Pushing Anscombe-Jahnke up the ladder

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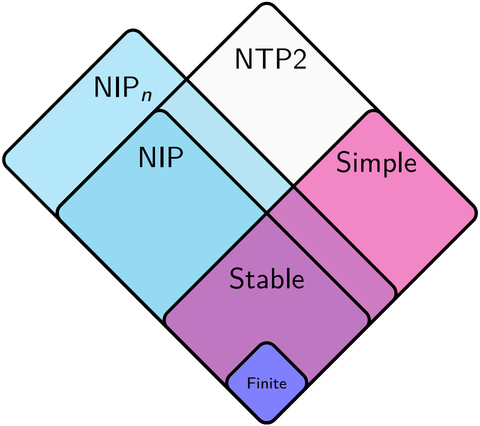
Complexity of first-order theories

What combinatorial truth patterns can a theory express?



Stable – no order property

A formula $\varphi(x, y)$ is said to have *the order property* (in M) if there is $(a_i)_{i\in\omega}, (b_i)_{i\in\omega} \in M$ such that $M \vDash \varphi(a_i, b_j)$ iff $i < j$.
A first-order theory T is said to be *stable* if no formula has the order property (in any $M \vDash T$). A structure M is said to be stable if $Th(M)$ is.

Examples of stable structures

- Finite structures, infinite sets without structure;
- $(\mathbb{Z}, \mathrm{succ})$;
- finitely generated free groups (Sela, 2006);
- separably closed fields (Wood, 79).

Examples of unstable structures

- Linear orders;
- $(\mathbb{Q}, +, \times)$ and $(\mathbb{R}, +, \times)$;
- ordered abelian groups and non-trivially valued fields.

NIP – no independence property

A formula $\varphi(x, y)$ is said to have *the independence property* (in M) if there are $(a_i)_{i\in\omega}, (b_J)_{J\subset\omega} \in M$ such that $M \models \varphi(b_J, a_i)$ iff $i \in J$.
A first-order theory T is said to be *NIP* if no formula has the independence property (in any $M \models T$). A structure M is said to be NIP if $Th(M)$ is.

Examples of NIP structures

- Stable structures;
- Linear orders,
- $(\mathbb{R}, +, \times)$ (Tarski, 48) and $(\mathbb{Q}_p, +, \times, v_p)$ (Macintyre, 76);
- ordered abelian groups (Gurevitch-Schmidt, 84);
- separably closed valued fields (Anscombe-Jahnke, 2019).

Examples of IP structures

- Simple unstable structures;
- the random graph;
- pseudo-algebraically closed fields (Duret, 80).

NIP_n – no independence property of order n

A formula $\varphi(x; y_1, \ldots, y_n)$ is said to have *the independence property of order n* (in M) if there are $(a_i^1)_{i\in\omega}, \ldots, (a_i^n)_{i\in\omega}, (b_J)_{J\subset\omega^n} \in M$ such that $M \models \varphi(b_J, a_{i_1}^1, \ldots, a_{i_n}^n)$ iff $(i_1, \ldots, i_n) \in J$. A first-order theory T is said to be NIP_n if no formula has IP_n (in any $M \models T$). A structure M is said to be NIP_n if $Th(M)$ is and strictly NIP_n if it is NIP_n and IP_{n-1}.

Examples of NIP_n structures

- Stable, NIP an NIP_k structures are NIP_n for $k < n$;
- The random n-hypergraph is strictly NIP_n.
- $(\mathbb{F}_p^{<\omega}, \mathbb{F}_p, 0, +, \times)$, with $(a_i) \times (b_i) = \sum a_i b_i \in \mathbb{F}_p$, is strictly NIP_2 (Hempel, 2016).
- Strictly NIP_n pure groups are known (Chernikov-Hempel, 2019).

Example of IP_n structure

- PAC fields are IP_n for all n (Hempel, 2016).

