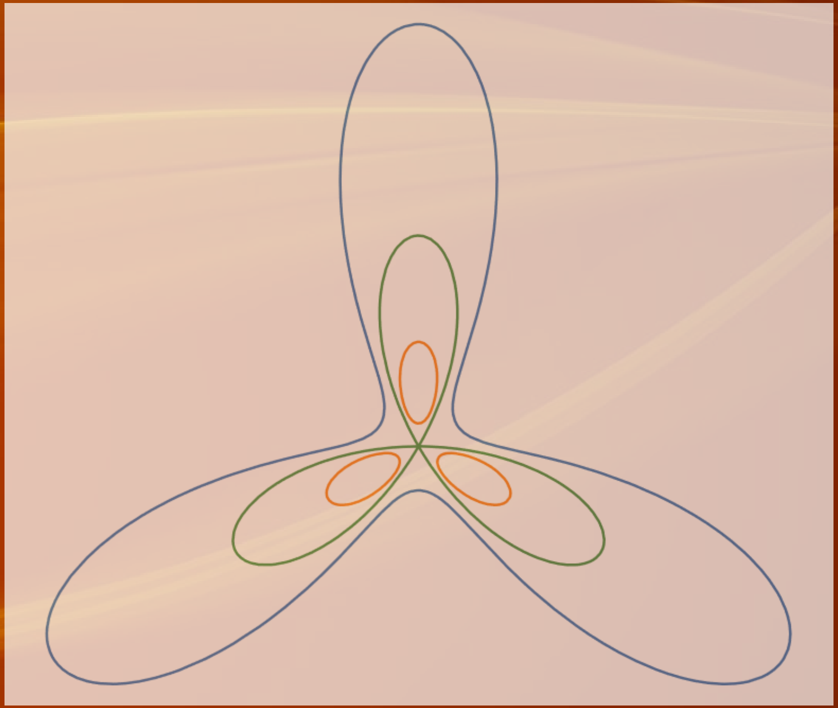


Classification of Normal Domains using Valuations

Hagen Knaf, March 2022

Overview

1. The Classification Problem(s)
2. Divisorial Ideals
3. Classes of Normal Domains

Otto Endler (left)
Wolfgang Krull (2nd from the right)
1967, Pocos de Caldas, Brasil

The Classification Problem(s)

Some Basics

- In the sequel we consider domains R with field of fractions K.
- An element $x \in L$ of an extension field L of K is called *integral over* R, if $p(x) = 0$ for a monic polynomial $p \in R[X] \setminus 0$.
- The set $\widetilde{R}_L$ of elements $x \in L$ that are integral over R is a ring extension of R with field of fractions equal to L.
- The domain R is called *integrally closed* or *normal* if $\widetilde{R}_K = R$.
- Every valuation domain is normal.
- The intersection $\bigcap_{i \in I} R_i$ of a family $(R_i)_{i \in I}$ of normal domains is normal.

The Classification Problem(s)
Some Basics

- In the sequel for a valuation $v : K \to \Gamma \cup \infty$ the symbols O_v, M_v and Kv will denote the valuation ring of v, its maximal ideal and the residue field O_v/M_v.

- Furthermore let $S(R)$ be a set of representatives of the equivalence classes of valuations v of K with the property $R \subseteq O_v$ – the so-called *Riemann-Zariski-space of* R.

- For $v \in S(R)$ the prime ideal $z(v) := M_v \cap R$ is called the *center of* v *on* R.

Proposition: *For every prime ideal* P *of the domain* R *there exists a valuation* v *of* K *with the property* $z(v) = P$.

- In general $R_{z(v)} \neq O_v$. If equality holds v is called *essential for* R.

- v is said to be *well-centered on* R if $v(R) = v(K)^{\geq 0} \cup \infty$.

The Classification Problem(s)

Theorem: *The domain R is normal if and only if* $R = \bigcap_{v \in S(R)} O_v$.

