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Henselian Function Fields and Tame Fields

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Preface

This is an extended version of my doctoral thesis "Henselian function fields" (Heidelberg 1989). I have added results that I obtained during the last year and which round up the results of my thesis. In particular, the structure theorems on henselian rationality of immediate henselian function fields over tame fields of mixed characteristic and over separably tame fields and their applications to the model theory of these fields were added. These are analogues to results about tame fields of equal characteristic as presented in my thesis, but they needed new ideas and improvements of the original concepts (e.g. the notion of the "relative approximation degree"). Furthermore, I added criteria for Ax–Kochen–Ershov–classes which are meant to establish a unified approach to the model theory of valued fields; it has turned out that some concepts introduced in my thesis and in particular the notion of a tame field serve such a unification well.

My work was originally initiated by the question whether the (valued) power series field $\mathbb{F}_p((t))$ has a decidable theory. Viewing the solution of this problem as very remote and difficult, my own interest was rather to expand on the existing model theory of valued fields and to analyse the valuation theoretical problems related to the open questions. Even this extended version of my thesis is a summary of approaches and results in different directions and may thus show a somewhat inhomogeneous appearance. I want to apologize for that by expressing my hope that these results are not an endpoint, but a basis for new investigations in the model theory and algebra of valued fields.

With regard to the problem mentioned above concerning the theory of $\mathbb{F}_p((t))$ and of other valued fields of characteristic $p > 0$ which are not covered by earlier model theoretical results, I want to call special attention to one positive and one negative result. The positive result states that tame fields have a good model theory, which may be expressed by saying that they satisfy an Ax–Kochen–Ershov–principle. In positive characteristic, these fields are just the algebraically maximal perfect fields. The tame fields constitute a bigger and more natural elementary class than the algebraically maximal Kaplansky fields, which in some sense represented the limit of known model theory for valued fields when I started my thesis. The negative result states that algebraically complete fields in general do not satisfy the Ax–Kochen–Ershov–principle and that the axiom system

– (K, v) is algebraically complete
– $\mathrm{char}(K) = p > 0$
– $v(K)$ is a $\mathbb{Z}$–group
– $\overline{K} = \mathbb{F}_p$

is not complete and is thus (unfortunately) not an axiomatization of the theory of $\mathbb{F}_p((t))$; on the other hand, this negative result also carries an idea how the axiom system may possibly be completed.

At several instances throughout my work it turned out that an advance in the model theory of valued fields is based on new valuation theoretical results. In the preparation of my thesis, these valuation theoretical results step by step gained more independent interest which culminated in an unexpected discovery of close connections between the work of M. Matignon, J. Ohm and other authors, dealing with nonarchimedean analysis and valued function fields, and my results which were inspired by model theoretical questions. The valuation theoretical part of my work may be characterized as follows: it is a study of the "defect" of valued field extensions, i.e. of finite extensions (for which the defect is classically defined), but also of certain valued function fields for which it may also be defined. Fields which do not admit finite extensions with nontrivial defect are called “defectless”, and the following is a main question studied in my work: when does an extension of a defectless field inherit this property? Extensions with nontrivial defect are closely related to immediate extensions of valued fields, for which the theory of pseudo Cauchy sequences as developped by Ostrowski and Kaplansky, is a good tool. But for the aims of my work, it has to be very much refined, and I take the occasion to introduce the more advanced notion of “approximation types”.

In this work, many concepts and proofs will not have reached their optimal form, and hurrying to finish this print, I will certainly have failed to notice a lot of typing errors and more serious mistakes. I want to apologize for this. Nobody is defectless.

Heidelberg, August 1990 — Franz–Viktor Kuhlmann

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Contents

1 Introduction.

1.1 Results on the algebra of valued fields.

In the theory of algebraic function fields (which we will simply call *function fields*), the valued function fields (where the valuation is not necessarily trivial on the ground field) play an important role; for instance, constant reduction of function fields is given by valuations with certain properties that we will discuss later. One can read off important information about a valued function field from its henselizations, hence henselizations of valued function fields appear in a natural way when valued function fields are studied; in particular, if the valuation is nontrivial on the ground field or the transcendence degree is greater than 1 since then the completion is in general not an adequate device for the study of such valued function fields. For example, the question may arise whether a given valued function field F has the property that every finite extension E of F is *defectless*, i.e. satisfies the basic equality

$$n = \sum e_i f_i \tag{1}$$

where $n = [E : F]$ is the degree of the extension and e_i, f_i are ramification index and inertia degree of the finitely many prolongations of the valuation of F to the extension field E. If a valued field has this property, then we will call it a *defectless field*. Note that every valued field of residual characteristic 0 is a defectless field; this follows from the Lemma of Ostrowski, cf. Lemma 2.4. Now if F^h is a henselization of F, then it is known that F is a defectless field if and only F^h is a defectless field. Though F^h has the disadvantage that it is not itself a function field (if the given valuation is nontrivial), it has the advantage that it satisfies Hensel's Lemma and that the valuation admits a unique prolongation to every algebraic extension. These facts can be very helpful for the study of the given valuation. Note that a henselization is an immediate extension; an extension $L|K$ of valued fields is called *immediate*, if $v(K) = v(L)$ and $\overline{K} = \overline{L}$. Here $v(K)$, $v(L)$ denote the value groups and $\overline{K}$, $\overline{L}$ denote the residue fields of K and L; we will also write K/v instead of $\overline{K}$ if several valuations are considered. Henselian defectless fields are the same as *algebraically complete* fields, i.e. fields that have a unique prolongation of the valuation to every finite extension field and which moreover satisfies the basic equality

$$n = e \cdot f \tag{2}$$

which is just the version of (1) for the case of a unique prolongation.

Henselizations of valued function fields will be called "henselian function fields" though they are in general not function fields, as we have mentioned already. Both, henselian function fields and defectless fields will play the central role in our investigations. Without any additional assumptions there is no hope for a simple structure theory of henselian function fields. But under certain assumptions which are common in the theory of valued function fields, we will prove that henselian function fields have a rather simple structure.

Let us discuss two special cases to illustrate the significance of defectless fields and the possible assumptions on the henselian function field $F|K$. The valuation on both F and K will be denoted by v. The first special case is given by the hypothesis that $F|K$ is a henselian function field of transcendence degree 1 and that $F|K$ is an immediate extension. Let us take an arbitrary element $x \in F$ which is transcendent over K. Choosing a henselization $K(x)^h$ of $K(x)$ inside the henselian field F, we find that $F|K(x)^h$ is a finite extension which is immediate since $F|K$ is immediate. Note that the henselian field $K(x)^h$ admits a unique prolongation of the valuation v to the algebraic extension field F. If in this situation we would know that $K(x)^h$ is a defectless field (e.g. if $\mathrm{char}(\overline{K(x)^h}) = 0$), then the finite immediate extension $F|K(x)^h$ would necessarily be trivial:

$$[F : K(x)^h] = e \cdot f = (v(F) : v(K(x)^h) \cdot [\overline{F} : \overline{K(x)^h}] = 1 \cdot 1 = 1\ ;$$

here we have used equation (2). Consequently,

$$F = K(x)^h\ ,$$

the henselization of a rational function field; such a henselian function field will be called *henselian rational function field*. The assumption "$\mathrm{char}(\overline{K(x)}) = 0$" may also be expressed as an assumption on the ground field K; indeed, it holds if

– $\mathrm{char}(\overline{K}) = 0$.

There are other possible sufficient assumptions on K:

– (K, v) is a (formally) $\wp$–adic field,

or, more generally,

– (K, v) is a finitely ramified field,

i.e. $v(K)$ admits a least positive element and there is a prime p such that $v(p \cdot 1_K)$ is a multiple of this element; in this case, $\mathrm{char}(K) = 0$ and $\mathrm{char}(\overline{K}) = p$. The structure of these fields is slightly more general than that of formally $\wp$–adic fields. The difference is that the residue fields may be arbitrary not necessarily finite fields of characteristic $p > 0$. It is well known that all finitely ramified fields are defectless. On the other hand, it can be read off from the definition that an immediate extension of a finitely ramified field is again finitely ramified, so if we assume that the ground field K is finitely ramified, the same will hold for the immediate extension $K(x)^h$ and again, we obtain that $K(x)^h$ is defectless and that $F = K(x)^h$. Note that the henselian field F contains a henselization K^h of K, so we may assume from the start that the ground field K is henselian.

Other possible assumptions on the ground field may fail in this situation, even if they are as common as the following one:

– K is algebraically closed.

