The Newton Polygon and values of roots of polynomials

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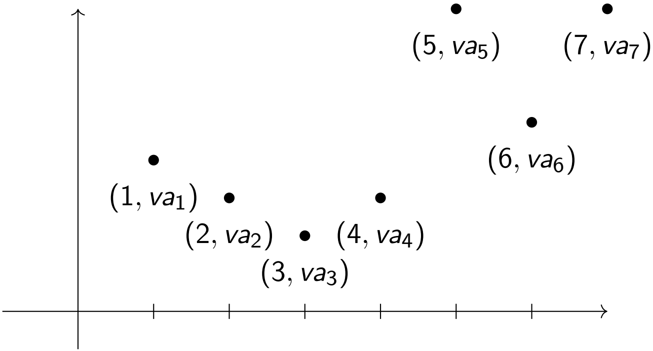
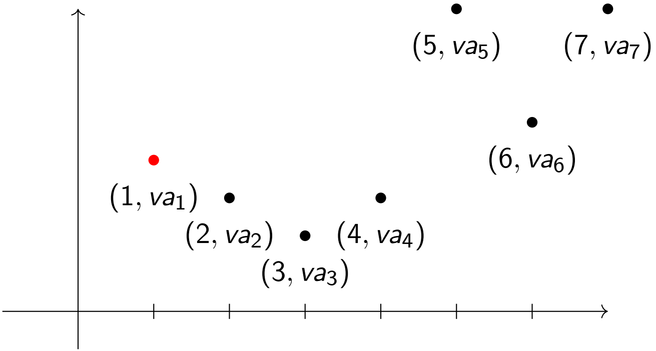
Building the Newton Polygon

Step 1: Take $f(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$. For each i such that $a_i \neq 0$, draw the points (i, va_i) in the Cartesian product $\mathbb{R} \times \Gamma$.



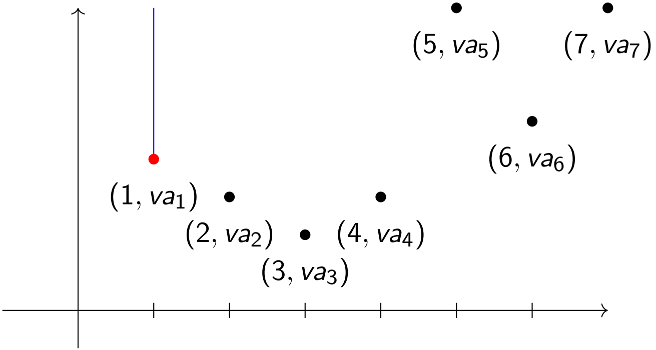
Building the Newton Polygon

Step 2: The leftmost point (that is, the point with the smallest first coordinate) is always a *vertex* of the Newton Polygon.



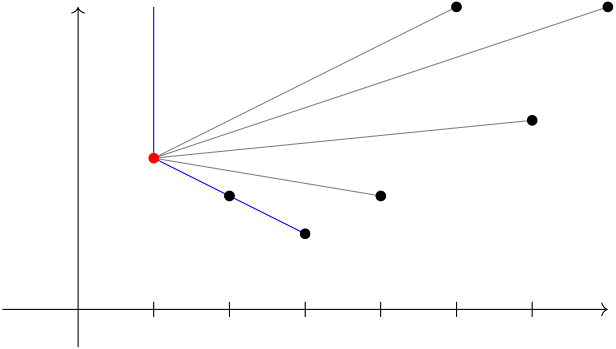
Building the Newton Polygon

Step 2.5: If 0 is a root of f, then the vertical line coming from the leftmost point will be the first *face* of the Newton Polygon.