It suggests itself to consider »natural« subsets $T \subset S(R)$ such that

$$R = \bigcap_{v \in T} O_v;$$

we then say that T is a *defining set of* R.

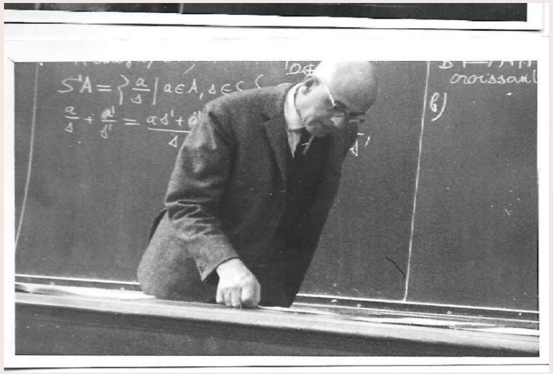
At least three different problems can be posed in this context:

1. Is there a defining subset $T \subseteq S(R)$ such that all of the value groups $v(K)$ resp. all of the valuation rings O_v possess a given property $\mathcal{P}$?
2. Is there a defining subset $T \subseteq S(R)$ such that all $v \in T$ are well-centered on R resp. are essential for R? Main focus
3. Is there an irredundant defining subset $T \subseteq S(R)$?

The Classification Problem(s)
Examples

Factorial domains

- Let R be a factorial domain and $P \subset R$ a set of representatives of all of its prime elements.
- For $p \in P$ let $v_p : K \to \mathbb{Z} \cup \infty$ be the discrete valuation determined by prime factorization.
- $O_{v_p} = R_{(p)}$.
- $R = \bigcap_{p \in P} O_{v_p}$.
- The set $\{v_p : p \in P\} \subset S(R)$ is irredundant.



Pierre Samuel, 1966

The Classification Problem(s)

Examples

The Integral Closure of a Valuation Domain

- Let v be a valuation of the field K.
- Let R be the integral closure $(\widetilde{O_v})_L$ of O_v in an algebraic extension field L of K.
- For every maximal ideal M of R the localization R_M is the valuation ring O_w of an extension w of v to L.
- For every extension w of v to L the center $z(w)$ of w on R is a maximal ideal with the property $R_{z(w)} = O_w$.
- $R = \bigcap_{w|v} O_w$.
- $\dim(O_w) = \dim(O_v)$ for all $w|v$.
- The set of all extensions w of v to L is irredundant.

Divisorial Ideals
Fractional Ideals

- Let R be a domain with field of fractions K.
- A *fractional R-ideal* is an R-submodule $I \subseteq K$, $I \neq 0$, with the property $sI \subseteq R$ for some $s \in R \setminus 0$.
- If I and J are fractional R-ideals, then $I + J$, $I \cdot J$ and $I \cap J$ are fractional R-ideals as well.
- The sets $F(R)$ and $F_{\text{fin}}(R)$ of (finitely generated) fractional R-ideals form commutative monoids with respect to ideal multiplication.
- For $I, J \in F(R)$ the R-module $(I : J) := \{x \in K : xJ \subseteq I\}$ is a fractional R-ideal.
- An element $I \in F(R)$ is invertible if and only if $I \cdot (R : I) = R$.
- Every invertible element I of $F(R)$ is finitely presented:

$$R^m \to R^n \to I \to 0.$$

Divisorial Ideals
Prüfer Domains

Theorem: *For a domain R the following properties are equivalent:*

1. $F_{\text{fin}}(R)$ *is a group.*
2. *Every localization* R_p, $p \in \operatorname{Spec}(R)$, *is a valuation ring.*

- A domain with the properties mentioned in the theorem is called *Prüfer domain.*

  The noetherian Prüfer domains are precisely the *Dedekind domains.*

- For a Prüfer domain $T := \{v \in S(R) : z(v) \text{ is maximal}\}$ is a defining set.

  All $v \in T$ are essential for R and T is irredundant.

  For a Dedekind domain all $v \in T$ are discrete.

