Real hyperfields

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Hyperoperation

Let H be a nonempty set and $\mathcal{P}^*(H)$ the family of nonempty subsets of H. A *hyperoperation* $+$ is a function:

$$+ : H \times H \longrightarrow \mathcal{P}^*(H)$$

$$(x, y) \mapsto x + y.$$

For a subset $A \subseteq H$ and $x \in H$ we have

$$A + x := \bigcup_{a \in A} a + x \quad \text{and} \quad x + A = \bigcup_{a \in A} x + a.$$

Canonical hypergroup

Definition

A *canonical hypergroup* is a tuple $(H, +, 0)$, where $+$ is a hyperoperation on H and $0 \in H$ is an element such that the following axioms hold:

(H1) the hyperoperation $+$ is associative, i.e., $(x+y)+z = x+(y+z)$ for all $x, y, z \in H$,

(H2) $x + y = y + x$ for all $x, y \in H$,

(H3) for every $x \in H$ there exists a unique $-x$ such that $0 \in x + (-x) =: x - x$,

(H4) $z \in x + y$ implies $y \in z - x$ for all $x, y, z \in H$.

Hyperrings and hyperfields

Definition

A *(commutative) hyperring with unity* is a tuple $(R, +, \cdot, 0, 1)$ which satisfies the following axioms:

(R1) $(R, +, 0)$ is a canonical hypergroup,

(R2) $(R, \cdot, 1)$ is a commutative monoid and $x \cdot 0 = 0$ for all $x \in R$,

(R3) the operation $\cdot$ is distributive with respect to the hyperoperation $+$. That is, for all $x, y, z \in R$,

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

If $(R, +, \cdot, 0, 1)$ is a hyperring with unity and $(R \setminus \{0\}, \cdot, 1)$ is a group, then $(R, +, \cdot, 0, 1)$ is called a *hyperfield*.

Hyperideal

Definition

Let R be a hyperring.

1. A nonempty subset $I \subseteq R$ is a *hyperideal* if for all $a, b \in I$ and for all $r \in R$ we have $a + b \subseteq I$, $-a \in I$ and $ar \in I$.
2. A hyperideal $I \subsetneq R$ is *maximal* if I satisfies the following property: if $J \subseteq R$ is a hyperideal of R such that $I \subsetneq J$, then $J = R$.

Valuation in hyperfields

Definition

Take a hyperfield F and an ordered abelian group Γ (written additively). The map $v : F \to \Gamma \cup \{\infty\}$ is called a *valuation on* F if it has the following properties:
(V1) $v(a) = \infty$ iff $a = 0$;
(V2) $v(ab) = v(a) + v(b)$;
(V3) $c \in a + b \Rightarrow v(c) \geqslant \min\{v(a), v(b)\}$.

Definition

Let F be a hyperfield. A subhyperring $\mathcal{O}$ of F is called a *valuation hyperring* if for all $x \in F$ we have that either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Proposition

Let v be a valuation on a hyperfield F. Then $\mathcal{O}_v := \{x \in F \mid v(x) \geqslant 0\}$ is a valuation hyperring of F and $\mathcal{M}_v := \{x \in F \mid v(x) > 0\}$ its unique maximal hyperideal. The quotient $\mathcal{O}_v / \mathcal{M}_v$ is a hyperfield, called *the residue hyperfield*.

Factor hyperfields

The following construction comes from Krasner.
Let K be a field and T a subgroup of $K^\times$. We consider the equivalence relation:

$$x \sim y \quad \text{if and only if} \quad x = yt \text{ for some } t \in T.$$

Denote by $[x]_T$ the equivalence class of x and by $K_T := K^\times / T \cup \{[0]_T\}$ the set of all equivalence classes. The set K_T with the operations

$$[x]_T + [y]_T = \{[x + yt]_T \mid \ t \in T\};$$

$$[x]_T \cdot [y]_T = [xy]_T,$$

is a hyperfield, called *a factor hyperfield (quotient hyperfield)*. The neutral element of the multivalued addition in K_T is $[0]_T = \{0\}$ and the additive inverse of $[x]_T \in A_K$ is $[-x]_T$.

Examples of hyperfields

Consider the field of real numbers $\mathbb{R}$ with its multiplicative subgroup $(\mathbb{R}^\times)^2$. We can identify the factor hyperfield $\mathbb{R}_{(\mathbb{R}^\times)^2}$ with the set $H = \{-1, 0, 1\}$. The hyperaddition works here as follows:

$$\begin{aligned}
(-1) + (-1) = (-1) + 0 = 0 + (-1) &= \{-1\} \\
0 + 0 &= \{0\} \\
1 + 1 = 1 + 0 = 0 + 1 &= \{1\} \\
1 + (-1) = (-1) + 1 &= \{-1, 0, 1\}.
\end{aligned}$$

This hyperfield is called the *sign hyperfield*.