Note that every algebraically closed valued field is trivially a defectless field. Another possible condition is known from the article "Maximal fields with valuations" of Kaplansky [KAP1]:

– (K, v) is a henselian defectless Kaplansky–field;

a Kaplansky–field is a valued field satisfying Kaplansky's hypothesis (A), cf. [KAP1], p. 312. This hypothesis (A) was later shown by Whaples [WHA] and, independently, by Delon [DEL1] to mean that

(Kf 1) the value group $v(K)$ is p–divisible ($p = \mathrm{char}(\overline{K}) > 0$),

(Kf 2) the residue field $\overline{K}$ is perfect (i.e. it allows no inseparable algebraic extensions),

(Kf 3) every finite separable extension of the residue field has degree not divisible by p.

In general, henselian Kaplansky–fields are not defectless fields. Even if the ground field K is a henselian defectless Kaplansky–field, an immediate henselian function field may not be defectless, though it is also a Kaplansky–field (since (Kf 1), (Kf 2) and (Kf 3) are conditions only on the value group and the residue field). Similarly, there exist immediate henselian function fields over algebraically closed ground fields of positive residual characteristic which are not defectless fields. In both cases it turns out that it is not a sufficient approach if we choose the transcendental element x without any care. We will show that it is indeed possible to choose x well; we will do this e.g. under the assumption that K is a perfect henselian defectless field of positive characteristic. This class of fields includes the algebraically closed valued fields of positive characteristic and the henselian defectless Kaplansky–fields of positive characteristic. Recall that "henselian defectless" is the same as "algebraically complete". For perfect fields of positive characteristic and for Kaplansky–fields of positive characteristic, "algebraically complete" is equivalent to *algebraically maximal* which means that the valued field has no immediate algebraic extensions. Our result for algebraically maximal perfect fields of positive characteristic is the following:

Let K be an algebraically maximal perfect field of positive characteristic and F an immediate henselian function field over K. If its transcendence degree over K is 1, then F is a henselian rational function field over K. In the general case of transcendence degree ≥ 1, there is a finite immediate extension F' of F such that F' is a henselian rational function field over K.

In section 10 we will show that this theorem does not remain true if the condition "perfect" is dropped without a substitute, cf. Corollary 10.9. Note that if we replace the condition on the characteristic by "$\mathrm{char}(\overline{K}) = 0$" then the theorem remains true: for every transcendence basis $\mathcal{T}$ of $F|K$ we have $F = K(\mathcal{T})^h$. This is proved as we described above in the case of transcendence degree 1. In contrast to this short proof, the proof of the above result requires the entire section 7.1, including material from other sections, in particular from section 6. In section 7.1 we will give normal forms for certain classes of immediate extensions. Since immediate algebraic extensions are purely wild, we will start from a description of the structure of minimal purely wild algebraic extensions of henselian fields of positive characteristic given by F. Pop, cf. Lemma 7.3. The theory of purely wild algebraic extensions was developed by M. Pank [PAN1], [PAN2] and is summarized in [KPR]. We will use it at several points as a basic tool for our investigations. In particular, it is used in section 6 where algebraically complete perfect fields of positive characteristic are characterized to be just the tame fields of positive characteristic (cf. Lemma 6.2). A field K is called *tame* if its algebraic closure is a tame extension in the sense of [KPR], and this holds if and only if its (absolute) ramification field K^r is algebraically closed. The class of tame fields comprises almost all assumptions on K that we have discussed so far:

- algebraically closed valued fields
- henselian fields of residual characteristic 0
- algebraically maximal Kaplansky fields
- algebraically maximal perfect fields of positive characteristic

are tame fields. The henselian formally $\wp$–adic fields and the henselian finitely ramified fields are missing in this list. But their valuation admits a finest coarsening under which they become a tame field having a residue field of characteristic 0 with a discrete henselian valuation of residual characteristic $p > 0$; this special structure guarantees that they are defectless fields.

Generalizing the proof given in section 7.1, we will prove that the above result holds for all tame fields:

(**Theorem 7.1**) *Let K be a tame field and F an immediate function field over K. If its transcendence degree over K is 1, then F^h is a henselian rational function field over K^h. In the general case of transcendence degree ≥ 1, given any immediate extension N of F which is a tame field (such an extension does always exist), there is a finite immediate extension F_1 of F within N such that F_1^h is a henselian rational function field over K.*

For this generalization, we have to treat the case of a tame ground field (K, v) of "mixed characteristic", i.e. $\mathrm{char}(K) = 0$ and $\mathrm{char}(\overline{K}) = p > 0$; this is done in section 7.3. Since in the mixed characteristic case, we do not know an analogue of the lemma of Pop, we start the proof assuming K to be algebraically closed, and then we involve a "pull down principle for henselian rationality through tame extensions" which we prove in section 7.2. This in turn uses a valuation theoretical property that characterizes the Galois groups of tame Galois extensions, cf. Theorem 6.24. In both characteristic cases, our proof requires an important property of tame fields:

(**Lemma 6.6**) *Let L be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $\overline{L}|\overline{K}$ is an algebraic extension, then K is also a tame field and moreover, $v(K)$ is pure in $v(L)$ and $\overline{K} = \overline{L}$.*

This lemma is used several times in the proof of Theorem 7.1 and in the applications of this theorem which we will describe in sections 8 and 9. It allows us to break up a given transcendental extension of tame fields into an immediate and an "anti–immediate" extension of tame fields (the meaning of the latter notion is explained below). Moreover, it shows the immediate extension to be a tower of extensions of transcendence degree 1 of tame fields. Note that the lemma does not remain true if one replaces "tame" by "algebraically complete" or drops the condition on the residue fields.

The above results may be further generalized to separably tame fields; these are henselian fields for which every separable–algebraic extension is tame. In characteristic 0, they coincide with the tame fields; but in positive characteristic, they form a bigger (elementary) class. They are connected to tame fields as follows:

(K, v) is a separably tame field if and only if its perfect closure $(\sqrt{K}, v)$ is a tame field. In this case, $(\sqrt{K}, v)$ lies in the completion of (K, v).

See Lemmata 6.13 and 6.14.

The second special case of henselian function fields is somewhat the complement of the first: it is the "anti–immediate" case. We have to make this precise. The assumption will be that the henselian function field $F|K$ *has no transcendence defect*, i.e.

$$\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K}) + \mathrm{rr}(v(F)/v(K))$$

where $\mathrm{rr}(v(F)/v(K))$ denotes the rational rank (i.e. the maximal number of rationally independent elements) of the abelian group $v(F)/v(K)$. The assumption is fulfilled if constant reduction of function fields is studied: in this case we have by definition that

$$\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K})$$

which implies $\mathrm{rr}(v(F)/v(K)) = 0$ (cf. Lemma 2.20). For tame fields, extensions without transcendence defect are indeed the complement to immediate extensions, in the following sense: with the help of Lemma 6.6 it can be shown that every extension $L|K$ of tame fields admits a subextension $L'|K$ such that L' is a tame field, $L'|K$ is an extension without transcendence defect, and $L|L'$ is an immediate extension; this is the content of Corollary 6.8. In this situation, every henselization of a finitely generated subextension L_0 of $L'|K$ is a henselian function field without transcendence defect; we will use this fact and Corollary 6.8 when studying the model theory of tame fields, as we will describe below.

For henselian function fields without transcendence defect, we will prove in section 3:

(**Theorem 3.1**) *Let $F|K$ be a henselian function field without transcendence defect. If K is a defectless field then F is a defectless field.*

With the help of this theorem it is easy to show that a henselian function field $F|K$ without transcendence defect over a defectless ground field K is not only itself a defectless field, but also that it has a simple structure: it is a finite defectless extension of a henselian rational function field (cf. Theorem 3.2). Depending on the value group and the residue field of F, the structure may even be much simpler. If for instance, $\overline{F} = \overline{K}$ or $\overline{F}|\overline{K}$ is a rational function field and if $v(K)$ is pure in $v(F)$, then F is a henselian rational function field (cf. Theorem 3.4). If $\overline{F}|\overline{K}$ is separable and $v(K)$ pure in $v(F)$ then F^h is "almost" a henselian rational function field: it is a finite inert extension of a henselian rational function field (cf. Theorem 3.4). These results together with Theorem 7.1 generalize a theorem that was proved by M. Matignon in an unpublished paper [MAT1] and announced, together with a sketch of the proof, in [MAT1a], cf. Théorème 3, p. 7, and in [MAT1b]. (Matignon proves that the completion of F is the completion of a suitable rational function field, under the following additional assumptions: $v(F)$ is of rank 1, i.e. archimedean, $\overline{F} = \overline{K}$, and K is algebraically closed and complete with respect to the valuation v.)