Heinz Prüfer

Divisorial Ideals
v-Operation

- For $I \in F(R)$ define $I^v := (R : (R : I))$ ...
- ... or equivalently $I^v := \bigcap_{I \subseteq Rx} Rx$.
- $\cdot^v$ is a hull operator: $I \subseteq I^v$, $I \subseteq J$ implies $I^v \subseteq J^v$, $(I^v)^v = I^v$.
- An R-ideal I with the property $I = I^v$ is called *divisorial*.

  A divisorial R-ideal of the form $I = I_0^v$, I_0 finitely generated, is called *divisorially finitely generated*.

  Note that $(R : I_0)$ needs not be finitely generated if I_0 is finitely generated.
- The operation $I \odot J := (I \cdot J)^v$ gives the set $D(R)$ of divisorial R-ideals the structure of a commutative monoid.
- The set $D_{\text{fin}}(R)$ of divisorially finitely generated R-ideals is a submonoid of $(D(R), \odot)$.

Divisorial Ideals
Monoids of Ideals

- The group $(I(R), \cdot)$ of invertible R-ideals is a subgroup of $(D_{\mathrm{fin}}(R), \odot)$.
- Denoting by $P(R)$ the group of principal ideals of R one arrives at

$$\begin{array}{ccccccc} P(R) & \subseteq & I(R) & \subseteq & D_{\mathrm{fin}}(R) & \subseteq & D(R) \\ & & \cap & & & & \cap \\ & & F_{\mathrm{fin}}(R) & & \subseteq & & F(R) \end{array}$$

- Analyzing when two or more of these groups / monoids are equal or weaker, when they are groups, leads to various interesting classes of rings.

  Note that $D_{\mathrm{fin}}(R)$ and $D(R)$ in general are not submonoids of $F_{\mathrm{fin}}(R)$ and $F(R)$.

- Dedekind domains: $I(R) = F(R)$,

  Prüfer domains: $I(R) = F_{\mathrm{fin}}(R)$,

  Principal ideal domains: $P(R) = F(R)$,

  Bezout domains: $P(R) = F_{\mathrm{fin}}(R)$.

Robert Gilmer, 2013

Divisorial Ideals

Associated Prime Ideals

Divisorial ideals of a domain can be characterized using a certain subset of prime ideals of R.

Definition: *The prime ideal $P \subset R$ is associated to the principal ideal Ra if $(Ra : Rb) \subseteq P$ for some $b \in R \setminus Ra$ and P is minimal among the primes containing $(Ra : Rb)$. Let $P(R)$ denote the set of all prime ideals associated to some principal ideal of R.*

- If R is factorial, $P(R)$ consists of all prime ideals generated by prime elements of R.
- Every prime ideal that is minimal among the prime ideals containing a principal ideal Ra is an element of $P(R)$.
- Every prime ideal of height equal to 1 is an associated prime ideal.
- If R is noetherian and normal, $P(R)$ equals the set of all prime ideals of height equal to 1 (Krulls Principal Ideal Theorem).

Divisorial Ideals
Associated Prime Ideals

- $\forall I \in D(R) \quad I = \bigcap_{P \in P(R)} IR_P.$
- $\forall I, J \in D(R) \quad I = J \Leftrightarrow \forall P \in P(R)\ IR_P = JR_P.$

Theorem: *Let R be a domain for which $D_{\text{fin}}(R)$ is a group, then the local rings R_P, $P \in P(R)$, are valuation rings and R is normal.*

If all finitely generated ideals of R are finitely presented, the implication in the theorem is an equivalence.

