Krasners valued hyperfields and amc-structures

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Definition

Consider a valued field (K, v) with a valuation ring $\mathcal{O}$. For any nonnegative $\gamma \in vK$ we can set an $\mathcal{O}$-ideal

$$\mathcal{M}^\gamma := \{x \in \mathcal{O} \mid v(x) > \gamma\}.$$

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Then the quotient $K/(1+\mathscr{M}^{\gamma})$ together with the following operations induced from the field K:

1. multiplication: $[a]_{\gamma}[b]_{\gamma} = [ab]_{\gamma}$, and
2. hyperaddition: $[a]_{\gamma} + [b]_{\gamma} = \{[x+y]_{\gamma} \mid x \in [a]_{\gamma},\ y \in [b]_{\gamma}\}$

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We will denote it by $\mathcal{H}_\gamma(K)$ and call it the γ-**valued hyperfield** of the valued field (K, v).

Definition

Take any hyperfield F and an ordered abelian group Γ (written additively). A surjective map $v : F \to \Gamma \cup \{\infty\}$ is called a **valuation** on F if it satisfies the following conditions:

1. $v(a) = \infty \Leftrightarrow a = 0$,
2. $v(ab) = v(a) + v(b)$,
3. $c \in a + b \Rightarrow v(c) \geq \min\{v(a), v(b)\}$.

Fact

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Sketch of proof

If $[a]_\gamma = [b]_\gamma$, then there exists $m_\gamma \in \mathcal{M}^\gamma$ such that $b = a(1 + m_\gamma)$. Hence

$$v(b) = v(a(1 + m_\gamma)) = v(a) + v(1 + m_\gamma) = v(a).$$

To prove property 1 we just need to notice, that $[0]_\gamma = \{0\}$. Properties 2 and 3 follows from the properties of the valuation v on the field K.

Let us quickly recall that by the *amc*-structure of level γ we refer to the triple

$$K_\gamma = (\mathscr{O}^\gamma, G^\gamma, \Theta_\gamma),$$

where $\mathscr{O}^\gamma$ is the factor ring $\mathscr{O}/\mathscr{M}^\gamma$, G^γ is the quotient $K^\times/(1+\mathscr{M}^\gamma)$ and Θ_γ is a binary relation given by

$$\forall x \in \mathscr{O}^\gamma \, \forall y \in G^\gamma : \; \Theta_\gamma(x,y) \Leftrightarrow \exists z \in \mathscr{O} : z + \mathscr{M}^\gamma = x \wedge z(1+\mathscr{M}^\gamma) = y.$$

We will extend the meaning of the relation Θ_γ by fixing

$$\Theta_\gamma(x,0) \Leftrightarrow x = 0.$$

Definition

A subset A of a hyperring R is called a **subhyperring** of R if it is closed under multiplication and with the induced hyperaddition

$$a +_S b := (a +_R b) \cap S$$

for all $a, b \in S$, is itself a hyperring.

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Definition

A subhyperring $\mathcal{O}$ of a hyperfield F is called a **valuation hyperring** if for all $x \in F$ we have that either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Proposition

Let $v : F \to vF \cup \{\infty\}$ be a valuation on a hyperfield F. Then

$$\mathcal{O} := \{x \in F \mid vx \geq 0\}$$

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Lemma

Let $\mathcal{O}_{v_{\mathcal{H}}}$ denote the valuation hyperring of the valued hyperfield $(\mathcal{H}_\gamma(K), v_{\mathcal{H}})$ and $\mathcal{O}$ denote the valuation ring of the valued field (K, v). Then

$$\mathcal{O}_{v_{\mathcal{H}}} = \mathcal{H}_\gamma(\mathcal{O}).$$

1. γ -valued hyperfields and amc-structures
2. Valuation hyperrings
3. Hyperideals and quotient hyperfields
4. Final Theorem

Definition

A subhyperring I of a hyperring R is a **hyperideal** if it satisfies following conditions:

1. For all $a, b \in I$ one has that both $a - b \subseteq I$ and $ab \in I$,
2. for any $r \in R$ and $x \in I$ we have that $rx \in I$.