Theorem 3.1 is a generalization of a theorem of Grauert and Remmert [GR] which works with completions instead of henselizations and is restricted to the case of algebraically closed complete ground fields of rank 1 (the value group is archimedean). A generalization of the Grauert–Remmert Theorem was given by Gruson [GRU], an improved presentation of it can be found in the book [BGR] of Bosch, Güntzer and Remmert. The proof uses methods of nonarchimedean analysis. Further generalizations are due to M. Matignon and J. Ohm; see also [GMP]. In a recent paper [OHM3], Ohm arrived independently of our thesis at a version of Theorem 3.1 which is restricted to the case $\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K})$. The idea of Ohm's proof is to rule back the general case of valuations of arbitrary rank to the case of rank 1 and to give an auxiliary argument for the transition from completions to henselizations. In this way, he is able to deduce his theorem from [BGR], Proposition 3, p. 215 (more precisely, from the generalized version of this proposition which is proved but not stated in [BGR]).

In contrast to this approach, we will give in section 3 a new proof which replaces the analytic methods of [BGR] by valuation theoretical arguments. Such arguments may seem to be more adequate for a theorem that appears to be of valuation theoretical nature. Nevertheless, for the case of $\mathrm{trdeg}(\overline{F}|\overline{K}) > 0$ there appears an argument in our proof which may be rather unexpected. We have to use the existence of certain bases for the function field $\overline{F}|\overline{K}$ which we will call *Frobenius–closed* bases. These are bases where every p–th power of a basis element is also contained in the basis (here p denotes the characteristic of $\overline{K}$). The existence of such bases is shown at the end of section 3. It seems that the deployment of such bases is fit to replace arguments of algebraic geometry that were used in the "classical" approaches; it would certainly be an interesting task to determine the connection between both methods.

Our proof of Theorem 3.1 also uses reduction to valuations of rank 1; further reduction leads to Galois extensions of degree $p = \mathrm{char}(\overline{K}) > 0$ over a henselian "almost" rational function field F_0 over an algebraically closed ground field. Such a Galois extension is an Artin–Schreier–extension generated by an element ϑ with $\vartheta^p - \vartheta = a \in F_0$ if $\mathrm{char}(F) = p$, and a Kummer–extension (by a p–th root of an element $a \in F_0$) if $\mathrm{char}(F) = 0$. We deduce a normal form for a which allows us to read off that the extension is defectless. A normal form for a can also be read as a result on the structure of the group $F_0/\wp(F_0)$ resp. $F_0^\times/(F_0^\times)^p$. Such results can already be found in the work of Hasse, Whaples and others, in the article "Eliminating wild ramification" of H. Epp [EPP], and in Matignon's proof of Theorem 3.1. In the case of $\mathrm{char}(F) = 0$, we have taken over Matignon's approach to replace a former proof which worked with extensions of the form $\vartheta^p - \vartheta = a \in F_0$ but did not lead to a normal form result for a (since it was an indirect proof, replacing an extensions which was assumed to be immediate, by other immediate extensions).

A main application of generalizations of the Grauert–Remmert Theorem is that they enable us to define a *defect* for valued function fields, which has good properties if the function field has no transcendence defect. For an arbitrary algebraic extension $L|K$ of valued fields, the defect is defined to be

$$\mathrm{d}(L|K) = \mathrm{d}(L^h|K^h) = \frac{[L^h : K^h]}{(v(L) : v(K)) \cdot [\overline{L} : \overline{K}]}$$

if the extension $L^h|K^h$ is finite; here the henselization K^h is always chosen inside of L^h. The defect is a measure for the deviation from the property of being defectless. It is also possible to use the respective ramification fields in the place of the henselizations in order that the definition works in all cases where the extension $L^r|K^r$ of the ramification fields is finite; if in addition also $L^h|K^h$ is finite then both definitions coincide (cf. p. 21 and Lemma 2.11). Now the defect of a valued function field $F|K$ may be defined to be

$$\mathrm{d}(F|K) := \sup_{\mathcal{T}} \mathrm{d}(F|K(\mathcal{T})) ,$$

where the supremum runs over all transcendence bases $\mathcal{T}$ of $F|K$. For the case that $F|K$ has no transcendence defect, we will show in section 5 that this supremum is a finite number and moreover, that it is equal to $\mathrm{d}(F|K(\mathcal{T}))$ for every valuation transcendence basis $\mathcal{T}$ of $F|K$; this is the assertion of Theorem 5.4. A valuation transcendence basis is a transcendence basis with special valuation theoretical properties, comparable to the additional properties of a basis to be a valuation basis. Both, valuation bases and valuation transcendence bases are defined and their basic properties are outlined in section 2.

Again, Theorem 5.4 was independently obtained by Ohm [OHM3] in the case where $\mathrm{trdeg}(F|K) = \mathrm{trdeg}(\overline{F}|\overline{K})$. He calls this result "*Independence Theorem*" since it shows the independence of the defect from the chosen valuation transcendence basis. In section 5, we will carry through a closer investigation of the defect of function fields without transcendence defect, in particular, we will connect it to certain algebraic extensions of the ground field K and their defect, cf. Theorem 5.4 and Corollary 5.6.

Matignon [MAT2] and later Green, Matignon and Pop [GMP] use a different definition of the defect of valued function fields for the formulation and the proof of important genus reduction inequalities. These inequalities connect the genus of a function field $F|K$

of transcendence degree 1 (where K is the exact constant field) with the genera of the reduced function fields

$$F/v_i \,|\, K/v \;\; (1 \le i \le s)$$

where the valuations v_i are distinct prolongations of a given valuation v on K, all valuations being of rank 1 and such that all extensions $F/v_i \,|\, K/v$ are transcendental, hence finitely generated of transcendence degree 1. The defect used here coincides with the "henselian" defect that we have introduced above, if (K, v) is henselian (see [GMP] for details). Recent generalizations of Matignon's results to valuations of arbitrary rank (cf. [GMP]) have led to an increased interest in several possible notions of the defect for valuations of arbitrary rank (and the corresponding independence results).

Matignon and Ohm (cf. [MAT2],[OHM3]) have used a *completion defect* which measures the defect (for valuations of rank 1) using completions instead of henselizations; it can be analogously defined for valued function fields. This defect is always less or equal to the henselian defect. In [MAT2], Matignon has shown an independence result for the completion defect of valued function fields (for valuations of rank 1). Apart from the defect for general function fields without transcendence defect that we discussed already, we will study in section 5 a possible generalization of the completion defect. Since for valuations of arbitrary rank, the completion is in general not henselian, we will measure the defect over the completion of the henselization: we will define the completion defect to be

$$\mathrm{d_m}\,(L|K) := \mathrm{d}(L^{hc}|K^{hc}) \;,$$

where "$.^{hc}$" denotes the completion of a henselization. This defect coincides with Matignon's and Ohm's completion defect if the valuation is of rank 1.

We will characterize those extensions for which the completion defect is equal to the henselian defect, and compute the *defect quotient* which will be defined to be the quotient of the henselian defect and the completion defect:

$$\mathrm{d_c}\,(L|K) := \frac{\mathrm{d}(L|K)}{\mathrm{d_m}\,(L|K)} = \frac{[L:K]_{\mathrm{insep}}}{[L^c:K^c]_{\mathrm{insep}}}$$

where "$.^{c}$" denotes the completion (cf. Corollary 5.14). For the proof, a special characterization of elements of henselizations is required. We will discuss this when talking about approximation types.

The completion defect of valued function fields without transcendence defect may be defined similarly as it was done for the henselian defect. In subsection 5.2, we will prove the Independence Theorem for the completion defect and the defect quotient (Theorem 5.22). Furthermore, we will study the behaviour of henselian defect, completion defect and defect quotient under a given decomposition of the valuation (Corollary 5.8 and Corollary 5.33).

A valued field K is called *c–defectless* (resp. *q–defectless*) if every finite extension $L|K$ has trivial completion defect (resp. defect quotient). As an analogue to Theorem 3.1, we will prove for the defect quotient:

(**Theorem 5.20**) *Let F be a henselian function field without transcendence defect over K. Assume that K is a q–defectless field or that $v(K)$ is not cofinal in $v(F)$. In both cases, F is a q–defectless field.*

The properties of being "c–defectless" or "q–defectles" are weaker than "defectless". Another pair of weaker properties are *inseparably defectless* (= every finite purely inseparable extension is defectless) and *separably defectless* (= every finite separable extension is defectless). For the first property, a theorem which corresponds to Theorem 3.1 is the following:

(**Theorem 4.16**) *Let $F|K$ be a henselian function field without transcendence defect. If K is inseparably defectless then F is inseparably defectless.*
A special case of this theorem is actually one step in the proof of Theorem 3.1, but results from section 4 are needed to obtain it in full generality.

The corresponding theorem for the second property is:

(**Theorem 5.29**) *Let $F|K$ be a henselian function field without transcendence defect. If K is separably defectless and $v(K)$ is cofinal in $v(F)$ then F is separably defectless.*

For the proof of this theorem, we will first study the properties of separably defectless fields in section 4. This will enable us to characterize these fields as just being the c–defectless fields, and on the basis of this identification the theorem will be proved in section 5 (cf. Lemma 5.28 and Theorem 5.29). Since the notion "separably defectless" appears to be quite natural, this identification also shows that it is a reasonable approach to measure the defect over the completion of the henselization. This in turn means that completions are of interest even for valuations of arbitrary rank; indeed, they eliminate immediate extensions of a very special form (which in some sense are not as "wild" as the other immediate algebraic extensions of henselian fields).

