# On defectless and tame fields

Anna Rzepka

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# Motivation

$(K, v)$ a valued field $vK$ the value group, $Kv$ the residue field.

Open problems in valuation theory and related areas:

- local uniformization - a local version of resolution of singularities;
- model theory of valued fields, like AKE Principles

  $AKE^{\equiv}$ :

  $$vL \equiv vK \wedge Lv \equiv Kv \Rightarrow (L, v) \equiv (K, v)$$

  $AKE^{\prec}$ :

  $$(K, v) \subseteq (L, v) \wedge vK \prec vL \wedge Kv \prec Lv \Rightarrow (K, v) \prec (L, v)$$

- the structure of valued function fields

Obstacles:

- defect extensions
- purely inseparable extensions

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# defect extensions

If $(L|K, v)$ is a finite extension of valued fields and the extension of $v$ from $K$ to $L$ is unique, then

$$[L : K] = p^n (vL : vK)[Lv : Kv],$$

where $p = \mathrm{char} Kv$ if it is positive and $p = 1$ otherwise.

$d(L|K, v) := p^n$ - the **defect** of $(L|K, v)$.

If $p^n > 1$, then $(L|K, v)$ is called a **defect extension**. Otherwise it is called a **defectless extension**.

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# defectless fields

A henselian field $(K, v)$ is called defectless if every finite extension $(L|K, v)$ is defectless, i.e.,

$$[L : K] = (vL : vK)[Lv : Kv].$$

- Every henselian field of residue characteristic zero is defectless.
- Defectless fields may be imperfect, e.g. $(\mathbb{F}_p((t)), v_t)$.

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# tame fields

A henselian valued field $(K, v)$ of residue characteristic $p$ is called **tame** if the following conditions hold:

**(T1)** if $p > 0$, then $vK$ is $p$-divisible,

**(T2)** $Kv$ is perfect;

**(T3)** $(K, v)$ is defectless.

- Every henselian field of residue characteristic zero is tame.
- If $(K, v)$ is henselian of positive characteristic, then $(K, v)$ is tame if and only if it is defectless and perfect.

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- Tame fields satisfy $AKE^{\prec}$. Moreover, $AKE^{\equiv}$ holds for tame fields of equal characteristic (F.-V. Kuhlmann).
- The properties of tame fields play an important role in the proof of the fact that every place of an algebraic function field admits a local uniformization in a finite extension (H. Knaf, F.-V. Kuhlmann).

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- A finite extension of a defectless field is again a defectless field.

  This may not hold for infinite algebraic extensions: $(\mathbb{F}_p((t)), v_t)$ is a henselian defectless field, but the perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$ of $\mathbb{F}_p((t))$ (together with the unique extension of $v_t$) admits a defect extension generated by a root of $X^p - X - \frac{1}{t}$.

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**Theorem 1 (R.)**

*A henselian field $(K, v)$ of equal characteristic is tame if and only if every algebraic extension of $(K, v)$ is a defectless field.*

*Take a henselian field $(K, v)$ of mixed characteristic with an archimedean value group and perfect residue field. Then $(K, v)$ is tame if and only if every algebraic extension of $(K, v)$ is a defectless field.*

**Remarks on a proof:**

- If char$Kv = 0$ - trivial;
- If char$K = p > 0$, then one can give a short proof using basic facts from ramification theory.
- In both cases (equal positive and mixed characteristic) if $(K, v)$ is not tame (T1 or T2 does not hold) one can give a precise construction of an algebraic extension $(L, v)$ of $(K, v)$ (an infinite tower or a compositum of two infinite towers of extensions of degree $p$) and a Galois defect extension of $(L, v)$ of degree $p$.

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# compositum of valuations

Take a valued field $(K, v)$. A valuation $w$ of $K$ is a **coarsening** of $v$ if $\mathcal{O}_v \subseteq \mathcal{O}_w$.

Take a valuation $\bar{v}$ of $Kv$. Then by $v \circ \bar{v}$ we denote the valuation on $K$ whose valuation ring is of the following form

$$\mathcal{O}_{v\circ\bar{v}} := \{a \in \mathcal{O}_v : av \in \mathcal{O}_{\bar{v}}\}.$$

We call $w = v \circ \bar{v}$ the **composition of the valuations** $v$ and $\bar{v}$.

- The valuation $v$ is a coarsening of $w$.
- The value group $\bar{v}(Kv)$ of $(Kv, \bar{v})$ is isomorphic to a convex subgroup $H$ of $wK$ and $vK \cong wK/H$.
- $(K, w)$ is a henselian field if and only if $(K, v)$ and $(Kv, \bar{v})$ are henselian fields.
- Assume that $(K, w)$ is henselian. Then it is a defectless field if and only if $(K, v)$ and $(Kv, \bar{v})$ are defectless fields.

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# mixed characteristic -example

$(K, v_1)$ - a tame field, $\mathrm{char}K = 0$, $\mathrm{char}Kv_1 = p > 0$, $v_1K \subseteq \mathbb{R}$.

Take $\gamma > v_1K$ and $v_2$ the $x$-adic valuation on $K(x)$, $v_2x := \gamma$. Set $F$ to be the henselization of $K(x)$ with respect to $v_2$.

