Abhyankar places admit local uniformization in any characteristic

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Abstract

We prove that every place P of an algebraic function field $F|K$ of arbitrary characteristic admits local uniformization, provided that the sum of the rational rank of its value group and the transcendence degree of its residue field FP over K is equal to the transcendence degree of $F|K$, and the extension $FP|K$ is separable. We generalize this result to the case where P dominates a regular local Nagata ring $R \subseteq K$ of Krull dimension $\dim R \leq 2$, assuming that the valued field (K, v_P) is defectless, the factor group $v_P F/v_P K$ is torsion-free and the extension of residue fields $FP|KP$ is separable. The results also include a form of monomialization.

1 Introduction and main results

In [Z1], Zariski proved the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In [Z3], he uses this theorem to prove resolution of singularities for algebraic surfaces in characteristic 0. This result was generalized by Abhyankar to the case of positive characteristic [A4] and to the case of arithmetic surfaces over a Dedekind domain [A5]. More recently de Jong [dJ] proved that the singularities of an algebraic or arithmetic variety X can be resolved by successively applying a finite number of morphisms called alterations. An alteration $f : Y \to X$ is a composition $f = g \circ h$ of a birational morphism $h : X' \to X$ and a finite morphism $g : Y \to X'$. In general it leads to a finite extension $K(Y)|K(X)$ of function fields. Applying de Jong's results to a proper algebraic variety over a field of positive characteristic or to a proper arithmetic variety over a discrete valuation ring implies Local Uniformization in these cases, provided one allows finite extensions of the variety's function field or of the discrete valuation ring, respectively. In [K–K], we replace the application of de Jong's results by a purely valuation theoretical proof, thereby obtaining a more detailed description of the extensions of the function field.

As resolution of singularities in the birational sense in positive characteristic for arbitrary dimensions is still an open problem, one is interested in generalizations of the original Local Uniformization Theorem. In the present paper we investigate the class of so-called Abhyankar places of a function field $F|K$. We prove that Abhyankar places that are trivial on K admit local uniformization on algebraic varieties without any extension of F, provided they have a separable residue field extension. We also consider the case of Abhyankar places dominating a regular local ring $R \subset K$ and provide sufficient conditions for local uniformization on R-models of $F|K$. The core of our method is purely valuation-theoretical. Before stating the main results precisely, we have to introduce the necessary terminology.

Throughout this paper the term *function field* will always mean *algebraic function field*. We consider places P of a function field $F|K$ not necessarily inducing the identity on the constant field K. Places that do induce the identity on K (or an isomorphism) are called K-**trivial**. The valuation associated with P will be denoted by v_P, the v_P-value of an element a by $v_P a$ and consequently, the value group of v_P on F by $v_P F$. Places are considered to operate on the right: the residue of an element $a \in F$ is denoted by aP and consequently, FP stands for the residue field of F with respect to P. We frequently do not distinguish between a place P on the field F and its restrictions to subfields $E \subseteq F$. Since we are usually working with one fixed place, this does not lead to confusion. The valuation ring of P on a subfield $E \subseteq F$ is denoted by $\mathcal{O}_E$. If we have to distinguish between several places on the same field F, then we use the notation $\mathcal{O}_P$ for the valuation ring of P on F. The maximal ideal of the valuation rings $\mathcal{O}_P$ and $\mathcal{O}_E$ is denoted by $\mathcal{M}_P$ and $\mathcal{M}_E$, respectively. Finally, it should be mentioned that by abuse of language we refer to a pair (F, P) consisting of a field F and a place P of F as a valued field, and to a pair $(F|K, P)$, $F|K$ an extension of fields, as an extension of valued fields, or as a valued function field if $F|K$ is a function field.

For every place P of a function field $F|K$, we have the following inequality:

$$\operatorname{trdeg} F|K \; \geq \; \operatorname{trdeg} FP|KP \, + \, \dim_{\mathbb{Q}}((v_P F/v_P K) \otimes \mathbb{Q}) \,. \tag{1}$$

Note that $\dim_{\mathbb{Q}}(v_P F/v_P K \otimes \mathbb{Q})$ is the **rational rank** of the abelian group $v_P F/v_P K$, i.e., the maximal number of rationally independent elements in $v_P F/v_P K$. We call P an **Abhyankar place of $F|K$** if equality holds in (1).

In the context of local uniformization we shall be concerned with the prime factorization in a regular local ring and use the following terminology: Let $\mathcal{O}$ be a commutative ring and $H \subseteq \mathcal{O}$. An element $a \in \mathcal{O}$ is called an $\mathcal{O}$-**monomial in H** if

$$a = u \prod_{i=1}^{d} h_i^{\mu_i}, \; u \in \mathcal{O}^{\times}, \; h_i \in H, \; \mu_i \in \mathbb{N}_0, \; i = 1, \ldots, d,$$

holds, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We investigate local uniformization over regular base rings: let $(F|K, P)$ be a valued function field, and let $R \subseteq \mathcal{O}_P$ be a subring of K with $\operatorname{Frac} R = K$. Moreover, let $Z \subset \mathcal{O}_P$ be a finite set. The pair (P, Z) is called R-**uniformizable** if there exists an

2

integral separated R-scheme X of finite type with field of rational functions F — an R-**model of** F — such that P is centered in a regular point $x \in X$ and Z is a subset of the local ring $\mathcal{O}_{X,x}$ at x. If this holds, then we also use the phrase R-**uniformizable on** X. The place P is called R-uniformizable if (P, Z) is R-uniformizable for every finite set $Z \subset \mathcal{O}_P$. Note that including the finite set Z in the problem of uniformization allows to prove statements like: given an integral R-scheme X such that P is centered in the (singular) point $x \in X$ there exists an integral R-scheme Y and a morphism $\pi : U \to V$, where $U \subseteq Y$ and $V \subseteq X$ are open subschemes such that $x \in V$ and P is centered in a regular point $y \in \pi^{-1}x$.

Throughout the paper we restrict ourselves to the case of a *regular local ring* R with maximal ideal $M = \mathcal{M}_P \cap R$, i.e., we assume that $\mathcal{O}_P$ dominates R.

We prove a Local Uniformization Theorem for K-trivial Abhyankar places in arbitrary characteristic:

Theorem 1.1 *Let P be a K-trivial Abhyankar place of the function field $F|K$, and assume that $FP|K$ is separable. Take any finite set $Z \subset \mathcal{O}_P$. Then the pair (P, Z) is K-uniformizable on a variety X such that P is centered in a smooth point $x \in X$ and $\dim \mathcal{O}_{X,x} = \dim_{\mathbb{Q}}(v_P F \otimes \mathbb{Q})$. Moreover, X can be chosen such that all $\zeta \in Z$ are $\mathcal{O}_{X,x}$-monomials in $\{a_1, \ldots, a_d\}$ for some regular parameter system $(a_1, \ldots, a_d)$ of $\mathcal{O}_{X,x}$.*

One can also achieve that the $\mathcal{O}_{X,x}$-ideal generated by Z is principal, generated by each element of minimal value. For this, one applies the above theorem to the set $Z' = Z \cup \{\frac{a}{b} \mid a, b \in Z, va \geq vb\}$ in the place of Z.