Section 4 is devoted in particular to the characterization of henselian defectless fields. The result which is used for the proof of Theorem 5.29 is the following:

(**Theorem 4.19**) *Let K be a separable–algebraically complete (= separably defectless henselian) field of characteristic $p > 0$. If in addition $K^c|K$ is separable, then K is algebraically complete.*
A slightly different characterization is given by the following Corollary to Theorem 4.19 which we also involve in our study of separably tame fields:

(**Corollary 4.20**) *Let K be a henselian field of characteristic $p > 0$. If K is a separable–algebraically complete field, then K^c is an algebraically complete field, and vice versa.*

Furthermore, we will prove

(**Theorem 4.17**) *Let K be a separable–algebraically maximal field of characteristic $p > 0$. If in addition K is inseparably defectless, then K is algebraically complete.*
This theorem shows that a valued field is algebraically complete if and only if it is algebraically maximal and inseparably defectless (since the two latter properties are consequences of "algebraically complete").

The key to these theorems and to other results in section 4 is the closer investigation of immediate extensions of degree p (where $0 < p =$ the characteristic of the residue field of the given valued field). These extensions are crucial since every finite extension $L|K$ of a henselian field K, if lifted up through a tame extension $N|K$, will become an extension $L.N|N$ which is a tower of normal extensions of degree p where some of them are immediate if $L|K$ is not defectless (cf. Lemma 3.15). This reduction to normal extensions of degree p is also a main ingredient of the proof of Theorem 3.1 in section 3. If the characteristic of K is p, then normal extensions of degree p are either purely inseparable or Artin–Schreier–extensions. Both are studied in section 3 and 4. In the setting of section 3, we will deduce certain normal forms for Artin–Schreier–extensions in order to prove that they are defectless under the given assumptions. After gaining some familiarity with such Artin–Schreier–extensions, the significance of Frobenius closed bases is no longer as surprising as may appear at first sight.

The characterization of henselian defectless fields which is given by Theorem 4.17, is very helpful if one wants to construct henselian defectless fields, as it is done in section 10. The "classical" method of just taking maximally valued fields (which are always henselian defectless fields) is too coarse for certain purposes, and section 10 establishes an interesting example for this fact. Dealing now with nonperfect fields of characteristic $p > 0$, we will give a valuation theoretical axiom (suggested by L. van den Dries) for valued fields of p–degree 1 (cf. page 168):

$$\begin{array}{l}\forall x \exists y \exists x_0, ..., x_{p-1} : \\ \quad (x_0 = 0 \vee v(x_0) = 0) \;\wedge\; x = y^p - y + x_0 + t x_1^p + ... + t^{p-1} x_{p-1}^p \,. \end{array} \tag{3}$$

We will show that it is satisfied by all maximally valued fields of p–degree 1, hence in particular by the power series field $\mathbb{F}_p((t))$. On the other hand, we succeed in constructing an algebraically complete field (K, v) of p–degree 1 which does not satisfy this axiom: there is an element $x \in K$ which is not of the form

$$x = y^p - y + x_0 + t x_1^p + ... + t^{p-1} x_{p-1}^p \quad \text{with} \quad x_0 = 0 \vee v(x_0) = 0 \tag{4}$$

in K. But we will construct an immediate function field of transcendence degree 1 over (K, v) in which x is of the form (4). The model theoretical consequence of this will be discussed later. Here, we should state that it can be deduced that F cannot be a henselian rational function field over K, hence Theorem 7.1 does not remain true if "tame" (which is equivalent to "algebraically complete and perfect" if $\mathrm{char}(K) = p > 0$) is replaced just by "algebraically complete".

Finally, we want to mention valuation theoretical results obtained in section 9 by an application of one of the model theoretical results, namely the AKE–principle for tame fields (which we will describe later).

In the article [KP], Prestel and the author deduced results on places of function fields over ground fields of characteristic 0, using the AKE–principle for henselian fields of residual characteristic 0. By an application of the AKE-principle for tame fields, it is possible to generalize two results of [KP] to the case of perfect ground fields (of arbitrary characteristic). The generalization of the Main Theorem of [KP] reads as follows:

(**Theorem 9.1**) *Let $F|k$ be a function field in n variables with perfect ground field k. Let Q be a place of $F|k$ and $x_1, ..., x_m, x_{m+1}, ..., x_{m+s} \in F$. Then there exists a place P of $F|k$ with a finitely generated residue field over k such that*

$$\begin{array}{rcll} x_i Q &=& x_i P & \text{for} \quad 1 \leq i \leq m \,, \\ v_Q(x_i) &=& v_P(x_i) & \text{for} \quad m+1 \leq i \leq m+s \,. \end{array}$$

Moreover, if r_1 and d_1 are natural numbers satisfying

$$\dim(Q) \leq d_1 \leq n-1 \,, \qquad \mathrm{rr}(Q) \leq r_1 \leq n - d_1 \,,$$

then P may be chosen to satisfy in addition:

(1) $\dim(P) = d_1$ *and* FP *is a subfield of a purely transcendental extension of the perfect hull of* FQ*, finitely generated over* k*,*

(2) $\mathrm{rr}(P) = r_1$ *and* $v_P(F)$ *is a finitely generated subgroup of a discrete lexicographic extension of the p–divisible hull of* $v_Q(F)$*, where* $p = \mathrm{char}(k) > 0$ *or* $p = 1$ *if* $\mathrm{char}(k) = 0$*.*

For the other generalized result, see Theorem 9.2, p. 165. Note that the second important ingredient for the proof of these generalizations is Lemma 6.6 on relatively algebraically closed subfields of tame fields, which we have already introduced above.

1.2 Results on the model theory of valued fields.

The theory of henselian function fields that we have described so far, has interesting applications to the model theory of valued fields. We will consider *AKE–fields*, i.e. valued fields (K, v) that satisfy the following Ax–Kochen–Ershov–principle ("AKE–principle"):
if $(L, v)|(K, v)$ is an extension of valued field such that

$$v(K) \prec_\exists v(L) \quad \text{and} \quad \overline{K} \prec_\exists \overline{L}\ ,$$

i.e. $v(K)$ is existentially closed in $v(L)$ in the language of ordered groups resp. $\overline{K}$ is existentially closed in $\overline{L}$ in the language of fields, then

$$(K, v) \prec_\exists (L, v)$$

i.e. (K, v) is existentially closed in (L, v) in the language of valued fields.

For a submodel $\mathcal{N}$ of a model $\mathcal{M}$, the notion "$\mathcal{N} \prec_\exists \mathcal{M}$" means that every existential sentence with constants from $\mathcal{N}$, which holds in $\mathcal{M}$, will also hold in $\mathcal{N}$. An elementary class of valued fields, in which every member is an AKE–field, will be called *AKE–class*. This notion is basic, in so far as model completeness and completeness results may be derived from it. Indeed, if an elementary class $\mathcal{K}$ of valued fields is an AKE–class, then it is model complete, provided that the classes

$$\begin{aligned} v(\mathcal{K}) &= \{v(K) \mid (K, v) \in \mathcal{K}\} \\ \overline{\mathcal{K}} &= \{\overline{K} \mid (K, v) \in \mathcal{K}\} \end{aligned}$$

of value groups and residue fields are elementary model complete classes. Furthermore, in many cases completeness can be deduced from model completeness by means of prime models.

The following classes of valued fields were shown to be AKE–classes (in literature, this was usually implicitly proved by corresponding model completeness results):

a) The class of all algebraically closed valued fields (by A. Robinson [ROB]). This may be viewed as the beginning of the (explicit) model theory of valued fields. For applications of Robinson's result in number theory, see [ROQ2].

b) The class of all henselian fields of residual characteristic zero (by Ax and Kochen [AK1] and, independently, by Ershov [ER3]). An application of this result to diophantine problems concerning the p–adics was given by Ax and Kochen in [AK1]. A short proof, this time using the notation "existentially closed", and an application to places of function fields was given by Prestel and the author in [KP].

c) The class of all $\wp$-adically closed fields (by Ax and Kochen [AK2] for p–adically closed fields; cf. also Prestel and Roquette [PR]).

d) The class of all henselian valued fields of finite ramification (by Ershov [ER5]; cf. also Ziegler [ZIE]).

e) The class of all algebraically maximal Kaplansky–fields (by Ershov [ER4]; cf. also Ziegler [ZIE]).

f) The class of all separable–algebraically maximal Kaplansky–fields of characteristic $p > 0$; but here one has to add predicates for p–independence to the language of valued fields (by Delon, cf. [DEL1]); this means that the AKE–principle is restricted to separable extensions $L|K$. "separable–algebraically maximal" means that these fields have no separable–algebraic immediate extension. This result is interesting in so far as a weakening of the condition "algebraically maximal" may be compensated by additional predicates.