Simple – no tree propperty

A formula $\varphi(x, y)$ is said to have *the tree property* (in M) if there are $(a_s)_{s \in \omega^{<\omega}} \in M$ such that for each $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_0}), \varphi(x, a_{\sigma|_1}), \varphi(x, a_{\sigma|_2}), \dots\}$ is consistent, but for any $s \in \omega^{<\omega}$, $\{\varphi(x, a_{s0}), \varphi(x, a_{s1}), \varphi(x, a_{s2}) \dots\}$ is k-inconsistent. A first-order theory T is said to be *simple* if no formula has the tree property (in any $M \models T$). A structure M is said to be simple if $Th(M)$ is.

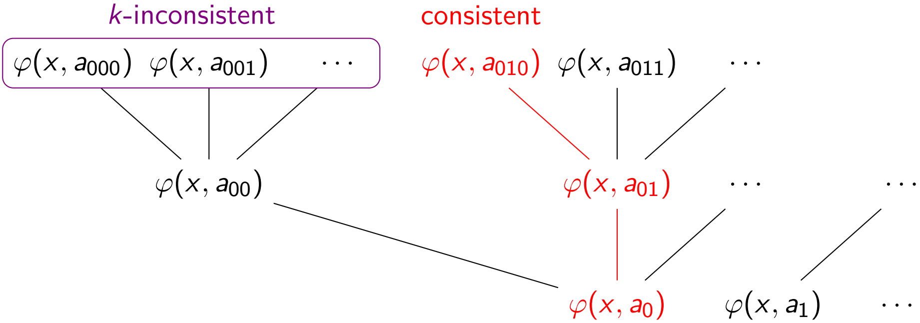
Examples of simple structures

- Stable structures;
- the random n-hypergraph;
- bounded PAC fields (Hrushovski 2002).

Examples of non-simple structures

- NIP unstable structures;
- unbounded PAC fields (Chatzidakis, 99);
- non-trivially valued fields.

The tree property



NTP2 – no tree property of order 2

A formula $\varphi(x, y)$ is said to have *the tree property of order 2* (in M) if there is $(a_{ij})_{i,j\in\omega} \in M$ such that:

- For any $i \in \omega$, $\{\varphi(x, a_{ij}) \mid j \in \omega\}$ is k-inconsistent,
- For any $f : \omega \to \omega$, $\left\{\varphi(x, a_{if(i)}) \mid i \in \omega\right\}$ is consistent.

A first-order theory T is said to be *NTP2* if no formula has the tree property of order 2. A structure M is said to be NTP2 if $Th(M)$ is.

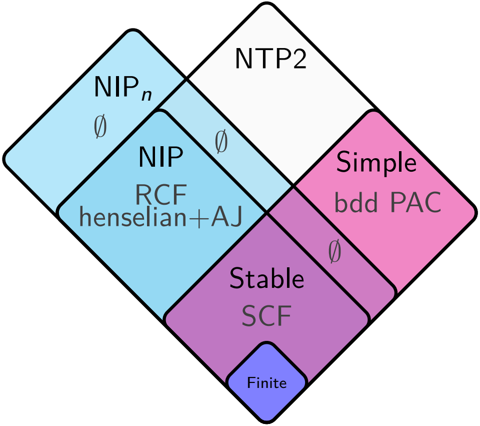
Examples of NTP2 structures

- Simple and NIP structures;
- bounded pseudo real closed and pseudo p-adically closed fields (Montenegro, 2014);

Examples of TP2 structures

- Unbounded PAC, PRC and PpC fields;
- ZFC, Peano...

Conjectures on complex (theories of) fields



Classification of NIP henselian valued fields

Theorem (Anscombe-Jahnke, 2019)

Let (K, v) be henselian. (K, v) is NIP iff k is NIP and:

1. $\mathrm{ch}(K, k) = (0, 0)$ or (p, p) and (K, v) is SAMK or trivial;
2. $\mathrm{ch}(K, k) = (0, p)$, (K, v_p) is finitely ramified and $(k_p, \overline{v})$ checks 1;
3. $\mathrm{ch}(K, k) = (0, p)$ and $(k_0, \overline{v})$ is AMK.