Theorem 1.1 is essentially proved by embedding F in the field of fractions of the strict henselization of the valuation ring $\mathcal{O}_{K(T)} = \mathcal{O}_P \cap K(T)$ for a suitable transcendence basis $T \subset F$ of $F|K$. The methods used in the proof of Theorem 1.1 are applicable even if P is not trivial on K. They then lead to a uniformization result for integral schemes of finite type over certain base rings $R \subset K$. For the definition of the notion "defectless", see Section 3.

Theorem 1.2 *Let P be an Abhyankar place of the function field $F|K$, which is non-trival on K. Assume that (K, P) is defectless, $FP|KP$ is separable and the group $v_P F/v_P K$ is torsion-free.*

Let $R \subset K \cap \mathcal{O}_P$, $\operatorname{Frac} R = K$, be a noetherian, regular local ring with maximal ideal $M = \mathcal{M}_P \cap R$ and of dimension $\dim R \leq 2$. Assume that R is a Nagata ring if $\dim R = 2$.

Then for every finite set $Z \subset \mathcal{O}_P$ the pair (P, Z) is R-uniformizable on an R-scheme X such that the center $x \in X$ of P on X satisfies:

- $\dim \mathcal{O}_{X,x} = \dim_{\mathbb{Q}}(v_P F/v_P K \otimes \mathbb{Q}) + 1$ *if $\dim R = 1$ or $\operatorname{trdeg}(KP|R/M) > 0$.*

- $\dim \mathcal{O}_{X,x} = \dim_{\mathbb{Q}}(v_P F/v_P K \otimes \mathbb{Q}) + 2$ *in the remaining cases.*

Moreover, X can be chosen such that all $\zeta \in Z$ are $\mathcal{O}_{X,x}$-monomials in $\{a_1, \ldots, a_d\}$ for some regular parameter system $(a_1, \ldots, a_d)$ of $\mathcal{O}_{X,x}$.

Some remarks concerning the condition $\dim R \leq 2$ may be helpful at this point. Recall that the domain R is called **Nagata**, if the integral closure of every factor ring R/p, $p \in \operatorname{Spec} R$, in every finite extension of $\operatorname{Frac} R/p$ is finite. In the case of $\dim R = 1$, the ring R is a discrete valuation ring and moreover, in the situation of the theorem, we have $R = \mathcal{O}_K$. It is well-known that then the property of being Nagata is equivalent to the valued field (K, P) being defectless. *We conclude that all base rings appearing in Theorem 1.2 are Nagata.*

A further important ring-theoretic notion that we will have to use is universal catenarity: the domain R is called **universally catenary** if every polynomial ring $R[X_1, \ldots, X_n]$, $n \in \mathbb{N}$, has the following property: for every pair of prime ideals $p, q \in R[X_1, \ldots, X_n]$ with $p \subset q$, all non-refinable chains of primes $p =: p_0 \subset p_1 \subset \ldots \subset p_\ell := q$ have a common finite length ℓ (depending on p, q).

Since Cohen-Macaulay rings are universally catenary, all base rings appearing in Theorem 1.2 are universally catenary.

Every universally catenary domain R satisfies the **altitude formula**: for every prime $q \in \operatorname{Spec} A$ of every domain A finitely generated over R the equation

$$\text{height } q + \operatorname{trdeg}(A/q \mid R/p) = \text{height } p + \operatorname{trdeg}(A \mid R), \ p := q \cap R, \tag{2}$$

holds.

The condition $\dim R \leq 2$ can be replaced by a more general but rather technical condition that ensures the existence of certain monoidal transforms of R along the valuation v_P.

In [K3] it is shown that the K-trivial Abhyankar places lie dense in the Zariski space of all K-trivial places of $F \mid K$, with respect to a "Zariski patch topology". This topology is finer than the Zariski topology (but still compact); its basic open sets are the sets of the form

$$\{P \mid P \text{ a place of } F \mid K \text{ such that } a_1 P \neq 0, \ldots, a_k P \neq 0 \, ; \, b_1 P = 0, \ldots, b_\ell P = 0\}$$

with $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in F \setminus \{0\}$. Theorem 1.1 thus yields:

Corollary 1.3 *The K-uniformizable places of $F \mid K$ lie dense in the Zariski space of $F \mid K$, with respect to the Zariski patch topology, provided that K is perfect.*

2 Valuation independence

The following theorem, together with Theorem 3.1 below, gives the motivation for the definition of the distinguished class of Abhyankar places. For its proof see [B], Chapter VI, §10.3, Theorem 1.

Theorem 2.1 *Let $(F \mid K, P)$ be an extension of valued fields. Take elements $x_i, y_j \in F$, $i \in I$, $j \in J$, such that the values $v_P x_i$, $i \in I$, are rationally independent over $v_P K$, and*

the residues $y_j P$, $j \in J$, are algebraically independent over KP. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.

Moreover, if we write

$$f \; = \; \sum_k c_k \prod_{i \in I} x_i^{\mu_{k,i}} \prod_{j \in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$v_P f \; = \; \min_k \left(v_P \left(c_k \prod_{i \in I} x_i^{\mu_{k,i}} \prod_{j \in J} y_j^{\nu_{k,j}} \right) \right) \; = \; \min_k \left(v_P c_k + \sum_{i \in I} \mu_{k,i} v_P x_i \right) .$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$v_P K(x_i, y_j \mid i \in I, j \in J) \; = \; v_P K \oplus \bigoplus_{i \in I} \mathbb{Z} v_P x_i$$

$$K(x_i, y_j \mid i \in I, j \in J)P \; = \; KP(y_j P \mid j \in J) .$$

It also implies that the valuation v_P and the place P on $K(x_i, y_j \mid i \in I, j \in J)$ are uniquely determined by their restrictions to K, the values $v_P x_i$ and the residues $y_j P$.

Every finite extension L of the valued field (K, P) satisfies the **fundamental inequality** (cf. [En]):

$$[L : K] \; \geq \; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{3}$$

where $P_1, \ldots, P_{\mathrm{g}}$ are the distinct extensions of P from K to L, $\mathrm{e}_i = (v_{P_i} L : v_P K)$ are the respective ramification indices and $\mathrm{f}_i = [LP_i : KP]$ are the respective inertia degrees. Note that $\mathrm{g} = 1$ if (K, P) is henselian.

Corollary 2.2 *Let $(F|K, P)$ be an extension of valued fields of finite transcendence degree. Then the following inequality holds:*

$$\mathrm{trdeg}\, F|K \; \geq \; \mathrm{trdeg}\, FP|KP \; + \; \dim_{\mathbb{Q}}((v_P F / v_P K) \otimes \mathbb{Q}) . \tag{4}$$

If in addition $F|K$ is a function field, and if equality holds in (4), then the extensions $v_P F|v_P K$ and $FP|KP$ are finitely generated. In particular, if P is trivial on K, then $v_P F$ is a product of finitely many copies of $\mathbb{Z}$, and FP is again a function field over K.