Lemma

Let (K, v) be a valued field and $0 \leq \gamma \in vK$. Then $\mathscr{H}_\gamma(\mathscr{M}^\gamma)$ is a hyperideal of $\mathscr{H}_\gamma(\mathscr{O})$.

Let x, y be elements of the hyperring R. Set I to be a hyperideal of R and let $x + I$ denote the union $\bigcup_{a \in I} x + a$. We introduce the following relation:

$$x \sim_I y \Leftrightarrow x + I = y + I.$$

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The relation $\sim_I$ is an equivalence relation.
The proof of this fact, as well as the proofs of the next two lemmas can be found in work *Algebraic geometry over hyperrings* by Jaiung Jun.

Lemma

Let R be a hyperring and I a hyperideal of R. Then for all $x, y \in R$,

$$x \sim_I y \Leftrightarrow (x - y) \cap I \neq \emptyset.$$

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Remark

For any $[a]_\gamma, [b]_\gamma \in \mathscr{H}_\gamma(\mathscr{O})$ we have that $[a]_\gamma \sim_{\mathscr{H}_\gamma(\mathscr{M}^\gamma)} [b]_\gamma$ is equivalent to $[a]_\gamma - [b]_\gamma \subseteq \mathscr{H}_\gamma(\mathscr{M}^\gamma)$.

Let us denote by $[x]_I$ the equivalence class of $x \in R$ under the relation $\sim_I$. We will denote the set of all equivalence classes of $\sim_I$ on R by

$$R/I := \{[x]_I \mid x \in R\}.$$

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Lemma

The set R/I together with hyperaddition

$$[a]_I + [b]_I := \{[c]_I \mid c \in a + b\}$$

and a multiplication

$$[a]_I \cdot [b]_I := [ab]_I.$$

forms a **hyperring**.

We call this hyperring a **quotient hyperring** of R modulo I.

Lemma

For all $a, b \in \mathscr{H}_\gamma(\mathscr{O})/\mathscr{H}_\gamma(\mathscr{M}^\gamma)$ we have that $a + b$ is a singleton.

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Remark

Any ring (field) can be also viewed as a hyperring (hyperfield).

Definition

Consider two hyperrings R, S and a mapping $\sigma : R \to S$. If for any $x, y \in R$, σ satisfies

1. $\sigma(0_R) = 0_S$
2. $\sigma(x \cdot_R y) = \sigma(x) \cdot_S \sigma(y)$
3. $\sigma(x +_R y) \subseteq \sigma(x) +_S \sigma(y)$,

then we call σ a **homomorphism** of hyperrings.
If additionally σ satisfies

$$\sigma(x +_R y) = \sigma(x) +_S \sigma(y)$$

then we call σ a **strict homomorphism** of hyperrings.
If a strict homomorphism σ is also a bijective map, then we call it an **isomorphism** of hyperrings.

Proposition

The map

$$\sigma : \mathcal{H}_\gamma(\mathcal{O})/\mathcal{H}_\gamma(\mathcal{M}^\gamma) \to \mathcal{O}/\mathcal{M}^\gamma$$
$$[x]_{\gamma,\mathcal{H}_\gamma(\mathcal{M}^\gamma)} \mapsto x + \mathcal{M}^\gamma$$

is an isomorphism of hyperrings.

Sketch of the proof

To show, that σ is independent of the chocie of the representatives we consider two distinct $x, y \in \mathscr{O}$ such that $[x]_\gamma \sim_{\mathscr{H}_\gamma(\mathscr{M}^\gamma)} [y]_\gamma$. Hence $[x]_\gamma - [y]_\gamma \subseteq \mathscr{H}_\gamma(\mathscr{M}^\gamma)$, and since $[x - y]_\gamma \in [x]_\gamma - [y]_\gamma$ we obtain $x - y \in \mathscr{M}^\gamma$.

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Surjectivity of σ follows from the fact that for any $x \in \mathscr{O}$ we have $[x]_\gamma \in \mathscr{H}_\gamma(\mathscr{O})$ and also $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} \in \mathscr{H}_\gamma(\mathscr{O})/\mathscr{H}_\gamma(\mathscr{M}^\gamma)$. This implies that $\sigma[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = x + \mathscr{M}^\gamma$.