As $Fv_2 = K(x)v_2 = K$, we may consider the compositum $v = v_2 \circ v_1$ and the valued field $(F, v)$.

$$v\left(\sum_{i=1}^{n} a_i x^i\right) = \min_{1 \leq i \leq n} v_1(a_i) + i\gamma.$$

$(F, v_2)$ is heselian, $(K, v_1)$ is henselian $\Rightarrow$ $(F, v)$ is henselian.

Take an algebraic extension $(L, v')$ of $(F, v)$. Then $v' = v_2' \circ v_1'$, where $v_i'$ is the unique extension of $v_i$.

$(K, v_1)$ is tame, $Lv_2'|K$ algebraic, so $(Lv_2', v_1')$ is a defectless field; $(L, v_2')$ is defectless, as $\mathrm{char}Lv_2' = 0 \Rightarrow (L, v')$ is defectless.

$vF = \mathbb{Z}\gamma + vK \neq pvF$. Hence $(F, v')$ is not a tame field.

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# mixed characteristic -example

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Take $\gamma > v_1K$ and $v_2$ the $x$-adic valuation on $K(x)$, $v_2x := \gamma$. Set $F$ to be the henselization of $K(x)$ with respect to $v_2$.

As $Fv_2 = K(x)v_2 = K$, we may consider the compositum $v = v_2 \circ v_1$ and the valued field $(F, v)$.

$$v\left(\sum_{i=1}^{n} a_i x^i\right) = \min_{1\leq i\leq n} v_1(a_i) + i\gamma.$$

$(F, v_2)$ is heselian, $(K, v_1)$ is henselian $\Rightarrow$ $(F, v)$ is henselian.

Take an algebraic extension $(L, v')$ of $(F, v)$. Then $v' = v_2' \circ v_1'$, where $v_i'$ is the unique extension of $v_i$.

$(K, v_1)$ is tame, $Lv_2'|K$ algebraic, so $(Lv_2', v_1')$ is a defectless field; $(L, v_2')$ is defectless, as $\mathrm{char}Lv_2' = 0 \Rightarrow (L, v')$ is defectless.

$vF = \mathbb{Z}\gamma + vK \neq pvF$. Hence $(F, v')$ is not a tame field.

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$(K, v)$ - of residue characteristic $p > 0$
Denote by $(vK)_{vp}$ the smallest convex subgroup of $vK$ that contains $vp$ if char$K = 0$, and set $(vK)_{vp} = vK$ otherwise.

We call$(K, v)$ **a quasi-tame field** if the following conditions hold

(QT1) if $p > 0$ then the group $(vK)_{vp}$ is $p$-divisible,

(QT2) the residue field $Kv$ is perfect,

(QT3) $(K, v)$ is a defectless field.

- Every tame field is a quasi-tame field.
- If char$K > 0$ or $vK$ archimedean, then $(K, v)$ is quasi-tame if and only if it is tame.

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$(K, v)$ - a henselian field of characteristic 0 and residue characteristic $p$.
$v = v_2 \circ v_1$, where $v_2$ is the finest coarsening of $v$ that has residue characteristic 0.

- Both fields $(K, v_2)$ and $(Kv_2, v_1)$ are henselian.

**Proposition 1**

*Under the above assumptions:*

(1) *$(K, v)$ is quasi-tame if and only if $(Kv_2, v_1)$ is tame;*

(2) *every algebraic extension of $(K, v)$ is a defectless field if and only if every algebraic extension of $(Kv_2, v_1)$ is a defectless field.*

**Theorem 2 (R.)**

*Take a henselian field $(K,v)$ of characteristic $0$ and positive residue characteristic $p$.*
*Assume that the residue field of $(K,v)$ is perfect or $K$ contains an element $d_0 \in K$ such that $-vp \leq vd_0 < 0$ and for every $n \in \mathbb{N}$ the polynomial $X^{p^n} - d_0$ admits a root in $K$.*
*Then $(K,v)$ is quasi-tame if and only if every algebraic extension of $(K,v)$ is a defectless field.*

It is an open question if there is a henselian field $(K, v)$ of characteristic 0 and positive residue characteristic $p$ satisfying the following conditions:

(1) $vK = (vK)_{vp}$ is $p$-divisible;

(2) $Kv$ is imperfect

(3) for every sequence of elements $d_i$ satisfying

$$d_0 \in K \text{ with } -vp \leq vd_0 \leq 0 \text{ and } d_i^p = d_{i-1},\ i \in \mathbb{N}. \quad (1)$$

each of the extensions $(K(d_i)|K, v)$ is defectless and the valued field $(K(d_i|i \in \mathbb{N}), v)$ is tame.

A nontrivially valued field $(K, v)$ is called **semitame** if

**(A)** if $\mathrm{char} Kv = p > 0$, then the following homomorphism is surjective

$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}}.$$

**(B)** if $\mathrm{char} Kv = p > 0$, then the value group $vK$ is $p$-divisible.

A nontrivially valued field $(K, v)$ is called **generalized deeply ramified** (in short is a **gdr field**) if

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$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}}.$$

**(B)** if $\mathrm{char}\, Kv = p > 0$, then $vp$ is not the smallest positive element in the value group $vK$.

**Proposition 2**

*Take a nontrivially valued henselian field $(K, v)$.*

1) *If $(K, v)$ is semitame and quasi-tame, then it is tame.*
2) *If $(K, v)$ is quasi-tame then it is a gdr field.*

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THANK YOU
FOR YOUR ATTENTION!