Proof: Choose elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau \in F$ such that the values $v_P x_1, \ldots, v_P x_\rho$ are rationally independent over $v_P K$ and the residues $y_1 P, \ldots, y_\tau P$ are algebraically independent over KP. Then by the foregoing theorem, $\rho + \tau \leq \mathrm{trdeg}\, F|K$. This proves that $\mathrm{trdeg}\, FP|KP$ and the rational rank of $v_P F / v_P K$ are finite. Therefore, we may choose the elements x_i, y_j such that $\tau = \mathrm{trdeg}\, FP|KP$ and $\rho = \dim_{\mathbb{Q}}((v_P F / v_P K) \otimes \mathbb{Q})$ to obtain inequality (4).

Assume that this is an equality. This means that for $F_0 := K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$, the extension $F|F_0$ is algebraic. Since $F|K$ is finitely generated, it follows that $F|F_0$ is finite. By the fundamental inequality (3), this yields that $v_P F|v_P F_0$ and $FP|F_0 P$ are finite extensions. Since already $v_P F_0 | v_P K$ and $F_0 P | KP$ are finitely generated by the foregoing theorem, it follows that also $v_P F | v_P K$ and $FP|KP$ are finitely generated. $\quad\square$

If equality holds in (4) we will either say that $(F|K, P)$ **is without transcendence defect** or as already defined earlier that P is an Abhyankar place of $F|K$.

3 Inertially generated function fields

In this section we provide the valuation-theoretic core of the present paper. It is a generalization of the "Grauert–Remmert Stability Theorem" and is proved in [K4]. To state it, we introduce a fundamental notion: a valued field (K, P) is called **defectless** (or **stable**) if equality holds in the fundamental inequality (3) for every finite extension $L|K$. If char $KP = 0$, then (K, P) is defectless (this is a consequence of the "Lemma of Ostrowski", cf. [En], [R]).

Theorem 3.1 (Generalized Stability Theorem) *Let $(F|K, P)$ be a valued function field without transcendence defect. If (K, P) is a defectless field, then also (F, P) is a defectless field.*

In what follows we consider concepts like the henselization of a valued field (K, P), where one has to fix an extension of P and v_P to the algebraic closure of K. Thus, whenever we talk of a valued field (K, P), we will implicitly assume the valuation v_P and the place P to be extended to the algebraic closure of K, the extensions denoted by v_P and P again. Therefore, we will talk of *the* henselization K^h, and of *the* **absolute inertia field** K^i **of** K, which we define to be the inertia field of the normal separable extension $K^{\mathrm{sep}}|K$ with respect to the given valuation v_P; here, K^{sep} denotes the separable-algebraic closure of K.

The following lemma is proved in [K4] (and partially also in [En]):

Lemma 3.2 *A valued field (K, P) is defectless if and only if its henselization (K^h, P) is.*

An extension $(L|K, P)$ of valued fields is called **immediate** if the canonical embeddings $KP \to LP$ and $v_P K \to v_P L$ are onto.

Corollary 3.3 *If (K, P) is defectless, then (K^h, P) does not admit proper immediate algebraic extensions.*

Proof: If (K, P) is defectless, then so is (K^h, P), by the foregoing lemma. Suppose that $(L|K^h, P)$ is a finite immediate algebraic extension. Hence, $(v_P L : v_P K^h) = 1 = [LP : K^h P]$. Since (K^h, P) is a henselian field, there is a unique extension of v_P from K^h to

L. Since (K^h, P) is defectless, we have that $[L : K^h] = (v_P L : v_P K^h)[LP : K^h P] = 1$, showing that $L = K^h$. As every proper immediate algebraic extension would contain a proper immediate finite extension, it follows that (K^h, P) does not admit any proper immediate algebraic extension. □

¿From these facts, we deduce:

Theorem 3.4 *Assume that $(F|K, P)$ is a valued function field without transcendence defect such that $FP|KP$ is a separable extension, (K, P) is a defectless field and $v_P F / v_P K$ is torsion-free. Then $(F|K, P)$ is* **inertially generated***, by which we mean that there is a transcendence basis $T = \{x_1, \ldots, x_\rho, y_1, \ldots, y_\tau\}$ such that*

a) $v_P F = v_P K \oplus \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$,

b) $y_1 P, \ldots, y_\tau P$ *is a separating transcendence basis of $FP|KP$,*

c) (F, P) *lies in the absolute inertia field of $(K(T), P)$.*

Assertion c) holds for each transcendence basis T which satisfies assertions a) and b).

Proof: By Corollary 2.2, the factor group $v_P F / v_P K$ and the residue field extension $FP|KP$ are finitely generated. We choose $x_1, \ldots, x_\rho \in F$ such that $v_P F = v_P K \oplus \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$, where $\rho = \dim_{\mathbb{Q}}((v_P F / v_P K) \otimes \mathbb{Q})$. Since $FP|KP$ is a finitely generated separable extension, it is separably generated. Therefore, we can choose $y_1, \ldots, y_\tau \in F$ such that $FP|KP(y_1 P, \ldots, y_\tau P)$ is separable-algebraic, where $\tau = \text{trdeg}\, FP|KP$. We set $T := \{x_1, \ldots, x_\rho, y_1, \ldots, y_\tau\}$ and $F_0 := K(T)$.

Now we can choose some $a \in FP$ such that $FP = KP(y_1 P, \ldots, y_\tau P, a)$. Since a is separable-algebraic over $KP(y_1 P, \ldots, y_\tau P)$, by Hensel's Lemma there exists an element η in the henselization of (F, P) such that $\eta P = a$ and that the reduction of the minimal polynomial of η over F_0 is the minimal polynomial of a over $KP(y_1 P, \ldots, y_\tau P)$. Then η lies in the absolute inertia field of F_0. Now the field $F_0(\eta)$ has the same value group and residue field as F, and it is contained in the henselization F^h of F. As henselizations are immediate extensions and the henselization $F_0(\eta)^h$ of $F_0(\eta)$ can be chosen inside of F^h, we obtain an immediate algebraic extension $(F^h | F_0(\eta)^h, P)$. On the other hand, (K, P) is assumed to be a defectless field. By construction, $(F_0|K, P)$ is without transcendence defect, and the same is true for $(F_0(\eta)|K, P)$ since this property is preserved by algebraic extensions. Hence we know from Theorem 3.1 that $(F_0(\eta), P)$ is a defectless field. Now Corollary 3.3 shows that the extension $F^h|F_0(\eta)^h$ must be trivial. Therefore, F is contained in $F_0(\eta)^h$, which in turn is a subfield of the absolute inertia field of F_0. This proves our theorem. □

Theorem 3.4 is central in the proof of the uniformization results presented in this paper. Its importance is based on the fact that the valuation ring $\mathcal{O}_{K(T)^i} = \mathcal{O}_P \cap K(T)^i$ is the strict henselization of the valuation ring $\mathcal{O}_{K(T)}$ and therefore the ring extension $\mathcal{O}_{K(T)^i} | \mathcal{O}_{K(T)}$ is local-ind-étale; see [Ray], Ch. X. Using this property, one can construct an extension $B|A$ of finitely generated R-algebras, $R = K$ or R the local base ring appearing in Theorem 1.2, such that $B \subset \mathcal{O}_P$, $\text{Frac}\, B = F$, $\text{Frac}\, A = K(T)$, and $B_q | A_q$

is an étale extension for $q := \mathcal{M}_P \cap A$. In order to prove Theorems 1.1 and 1.2, and ignoring the requirement for the elements $\zeta \in Z$ for the moment, by the permanence of smoothness and regularity under étale extension it therefore suffices to construct A in such a way that A_q is smooth over K, or is a regular local ring, respectively. This is done in the next section.

