Structures for relative quantifier elimination in valued fields

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Some history

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- MacIntyre (1976) shows quantifier elimination for the theory of p-adically closed fields in the language of valued fields extended with power predicates:

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$$P_n(x) \longleftrightarrow \exists y : y^n = x.$$

- Pas (1989) shows quantifier elimination for the theory of henselian valued fields of residue characteristic 0, relative to the value group and the residue field, in the Denef-Pas language with angular component map.

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- In 1991, Basarab obtains quantifier elimination for the theory of henselian valued fields of characteristic 0, relative to the mixed structures.
- In 1994, Kuhlmann simplifies the structures of Basarab introducing the structures of additive and multiplicative congruences (amc-structures).
- In 2011, Flenner simplifies the structures introduced by Kuhlmann even further, with the leading term structures (RV-structures).

amc-structures

Definition (Kuhlmann)

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- The residue ring $\mathcal{O}^\gamma := \mathcal{O}_v/\mathcal{M}^\gamma$.
- The multiplicative quotient group $G^\gamma := K^\times/1 + \mathcal{M}^\gamma$.
- A binary relation

$$\Theta_\gamma := \{(x, y) \in \mathcal{O}^\gamma \times G^\gamma \mid \exists z \in \mathcal{O}_v : \pi_\gamma z = x \wedge \pi_\gamma^* z = y\}.$$

where π_γ and π_γ^* are the canonical epimorphisms onto $\mathcal{O}^\gamma$ and G^γ, respectively.

RV-structures

Definition (Flenner)

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$. The *RV-structure of level γ of* (K, v) is

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with its multiplicative structure and a ternary relation

$$\oplus_{\gamma}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \iff \exists x, y, z \in K : rv_{\gamma}(x) = \mathbf{x} \wedge rv_{\gamma}(y) = \mathbf{y} \wedge rv_{\gamma}(z) = \mathbf{z} \wedge x + y = z$$

Graded rings

Definition

Let (K, v) be a valued field. The *graded ring associated to* (K, v) is

$$\mathrm{gr}_v(K) := \bigoplus_{\gamma \in vK} \mathcal{P}^\gamma / \mathcal{M}^\gamma,$$

where $\mathcal{P}^\gamma := \{x \in K \mid vx \geq \gamma\}$.

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The set

$$H(\mathrm{gr}_v(K)) := \bigcup_{\gamma \in vK} \mathcal{P}^\gamma / \mathcal{M}^\gamma$$

is called the *set of homogeneous elements of* $\mathrm{gr}_v(K)$.

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Recall from the talk of H. Stojałowska that a hyperfield is a tuple $(F, +, \cdot, 0, 1)$, where $(F \setminus \{0\}, \cdot, 1)$ is a group and $(F, +, 0)$ is a canonical hypergroup.

Hyperfields

Recall from the talk of H. Stojałowska that a hyperfield is a tuple $(F, +, \cdot, 0, 1)$, where $(F \setminus \{0\}, \cdot, 1)$ is a group and $(F, +, 0)$ is a canonical hypergroup.
In particular, $+ : F \to \mathcal{P}^*(F)$ is an associative, commutative, invertible and reversible multivalued operation and $\cdot$ distributes over $+$.

The factor construction

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The hyperoperation is defined as follows:

$$[x]_T + [y]_T := \{[x + yt]_T \in K_T \mid t \in T\}.$$

Valuations on hyperfields

Definition (Davvaz and Salasi, 2006)

Take a hyperfield F and an ordered abelian group Γ (written additively). A surjective map $v : F \to \Gamma \cup \{\infty\}$ is called a *valuation on F* if it has the following properties:

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If v is a valuation on a hyperfield F we call (F, v) a *valued hyperfield*.

The γ-valued hyperfields

Definition

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The hyperoperation $+$ can be encoded by a ternary relation symbol:

$$r_+(x, y, z) \iff z \in x + y.$$

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As sets, $\mathcal{H}_\gamma(K)$ and the RV-structure of level γ of (K, v) are the same thing.

Valued hyperfields and RV-structures

Let (K, v) be a valued field and $\gamma \in vK_{\geq 0}$.
As sets, $\mathcal{H}_\gamma(K)$ and the RV-structure of level γ of (K, v) are the same thing.
The relation which encodes the hyperoperation of $\mathcal{H}_\gamma(K)$ is the same thing as Flenner's relation $\oplus_\gamma$.

Theorem

Let (K, v) and (L, w) be valued fields, $\gamma \in vK_{\geq 0}$ and $\delta \in wL_{\geq 0}$. The valued hyperfields $(\mathcal{H}_\gamma(K), v_\gamma)$ and $(\mathcal{H}_\delta(L), w_\delta)$ are isomorphic if and only if the amc-structures K_γ and L_δ are isomorphic.

Regarding the graded rings, one has

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as subsets of K, for all $x \in K^\times$.
There is a language $\mathcal{L}_{gr}$ extending the language of rings such that $\mathrm{gr}_v(K)$ is an $\mathcal{L}_{gr}$-structure and the hyperfield structure of $\mathcal{H}_0(K)$ is interpretable in $\mathrm{gr}_v(K)$.

Residue characteristic 0

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- $\mathrm{gr}_w(L) \equiv_{\mathrm{gr}_v(K)} \mathrm{gr}_u(F)$ *as $\mathcal{L}_{gr}$-structures.*

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- $\mathcal{H}_{n \cdot vp}(L) \equiv_{\mathcal{H}_{n \cdot vp}(K)} \mathcal{H}_{n \cdot vp}(F)$ *for all* $n \in \mathbb{N}$;

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Remark

The graded rings are not sufficient in the mixed characteristic case.

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