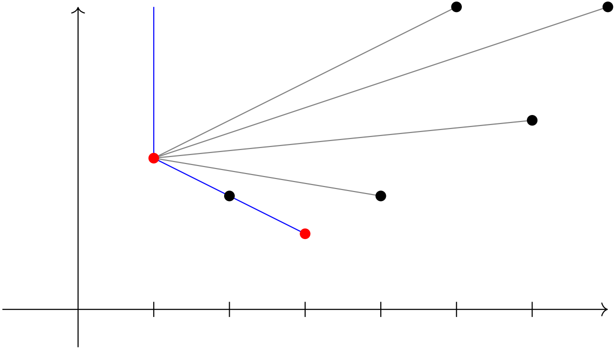


Building the Newton Polygon

Step 3: Consider the finite set of segments connecting the first vertex with each of the other points and choose the one with the smallest *slope*.

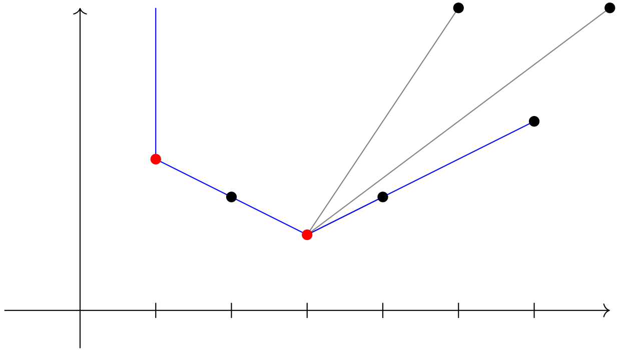


Step 4: Among all the points located on the chosen segment, we choose the rightmost one to be the next *vertex*.



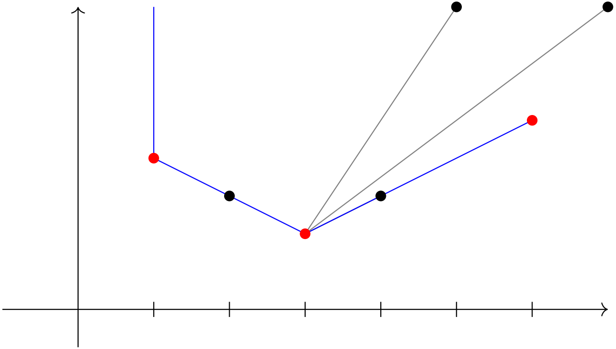
Building the Newton Polygon

Step 5: We repeat step 3 with the next vertex, considering only the points located to the right of this point.

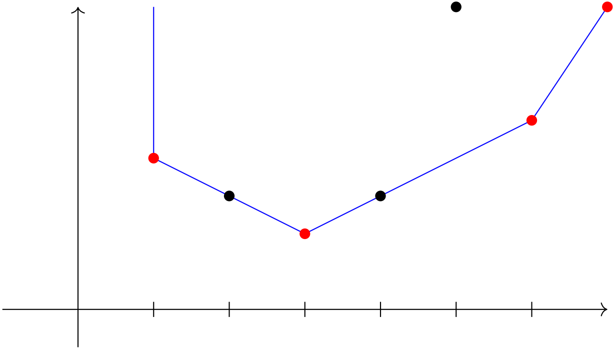


Building the Newton Polygon

Step 6: We repeat step 4 to choose the next vertex as the rightmost point on the chosen segment.



Step 7: We continue with step 3 and 4 until we reach the rightmost point. This point will also always be the *vertex*.



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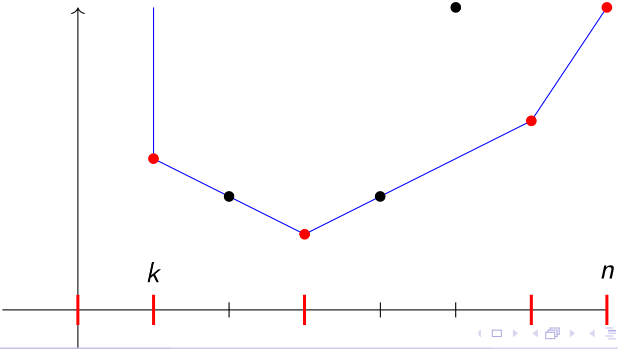
The function given by the resulting graph is called *the Newton Polygon* and denoted by NP_f.