Theorem: *For a noetherian domain R the monoid $D(R)$ is a group if and only if R is normal. The local rings R_P, $P \in P(R)$, then are discrete valuation rings.*

Some Classes of Normal Domains
Krull Domains

Definition: *A domain R is called Krull domain if there exists a defining set $T \subseteq S(R)$ of discrete valuations with the property: for every $x \in R \setminus 0$ the set $\{v \in T : v(x) \neq 0\}$ is finite.*

- A defining set $T \subseteq S(R)$ with the property mentioned in the theorem is called *of finite character*.
- Every noetherian normal domain is a Krull domain.
- The defining family T in the definition can be chosen to be essential:

$$T := \{v \in S(R) : z(v) \cap R \text{ has height equal to } 1\}.$$

Robert Fossum, 1986
Oberwolfach, Germany

Some Classes of Normal Domains
Krull Domains

Proposition: *Let R be a Krull domain with field of fractions K, then:*

- $R \cap K_0$ *is a Krull domain for every subfield* $K_0 \subset K$,
- *the integral closure* $\widetilde{R}_L$ *in every finite extension* $L|K$ *is a Krull domain,*
- *the polynomial ring* $R[X]$ *is a Krull domain,*
- *every localisation* $S^{-1}R$ *is a Krull domain.*

The following famous and deep theorem underlines the importance of the class of Krull domains:

Theorem (S. Mori, M. Nagata): *The integral closure of a noetherian domain in its field of fractions is a Krull domain.*

Note that the integral closure need not be noetherian if the dimension of the domain is bigger than 2.

Some Classes of Normal Domains

Rings of Krull Type

Definition: *A domain R is called ring of Krull type if there exists an essential defining family $T \subseteq S(R)$ of finite character.*

- Rings of Krull type may be viewed as a non-discrete generalization of Krull rings.

Proposition: *Let R be a ring of Krull type with field of fractions K, then:*

- *the integral closure $\widetilde{R}_L$ in every finite extension $L|K$ is a ring of Krull type,*
- *the polynomial ring $R[X]$ is a ring of Krull type,*
- *every localisation $S^{-1}R$ is a ring of Krull type.*

Some Classes of Normal Domains

Rings of Krull Type

What makes rings of Krull type particularly interesting is the possibility to describe the monoid $D_{\text{fin}}(R)$ using the valuations of a defining set of R:

- $R = \bigcap_{v \in T} O_v$, where $T \subseteq S(R)$ is an essential defining set.
- The map
$$\phi : D_{\text{fin}}(R) \to \bigoplus_{v \in T} v(K),\ I \mapsto (\min(v(x) : x \in I \setminus 0))_{v \in T}$$
is a well-defined, injective group homomorphism.
- In general it is not surjective.
- Using ϕ one can prove that $D_{\text{fin}}(R)$ actually is a group.

Some Classes of Normal Domains

Prüfer-v-Multiplication Domains

Definition: *A domain R for which $D_{\text{fin}}(R)$ is a group is called Prüfer-v-multiplication domain.*

- A Prüfer-v-multiplication domain possesses the essential defining set

$$T := \{v \in S(R) : O_v = R_P, P \in P(R)\},$$

which in general is hard to determine.

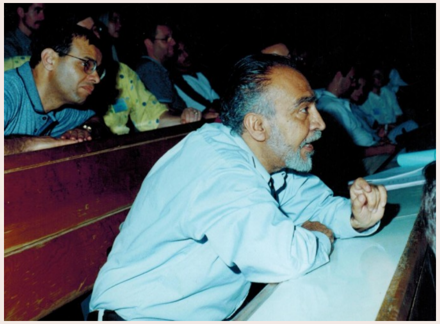
Theorem: *A domain R is a Prüfer-v-multiplication domain if and only if it is defined by an essential subset $T \subseteq S(R)$ and all ideals $Rx \cap Ry$, $x, y \in R$, are finitely generated.*

- Every finitely generated, normal O_v-algebra is a Prüfer-v-multiplication domain.