4 Abhyankar places on rational function fields

In this section, uniformization of Abhyankar places on rational function fields of the type appearing in Theorem 3.4 is investigated. Throughout this section, let $(F|K, P)$ be a rational function field equipped with a place P subject to the following conditions:

(T) *There exists a transcendence basis $T = (x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$ of $F|K$ such that:*

- *$F = K(T)$,*

- *$FP = KP(y_1 P, \ldots, y_\tau P)$, with $y_1 P, \ldots, y_\tau P$ algebraically independent over KP,*

- *$v_P F = v_P K \oplus \mathbb{Z} v_P x_1 \oplus \ldots \oplus \mathbb{Z} v_P x_\rho$.*

In particular, P is an Abhyankar place of $F|K$.

We fix a finite set $Z \subset \mathcal{O}_P$ and a regular local ring $R \subseteq K$ such that $\operatorname{Frac} R = K$ and $\mathcal{M}_P \cap R = M$, where M is the maximal ideal of R. The case of $R = K$ is included.

In the case of $R \neq K$, we cannot prove uniformization over the base ring R for an arbitrary pair (P, Z) of given data. Instead, we have to impose rather technical conditions on the pair (R, Z). These conditions involve the notion of a monoidal transform of R: assume for the moment that $\dim R > 0$ holds, and let v be a valuation of the field K such that $R \subseteq \mathcal{O}_v$ and $M = R \cap \mathcal{M}_v$. Let $p \in \operatorname{Spec} R$ be a prime of height $(p) \geq 1$. The monoidal transform of R along v with center p is the local ring

$$R_1 := R[x^{-1}p]_{\mathcal{M}_v \cap R[x^{-1}p]},$$

where $x \in p$ satisfies $vx = \min\{va \mid a \in p\}$. If $p = M$, then the monoidal transform is also called quadratic transform. It is well-known that R_1 does not depend on the choice of x and that it is a regular local ring of dimension $\dim R_1 \leq \dim R$ provided that p is generated by a part of a regular system of parameters of R. Moreover, $R_1 \subseteq \mathcal{O}_v$ and $M_1 = R_1 \cap \mathcal{M}_v$ for the maximal ideal M_1 of R_1. Every member of a finite chain $R =: R_0 \leq R_1 \leq \ldots \leq R_t$ of local rings, where R_{i+1} is a monoidal transform of R_i along v with center $p_i \in \operatorname{Spec} R_i$, is called an iterated monoidal transform of R along v.

The properties of (R, Z) we are interested in can now be formulated as follows:

(NC) *There exists an iterated monoidal transform R' of R along the valuation $v_P|_K$ such that the regular local ring R' admits a regular parameter system $(t_1, \ldots, t_d)$ with the*

following property: every $\zeta \in Z$ admits a representation

$$\zeta \;=\; \frac{\sum\limits_{i=1}^{N} r_i \, \underline{x}^{\underline{\mu}_i} \underline{y}^{\underline{\nu}_i}}{\sum\limits_{i=1}^{N} s_i \, \underline{x}^{\underline{\kappa}_i} \underline{y}^{\underline{\lambda}_i}} \;,\;\; r_i, s_i \in R', \tag{5}$$

with $\underline{x} := (x_1, \ldots, x_\rho)$, $\underline{y} := (y_1, \ldots, y_\tau)$, $\underline{\mu}_i$, $\underline{\kappa}_i \in \mathbb{N}_0^\rho$, $\underline{\nu}_i$, $\underline{\lambda}_i \in \mathbb{N}_0^\tau$, where the prime factorizations in R' of the coefficients r_i, s_i have the form

$$u \prod_{i=1}^{d} t_i^{\epsilon_i}, \;\; u \in (R')^{\times}, \; \epsilon_i \in \mathbb{N}_0 \,, \tag{6}$$

i.e., the coefficients r_i, s_i are R'-momomials in $\{t_1, \ldots, t_d\}$.

(V) *The values $v_P t_1, \ldots, v_P t_\delta$ of those parameters $t_1, \ldots, t_\delta$ actually occurring in at least one of the prime factorizations (6), are rationally independent. Without loss of generality, we assume here that these parameters are the first δ of the complete set.*

The main result of this subsection now reads as follows:

Theorem 4.1 *Let $(F|K, P)$ be a valued rational function field satisfying the requirement (T). Let $R \subseteq K$ be a regular local ring dominated by $\mathcal{O}_P$ and let $Z \subset \mathcal{O}_P$ be a finite set such that the pair (R, Z) fulfills the requirements (NC) and (V). Then there exist an iterated monoidal transform R' of R along $v_P|_K$ and elements $x_1', \ldots, x_{\rho+\delta}' \in \mathcal{O}_P$, $\delta \leq \dim R'$, such that the localization of the R'-algebra $A := R'[x_1', \ldots, x_{\rho+\delta}', y_1, \ldots, y_\tau]$ at the prime $q := \mathcal{M}_P \cap A$ is a regular ring having the properties: $\dim A_q = \dim R' + \rho$, the elements $x_1', \ldots, x_{\rho+\delta}'$ are a part of a regular parameter system of A_q, $Z \subset A_q$, and every element of Z is an A_q-monomial in $\{x_1', \ldots, x_{\rho+\delta}'\}$.*

In particular, there exists an R-model X of F such that P is centered in a regular point $x \in X$ with the properties $Z \subset \mathcal{O}_{X,x}$ and $\dim \mathcal{O}_{X,x} \leq \dim R + \rho$. Moreover, if $R = K$ then the model X can be chosen such that $X \cong \mathbb{A}_K^{\rho+\tau}$ and $\dim \mathcal{O}_{X,x} = \rho$.

The following lemma is applied in the proof of Theorem 4.1; it was proved (but not explicitly stated) by Zariski in [Z1] for subgroups of $\mathbb{R}$, using the algorithm of Perron. We leave it as an easy exercise to the reader to prove the general case by induction on the rank of the ordered abelian group. However, an instant proof of the lemma can also be found in [El] (Theorem 2.2).