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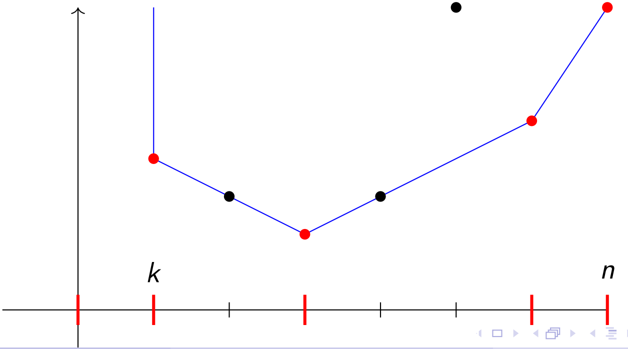

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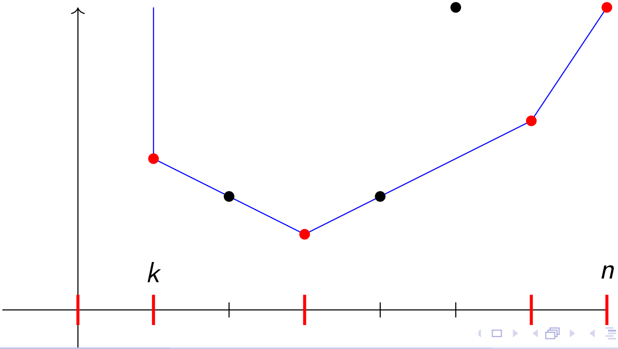

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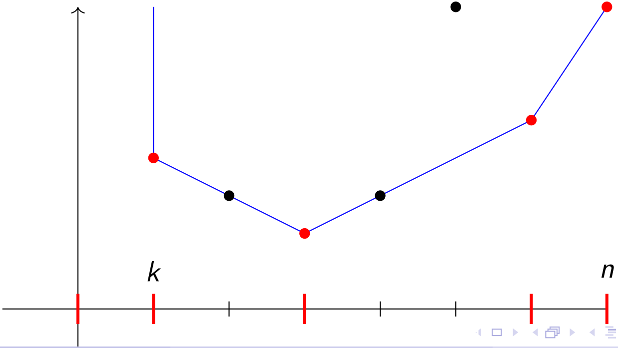

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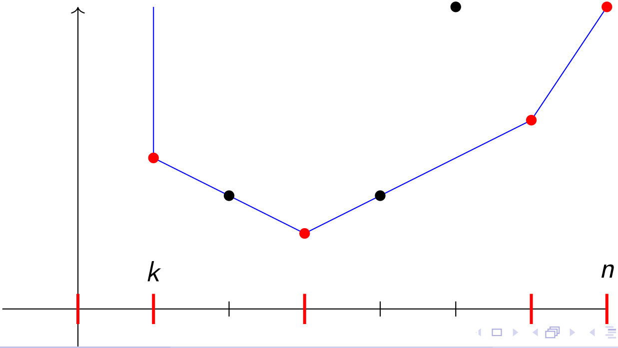
(d) NP_f is the largest function for which points (a)–(c) hold.



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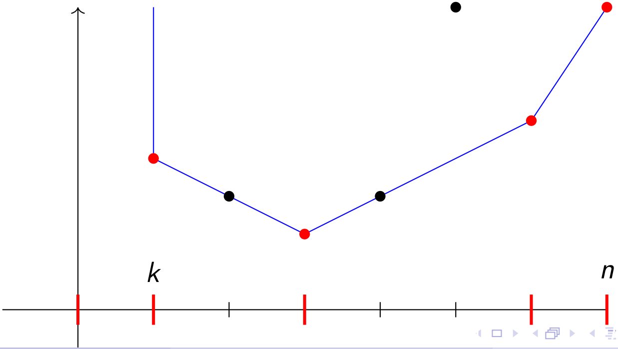
The segments of the graph of NP_f are called *the faces of the Newton Polygon*. If 0 is a root of f, then the first face is informally a 'segment' from $(0, \infty)$ to (k, va_k).



