Abstract evolution systems
(joint work with Wiesław Kubiś)

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Second Graduate Students' Workshop on
Algebra, Logic and Analysis
March 2022

Plan

1. The notion
2. Examples
3. Properties
4. A theorem
5. Evolving vs. rewriting

Category theory

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topological spaces	+	continuous functions
vector spaces	+	linear mappings

Definition

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$$\mathcal{E} = \langle \mathcal{V}, \mathcal{T}, \theta \rangle,$$

where $\mathcal{T}$ is a distinguished class of $\mathcal{V}$-arrows called **transitions** and θ is a fixed $\mathcal{V}$-object called the **origin**.

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the initial object?

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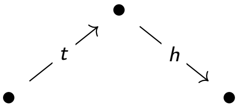

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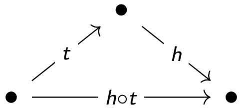

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Both cases are not particularly interesting.
Let $\mathcal{V} = Sets$. The class $\mathcal{T}$ can also consist of one-point extensions, namely

$$f \in \mathcal{T}(X) \overset{df}{\Leftrightarrow} f \colon X \overset{1-1}{\to} Y \ \wedge \ |Y \setminus f[X]| \leq 1,$$

where $\mathcal{T}(X)$ is the set of all transitions with domain X.

Evolution

Definition

We are interested in investigating **evolutions**, namely sequences of objects and transitions

$$\theta = A_0 \to A_1 \to A_3 \to \dots$$

Examples

Let $\mathcal{V}$ be a category of finitely generated structures in a fixed first-order language. Let $\mathcal{T}$ consists of all isomorphisms and embeddings of the form $f\colon X \to Y$ where Y is generated by $f[X] \cup \{r\}$ for some $r \in Y$.

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- the class of all finite fields with θ being p-element field

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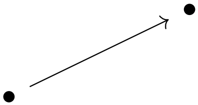

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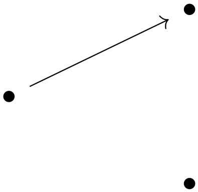

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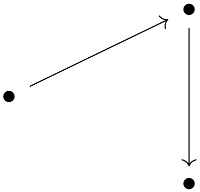

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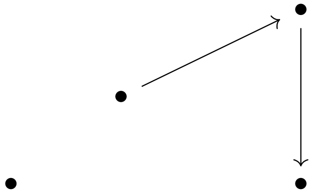

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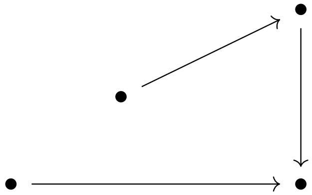

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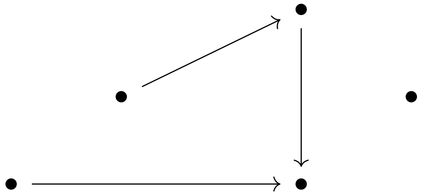

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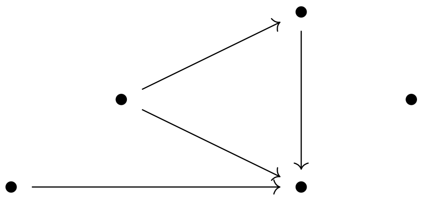

Let $\mathcal{V}$=LinSets be a category of linearly ordered sets. Consider an evolution system where transitions are strictly order preserving functions and the empty set or any singleton to be the origin. At each step we can add just one point in different ways:

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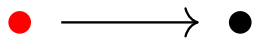

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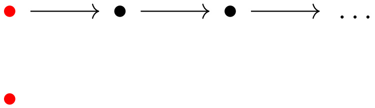

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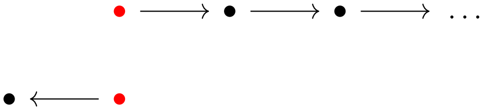

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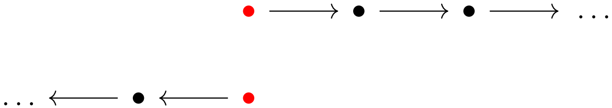

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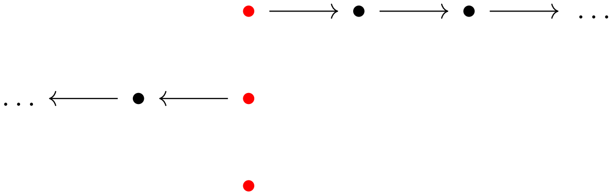