"$\Leftarrow$" is by CHIPS transfer theorem and "$\Rightarrow$" is because:

- k is interpretable and thus NIP;
- Infinite NIP fields of characteristic p are Artin-Schreier closed (Kaplan-Scanlon-Wagner, 2011), so if $\mathrm{ch}(K) = p$ it is SAMK: no separable alebraic immediate extension, Γ p-div, k p-closed.
- In mixed characteristic we decompose around $v(p)$: $K \xrightarrow{\Gamma/\Delta_0} k_0 \xrightarrow{\Delta_0/\Delta_p} k_p \xrightarrow{\Delta_p} k$. Convex subgroups are externally definable, so by Shelah's expansion theorem the structure (K, v, v_0, v_p) remains NIP and we work part by part.

Anscombe-Jahnke for NIP_n and NTP2

To prove Anscombe-Jahnke for NIP_n:

- Infinite NIP_n fields are AS-closed (Hempel, 2016), so we can do "$\Rightarrow$" of the equicharacteristic case, but
- We need a NIP_n transfer theorem,
- For mixed characteristic, when doing the decomposition, we can't use Shelah's expansion theorem.

As for NTP2:

- Transfer works in the same cases, but
- NTP2 fields are not AS-closed, they are only AS-finite (Chernikov-Kaplan-Simon, 2013),
- We also can't use Shelah's expansion theorem.

Chernikov-Hils Im Plus SE (CHIPS) transfer

Chernikov-Hils isolated 2 conditions which gives NTP2 transfer; they have since been adapted to NIP transfer by Jahnke-Simon and to NIP_n transfer:

(SE): The residue field and the value group are stably embedded.

(Im): For any model K and any singleton b (from a model $K^* \succcurlyeq K$) such that $K(b)/K$ is immediate, we have that $tp(b/K)$ is implied by instances of NTP2 formulas, that is, there is $p \subset tp(b/K)$ preserved under conjunctions and such that:

- any formula $\varphi(x, y) \in p$ – where x is the cast for b and y for (a finite subtuple of) K – is NTP2,
- $\psi(b, K)$ holds iff $p \vdash \psi$.

Anscombe-Jahnke

SAMK fields and unramified fields have NIP CHIPS.

It directly implies that they have NTP2 CHIPS, and with a bit of work, it is possible to prove they have NIP_n CHIPS; with a bit more work we have NTP2 and NIP_n transfer in all cases of Anscombe-Jahnke.

Localising KSW-H

Proof scheme of Artin-Schreier closure of NIP$_n$ fields:

- If K is NIP$_n$, then any definable family of additive subgroups checks Baldwin-Saxl-Hempel's condition,
- $H_{a_1,\dots,a_n} = \{a_1 \cdots a_n(t^p - t) \mid t \in K\}$ is a such a definable family,
- If $H_{a_1,\dots,a_n}$ checks Baldwin-Saxl-Hempel's condition, then K is AS-closed.

Baldwin-Saxl (75), Hempel (2016)

Let $H_{a_1,\dots,a_n} = \{x \in M \mid M \vDash \varphi(x, a_1, \dots, a_n)\}$ be a definable family of subgroups (of a definable group of M). φ is NIP$_n$ iff there is $N \in \omega$ such that for any $d > N$, for any $(a^i_j)^{i\leqslant n}_{j\leqslant d}$, there is $\overline{k} = (k_1, \dots k_n)$ with $\bigcap_{\overline{j}\in d^n} H_{a^1_{j_1},\dots,a^n_{j_n}} = \bigcap_{\overline{j}\neq\overline{k}} H_{a^1_{j_1},\dots,a^n_{j_n}}$.