Examples of hyperfields

Consider the following cartesian product $F = \{-1, 1\} \times \Gamma$, where Γ is an ordered abelian group. Denote $F := \{-1, 1\} \times \Gamma \cup \{0\}$. Then the tuple $(F, +, \cdot, 0, (1, 0))$ is a hyperfield, where the hyperaddition is defined as follows:

$$
\begin{aligned}
x + 0 = 0 + x &= \{x\} && x \in F \\
(1, \gamma_1) + (1, \gamma_2) &= \{(1, \min(\gamma_1, \gamma_2))\} && \gamma_1, \gamma_2 \in \Gamma \\
(-1, \gamma_1) + (-1, \gamma_2) &= \{(-1, \min(\gamma_1, \gamma_2))\} && \gamma_1, \gamma_2 \in \Gamma \\
(1, \gamma_1) + (-1, \gamma_2) &= \{(1, \gamma_1)\} && \gamma_1 < \gamma_2 \in \Gamma \\
(1, \gamma_1) + (-1, \gamma_2) &= \{(-1, \gamma_2)\} && \gamma_1 > \gamma_2 \in \Gamma \\
(1, \gamma) + (-1, \gamma) &= \{(e, \delta) \mid e \in \{-1, 1\}, \delta \geqslant \gamma\} \cup \{0\} && \gamma \in \Gamma
\end{aligned}
$$

The result of multiplication by 0 is obvious. For nonzero elements of F we define:

$$(s_1, \gamma_1) \cdot (s_2, \gamma_2) = (s_1 \cdot s_2, \gamma_1 + \gamma_2).$$

During this talk we will call it *the hyperfield of pairs*.

Artin-Schreier theory for hyperfields

Let F be a hyperfield. A subset $P \subseteq F$ is an *ordering* of F if

$$P + P \subseteq P, \quad P \cdot P \subseteq P, \quad P \cap -P = \emptyset, \quad P \cup -P = F^{\times}.$$

A subset $T \subseteq F$ is called a *preordering* in F if

$$T + T \subseteq T, \quad T \cdot T \subseteq T, \quad (F^{\times})^2 \subseteq T, \quad -1 \notin T.$$

We say that a preordering $T \subseteq F$ is *maximal* if $T \subseteq S$ for some preordering S in F implies that $S = T$. The set of all orderings of a hyperfield F will be denoted by $\mathcal{X}(F)$ and the set of all orderings of F containing a subset $T \subset F$ by $\mathcal{X}(F \mid T)$.
A hyperfield F is *real* if $\mathcal{X}(F) \neq \emptyset$.

Example

We observe that the sign hyperfield is real with the ordering $P = \{1\}$ and the hyperfield of pairs is real with the ordering $P = \{(1, \gamma) \in F \mid \gamma \in \Gamma\}$.

Artin-Schreier theory for hyperfields

Theorem (M. Marshall)

Let F be a hyperfield.

1. *Every maximal preordering T of F is an ordering.*
2. *F is real if and only if $-1 \notin \Sigma(F^\times)^2$.*
3. *For every preordering T of F we have*

$$T = \bigcap_{P \in \mathcal{X}(F|T)} P.$$

Compatibility between ordering and valuation

If K is a field, then with the ordering P one can associate a linear order defined by

$$a < b \Leftrightarrow b - a \in P.$$

In the hyperfield case, the relation $a < b \Leftrightarrow b - a \subseteq P$ does not have to be a linear order. However, it defines on F a strict partial order, which does not have to be compatible with the hyperaddition.

Let K be a field. A valuation v of K is *compatible* with an ordering $<$ if the valuation ring $\mathcal{O}_v$ of v is convex with respect to $<$. The partial order $<$ of a real hyperfield F for an ordering P of F allows us to consider the notion of convexity in F. Hence the questions arise:

How can we define compatibility between an ordering and a valuation in hyperfields?
Can we use the notion of convexity to define it?

Construction of $A(P)$ and $I(P)$

Every real field K has characteristic 0, so it contains the rationals. If P is an ordering of K, then the set

$$A(P) := \{a \in K \mid n \pm a \in P \text{ for some } n \in \mathbb{N}\}$$

is a valuation ring of K (associated with the natural valuation of P) with the maximal ideal

$$I(P) := \{a \in K \mid \frac{1}{n} \pm a \in P \text{ for all } n \in \mathbb{N}\}.$$

We wish to construct $A(P)$ and $I(P)$ in real hyperfields.

Construction of $A(P)$ and $I(P)$

But how can we construct $A(P)$ and $I(P)$ if a hyperfield does not have to contain the rationals?
Let F be a hyperfield with ordering P. For $n \in \mathbb{N}$, define

$$I_n := \underbrace{1 + \ldots + 1}_{n \text{ times}}.$$

Define

$$A(P) := \{a \in F \mid (I_n \pm a) \cap P \neq \emptyset \text{ for some } n \in \mathbb{N}\}$$
$$I(P) := \{a \in F \mid (1 \pm I_n \cdot a) \subseteq P \text{ for all } n \in \mathbb{N}\}.$$

Proposition

The set $A(P)$ is a valuation hyperring in F with its unique maximal hyperideal $I(P)$.

Definition

An ordering P of a hyperfield F is called *archimedean* if $A(P) = F$.