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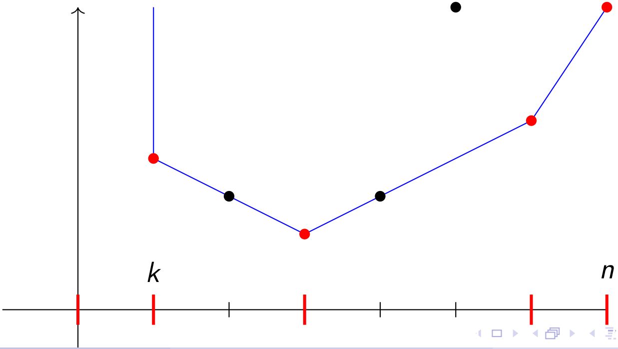

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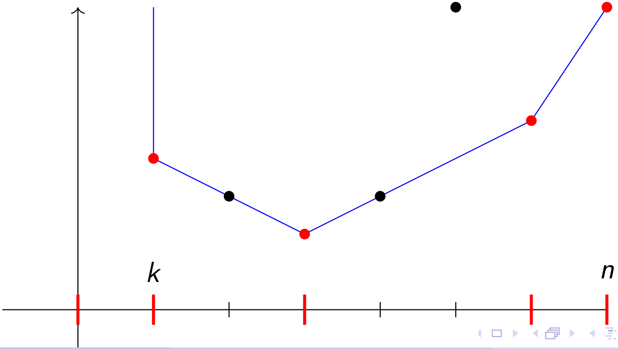

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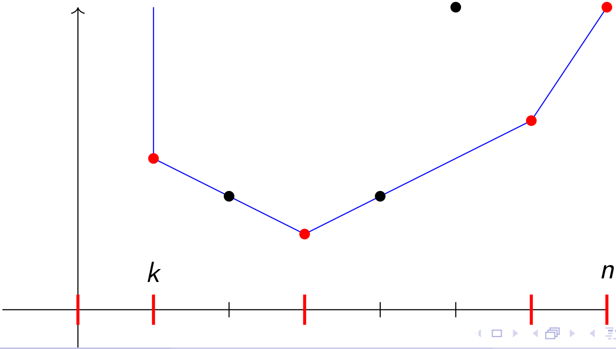

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Theorem 1

Take a polynomial $f \in K[x]$. If the Newton Polygon of f has a face of length k with slope $-\gamma$, then f has exactly k many roots of value γ (counted with multiplicity).

'Continuity' of Newton Polygons

Denote the distinct values of the roots of a polynomial f by $\gamma_1, \ldots, \gamma_s$, with $\gamma_i < \gamma_{i+1}$ for $1 \leq i < s$. We define k_i to be the number of roots of f of value strictly greater than γ_i.

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Theorem 2

Consider polynomials $f, g \in K[x]$ with f monic and $\deg g \geq \deg f =: n$. Fix some $\varepsilon \geq 0$, assume that $v(f-g) > n\varepsilon$ and that the set

$$\{\ell \in \{1 \ldots, s\} \mid \gamma_\ell \leq \varepsilon\}$$

is nonempty. If ℓ_ε is the maximum of this set, then $\mathrm{NP}_f(k) = \mathrm{NP}_g(k)$ for $k \in [k_{\ell_\varepsilon}, n]$.

Applications

For $\gamma \in \Gamma$ let $n_s(f, \gamma)$ be the number of roots of f with value γ, and let $n_b(f, \gamma)$ be the number of roots of f with value strictly greater than γ.

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(f) g has $(\deg g - n)$ many roots of value $< \gamma_1$ if and only if the point (n, vb_n) is a vertex of NP_g.

Applications

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Theorem 4

Let (K, v) be a valued field and take $f, g \in K[x]$, with f monic and $\deg g \geq \deg f =: n$. Take any $\varepsilon \in vK$ large enough and assume that

$$v(f-g) > n\varepsilon - \deg(f-g) \min_{1 \leq i \leq n} \{\min\{v\alpha_i, 0\}\}.$$

Then, after suitably rearranging indices, for every $k \in \{1, \ldots, n\}$ we have that $v(\alpha_k - \beta_k) > t_k \varepsilon$.

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