Local KSW-H

Let K be infinite and of characteristic p.

$\varphi(x; y_1, \dots, y_n) : \exists t\, x = y_1 \cdots y_n(t^p - t)$ is NIP$_n$ (in K) iff K is AS-closed.

Localising CKS

Proof scheme of CKS:

- If K is NTP2, then any definable family of additive subgroups checks a certain chain condition,
- $H_a = \{a(x^p - x) \mid x \in K\}$ is a such a definable family,
- If H_a checks this chain condition, then K is p-finite.

CKS chain condition

Let $H_a = \{x \in M \mid M \vDash \varphi(x, a)\}$ be a definable family of subgroups (of a definable group of M). $\psi(x, y, z) : \exists t\, x \in zH_y$ is NTP2 iff for any $(a_i)_{i \in \omega}$, there is j such that $\left[\bigcap_{i \neq j} H_{a_i} : \bigcap_{i \in \omega} H_{a_i}\right]$ is finite.

Local CKS

Let K be of characteristic p. $\psi(x, y, z) : \exists t\, x = y(t^p - t) + z$ is NTP2 iff K is AS-finite.

Artin-Schreier lifting

The formulas we obtained are positive existential formulas. If they witness a pattern in a residue field, we can lift this pattern by henselianity.

Lifting Artin-Schreier complexity

Let (K, v) be henselian of mixed characteristic. If k is infinite & not AS-closed, then K has IP_n as a pure field; if k is not AS-finite, then K has TP2 as a pure field.

Thus, when doing the decomposition $K \xrightarrow{\Gamma/\Delta_0} k_0 \xrightarrow{\Delta_0/\Delta_p} k_p \xrightarrow{\Delta_p} k$, we can say, to some extent, that relevant parts are p-closed/p-div/defectless without adding the intermediate valuations in the language. As a consequence, we have "$\Rightarrow$" of Anscombe-Jahnke for NIP_n, and we obtain strong conditions on NTP2 henselian valued fields.

Summary of NIP_n results

NIP_n Anscombe-Jahnke

Let (K, v) be henselian. (K, v) is NIP_n iff k is NIP_n and:

1. $\text{ch}(K, k) = (0, 0)$ or (p, p) and (K, v) is SAMK or trivial;
2. $\text{ch}(K, k) = (0, p)$, (K, v_p) is finitely ramified and $(k_p, \overline{v})$ checks 1;
3. $\text{ch}(K, k) = (0, p)$ and $(k_0, \overline{v})$ is AMK.

Corollaries

- NIP_n henselian valued fields with NIP residues are NIP;
- in particular, algebraic extensions of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ are NIP_n iff they are NIP.
- With the help of Jahnke-Koenigsmann definability results, we obtained that if K is NIP_n, if v is henselian and if $\text{ch}(k_v) = p$, then (K, v) is NIP_n.

Summary of NTP2 results

NTP2 transfer

Let (K, v) be henselian. If k is NTP2 and if:

1. $\mathrm{ch}(K, k) = (0, 0)$ or (p, p) and (K, v) is SAMK or trivial;
2. $\mathrm{ch}(K, k) = (0, p)$, (K, v_p) is finitely ramified and $(k_p, \overline{v})$ checks 1;
3. $\mathrm{ch}(K, k) = (0, p)$ and $(k_0, \overline{v})$ is AMK.

then (K, v) is NTP2.

NTP2 consequences

Let K be NTP2 and v be henselian. Then (K, v) is either

1. of equicharacteristic 0, hence tame, or
2. of equicharacteristic p and semitame, or
3. of mixed characteritic with $(k_0, \overline{v})$ semitame, or
4. of mixed characteristic with v_p finitely ramified and $(k_p, \overline{v})$ semitame.

In particular, (K, v) is gdr.

Thank you for your attention!