In addition to these results, we will prove in section 8:

(**Theorem 8.9**) *Every tame field is an AKE–field.*

(**Theorem 8.20**) *Let (K, v) be a separably tame field and $(L, v)|(K, v)$ a separable extension. If $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, then $(K, v) \prec_\exists (L, v)$.*

Direct proofs for all above classes to be AKE–classes may be carried out by means of embedding lemmata. It is deducible from general model theory that (K, v) is existentially closed in (L, v) if and only if (L, v) is embeddable over (K, v) into every $|L|^+$–saturated elementary extension $(K, v)^* = (K^*, v^*)$ of (K, v); by an "embedding of (K, v)" we will always mean a valuation–preserving embedding. This makes sense since we assume $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$ which by the same principle as we have sketched for valued fields, shows that $v(L)$ may be viewed as embedded into $v^*(K^*)$ over $v(K)$, and L/v as embedded over K/v into K^*/v^*; here "$/v$" and "$/v^*$" denote the respective residue fields. (Note that $v^*(L^*)$ is $|L|^+$–saturated and thus also $|v(L)|^+$–saturated, and an analogue holds for L^*/v^*). Furthermore, it is deducible from general model theory that (L, v) is embeddable in this way if already every finitely generated subextension (L_0, v) of $(L, v)|(K, v)$ is embeddable over (K, v) into $(K, v)^*$. Now a finitely generated extension $L_0|K$ is nothing else but a function field over K, so at this point, we encounter anew valued function fields.

On the other hand, it can be shown that a valued field will only satisfy the AKE–principle if it is henselian (more precisely, it must even be algebraically complete, cf. Lemma 10.2). In the above approach this means that (K, v) and also $(K, v)^*$ (being an elementary extension of (K, v)) will be henselian. In this situation, we will ask for an embedding of a henselization $(L_0, v)^h$ of (L_0, v) over (K, v) into $(K, v)^*$; such embedding would yield the desired embedding of (L_0, v) just by restriction. Though we are now dealing with the larger field $(L_0, v)^h$, we gain the possibility of using helpful structure theorems for the henselian function field $(F, v) = (L_0, v)^h$.

To illustrate this, we want first to sketch the case where the extension $(F, v)|(K, v)$ is immediate: let us assume $v(K) = v(F)$ and $\overline{K} = \overline{F}$, which trivially yields that the conditions "$v(K) \prec_\exists v(F)$" and "$\overline{K} \prec_\exists \overline{F}$" are fulfilled. In addition, let us assume that the transcendence degree of $F|K$ is 1. In order to solve the embedding problem, a suitable property has to be imposed on (K, v) (as we have mentioned already, (K, v) has to be algebraically complete if the AKE–principle should be satisfied in the simple language of valued fields).

Let us now assume that (K, v) is a tame field. By Theorem 7.1 that we have cited above, we know that (F, v) is a henselian rational function field $(K(x), v)^h$ for a suitable

$x \in F$ which is transcendent over K. Using that $(K,v)^*$ is highly enough saturated by assumption, it is possible to embed $(K(x),v)$ over (K,v) into $(K,v)^*$, cf. Embedding Lemma III, p. 149. Such embedding can be prolongated to an embedding of $K(x)^h$; this follows from the fact that $(K,v)^*$ is henselian and the universal property of henselizations (they are "minimal" henselian extensions, i.e. they admit a valuation preserving embedding into every other henselian extension field). The henselization may be viewed as a certain closure which is unique up to valuation preserving isomorphism. It is interesting to see that apart from the above approach which uses the henselian rationality, the embedding problem for the mentioned classes a) – e) may as well be solved by using the uniqueness (up to valuation–preserving isomorphism) of suitable closures:

– the algebraic closure in the case of algebraically closed valued fields,
– the henselization for the classes as described in b), c), d),
– the henselization and its maximal algebraic purely wild extensions in the case of Kaplansky–fields.

(A similar situation is found for real fields: the uniqueness of the real closure may serve to prove an embedding lemma from which model completeness can be deduced.)

The solution of the special embedding problem yields that tame fields are existentially closed in every immediate extension of transcendence degree 1 (cf. Corollary 8.7). On the other hand, we are able to show that this does not remain true if "tame" is replaced by "algebraically complete". This is done in section 10 by using the example that we have introduced above. Recall that this example consists of an algebraically complete field (K,v) of p–degree 1, admitting an element $x \in K$ which is not of the form (4), together with an immediate function field (F,v) of transcendence degree 1 over K in which x is of the form (4). The fact that x is of the form (4) is an existential sentence with just two constants, namely x and t, both of them being elements of K. Since this sentence holds in F but not in K, we see that (K,v) is not existentially closed in the immediate function field (F,v). Hence we conclude that

in general, algebraically complete fields do not satisfy the AKE–principle.

Moreover, since every maximally valued field (of p–degree 1) satisfies the axiom (3) as we mentioned above, (K,v) is an example of an algebraically complete field which does not have the property of admitting a maximal immediate extension in which it is existentially closed. On the other hand, the latter property can be shown to imply "algebraically complete" (cf. Lemma 10.2), so it is stronger than "algebraically complete". It is an open question whether this property allows an algebraic axiomatization.

For the proof of the AKE–principle for tame fields, it is necessary to deal also with the case of henselian function fields without transcendence defect. From Theorem 3.1 we will deduce in section 8 the AKE–principle for extensions without transcendence defect of algebraically complete fields:

(**Theorem 8.4**) *If (L,v) is an extension without transcendence defect of the algebraically complete field (K,v) then the "side conditions"*

$$v(K) \prec_\exists v(L) \quad \textit{and} \quad \overline{K} \prec_\exists \overline{L}$$

imply

$$(K,v) \prec_\exists (L,v)\ .$$

Note that this theorem is not only of interest for the proof of the AKE–principle for tame fields: contrary to the case of immediate extensions, it works already for all algebraically complete ground fields. There is no elementary axiomatization of the class of extensions without transcendence defect of algebraically complete fields, hence at this point it turns out to be adequate to investigate a single extension rather than all extensions in an elementary class. Moreover, the notion "$\prec_\exists$" appears to be more adequate than "$\prec$", since Theorem 8.4 becomes false if we replace $\prec_\exists$ by $\prec$. As an example for this fact, we can use again the algebraically complete field (K,v) which does not satisfy the axiom (3). Indeed, we will construct (K,v) to be an extension without transcendence defect of the henselian field $(\mathbb{F}_p(t),v)^h$ (valued such that $v(t)$ is the smallest positive element in the value group), and this extension even satisfies

$$v(\mathbb{F}_p(t)^h) = \mathbb{Z} \quad\prec\quad v(K)$$
$$\overline{\mathbb{F}_p(t)^h} = \mathbb{F}_p \quad\prec\quad \overline{K}\;;$$

on the other hand, we will show that $(\mathbb{F}_p(t),v)^h$ satisfies (3); this will be done in Lemma 10.3. Since (K,v) does not satisfy (3), we have $(\mathbb{F}_p(t),v)^h \not\prec (K,v)$.

Let us return again to the class of tame fields. This class appears at the first glimpse to be only a slight generalization of the class of all algebraically maximal Kaplansky–fields, since only condition (Kf 3) is dropped. Nevertheless, this turns out to be a qualitative difference in two respects. Firstly, it allows a fruitful application to places of function fields over perfect fields of positive characteristic, whereas to our knowledge, the AKE–principle for algebraically maximal Kaplansky–fields has found no applications to algebra (except for the subclass of algebraically closed valued fields which is already covered by Robinson's model completeness result). Though the very special form of Kaplansky–fields has evoked a nice structure theory (Kaplansky [KAP1]) and a nice Galois theoretical interpretation (cf. [KPR]), it seems that for applications, these valued fields are of minor interest.

Secondly, with tame fields of positive characteristic we encounter the first AKE–class of fields whose maximal immediate extensions are in general not unique up to isomorphism. We may thus conclude that this property is not necessary for fields to be AKE–fields, and it is adequate to replace its use by a good structure theory for henselian function fields.

Thirdly, our results on tame fields enable us to transfer our approach for tame fields to the classes a) – e) mentioned above. This gives a unified proof for the fact that all these classes are AKE–classes, see section 8.4. The proof is based on criteria for AKE–classes that reflect main properties of tame fields which we have used in our proof of Theorem 8.9.

Finally, let us point out that model theoretical questions about valued fields have led to the discovery of the interesting new axiom (3) for valued fields, and a whole scheme of axioms related to it, which remain to be studied carefully. Since these axioms show a certain relation to the theory of purely wild extensions of henselian fields, there is some hope that they are fit to replace the insufficient axiom "algebraically complete" in the case of nonperfect fields. But also from the valuation theoretical viewpoint, the investigation of fields satisfying these axioms should be interesting and fruitful.

