Diophantine problems over tamely ramified fields

Kartas Konstantinos

University of Oxford

Szczecin 2022

Table of Contents

Theorem

- *Assume resolution of singularities.*
- $\mathcal{O}_F$ *an excellent DVR.*
- (K, v), (L, w) *are henselian and tamely ramified over* (F, v_0).
- $RV(K) \equiv_{\exists, RV(F)} RV(L)$.

Then $K \equiv_{\exists, F} L$ *in* L_{rings}.

cf.

(1) Known (unconditionally) in equal characteristic 0 and *unramified* mixed characteristic. (Ax-Kochen/Ershov '66)
(2) Recently proved (unconditionally) for *finite* tame ramification in mixed characteristic (with perfect residue fields). (J. Lee '21)
(3) Encompasses the *conditional* existential decidability results for $\mathbb{F}_p((t))$. (Denef-Schoutens '03)
(4) No finiteness restriction on ramification.
(5) No restriction on the characteristic.

Overview

Some applications

Let $l \neq p$ be prime.

Corollary

- *Assume resolution of singularities.*

(1) Then $\mathbb{Q}_p(p^{1/l^\infty})$ is existentially decidable in L_{rings}.
(2) Then $\mathbb{F}_p((t))(t^{1/l^\infty})$ is existentially decidable in L_t.

Corollary

- *Assume resolution of singularities.*

(1) Then $\mathbb{Q}_p^{tr}$ is existentially decidable in L_{rings}.
(2) Then $\mathbb{F}_p((t))^{tr}$ is existentially decidable in L_t.

Corollary

$\mathbb{F}_p((t))(t^{1/l^\infty}) \preceq_1 \mathbb{F}_p((t^{\Gamma_l}))$, *where* $\Gamma_l = \frac{1}{l^\infty}\mathbb{Z}$.

cf.

$\mathbb{F}_p((t))(t^{1/p^\infty}) \not\preceq_1 \mathbb{F}_p((t^{\Gamma_p}))$. (Abhyankar '56)

Table of Contents

RV-structures

Motivating example

$K = k((t))$.

Idea: Define "leading term arithmetic" in K, i.e., we want to do arithmetic by forgetting higher order terms:

1. Define $x \sim y$ when their leading terms are equal.

Example

$\mathbf{2t} + t^2 \sim \mathbf{2t} + t^3$ because their leading term is $2t$.

rv(x)= "equivalence class of x"; $\mathrm{RV}(K) = \{\mathrm{rv}(x) : x \in K\}$.

2. Define multiplication on $\mathrm{RV}(K)$.
   (does not depend on representatives)
3. Try to define addition on $\mathrm{RV}(K)$:

Example

(1) $\mathrm{rv}(\mathbf{2t} + t^3) + \mathrm{rv}(\mathbf{3t} + 2t^4) = \mathrm{rv}(\mathbf{5t} + t^3 + 2t^4)$. (OK)
(2) $\mathrm{rv}(\mathbf{t} + t^2) + \mathrm{rv}(-\mathbf{t}) = \mathrm{rv}(\mathbf{t^2})$ but $\mathrm{rv}(\mathbf{t} + t^3) + \mathrm{rv}(-\mathbf{t}) = \mathrm{rv}(\mathbf{t^3})$!

Moral: Addition has to be multi-valued! (i.e., relation)

RV-structures

Setting:

- (K, v) a valued field; $\mathcal{O}_K$ valuation ring; $\mathfrak{m}$ maximal ideal.
- $k = \mathcal{O}_K/\mathfrak{m}$ residue field.
- Γ value group.