Lemma 4.2 *Let Γ be a finitely generated ordered abelian group. Take any non-negative elements $\alpha_1, \ldots, \alpha_\ell \in \Gamma$. Then there exist positive elements $\gamma_1, \ldots, \gamma_\rho \in \Gamma$ such that $\Gamma = \mathbb{Z}\gamma_1 \oplus \ldots \oplus \mathbb{Z}\gamma_\rho$ and every α_i can be written as a sum $\sum_j n_{ij}\gamma_j$ with non-negative integers n_{ij}.*

We now turn to the

Proof of Theorem 4.1: Since iterated monoidal transforms of a local ring R are essentially of finite type over R, the existence of the scheme X is a consequence of the first part of the theorem.

To simplify notation, we replace R with an iterated monoidal transform R' of R along the valuation $v_P|_K$ having the properties (NC) and (V) for a specific regular parameter system $t_1, \ldots, t_d \in M$, M the maximal ideal of R'. Recall that R and R' are universally catenary and that $\dim R' \leq \dim R$.

The coefficients appearing in the representations (5) can now be replaced by their prime factorizations in R:

$$\zeta = \frac{\sum\limits_{i=1}^{N} u_i \, \underline{t}^{\underline{\epsilon}_i} \underline{x}^{\underline{\mu}_i} \underline{y}^{\underline{\nu}_i}}{\sum\limits_{i=1}^{N} v_i \, \underline{t}^{\underline{\delta}_i} \underline{x}^{\underline{\kappa}_i} \underline{y}^{\underline{\lambda}_i}} \, , \quad u_i, v_i \in R^{\times}, \tag{7}$$

where $\underline{t} = (t_1, \ldots, t_\delta)$ are those regular parameters among the $t_1, \ldots, t_d$ that actually occur in at least one of the prime factorizations (6) of the coefficients. Note that in our notation in (7) the dependence of the coefficients and exponents on ζ does not explicitly appear, in order to avoid overloading our notation. We use this simplification throughout the present proof.

Next, one divides numerator and denominator by a monomial in $\underline{t}$ and $\underline{x}$ with least v_P-value among the monomials appearing in the denominator. One can assume that this monomial is the first one in the denominator, thus obtaining the expression:

$$\zeta = \frac{\sum\limits_{i=1}^{N} u_i \, \underline{t}^{\underline{\epsilon}_i - \underline{\delta}_1} \underline{x}^{\underline{\mu}_i - \underline{\kappa}_1} \underline{y}^{\underline{\nu}_i}}{\sum\limits_{i=1}^{N} v_i \, \underline{t}^{\underline{\delta}_i - \underline{\delta}_1} \underline{x}^{\underline{\kappa}_i - \underline{\kappa}_1} \underline{y}^{\underline{\lambda}_i}} \, , \quad u_i, v_i \in R^{\times}. \tag{8}$$

By construction, the monomial expressions $\underline{t}^{\underline{\delta}_i - \underline{\delta}_1} \underline{x}^{\underline{\kappa}_i - \underline{\kappa}_1}$ all have non-negative v_P-value. Since $v_P(\zeta) \geq 0$ and since P is an Abhyankar place the same holds for the monomial expressions $\underline{t}^{\underline{\epsilon}_i - \underline{\delta}_1} \underline{x}^{\underline{\mu}_i - \underline{\kappa}_1}$ appearing in the numerator; see Theorem 2.1.

Among the monomials in $\underline{t}$ and $\underline{x}$ appearing in the numerator of (8) we choose the unique one having least v_P-value, $h_\zeta := \underline{t}^{\underline{\epsilon}_m - \underline{\delta}_1} \underline{x}^{\underline{\mu}_m - \underline{\kappa}_1}$ say, and define the finite sets

$$\begin{aligned} H_\zeta \, := \, & \{h_\zeta\} \; \cup \; \{\underline{t}^{\underline{\delta}_i - \underline{\delta}_1} \underline{x}^{\underline{\kappa}_i - \underline{\kappa}_1} \mid i = 1, \ldots, N\} \\ & \cup \; \{\underline{t}^{\underline{\epsilon}_i - \underline{\epsilon}_m} \underline{x}^{\underline{\mu}_i - \underline{\mu}_m} = \tfrac{\underline{t}^{\underline{\epsilon}_i - \underline{\delta}_1} \underline{x}^{\underline{\mu}_i - \underline{\kappa}_1}}{h_\zeta} \mid i = 1, \ldots, N\}. \end{aligned}$$

The finite set

$$H := \{t_1, \ldots, t_\delta, x_1, \ldots, x_\rho\} \cup \bigcup_{\zeta \in Z} H_\zeta$$

then consists of elements with non-negative v_P-value only.

Let $G \subset K(x_1, \ldots, x_\rho)^{\times}$ be the group (freely) generated by the elements $t_1, \ldots, t_\delta$ and $x_1, \ldots, x_\rho$. By assumption, the valuation v_P induces an isomorphism

$$v_P : \, G \; \to \; \bigoplus_{i=1}^{\delta} \mathbb{Z} v_P t_i \oplus \bigoplus_{i=1}^{\rho} \mathbb{Z} v_P x_i \subseteq v_P F.$$

10

Applying Lemma 4.2 yields a basis $(v_P x_1', \ldots, v_P x_{\rho+\delta}')$ of the group $v_P G$ consisting of positive elements such that every $v_P h \in v_P H \subset v_P G$ can be expressed as a linear combination with non-negative coefficients. It follows that every $h \in H$ can be expressed in the form

$$h = \prod_{i=1}^{\rho+\delta} {x_i'}^{\mu_i'}, \ \mu_i' \in \mathbb{N}_0 . \tag{9}$$

By definition of the set H_ζ the monomials in $\underline{t}$ and $\underline{x}$ appearing in the numerator of (8) have the form hh_ζ for some $h \in H_\zeta$. Therefore, each of these monomials has the form

$$\underline{t}^{\underline{\epsilon}_i - \underline{\delta}_1} \underline{x}^{\underline{\mu}_i - \underline{\kappa}_1} = (\prod_{i=1}^{\rho+\delta} {x_i'}^{\beta_i})(\prod_{i=1}^{\rho+\delta} {x_i'}^{\alpha_i}), \ \beta_i \in \mathbb{N}_0 , \tag{10}$$

where

$$h_\zeta = \prod_{i=1}^{\rho+\delta} {x_i'}^{\alpha_i}, \ \alpha_i \in \mathbb{N}_0 . \tag{11}$$

Substituting (9) and (10) in (8) yields

$$\zeta = \frac{\sum\limits_{i=1}^{N} u_i \, \underline{x'}^{\underline{\beta}_i} \, \underline{y}^{\underline{\nu}_i}}{\sum\limits_{i=1}^{N} v_i \, \underline{x'}^{\underline{\kappa}_i'} \, \underline{y}^{\underline{\lambda}_i}} \, (\prod_{i=1}^{\rho+\delta} {x_i'}^{\alpha_i}) , \ \ u_i, v_i \in R^\times, \tag{12}$$

with non-negative exponents $\underline{\beta}_i, \underline{\kappa}_i' \in \mathbb{N}_0^{\rho+r}$.