1.3 A valuation theoretical tool of independent interest: approximation types.

We will use the new notion of approximation types at several instances. We use them instead of pseudo Cauchy sequences which have been developed by Ostrowski [OS] and Kaplansky [KAP1]. The concept of approximation types is more intrinsic than the one of pseudo Cauchy sequences, and they eliminate the arbitrariness that lies in the choice of one single pseudo Cauchy sequence. Moreover, they facilitate computations as well as the comparison of two different elements in a valued field extension concerning their approximation by elements from the ground field.

One basic advantage of approximation types in comparison to pseudo Cauchy sequences is that they allow the treatment of elements in valued field extensions with a uniform machinery, whether the extension is immediate or not. For instance, an element in a valued field extension $L|K$ may be the limit of a pseudo Cauchy sequence over K, whereas over an intermediate field K' of $L|K$ it may not. In this situation, pseudo Cauchy sequences are not adequate for a proper analysis. Furthermore, only pseudo Cauchy sequences which cannot be prolongated, carry the full information. For the formulation of certain properties, one would thus be forced to restrict the scope to those pseudo Cauchy sequences which are maximal, i.e. which cannot be prolongated. In contrast to pseudo Cauchy sequences, the definition of approximation types guarantees in itself the maximality, the full information.

In section 2, we will give a purely valuation theoretical definition of approximation types. They may be viewed as filters which do not forget the values, or as a collection of cosets. In section 11, we will give a full presentation of the theory of approximation types as far as we were able to develop it. Since we are trying by this to replace a classical notion, we did not respect the frame that is set by applications in other chapters; in other words, we have developed the major part of the theory for its independent interest. It covers the theory of pseudo Cauchy sequences and exceeds it at several points.

Starting from the abstract valuation theoretical definition of approximation types, we will show in section 11 how approximation types can be determined by sets of simple formulas, which may be considered as subsets of 1–types over the valued ground field. We will also show the relation to the reduct of these types to the language of valued vector spaces over the given ground field. This idea is recovered later when a transfer of Kaplansky's theory of pseudo Cauchy sequences leads to the determination of valued vector spaces whose valuation is fixed by the approximation type. The interpretation of approximation types as subsets of 1–types also enables us to show that every approximation type over a valued ground field K is realized by an element x in a valued field extension, so an approximation type is no longer as abstract as appeared by its definition. In certain cases (e.g. if the approximation type is immediate, cf. section 2), we may even assume that (K, v) is existentially closed in $(K(x), v)$.

The immediate approximation types are the analogue to pseudo Cauchy sequences. For them we will define a degree and the distinction whether they are algebraic or transcendent. Parallel to the theory of Kaplansky, we will show that algebraic approximation types are realized in certain immediate algebraic extensions, and that transcendent approximation types are realized in certain immediate transcendent extensions. As to these results, they are merely a transposition of Kaplansky's work, just a bit more detailed. Then *associated minimal polynomials* may be associated to algebraic approximation types; these are the potential minimal polynomials for those immediate algebraic extensions in which the approximation type can be realized. In his Lemma 10 in [KAP1] which Kaplansky states without using it further, the idea is expressed that for such associated minimal polynomials, interesting normal forms can be found. However, Kaplansky's hypothesis in Lemma 10 is quite restrictive. Developing Kaplansky's approach further, we will give normal forms in full generality.

One reason for the deployment of approximation types is to distinguish properties of elements in valued field extensions over a common ground field by their approximation types over that ground field. This principle will be used in section 4 for the classification of immediate extensions of degree p of a henselian ground field. A second application is the proof that approximation types of elements in henselizations over a valued field have a special structure. This leads to the following lemma which in turn is used for the study of the completion defect:

(**Lemma 11.92**) *Let* $x, y \in \tilde{K}$, $y \notin K$. *Assume* $\mathrm{appr}(y, K) = \mathrm{appr}(x, K)$ *and* $x \in K^h$. *Then*

$$[K^h(y) : K^h] < [K(y) : K] \;.$$

In particular, $K(y)|K$ *is not purely inseparable.*

A particularly interesting application of the theory of immediate approximation types is the following. In our study of immediate henselian function fields it is a question of central importance whether a given immediate henselian function field of transcendence degree 1 is henselian rational and which are its henselian generators. This amounts to the determination of the degree

$$[K(x)^h : K(y)^h]$$

for elements $y \in K(x)^h \setminus K$ which are not necessarily polynomials in x. Under certain hypotheses, we can determine this degree by relating it to the *relative approximation degree* $\mathbf{h}_K(x : y)$ which we define in section 11.3. Assume that (K, v) is algebraically maximal or just that the approximation type $\mathrm{appr}(x, K)$ is transcendental. If y is a polynomial $f(x)$, then by a combination of Kaplansky's results with Hensel's Lemma, we get

$$[K(x)^h : K(f(x))^h] \leq \mathbf{h}_K(x : f(x))$$

(cf. Lemma 11.58). If moreover we assume that the rank of (K, v) is 1 and that x does not lie in the completion of (K, v), then for every $y \in K(x)^h$, we can prove

$$[K(x)^h : K(y)^h] \leq \mathbf{h}_K(x : y) \;,$$

cf. Lemma 11.60, and

$$K(x)^h = K(y)^h \iff \mathbf{h}_K(x : y) = 1 \;,$$

cf. Corollary 11.64. These results and the further theory of relative approximation degrees constitute the main ingredient in the proof of the pull down principle for henselian rationality through tame extensions (see section 7.2).

Through all of these examples, it appears that the tool of approximation types is suitable to describe properties and situations which had been remote before. We hope that the approaches and tools as well as the new results that we introduce in this thesis, will help to get access to some of the unsolved problems in valuation theory and the model theory of valued fields.

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2 Preliminaries.

Throughout our investigations we will work in a large enough fixed algebraically closed topologically complete valued field (Ω, v) that contains all other appearing valued fields. In doing so, every appearing field (K, v) has a unique henselization (K^h, v) which is the relative henselization inside the henselian field (Ω, v). In this sense we will speak of *the* henselization of a field K. In an analogous way we will speak of *the* completion K^c of K, assuming for the moment that this completion is the unique one in Ω, i.e. that $v(F)$ is cofinal in $v(\Omega)$. This guarantees that for every extension $L|K$ of valued fields we will have $K^h \subset L^h$ and moreover that for every extension $L|K$ with $v(K)$ being cofinal in $v(L)$ we will have $K^c \subset L^c$. For general valuation theory and in particular for the definition and the properties of *henselization* and *completion*, see [RIB1].

Here are some basic definitions of objects that we will deal with:
An extension field F of a valued field K will be called a *henselian function field* if it is the henselization of an ordinary valued function field F_0 over K. If F is the henselization of a rational function field F_0 over K then F will be called a *henselian rational function field* over K. Furthermore, F will be called a *subhenselian function field* over K if its henselization is a henselian function field over K, i.e. if F is an intermediate field between a suitable function field and its henselization. Finally, an algebraic extension $L|K$ will be called *h–finite* if the extension $L^h|K^h$ is finite. For h–finite extensions and subhenselian function fields we note the following easy observations:

Lemma 2.1 *Let E and F be subhenselian function fields over K. Then for every transcendence basis $\mathcal{T}$ of $F|K$, the extension $F|K(\mathcal{T})$ is h–finite. If E is algebraic over F, then $E|F$ is an h–finite extension. Any h–finite purely inseparable extension is finite.*

For a finite extension $L|K$ over a henselian field K we will denote by $\mathrm{d}(L|K)$ the ordinary defect of the extension $L|K$, i.e.

$$\mathrm{d}(L|K) \;=\; \frac{[L:K]}{\left[\overline{L}:\overline{K}\right]\cdot(v(L):v(K))}\;.$$

Here, the prolongation of the valuation v from K to L is uniquely determined since K is assumed to be henselian. But also in general the prolongation is fixed by our convention to work in a comprising valued field Ω, thus it is convenient to take the following definition for the defect of an extension of not necessarily henselian fields: for an h–finite extension $L|K$ over an arbitrary, not necessarily henselian field K we define the *defect* of $L|K$ to be

$$\mathrm{d}(L|K) = \mathrm{d}(L^h|K^h)\;.$$

In the literature, it is sometimes called the *henselian defect*. An h–finite extension $L|K$ is called *defectless* if $\mathrm{d}(L|K) = 1$. Note that in the case of a finite normal extension $L|K$, this extension is defectless in the sense defined here if and only if it is defectless in the classical valuation–theoretical sense, i.e. if

$$[L:K] = e\cdot f\cdot g$$

where g is the number of different prolongations of the valuation v from K to L, and e resp. f are the ramification index resp. residue degree common to all these prolongations. The valued field K is called a *defectless field* if every finite extension of K is defectless. Similarly, it is called *separably defectless* if every finite separable extension of K is defectless, and *inseparably defectless* if every finite purely inseparable extension of K is defectless.