Consider the exact sequence of abelian groups

$$0 \to k^\times \xrightarrow{\iota} K^\times/(1+\mathfrak{m}) \xrightarrow{v} \Gamma \to 0$$

1. Define $\mathrm{RV}(K^\times) = K^\times/(1+\mathfrak{m})$. (Leading term structure)
2. Define the multiplicative monoid $\mathrm{RV}(K) = \mathrm{RV}(K^\times) \cup \infty$.
3. $\oplus(a, b, c)$ holds if there are $x, y, z \in K$ such that $\mathrm{rv}(x) = a, \mathrm{rv}(y) = b, \mathrm{rv}(z) = c$ and $x + y = z$.
4. We also equip $\mathrm{RV}(K)$ with a binary relation $va \leq vb \iff vx \leq vy$, where $\mathrm{rv}(x) = a$ and $\mathrm{rv}(y) = b$.

Table of Contents

Tamely ramified extensions

Algebraic case

Setting:

- $(L, w)/(K, v)$ **finite** valued field extension.
- l/k residue field extension.
- Δ/Γ value group extension.

Definition

$(L, w)/(K, v)$ is said to be *tamely ramified* if

1. l/k is separable.
2. $p \nmid [\Delta : \Gamma]$, where $p = \mathrm{char}(k)$.
3. $(L, w)/(K, v)$ is *defectless*.

Example

The extension $(\mathbb{Q}_p(p^{1/n}), v_p)/(\mathbb{Q}_p, v_p)$ is tamely ramified if and only if $p \nmid n$.

Remark: Our base field will always be defectless, so (3) can be ignored.

Tamely ramified extensions

Transcendental case

Setting:

- $(L, w)/(K, v)$ valued field extension.
- l/k residue field extension.
- Δ/Γ value group extension.

Definition

$(L, w)/(K, v)$ is said to be *tamely ramified* if

1. l/k is separable. (not necessarily algebraic)
2. Δ/Γ is p-torsion-free, where $p = \mathrm{char}(k)$.
3. $(L_1, w)/(K, v)$ is *defectless*, for every finite subextension L_1/K.

Example

(*a*) $(\mathbb{Q}_p(p^{1/l^\infty}), v_p)/(\mathbb{Q}, v_p)$ is tamely ramified, $l \neq p$ prime.
(*b*) $(\mathbb{F}_p((t^\Gamma)), v_t)/(\mathbb{F}_p(t), v_t)$ is tamely ramified if and only if 1 is not p-divisible in Γ.

Table of Contents

Some geometry

Regular schemes and normal crossings

Informally: Regular = non-singular.

Definition

- X be a Noetherian scheme.

(*a*) X is regular at $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring.
(*b*) X is regular if it is regular at all $x \in X$.

Regular system: A minimal set of generators of $\mathfrak{m}_{X,x}$ is called a *regular system* of parameters at x.

Example

(1) $X = \mathrm{Spec}(k[x, y]/(y^2 - x))$ is regular.
(2) $X = \mathrm{Spec}(k[x, y]/(y^2 - x^3))$ is singular at $x = y = 0$.

Some geometry

Regular schemes and normal crossings

Setting:

- R a DVR; t a uniformizer; κ the residue field.
- $X \to \mathrm{Spec} R$ a morphism.
- $X_s = X \times_{\mathrm{Spec} R} \mathrm{Spec}(\kappa)$ is the special fiber. (set $t = 0$)

Informally: X_s *locally* looks like a union of smooth varieties crossing transversely.

Definition

(*a*) $X \to \mathrm{Spec} R$ has strict normal crossings at $x \in X$ if there exists a regular system of parameters $f_1, \ldots, f_n$ of X at x and $h \in \mathcal{O}_{X,x}^{\times}$ such that $t = h \cdot f_1^{e_1} \cdot f_2^{e_2} \cdot \ldots \cdot f_m^{e_m}$.
(*b*) $X \to \mathrm{Spec} R$ has strict normal crossings if it has strict normal crossings at all $x \in X$.

Resolution of singularities

Definition

- R a discrete valuation ring.
- $K = \mathrm{Frac}(R)$.

R is called *excellent* if $\widehat{K}/K$ is separable.