We set $A := R[x_1', \ldots, x_{\rho+\delta}', y_1, \ldots, y_\tau] \subseteq \mathcal{O}_P$ and claim: A_{q_A}, where $q_A := \mathcal{M}_P \cap A$, is a regular ring of dimension $\rho + \dim R$, $Z \subset A_{q_A}$ and all $\zeta \in Z$ are A_{q_A}-monomials in $\{x_1', \ldots, x_{\rho+\delta}'\}$.

Let $B := R[x_1', \ldots, x_{\rho+\delta}'] \subseteq A$ and consider the ideals

$$J := \sum_{i=1}^{\rho+\delta} B x_i' + \sum_{i=\delta+1}^{d} B t_i \subseteq q_B := \mathcal{M}_P \cap B.$$

Due to the equations (9) for the elements t_i, $i = 1, \ldots, \delta$, one gets $J \cap R = M$. Hence, $B/J = R/M$ and therefore, $q_B = J$ is a maximal ideal of B generated by $\rho + d$ elements, $d = \dim R$.

According to the altitude formula in finitely generated integral domains over universally catenary rings, one has:

$$\begin{aligned} \text{height } q_B &= \text{height } M + \text{trdeg}\,(B|R) - \text{trdeg}\,(B/q_B|R/M) \\ &= d + \rho, \end{aligned}$$

where we use that due to the equations (9) for the elements x_i the relation $\text{Frac}\,B = K(x_1, \ldots, x_\rho)$ holds. Consequently, B_{q_B} is a regular local ring of Krull dimension $d + \rho$.

Considering that $A_{q_A} = B_{q_B}[y_1, \ldots, y_\tau]_p$, where $p = \mathcal{M}_P \cap B_{q_B}[y_1, \ldots, y_\tau]$, and that $B_{q_B}[y_1, \ldots, y_\tau]$ is a polynomial ring over B_{q_B}, we get the desired regularity of A_{q_A} because a polynomial ring over a regular ring is regular.

It remains to calculate the dimension of A_{q_A}. By the choice of the transcendence basis T we have that $B_{q_B}[y_1, \ldots, y_\tau]/p = R/M[y_1 + p, \ldots, y_\tau + p]$ is a polynomial ring in the variables $y_i + p$, $i = 1, \ldots, \tau$. Thus $p = q_B[y_1, \ldots, y_\tau]$ and therefore, height $p =$ height q_B, where the latter equation holds in any polynomial ring over a noetherian ring. Consequently, A_{q_A} is a local ring of Krull dimension $\rho + d$.

If $R = K$ is a field, then the K-algebra B is a polynomial ring in the variables $x'_1, \ldots, x'_\rho$ and therefore, A_{q_A} is a localization of the polynomial ring $K[x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau]$.

Turning to the claim about Z we first observe that the denominator of the expression (12) has v_P-value 0, which implies the inclusion $Z \subset A_{q_A}$. The numerator of the first factor in (12) has v_P-value 0 too, because its mth summand has v_P-value 0. Thus this first factor is a unit in A_{q_A}, which verifies the claim. $\qquad\square$

We finish this section with a discussion of the requirements (NC) and (V) in various cases to give examples in which Theorem 4.1 applies.

(1) $\dim R \le 1$: In this case, R is either the field K itself or a discrete valuation ring. In the first case, the coefficients appearing in the representations (5) are units, hence (NC) and (V) are trivially satisfied. In the second case, every prime element $t \in R$ satisfies the requirements (NC) and (V), so $R' = R$.

(2) $\mathcal{O}_K$ is a discrete valuation ring: It is known that in this case, $\mathcal{O}_K = \cup_{k=0}^\infty R_k$, where R_k is the quadratic transform of R_{k-1} along $v_P|_K$ — see for example [HRW]. Given a finite set $Z' \subset \mathcal{O}_K$, one chooses prime factorizations $z = u_z t^{\epsilon_z}$, $u_z \in \mathcal{O}_K^\times$, t a prime element, for all $z \in Z'$. It is now easy to verify that these prime factorizations remain valid in some R_ℓ, thus showing that (NC) and (V) are satisfied for arbitrary finite sets $Z \subset \mathcal{O}_P$.

(3) $\dim R = 2$: One applies Abhyankar's results [A6]. If $\mathrm{trdeg}\,(KP|R/M) > 0$, then the valuation $v_P|_K$ is discrete and $\mathcal{O}_K$ is an iterated quadratic transform of R, thus we can take $R' = R$. In the case of $\mathrm{trdeg}\,(KP|R/M) = 0$, due to [A6], §4, Thm. 2, for every finite set $Z' \subset \mathcal{O}_K$ there exists a 2-dimensional monoidal transform R' with a regular parameter system (t_1, t_2) such that the elements $z \in Z'$ have prime factorizations of the type (6) with property (V), where $\delta \in \{1, 2\}$ depending on the rational rank of $v_P K$. In the case of $\dim \mathcal{O}_K = 2$ one has to assume in addition that R is a Nagata ring. It follows that Theorem 4.1 applies for 2-dimensional regular local Nagata rings R.

(4) $\dim R = 3$: In this case, the following two results are relevant in the current context:

If R is excellent and char $R = $ char R/M holds, then every finite set $Z' \subset \mathcal{O}_K$ is contained in an iterated monoidal transform R' of R along $v_P|_K$ such that the elements $z \in Z'$ have prime factorizations of the form (6) - [A3], (5.2.3). Thus (NC) is satisfied for every finite set $Z \subset \mathcal{O}_P$.

D. Fu proved ([F], Prop. 3.5) that (NC) and (V) are satisfied for every finite set $Z \subset \mathcal{O}_P$, provided that R is essentially of finite type over an algebraically closed field k and $KP = k$.

5 The general case

In this section we provide the proofs for Theorems 1.1 and 1.2. The ingredients are already given in Sections 3 and 4. We need one more fact — an ascend property for R-uniformizability — to bring them together.

Consider a finite extension $(F|E, P)$ of valued fields such that F is contained in the absolute inertia field of $(E, P|_E)$. It is then well-known that the extension $\mathcal{O}_F|\mathcal{O}_E$ is local-étale ([Ray], Ch. X, Thm. 1): $\mathcal{O}_F = A_q$ for an étale $\mathcal{O}_E$-algebra A and $q = A \cap \mathcal{M}_F$. According to [Ray], Ch. V, Thm. 1 we can assume that A is standard-étale, i.e.,

$$A = \mathcal{O}_E[x]_{g(x)}, \tag{13}$$

where $\mathcal{O}_E[x] = \mathcal{O}_E[X]/f\mathcal{O}_E[X]$ with a monic polynomial $f \in \mathcal{O}_E[X]$. Furthermore, $g \in \mathcal{O}_E[X]$ is chosen such that the image of the derivative f' under the natural morphism $\phi : \mathcal{O}_E[X] \to A$ is a unit.

Claim: In the definition of A we can assume f to be prime.