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To show that σ is injective we will prove that for any two $x, y \in K$, $x - y \in \mathscr{M}^\gamma$ forces $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}$. On the one hand, we have that $[x-y]_\gamma \in [x]_\gamma - [y]_\gamma$. On the other hand, we have assumed that $x - y \in \mathscr{M}^\gamma$, so we have $[x-y]_\gamma \in \mathscr{H}_\gamma(\mathscr{M}^\gamma)$. Hence $[x]_\gamma - [y]_\gamma \cap \mathscr{H}_\gamma(\mathscr{M}^\gamma) \neq \emptyset$, so $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}$.

Sketch of the proof

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Surjectivity of σ follows from the fact that for any $x \in \mathscr{O}$ we have $[x]_\gamma \in \mathscr{H}_\gamma(\mathscr{O})$ and also $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} \in \mathscr{H}_\gamma(\mathscr{O})/\mathscr{H}_\gamma(\mathscr{M}^\gamma)$. This implies that $\sigma[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = x + \mathscr{M}^\gamma$.
To show that σ is injective we will prove that for any two $x, y \in K$, $x - y \in \mathscr{M}^\gamma$ forces $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}$. On the one hand, we have that $[x-y]_\gamma \in [x]_\gamma - [y]_\gamma$. On the other hand, we have assumed that $x - y \in \mathscr{M}^\gamma$, so we have $[x-y]_\gamma \in \mathscr{H}_\gamma(\mathscr{M}^\gamma)$. Hence $[x]_\gamma - [y]_\gamma \cap \mathscr{H}_\gamma(\mathscr{M}^\gamma) \neq \emptyset$, so $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} = [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}$.
Finally, to show that σ is a strict homomorphism of hyperrings, we recall that for any $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}, [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} \in \mathscr{H}_\gamma(\mathscr{O})/\mathscr{H}_\gamma(\mathscr{M}^\gamma)$ the sum $[x]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)} + [y]_{\gamma,\mathscr{H}_\gamma(\mathscr{M}^\gamma)}$ is a singleton.

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2. The second component of *amc*-structures is $G^{\gamma} := K^{\times}/(1+\mathscr{M}^{\gamma})$. Here the correspondence with the γ-valued hyperfields is immediate, since G^{γ} is a reduct of the corresponding γ-valued hyperfield.
3. What remains is the third element of *amc*-structure, namely the relation Θ_{γ}.

Lemma

Let (K, v) be a valued field, γ a nonnegative element from its value group vK, K_γ the *amc*-structure of level γ and $\mathscr{H}_\gamma(K)$ the γ-valued hyperfield of the valued field K. Then for any $x \in \mathscr{O}$ and $y \in K^\times$ we have that $\Theta_\gamma(x + \mathscr{M}^\gamma, [y]_\gamma)$ holds in K_γ if and only if $[x]_\gamma \sim_{\mathscr{H}_\gamma(\mathscr{M}^\gamma)} [y]_\gamma$ holds in $\mathscr{H}_\gamma(K)$.

Lemma

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Proof

First let us assume $\Theta_\gamma(x + \mathscr{M}^\gamma, [y]_\gamma)$. Hence there exists some $z \in \mathscr{O}$ such that $z + \mathscr{M}^\gamma = x + \mathscr{M}^\gamma$ and $[z]_\gamma = [y]_\gamma$. Then $[x - z]_\gamma \in ([x]_\gamma - [y]_\gamma) \cap \mathscr{H}_\gamma(\mathscr{M}^\gamma)$, so $[x]_\gamma \sim_{\mathscr{H}_\gamma(\mathscr{M}^\gamma)} [y]_\gamma$.
For the converse, assume that $[x]_\gamma \sim_{\mathscr{H}_\gamma(\mathscr{M}^\gamma)} [y]_\gamma$. By the Remark we obtain $[x]_\gamma - [y]_\gamma \in \mathscr{H}_\gamma(\mathscr{M}^\gamma)$, so in particular $[x - y]_\gamma \in \mathscr{H}_\gamma(\mathscr{M}^\gamma)$. Hence $x + \mathscr{M}^\gamma = y + \mathscr{M}^\gamma$. This proves that $\Theta_\gamma(x + \mathscr{M}^\gamma, [y]_\gamma)$ holds in K_γ.