Conjecture (Log-Resolution)

- *R excellent DVR.*
- *X a reduced, flat scheme of finite type over R.*

Then there exists a blow-up morphism $f : \tilde{X} \to X$ in a nowhere dense center $Z \subsetneq X$ such that

1. *$\tilde{X}$ is a regular scheme.*
2. *$\tilde{X} \to \mathrm{Spec} R$ has strict normal crossings.*

cf. (ε% of literature):
(1) Known in residue char.0. (Hironaka '64).
(2) Known when $\dim X \leq 3$. (Cossart-Piltant '13)

Table of Contents

RV-Hensel's Lemma

Classical geometric Hensel's Lemma

Setting: R henselian local ring; κ residue field.
Slogan: Smooth κ-points of X_s lift to integral points of X.

Lemma (Hensel's Lemma)

- *$X \to \mathrm{Spec} R$ a smooth morphism.*
- *$x \in X_s(\kappa)$.*

Then there is $P \in X(R)$ specializing to x.

Motivation for RV-Hensel's Lemma:

- Log-Resolution *does not* guarantee that $X \to \mathrm{Spec} R$ is smooth.
- The best we can hope for is strict normal crossings.
- We need to prove a Hensel's Lemma version for strict normal crossings.

RV-Hensel's Lemma

Setting:
R DVR; κ residue field; π uniformizer.
$A \supseteq R$ henselian local ring; k residue field.
Slogan: RV-points lift to integral points (under SNC $+p \nmid e_1$).

Lemma (RV-Hensel's Lemma)

- *$X \to \mathrm{Spec} R$ finite type with strict normal crossings.*
- *$x \in X_s$ a closed point; $\kappa(x)/\kappa$ is separable.*
- *$\pi = h \cdot x_1^{e_1} \cdot \ldots \cdot x_n^{e_n}$, where $e_i \in \mathbb{Z}^{>0}$, $h \in \mathcal{O}_{X,x}^{\times}$ and $\{x_1, \ldots, x_n\}$ part of a regular system of parameters at x.*
- $\mathbf{p \nmid e_1}$.
- *There exist $a_1, \ldots, a_n \in A$ such that $\iota(\overline{h}) \cdot rv(a_1^{e_1} \cdot \ldots \cdot a_n^{e_n}) = rv(\pi)$.*

Then there is $P \in X(A)$ specializing to the RV-point.

Table of Contents

Main Theorem

Theorem

- *Assume Log-Resolution.*
- $\mathcal{O}_F$ *an excellent DVR.*
- (K, v), (L, w) *are henselian and tamely ramified over* (F, v_0).
- $RV(K) \equiv_{\exists, RV(F)} RV(L)$.

Then $K \equiv_{\exists, F} L$ *in* L_{rings}.

First Step: Geometric reformulation.
Second Step: Strict normal crossings case.
(via RV-Hensel's Lemma)
Third Step: General scheme case (via Log-Resolution).

Table of Contents

Applications

Corollary

- *Assume Log-Resolution.*
- *(K, v) henselian and tamely ramified over $(\mathbb{Q}, v_p)$ (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$).*
- *(K, v) admits a cross-section that extends a cross-section of $(\mathbb{Q}, v_p)$ (resp. $(\mathbb{F}_p(t), v_t)$ or $(\mathbb{Q}(t), v_t)$).*

Then K is existentially decidable in L_{rings} relative to k in L_{rings} and (Γ, vp) in L_{oag} with a constant for the value of p (resp. t).

Ramification fields:
$\mathbb{Q}_p^{tr}$= "maximal tamely ramified algebraic extension of $\mathbb{Q}_p$".
(similarly $\mathbb{F}_p((t))^{tr}$)

Example (Ramification fields)

(a) $\mathbb{Q}_p^{tr}$ is $\exists$-decidable in L_{rings}.
(b) $\mathbb{F}_p((t))^{tr}$ is $\exists$-decidable in L_t.

Remark: Also, $\mathbb{Q}_p(p^{1/l^\infty})$ and $\mathbb{F}_p((t))(t^{1/l^\infty})$, $l \neq p$ prime.