The Lemma of Gauß allows to factorize f as $\prod_{i=1}^{r} p_i^{\nu_i}$ with pairwise distinct, monic prime polynomials $p_i \in \mathcal{O}_E[X]$. Among them there is a unique p_j with $\phi(p_j) = p_j(x) = 0$. We consider the natural surjection $\psi : A \to (\mathcal{O}_E[X]/p_j\mathcal{O}_E[X])_{g(x)}$. The equation $\psi(\frac{h+f\mathcal{O}_E[X]}{g(x)^s}) = 0$ implies $h \in p_j\mathcal{O}_E[X]$. Since in A we have $p_j(x) = 0$, there exists $t \in \mathbb{N}$ such that $g^t p_j \in f\mathcal{O}_E[X]$, thus $g^t h \in f\mathcal{O}_E[X]$ and hence $\frac{h+f\mathcal{O}_E[X]}{g(x)^s} = 0$. Finally, for some $p^\star \in \mathcal{O}_E[X]$ we have $\phi(f') = \phi((p^\star p_j)') = \phi(p^\star p_j') \in A^\times$, which implies that $\phi(p_j') \in A^\times$. This proves that ψ is an isomorphism and hence the claim.

Next, we fix a set of structural constants determining A uniquely, which we shall use to define an étale algebra over a subring of $\mathcal{O}_E$. Let $h \in \mathcal{O}_E[X]$ be chosen such that $f'(x)^{-1} = \frac{h(x)}{g(x)^s}$, $s \in \mathbb{N}$, and let $C(f, g, h) \subset \mathcal{O}_E$ be the set of coefficients of the polynomials f, g and h.

Let $Z \subset \mathcal{O}_F$ be a finite set and split it as $Z = (Z \cap \mathcal{O}_F^\times) \cup (Z \cap \mathcal{M}_F)$. Since v_P is unramified in the extension $F|E$ we can write every $\zeta \in Z \cap \mathcal{M}_F$ in the form $\zeta = u_\zeta \zeta'$ with $u_\zeta \in \mathcal{O}_F^\times$ and $\zeta' \in \mathcal{M}_E$; let $Z' := \{\zeta' \mid \zeta \in Z \cap \mathcal{M}_F\}$.

For each $\xi \in Z^\times := (Z \cap \mathcal{O}_F^\times) \cup \{u_\zeta \mid \zeta \in Z \cap \mathcal{M}_F\}$ we choose a representation

$$\xi = \frac{a(x)}{b(x)} g(x)^k, \quad a, b \in \mathcal{O}_E[X], \ b(x) \notin q, \ k \in \mathbb{Z}. \tag{14}$$

Finally we define the finite set $C(Z^\times) \subset \mathcal{O}_E$ as the collection of the coefficients of the polynomials a, b appearing in the representations (14) for the elements ξ.

Lemma 5.1 *Let $(F|E, P)$ be a finite extension of valued fields such that F is contained in the absolute inertia field E^i of $(E, P|_E)$. Let $Z \subset \mathcal{O}_F$ be a finite set and let $R \subseteq \mathcal{O}_E$ be a regular local ring with maximal ideal $M = \mathcal{M}_E \cap R$.*

Let $A \subseteq \mathcal{O}_F$ be the étale $\mathcal{O}_E$-algebra (13) and assume that there exists a set of representations (14) for the elements $\xi \in Z^\times$, such that the pair $(P|_E, C(f, g, h) \cup C(Z^\times) \cup Z')$ is R-uniformizable. Then the pair (P, Z) is R-uniformizable, too.

Moreover, one can find R-models X of E and Y of F and a morphism $\pi : U \to V$, where U is an affine open neighborhood of the center y of P on Y, and V is an affine open neighborhood of the center x of $P|_E$ on X such that: x, y are regular points on the respective model, $\pi y = x$, and the extension $\mathcal{O}_{Y,y}|\mathcal{O}_{X,x}$ is local-étale.

In particular, $\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{X,x}$ holds. Furthermore, if $R = K$ is a field and $\mathcal{O}_{X,x}$ is smooth over K, then $\mathcal{O}_{Y,y}$ is smooth over K.

If for some regular system of parameters $(a_1, \ldots, a_d)$ of $\mathcal{O}_{X,x}$ every $\zeta' \in Z'$ is an $\mathcal{O}_{X,x}$-monomial in $\{a_1, \ldots, a_d\}$, then every $\zeta \in Z$ is an $\mathcal{O}_{Y,y}$-monomial in the regular system of parameters $(a_1, \ldots, a_d)$ of $\mathcal{O}_{Y,y}$.

Proof: By assumption there exists a finitely generated R-algebra $B \subseteq \mathcal{O}_E$ such that B_{q_B}, $q_B := \mathcal{M}_E \cap B$, is regular and contains the finite set $C(f, g, h) \cup C(Z^\times) \cup Z'$. Define $C := B[x]_{g(x)} \subseteq A$, where x is the element appearing in the definition (13) of A. We have $\operatorname{Frac} C = F$, and the B_{q_B}-algebra $C_{q_B} = B_{q_B}[x]_{g(x)}$ is standard-étale: this is a consequence of the construction of C_{q_B} once we have verified that $B_{q_B}[x] \cong B_{q_B}[X]/fB_{q_B}[X]$. So assume $h(x) = 0$ for some $h \in B_{q_B}[X]$; since f is the minimal polynomial of x over E we get $h = fh^\star$, $h^\star \in E[X]$. Now B_{q_B} is integrally closed in E and f is monic, thus the Lemma of Gauß yields $h^\star \in B_{q_B}[X]$.

Since regularity ascends in étale extensions, the domain C_{q_B} and hence also C_{q_C}, where $q_C := \mathcal{M}_F \cap C$, are regular. Moreover, we have $Z \subset C_{q_C}$ by construction. Étale extension preserves the Krull dimension and smoothness. Moreover, if $(a_1, \ldots, a_d)$ is a regular parameter system of B_{q_B}, then it is a regular parameter system for C_{q_C}, too. These facts yield the remaining assertions.   □

We are now prepared to prove our main results.

Proof of Theorems 1.1 and 1.2:

One starts by choosing a transcendence basis $T \subset F$ with the properties described in Theorem 3.4; in particular, the valued field (F, P) lies in the absolute inertia field of $(K(T), P|_{K(T)})$.

According to Lemma 5.1, the pair (P, Z) is R-uniformizable for a given finite set $Z \subset \mathcal{O}_P$ once the pair $(P|_{K(T)}, Z'')$ is R-uniformizable for a certain finite set $Z'' \subset \mathcal{O}_{K(T)}$ derived from Z and the extension $\mathcal{O}_P|\mathcal{O}_{K(T)}$ — see the discussion preceding Lemma 5.1. Moreover, the elements $\zeta \in Z$ possess the required factorization property once the elements of a certain subset $Z' \subseteq Z''$ possess this factorization property.