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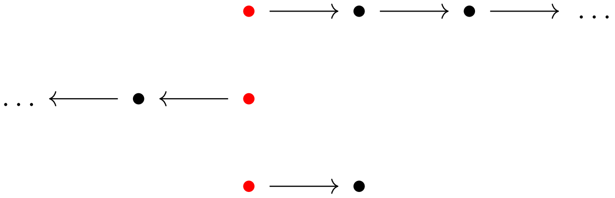

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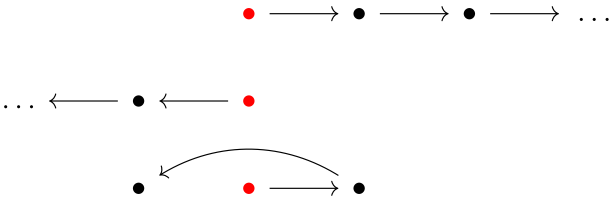

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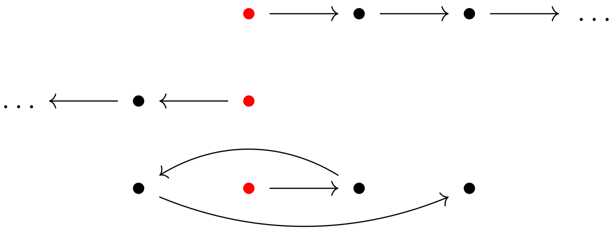

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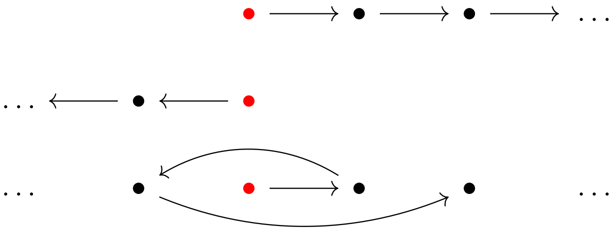

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Consider the category $\mathcal{E}^{fin}$ of finite objects and paths. It clearly is a subcategory of $\mathcal{E}$ and an evolution system itself.

Amalgamation property

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We say that $\mathcal{E}$ has the **finite** amalgamation property (**FAP**) if for every two transitions $f, g \in \mathcal{T}(\bullet)$, where $\bullet$ is a finite object, there exist two further transitions f', g' such that $f' \circ f = g' \circ g$.

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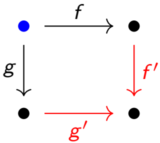

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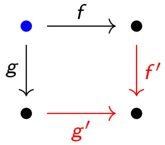

Remark

If $\mathcal{E}$ has the finite amalgamation property then $\mathcal{E}^{fin}$ has the amalgamation property.

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We say that an evolution

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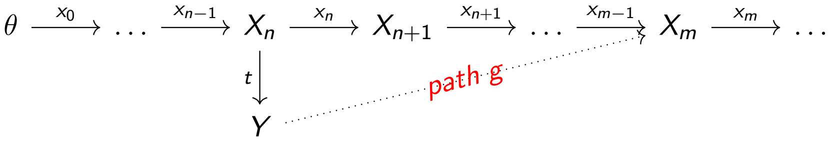

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Assume $\mathcal{E}$ is an essentially countable evolution system that has the finite amalgamation property. Then there exists a unique, up to an isomorphism, evolution with the absorption property.

The colimit of the evolution with the absorption property will be called the **Fráïssé limit** of $\mathcal{E}^{fin}$.

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- a framework for studying generic structures

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evolution generalises rewriting

Differences

rewriting system — evolution system

Differences

rewriting system	—	evolution system
poset	—	category

Differences

rewriting system	—	evolution system
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terminating	—	extending

Differences

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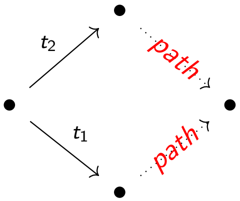
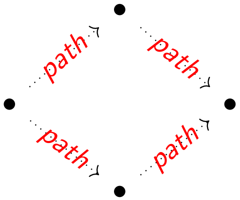
confluence	—	amalgamation

local confluence



confluence



A variant of Newman's lemma

Theorem

A locally confluent iso-stable terminating evolution system is confluent.

- W. Kubiś, R. *Abstract evolution systems*
https://arxiv.org/abs/2109.12600