For the most general definition of the defect see (9) on page 21 below. This definition will work for extensions $L|K$ for which the extension $L^r|K^r$ of their respective ramification fields is finite. It will be employed when we study the behaviour of the defect under decompositions of the valuation.

Finally, we recall the following definitions which we will frequently use. An extension $L|K$ of valued fields is called *immediate* if $v(L) = v(K)$ and $\overline{L} = \overline{K}$. (Note that in particular, the henselization of a valued field is an immediate extension.) We will need the following general lemma:

Lemma 2.2 *Let $(L,P)|(K,P)$ be a valued field extension and $P = Q\overline{Q}$ a decomposition of the place P. Then $(L,P)|(K,P)$ is immediate if and only if $v_Q(L) = v_Q(K)$ holds and $(LQ,\overline{Q})|(KQ,\overline{Q})$ is immediate.*

The proof is left to the reader.

A valued field K is called *algebraically maximal* if it admits no nontrivial immediate algebraic extension, and *maximal* or *maximally valued* if it admits no nontrivial immediate extensions at all. K is called *algebraically complete* if it is henselian and defectless, i.e. if every finite extension $L|K$ admits a unique prolongation of the valuation and satisfies $[L:K] = e \cdot f$.

2.1 Some generalities about the defect and defectless fields.

The following lemma is an easy consequence of our definition of the defect:

Lemma 2.3 *A valued field K is a defectless field if and only if its henselization is a defectless field. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

Furthermore we note the important

Lemma 2.4 (Lemma of Ostrowski)
The defect $\mathrm{d}(L|K)$ *is always a power of* p *where* $p = \mathrm{char}(\overline{K}) > 0$ *resp.* $p = 1$ *if* $\mathrm{char}(\overline{K}) = 0$. *In particular, every valued field of residue characteristic 0 is a defectless field.*

For the proof cf. [RIB1], p. 236, Théorème 2. In the literature it is even more usual to call e the defect of $L|K$ if $p^e = \mathrm{d}(L|K)$ (which turns the multiplicativity of the defect into additivity), but we suggest our use of the defect because it makes formulas easier and allows a generalization to valued vector spaces which will be developped in a subsequent paper.

Let us also recall

Lemma 2.5 *Every algebraically complete field is algebraically maximal since the defect of a finite immediate extension equals the degree of the extension. Every algebraically maximal field is henselian since the henselization is an immediate algebraic extension. For valued fields of residue characteristic 0, the properties "henselian", "algebraically maximal" and "algebraically complete" are equivalent by the Lemma of Ostrowski.*

A subset S of a valued extension field L of K will be called *linearly valuation–independent* over K if

$$v(\sum_{s\in S} c_s s) = \min_{s\in S} v(c_s s)$$

for every choice of coefficients $c_s \in K$, only finitely many of them being nonzero. In particular, a set S is linearly independent over K if it is linearly valuation–independent over K. A basis $\mathcal{B}$ of the valued field extension $L|K$ will be called *valuation basis* of $F|K$ if it is linearly valuation–independent over K. We note the following two well known lemmata:

Lemma 2.6 *Let $L|K$ be a valued field extension and*

$$\{z_1, ..., z_m, u_1, ..., u_n\} \subset L$$

such that the values $v(z_1), ..., v(z_m) \in v(L)$ belong to different cosets modulo $v(K)$ and that $u_1, ..., u_n$ are elements of $\mathcal{O}_L$ having residues $\overline{u_1}, ..., \overline{u_n} \in \overline{L}$ which are linearly independent over $\overline{K}$. Then $S = \{z_i u_j \mid 1 \le i \le m\,,\ 1 \le j \le n\}$ is linearly valuation–independent over K.

Lemma 2.7 *Assume that the finite valued field extension $L|K$ admits a unique prolongation of the valuation from K to L. Then $L|K$ is defectless if and only if it admits a valuation basis. If $L|K$ admits a valuation basis, then it also admits a valuation basis containing 1.*

The defect is multiplicative in the following sense:
Let $L|K$ and $M|L$ be h–finite extensions. The defect satisfies the following product formula

$$\mathrm{d}(M:K) = \mathrm{d}(M:L)\cdot \mathrm{d}(L:K) \tag{5}$$

which is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. This formula implies:

Lemma 2.8 *$M|K$ is defectless if and only if $M|L$ and $L|K$ are defectless.*

Corollary 2.9 *If K is a defectless field and if L is a finite extension of K then L is also defectless. Conversely, if there exists a finite extension L of K such that L is a defectless field and the extension $L|K$ is defectless, then K is a defectless field. The same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".*

For some proofs in the sequel we assume the reader to be familiar with tame and purely wild extensions of valued fields as described in [KPR]. Their definitions and some basic facts will be recalled at the beginning of section 6. We will now consider the behaviour of the defect when a finite extension $L|K$ of a henselian field K is shifted up through a tame extension $N|K$. We will frequently use the (absolute) *ramification field* K^r of the henselian field K, which we take to be the ramification field of the extension $K^{sep}|K$, where K^{sep} denotes the separable–algebraic closure of K. By Proposition 4.1 of [KPR] is the unique maximal tame algebraic extension of K; it is a normal separable extension of K. By Lemma 4.2 of [KPR], an algebraic extension $L|K$ is purely wild if and only if it is linearly disjoint from $K^r|K$. If K is not henselian, then K^r is defined to be the ramification field of K^h; since K^h is uniquely determined by working in a fixed valued universe, so is K^r. Moreover, we need the following information on K^r:

Lemma 2.10 *Let $L|K$ be algebraic. Then $L^r = L.K^r$; consequently, if $L \subset K^r$, then $L^r = K^r$. The separable–algebraic closure K^{sep} is a p–extension of K for $p = \max\{1, \mathrm{char}(\overline{K})\}$. Furthermore, $L \cap K^r|K$ is the maximal tame subextension of $L|K$, and $L|L \cap K^r$ is purely wild.*

Proof: For separable extensions, the first assertion follows from [END2], page 166, (20.15) b) (where we put $N = K^{sep}$ since we are considering the ramification field of the separable extension $K^{sep}|K$). Since a general algebraic extension can be viewed as a purely inseparable extension of a separable extension, it remains to show the first extension for a purely inseparable extension $L|K$. Here, it follows from the fact that $\mathrm{Gal}(K) \cong \mathrm{Gal}(L)$ and that by this isomorphism, the galois group of an intermediate field K' of $K^{sep}|K$ is isomorphic to the galois group of the intermediate field $L.K'$ of $L^{sep}|L$. The second assertion follows from [END2], p. 167, Theorem (20.18).

For the third assertion, the fact that K^r is the unique maximal tame extension of K shows that $L \cap K^r|K$ is the maximal tame extension contained in $L|K$. It remains to show that $L|L \cap K^r$ is purely wild. By what we have shown, $(L \cap K^r)^r = K^r$. Hence we have to show that $L|L \cap K^r$ is linearly disjoint from $K^r|L \cap K^r$. In view of the fact that K^r is a normal extension of K and thus also of $L \cap K^r$, the latter is equivalent to $L \cap K^r = L \cap K^r$ which is a triviality. □

The next lemma will be needed at several instances in our work. It shows the invariance of the defect under a lifting up through tame extensions.

Lemma 2.11 *Let K be a henselian field and N an arbitrary tame algebraic extension of K. If $L|K$ is a finite extension, then*

$$\mathrm{d}(L|K) = \mathrm{d}(L.N|N)\ .$$

In particular, $L|K$ is defectless if and only if $L.N|N$ is defectless. This implies: K is a defectless field if and only if N is a defectless field, and the same holds for "separably defectless" and "inseparably defectless" in the place of "defectless".

Proof: We put $L_0 := L \cap K^r$ where K^r denotes the ramification field of K. Since L_0 and N are subfields of K^r, we have $L_0^r = K^r$, $N^r = K^r$ and $(L_0.N)^r = K^r$ by Lemma 2.10. Again by Lemma 2.10, $L_0|K$ is the maximal tame subextension of $L|K$, and $L|L_0$ is purely wild. Thus $L|L_0$ is linearly disjoint from $K^r|L_0$ which yields that $L.N|L_0.N$ is linearly disjoint from $K^r|L_0.N$. This shows that $L.N|L_0.N$ is purely wild and that consequently $L_0.N|N$ is the maximal tame subextension of $L.N|N$ ($L_0.N|N$ is tame since $L_0.N \subset K^r = N^r$). Note that $L_0.N|L_0$ and $L_0.N|K$ are tame extensions because $L_0.N \subset K^r = L_0^r$.