Examples of archimedean and nonarchimedean orderings

Consider the sign hyperfield $F = \mathbb{R}_{(\mathbb{R}^\times)^2}$ with the ordering $P = \{1\}$. Observe that $1 + 1 = \{1\}$, so $I_n = \{1\}$ for every $n \in \mathbb{N}$. Hence

$$A(P) = \{a \in F \mid (1 \pm a) \cap P \neq \emptyset\} = F,$$

$$I(P) = \{a \in F \mid (1 \pm a) \subseteq P\} = \{0\}.$$

Hence P is an archimedean ordering in F.

Consider the hyperfield of pairs F. Then

$$A(P) = \{(s, \gamma) \in F \mid \gamma \geqslant 0\} \cup \{0\} \subsetneq F,$$

$$I(P) = \{(s, \gamma) \in F \mid \gamma > 0\} \cup \{0\}.$$

Hence P is a nonarchimedean ordering in F.

A valuation compatible with the ordering P

Proposition

Let F be a hyperfield with a valuation v and an ordering P. The following conditions are equivalent:

(i) $A(P) \subseteq \mathcal{O}_v$,

(ii) $\overline{P} := \{a + \mathcal{M}_v \mid a \in P \cap \mathcal{O}_v{}^{\times}\}$ is an ordering of $\mathcal{O}_v/\mathcal{M}_v$,

(iii) $1 + \mathcal{M}_v \subseteq P$,

(iv) if $(b+a) \cap P \neq \emptyset$ and $(b-a) \cap P \neq \emptyset$, then $v(a) \geqslant v(b)$.

Definition

A valuation v on a hyperfield F satisfying the equivalent conditions of the last proposition is called *a valuation compatible with the ordering P*.

Compatibility implies convexity

We have defined the notion of a valuation being compatible with an ordering. Can we say that v is compatible with the ordering P if and only if $\mathcal{O}_v$ is convex with respect to P?
The answer is **NO**.
However, we have the following lemma:

Lemma

Let v be a valuation on the real hyperfield F, compatible with an ordering P. Then $\mathcal{O}_v$ is convex with respect to P.

Lemma

Let P be an ordering of a field K and assume that a multiplicative subgroup T of K is contained in P. Consider the factor hyperfield $K_T = \{[a]_T \mid a \in K\}$.

1. *If $a \in P$, then $[a]_T \subseteq P$.*
2. *The set $P_T := \{[c]_T \mid c \in P\}$ is an ordering of K_T.*

The ordering P_T in K_T is called *an ordering induced by the ordering P.*

Lemma

Take a valued field (K, v) with the valuation ring $\mathcal{O}_v$ and let T be a subgroup of the group of units $\mathcal{O}_v^\times$. Take the factor hyperfield K_T. Then the map $v_T : [a]_T \mapsto v(a)$ is a valuation of K_T with value group $v(K^\times)$.

The valuation v_T is called *the valuation on K_T induced by v.*

Convexity does not imply compatibility

Proposition

Let v be a valuation on a real field K and let P be an ordering of K. Take $T = P \cap \mathcal{O}_v^\times$ and consider the factor hyperfield K_T with valuation v_T induced by v and ordering P_T induced by P. Then:
(i) The valuation v is compatible with P if and only if v_T is compatible with P_T.
(ii) If v is a rank 1 valuation which is not compatible with P, then any two distinct elements in K_T which are positive (with respect to P_T) are not comparable. In particular, $\mathcal{O}_{v_T}$ is convex with respect to P_T, while v_T is not compatible with P_T.

Baer-Krull Theorem for real hyperfields

We denote by $\mathcal{X}(F, v)$ the set of all orderings of F compatible with v.

Theorem

Let v be a valuation on a real hyperfield F with value group Γ. Then there is a bijection between the set $\mathcal{X}(F, v)$ and the set $\mathcal{X}(\overline{F}) \times \mathrm{Hom}(\Gamma, \{1, -1\})$.

Characteristic and C-characteristic

Definition

Let F be a hyperfield. A natural number $n \in \mathbb{N}$ is called *the characteristic of F* if n is the minimal number such that

$$0 \in \underbrace{1 + ... + 1}_{n \text{ times}}.$$

If there is no such number, then $charF = 0$.
A natural number $n \in \mathbb{N}$ is called *the C-characteristic of F* if n is the minimal number such that

$$1 \in \underbrace{1 + ... + 1}_{n+1 \text{ times}}.$$

If there is no such number, then $C\text{-}charF = 0$.

Every real hyperfield is of characteristic 0.

Results on C-characteristic in real hyperfields

Proposition

1. For every $n \in \mathbb{N}$ there exists an infinite real hyperfield F with $C\text{-}charF = n$.
2. For every even natural number $n > 1$ there exists a finite real hyperfield F with n elements such that $C\text{-}charF = 2$.
3. There exists a finite real hyperfield with $C\text{-}charF = 3$.

OPEN PROBLEM: Does there exist a finite hyperfield with $C\text{-}charF = n$ for every $n \in \mathbb{N}$?

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