The valued rational function field $(K(T), P|_{K(T)})$ satisfies the requirements (T) of Section 4. Points (1) and (3) of the discussion at the end of Section 4 show that the pair (R, Z'') fulfills the requirements (NC) and (V) for *every* finite set $Z'' \subset \mathcal{O}_{K(T)}$. An application of Theorem 4.1 thus yields R-uniformizability of $(P|_{K(T)}, Z'')$ and the factorization property for the elements $\zeta \in Z''$ for an arbitrary finite set Z''.

In the case of $R = K$, the pair $(P|_{K(T)}, Z'')$ is K-uniformizable on the affine space $X = \mathbb{A}_K^{\rho+\tau}$, $\rho = \dim_{\mathbb{Q}}(v_P F \otimes \mathbb{Q})$, due to Theorem 4.1. Moreover, the center $x \in X$ of $P|_{K(T)}$ satisfies $\dim \mathcal{O}_{X,x} = \rho$. Lemma 5.1 thus gives the remaining assertions of Theorem 1.1.

14

In the case of $R \neq K$, it remains to verify the dimension statements of Theorem 1.2. They are direct consequences of the remarks (1) and (3) made at the end of Section 4 combined with Lemma 5.1. □

As an immediate corollary of Theorem 1.1 we get:

Corollary 5.2 *Let the situation be as in Theorem 1.1, except for the separability of $FP|K$. Then there exists a finite purely inseparable extension $L|K$ such that $(\widehat{P}, Z)$, where $\widehat{P}$ is the unique extension of P to the constant extension $F.L$ of F, is L-uniformizable. All other assertions of Theorem 1.1 remain valid over L.*

Instead of stating the mere existence of an algebraic or arithmetic variety X on which the given place P can be uniformized as in Theorems 1.1 and 1.2, one can rather explicitly describe the structure of an affine scheme X which does the job. The description follows directly from the proof of the two theorems. In the case of a K-trivial place P, one can view this description as a structure theorem for the valued function field $(F|K, P)$.

Theorem 5.3 *Let $F|K$ be a function field and P an Abhyankar place of $F|K$ such that (K, P) is defectless, $FP|KP$ is separable and the group $v_P F/v_P K$ is torsion-free. Further, let $R \subset K \cap \mathcal{O}_P$, $\operatorname{Frac} R = K$, be a noetherian, regular local ring with $\dim R \leq 2$ and maximal ideal $M = \mathcal{M}_P \cap R$. Assume that R is a Nagata ring if $\dim R = 2$. Let $Z \subset \mathcal{O}_P$ be a finite set.*

Then there exists a transcendence basis $T' = (x'_1, \ldots, x'_\rho, y_1, \ldots, y_\tau) \subset \mathcal{O}_P$ of $F|K$, an iterated monoidal transform R' of R along $v_P|_K$ and a finitely generated $R'[T']$-algebra $A \subset \mathcal{O}_P$ having the properties:

1. *$y_1 P, \ldots, y_\tau P$ form a separating transcendence basis of $FP|KP$.*

2. *$v_P x'_1, \ldots, v_P x'_\rho$ are rationally independent elements.*

3. *$\operatorname{Frac} A = F$ and $Z \subset A_q$, where $q := \mathcal{M}_P \cap A$.*

4. *A_q is a regular local ring of Krull dimension $\rho + \dim R'$.*

If P is trivial on K, then the elements appearing in 2. form a basis of $v_P F$ over $\mathbb{Z}$, and the extension $A_q|K[T']_{q \cap K[T']}$ is local-étale.

References

[A1]   Abhyankar, S.: *Uniformization in p-cyclic extensions of a two dimensional regular local domain of residue field characteristic p*, Festschrift zur Gedächtnisfeier für Karl Weierstraß 1815–1965, Wissenschaftliche Abhandlungen des Landes Nordrhein-Westfalen **33** (1966), 243–317 (Westdeutscher Verlag, Köln und Opladen)

[A2]   Abhyankar, S. S.: *An algorithm on polynomials in one indeterminate with coefficients in a two dimensional regular local domain*, Ann. Mat. Pura Appl. (4) **71** (1966), 25–59

[A3]   Abhyankar, S.: *Resolution of singularities of embedded algebraic surfaces*, Academic Press, New York (1966); 2nd enlarged edition: Springer, New York (1998)

[A4]   Abhyankar, S.: *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. Math. **63** (1956), 491–526

[A5]   Abhyankar, S.: *Resolution of singularities of arithmetical surfaces*, Arithmetical Algebraic Geometry, Harper and Row, New York (1965)

[A6]   Abhyankar, S.: *Uniformization of Jungian Local Domains*, Math. Annalen **159** (1965), 1–43

[B]   Bourbaki, N.: *Commutative algebra*, Hermann, Paris (1972)

[dJ]   de Jong, A. J.: *Smoothness, semi-stability and alterations*, Publications mathématiques I.H.E.S. **83** (1996), 51–93

[El]   Elliott, G. A.: *On totally ordered groups, and K_0*, in: Ring Theory Waterloo 1978, eds. D. Handelman and J. Lawrence, Lecture Notes Math. **734**, 1–49

[En]   Endler, O.: *Valuation theory*, Springer, Berlin (1972)

[F]   Fu, D.: *Local Weak Simultaneous Resolution for High Rational Ranks*, J. Algebra **194** (1996), 614–630

[HRW]   Heinzer, W. – Rotthaus, C. – Wiegand, S.: *Approximating Discrete Valuation Rings by Regular Local Rings*, Proc. Am. Math. Soc. **129** No. 1 (2001), 37–43

[K–K]   Knaf, H. – Kuhlmann, F.–V.: *Every place admits local uniformization in a finite extension of the function field*, in preparation

[K1]   Kuhlmann, F.–V.: *On local uniformization in arbitrary characteristic*, The Fields Institute Preprint Series, Toronto (1997)

[K2]   Kuhlmann, F.–V.: *Valuation theoretic and model theoretic aspects of local uniformization*, in: Resolution of Singularities - A Research Textbook in Tribute to Oscar Zariski. Herwig Hauser, Joseph Lipman, Frans Oort, Adolfo Quiros (eds.), Progress in Mathematics Vol. **181**, Birkhäuser Verlag Basel (2000), 381–456

[K3]   Kuhlmann, F.–V.: *Places of algebraic function fields in arbitrary characteristic*, to appear in: Advances in Mathematics

[K4]   Kuhlmann, F.–V.: *Elimination of Ramification in Valued Function Fields I: The Generalized Stability Theorem*, in preparation

[R]   Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal (1964)

[Ray]   Raynaud, M.: *Anneaux Locaux Henséliens*, Lecture Notes in Math. **169**, Berlin-Heidelberg-New York (1970)

[Z1]   Zariski, O.: *Local uniformization on algebraic varieties*, Ann. Math. **41** (1940), 852–896

[Z2]   Zariski, O.: *The reduction of singularities of an algebraic surface*, Ann. Math. **40** (1939), 639–689

[Z3]   Zariski, O.: *A simplified proof for resolution of singularities of an algebraic surface*, Ann. Math. **43** (1942), 583–593

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