[a, b]$. A finite-valued function $\psi \in L^r\,[a, b]$ is said to be an L^r-major function of f if

1. $\psi\,(a) = 0$,
2. ψ is L^r-continuous on $[a, b]$,
3. except for at most a denumerable subset of $[a, b]$ we have

$$-\infty \neq \underline{D}_r\psi(x) \geq f(x). \tag{2}$$

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A function φ is an L^r-minor function of f if $-\varphi$ is an L^r-major function of $-f$.

P_r-integral

L. Gordon proved that for any L^r-major function ψ and any L^r-minor function φ of f, the function $\psi - \varphi$ is non-decreasing on $[a, b]$. This property allows us to define a Perron-type integral, the P_r-integral, in a standard way:

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Definition: P_r-integral

Suppose f is a function defined on $[a, b]$. If $\inf \psi(b)$ taken over all L^r-major functions of f equals $\sup \varphi(b)$ taken over all L^r-minor functions of f, then the common value, denoted by

$$(P_r) \int_a^b f$$

is called the P_r-integral of f on $[a, b]$, and f is said to be P_r-integrable on $[a, b]$.

HK_r-integral

Now we recall the definition of L^r-Henstock-Kurzweil-type integral.

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Definition: HK_r-integral (Musial and Sagher 2004)

A function $f : [a,b] \to \mathbb{R}$ is L^r-Henstock-Kurzweil integrable (HK_r-integrable) on $[a,b]$ if there exists a function $F \in L^r[a,b]$ so that for any $\varepsilon > 0$ there exists a positive function δ defined on $[a,b]$ so that for any finite collection of nonoverlapping tagged intervals $\mathcal{Q} = \{(x_i, [c_i, d_i]), 1 \le i \le q\}$ subordinate to δ we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} \left| F(y) - F(x_i) - f(x_i)(y - x_i) \right|^r dy \right)^{1/r} < \varepsilon.$$

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The function F in Definition is unique up to an additive constant, so we can consider the indefinite HK_r-integral.

$$F(x) = (HK_r) \int_a^x f, \text{ for each } x \in (a,b]$$

Some results

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The indefinite HK_r-integral F is L^r-continuous on $[a, b]$ with $F_r'(x) = f(x)$ a.e. on $[a, b]$

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P_r is contained in HK_r (Sagher Musial 2004)

If $f : [a,b] \to R$ is P_r-integrable then it is HK_r-integrable and the values of integrals coincide.

Charaterization of HK_r-integral

Charaterization of HK_r-integral

F is AC_r on $[a,b]$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a positive function δ defined on $[a,b]$ so that for any finite collection of nonoverlapping tagged intervals $\{(x_i, [c_i, d_i]), 1 \leq i \leq q\}$ subordinate to δ such that $\sum_{i=1}^{q}(d_i - c_i) < \eta$ we have

$$\sum_{i=1}^{q}\left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy\right)^{1/r} < \varepsilon.$$

$F \in ACG_r$ on $[a,b]$ if $[a,b]$ can be written as countable union of sets on each of which $F \in AC_r$.

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$F \in ACG_r$ on $[a,b]$ if $[a,b]$ can be written as countable union of sets on each of which $F \in AC_r$.

Characterization of HK_r-integral (Sagher Musial 2004)

A function f is HK_r-integrable on $[a,b]$ if and only if there exists $F \in ACG_r[a,b]$ such that $F_r' = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.

Main result

The converse of the theorem obtained in 2004 by Sagher and Musial that says P_r is contained in HK_r is not true, i.e., the class of P_r-integrable functions is strictly included in the class of HK_r-integrable functions.

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P_r is strictly included in HK_r (Musial, Skvortsov, Tulone 2022)

There exists a function which is HK_r-integrable on $[a, b]$ but which is not P_r-integrable on $[a, b]$

Main result

A function $f : [a, b] \to \mathbb{R}$ is L^r-variational integrable on $[a, b]$ if there exists a function $F \in L^r[a, b]$ with the following property: for each $\varepsilon > 0$ there exist a non-decreasing function φ defined on $[a, b]$ and a positive function δ defined on $[a, b]$ such that $\varphi(b) - \varphi(a) < \varepsilon$ and for any tagged interval $(x, [c, d])$ subordinate to δ, where $[c, d] \subseteq [a, b]$,

$$\left(\frac{1}{d-c}\int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy\right)^{1/r} < \varphi(d) - \varphi(c).$$

Main result

A function $f:[a,b]\to\mathbb{R}$ is L^r-variational integrable on $[a,b]$ if there exists a function $F\in L^r[a,b]$ with the following property: for each $\varepsilon>0$ there exist a non-decreasing function φ defined on $[a,b]$ and a positive function δ defined on $[a,b]$ such that $\varphi(b)-\varphi(a)<\varepsilon$ and for any tagged interval $(x,[c,d])$ subordinate to δ, where $[c,d]\subseteq[a,b]$,

$$\left(\frac{1}{d-c}\int_c^d |F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r}<\varphi(d)-\varphi(c)\,.$$

L^r is equivalent to HK_r-integral (Musial and Tulone 2022)

A function $f:[a,b]\to$ is L^r-Henstock-Kurzweil integrable on $[a,b]$ if and only if f is L^r-variational integrable on $[a,b]$.

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