Some Classes of Normal Domains

Prüfer-v-Multiplication Domains

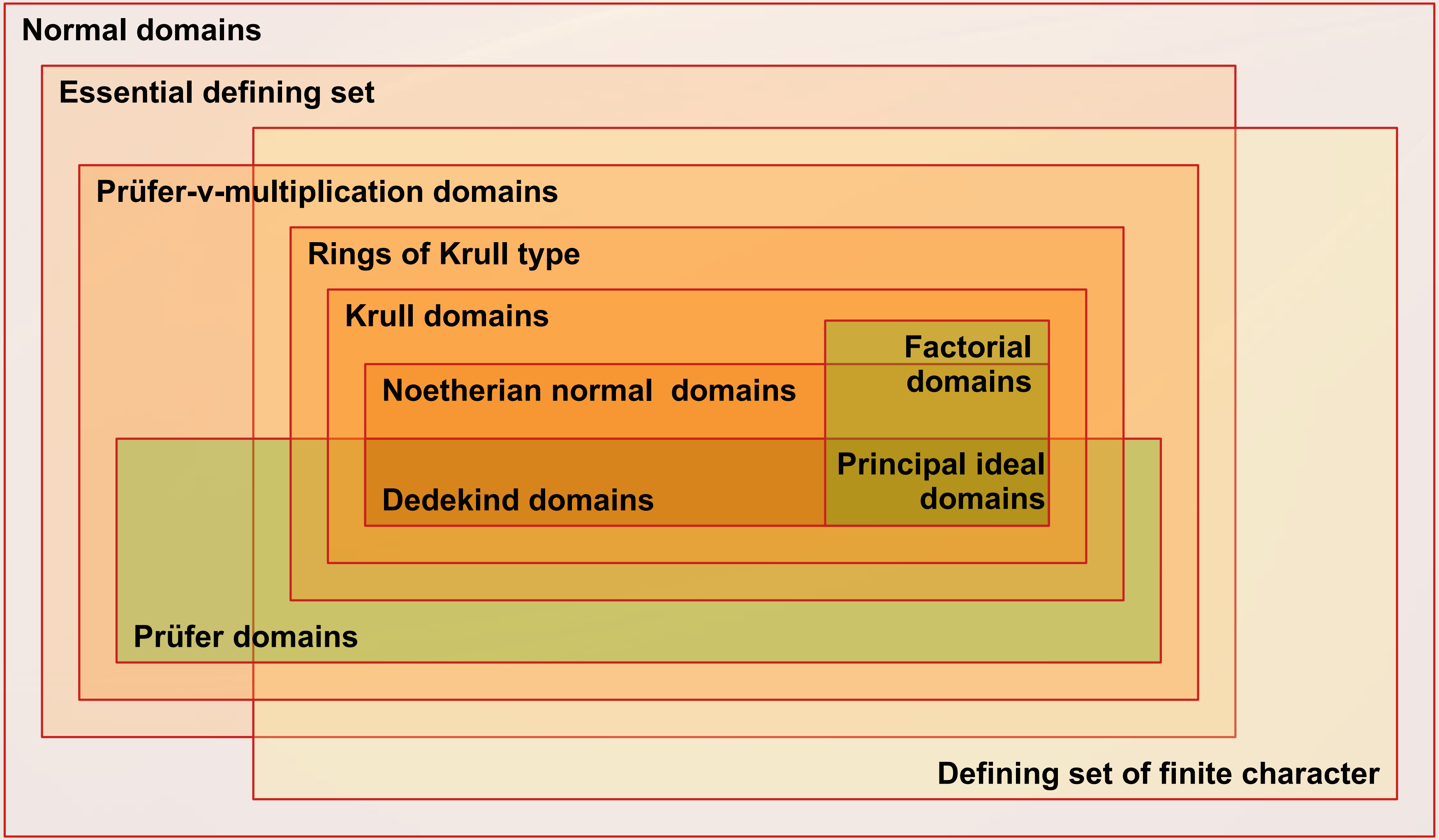
Theorem: *A domain R is a ring of Krull type if and only if it is a Prüfer-v-multiplication domain defined by an essential subset $T \subseteq S(R)$.*



Muhammad Zafrullah

Some Classes of Normal Domains

Relations among Classes



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