Definition

Consider two valued hyperfields $H = (H, v)$, $H' = (H', v')$ and an isomorphism of hyperfields $\sigma : H \to H'$.
Let $\mathcal{O}$, $\mathcal{O}'$ denote the valuation hyperrings of H, H' respectively.
We call σ an **isomorphism of valued hyperfields** if $\sigma(\mathcal{O}) = \mathcal{O}'$.

Definition

Consider two *amc*-structures K_γ and $L_{\gamma'}$. If there exist two maps $\sigma_r : \mathcal{O}_K^\gamma \to \mathcal{O}_L^{\gamma'}$, $\sigma_g : G_K^\gamma \to G_L^{\gamma'}$ such that σ_r is an isomorphism of rings and σ_g is an isomorphism of groups, and for every $x \in \mathcal{O}_K^\gamma$ and $y \in G_K^\gamma$ we have $\Theta_\gamma(x, y)$ if and only if $\Theta_{\gamma'}(\sigma_r(x), \sigma_g(y))$, then we say that K_γ and $L_{\gamma'}$ are **isomorphic**.
Whenever we will mention an isomorphism σ of *amc*-structures, we will in fact refer to a couple (σ_r, σ_g).

Theorem

Consider two valued fields (K, v) and (L, w) and nonnegative elements γ, γ' from vK and wL, respectively. Then $K_\gamma \simeq L_{\gamma'}$ if and only if $\mathscr{H}_\gamma(K) \simeq \mathscr{H}_{\gamma'}(L)$ as valued hyperfields.

Idea of the proof

Consider an isomorphism $\sigma : K_\gamma \to L_{\gamma'}$ of *amc*-structures. Define the map $\sigma_h : \mathscr{H}_\gamma(K) \to \mathscr{H}_{\gamma'}(L)$ in the following way:

$$\sigma_h([x]_\gamma) = \sigma_g(x(1 + \mathscr{M}_K^\gamma)) \text{ for all nonzero x, and}$$
$$\sigma_h([0]_\gamma) = [0]_{\gamma'}.$$

To obtain the first implication we have to show that σ_h is an isomorphism of valued hyperfields.

For the converse let us consider an isomorphism of valued hyperfields $\sigma : \mathscr{H}_\gamma(K) \to \mathscr{H}_{\gamma'}(L)$.

To obtain an isomorphism $\sigma_g : G_K^\gamma \to G_L^{\gamma'}$ it is enough to restrict σ to the nonzero elements of $\mathscr{H}_\gamma(K)$.

Now we have to construct the isomorphism $\sigma_r : \mathscr{O}_K^\gamma \to \mathscr{O}_L^{\gamma'}$.

We have already shown that it suffices to deduce from σ an isomorphism $\mathscr{H}_\gamma(\mathscr{O}_K)/\mathscr{H}_\gamma(\mathscr{M}_K^\gamma) \to \mathscr{H}_{\gamma'}(\mathscr{O}_L)/\mathscr{H}_{\gamma'}(\mathscr{M}_L^{\gamma'})$. The equality $\sigma(\mathscr{H}_\gamma(\mathscr{O}_K)) = \mathscr{H}_{\gamma'}(\mathscr{O}_L)$ follows from the fact that σ is value preserving.

Let us define the mapping

$$\begin{aligned} \bar{\sigma} : \mathscr{H}_\gamma(\mathscr{O}_K)/\mathscr{H}_\gamma(\mathscr{M}_K^\gamma) &\to \mathscr{H}_{\gamma'}(\mathscr{O}_L)/\mathscr{H}_{\gamma'}(\mathscr{M}_L^{\gamma'}) \\ [x]_\gamma + \mathscr{H}_\gamma(\mathscr{M}_K^\gamma) &\to \sigma[x]_\gamma + \mathscr{H}_{\gamma'}(\mathscr{M}_L^{\gamma'}). \end{aligned}$$

It remains to show, that $\bar{\sigma}$ is an isomorphism of hyperrings.

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