Both extensions $L_0|K$ and $L_0.N|N$ are defectless, since they are tame; hence

$$\mathrm{d}(L|K) = \mathrm{d}(L|L_0) \ \text{ and } \ \mathrm{d}(L.N|N) = \mathrm{d}(L.N|L_0.N)\ .$$

It remains to show $\mathrm{d}(L|L_0) = \mathrm{d}(L.N|L_0.N)$. Now $L|L_0$ (being a purely wild extension) is linearly disjoint from the tame subextension $L_0.N|L_0$ of $K^r|L_0$. In view of $L.N = L.(L_0.N)$ we thus find

$$[L.N : L_0.N] = [L : L_0]\ . \tag{6}$$

Since $L|L_0$ is purely wild, $v(L)/v(L_0)$ is a p-group and $\overline{L}\,|\,\overline{L_0}$ is purely inseparable. On the other hand, since $L_0.N|L_0$ is tame, $v(L_0.N)/v(L_0)$ has no elements of order divisible by p, and $\overline{L_0.N}\,|\,\overline{L_0}$ is separable. This shows that

$$\begin{array}{rcl}(v(L)+v(L_0.N):v(L_0.N)) & = & (v(L):v(L_0))\\ [\overline{L}\,.\,\overline{L_0.N}:\overline{L_0.N}] & = & [\overline{L}:\overline{L_0}]\end{array}$$

Consequently, we find

$$(v(L.N):v(L_0.N))\geq(v(L)+v(L_0.N):v(L_0.N))=(v(L):v(L_0)) \tag{7}$$
$$[\overline{L.N}:\overline{L_0.N}]\geq[\overline{L}\,.\,\overline{L_0.N}:\overline{L_0.N}]=[\overline{L}:\overline{L_0}] \tag{8}$$

On the other hand, if “>” would hold in one of these inequalities, then $v(L.N_0)\neq v(L)+v(N_0)$ or $\overline{L.N_0}\neq\overline{L}\,.\,\overline{N_0}$ would hold already for a finite tame subextension $N_0|L_0$ of $L_0.N|L_0$. But since $v(N_0)$ is linearly disjoint from $v(L)$ over $v(L_0)$ (in the group theoretical sense) and $\overline{N_0}$ is linearly disjoint from $\overline{L}$ over $\overline{L_0}$, we have

$$\begin{array}{rcl}(v(L)+v(N_0):v(L)) & = & (v(N_0):v(L_0))\\ [\overline{L}\,.\,\overline{N_0}:\overline{L}] & = & [\overline{N_0}:\overline{L_0}]\;.\end{array}$$

Using this and the fact that $N_0|L_0$ is tame, hence defectless and linearly disjoint from $L|L_0$, we compute:

$$\begin{array}{rcl}[L.N_0:L] & \geq & (v(L.N_0):v(L))\cdot[\overline{L.N_0}:\overline{L}]\\ & \geq & (v(L)+v(N_0):v(L))\cdot[\overline{L}\,.\,\overline{N_0}:\overline{L}]\\ & = & (v(N_0):v(L_0))\cdot[\overline{N_0}:\overline{L_0}]=[N_0:L_0]\geq[L.N_0:L]\;,\end{array}$$

showing that “=” holds everywhere and that

$$v(L.N_0)=v(L)+v(N_0)\;,\;\;\overline{L.N_0}=\overline{L}\,.\,\overline{N_0}\;.$$

We have shown that “=” holds in (7) and (8). From this and (6) we deduce

$$\begin{array}{rcl}\mathrm{d}(L.N|L_0.N) & = & \dfrac{[L.N:L_0.N]}{(v(L.N):v(L_0.N))\cdot[\overline{L.N}:\overline{L_0.N}]}\\ & = & \dfrac{[L:L_0]}{(v(L):v(L_0))\cdot[\overline{L}:\overline{L_0}]}=\mathrm{d}(L|L_0)\;.\end{array}$$

It remains to show the second assertion. Assume that N is a defectless field and let $L|K$ be an arbitrary finite extension. Then by hypothesis, $L.N|N$ is defectless; hence by what we have shown, $L|K$ is defectless. Since $L|K$ was arbitrary, K is shown to be a defectless field. Note that $L.N|N$ is separable if $L|K$ is separable, and $L.N|N$ is purely inseparable if $L|K$ is.

Conversely, assume that K is a defectless field. Since any finite extension $N'|N$ is contained in an extension $L.N|N$ where $L|K$ is a finite and, by hypothesis, defectless extension, we see that by what we have shown, $L.N|N$ and by virtue of Lemma 2.8 also its subextension $N'|N$ are defectless. Note that $L|K$ can be chosen to be separable if $N'|N$ is separable, and to be purely inseparable if $N'|N$ is purely inseparable. This completes the proof of our lemma. □

Corollary 2.12 *Let K be a henselian field and N an arbitrary tame algebraic extension of K. If $L|K$ is an immediate algebraic extension, then $L.N|N$ is immediate too.*

Proof: It suffices to show that any finite subextension of $L.N|N$ is immediate. Since any such extension is contained in an extension $L'.N|N$ where $L'|K$ is a finite subextension of $L|K$ (hence immediate), we may w.l.o.g. assume from the start that $L|K$ is finite. Since it is immediate and K is henselian, it must be purely wild. Consequently, it is linearly disjoint from the tame extension $N|K$. This shows $[L.N : N] = [L : K]$. From the preceeding lemma we infer $\mathrm{d}(L.N|N) = \mathrm{d}(L|K)$. Both equations together show that $L.N|N$ must be immediate. □

Moreover, Lemma 2.11 can be used to define a generalization of our notion of the defect. Indeed, if $L|K$ is an algebraic extension of valued fields such that the corresponding extension $L^r|K^r$ of their ramification fields is finite, then we define the *defect* of $L|K$ to be

$$\mathrm{d}(L|K) = \mathrm{d}(L^r|K^r)\ . \tag{9}$$

We have to show that this definition coincides with the original definition if $L|K$ is h–finite. The ramification field K^r of K contains the decomposition field of K which is just the henselization K^h of K, and K^r is at the same time the ramification field of K^h. The same holds for L. Since by definition, the defect of $L|K$ is equal to the defect of $L^h|K^h$ we may thus replace K and L by their henselizations and assume from now on that $L|K$ is an extension of henselian fields. Now Lemma 2.11 shows

$$\mathrm{d}(L|K) = \mathrm{d}(L.K^r|K^r)\ .$$

From Lemma 2.10 we infer $L.K^r = L^r$, which shows

$$\mathrm{d}(L|K) = \mathrm{d}(L^r|K^r)\ ,$$

as desired.

In the next lemma, the relation between immediate and defectless extensions is considered.

Lemma 2.13 *Let K be a valued field and $F|K$ an arbitrary immediate extension. If $L|K$ is a finite defectless extension admitting a unique prolongation of the valuation, then $F.L|F$ is a finite defectless extension and $F.L|L$ is immediate. Moreover,*

$$[F.L : F] = [L : K]\ ,$$

i.e. F is linearly disjoint from L over K.

Proof: $v(F.L)$ includes $v(L)$ and $\overline{F.L}$ includes $\overline{L}$. On the other hand we have $v(F) = v(K)$ and $\overline{F} = \overline{K}$ by hypothesis. Hence

$$[F.L : F] \geq (v(F.L) : v(F)) \cdot [\overline{F.L} : \overline{F}] \geq (v(L) : v(K)) \cdot [\overline{L} : \overline{K}] = [L : K] \geq [F.L : F]\ ,$$

hence "=" holds everywhere. This shows the asserted equation and that $F.L|F$ is defectless. Furthermore it follows that $v(F.L) = v(L)$ and $\overline{F.L} = \overline{L}$ expressing the fact that $F.L|L$ is immediate. □

As an immediate consequence we get:

Corollary 2.14 *If K is an inseparably defectless field then every immediate extension is separable. If K is a henselian defectless field then every immediate extension is regular.*

Now we will consider composite valuations. Let us assume the following situation:

$L\|K$	an h–finite extension of valued fields,
v	a valuation on L,
$P = P_v$	the associated place,
$P = Q\overline{Q}$	a decomposition of P,

where Q is a coarsening of P and $\overline{Q}$ is a place on the residue field LQ. If $L|K$ is of finite degree, then $LQ|KQ$ is of finite degree too. Let "$.^h$" denote the henselization with respect to P. To study the behaviour of the defect under such decompositions of the place, we need the following lemmata about henselian valuations and henselizations:

Lemma 2.15 *(K,P) is henselian if and only if (K,Q) and $(KQ,\overline{Q})$ are henselian.*

Proof: For the proof of "and only if" see [RIB1], p. 210, Proposition 9. The other implication is to be found in [RIB1], p. 211, Proposition 10. □

Lemma 2.16 *Let (K,P) be a valued field, (K^h,P) its henselization and $P = Q\overline{Q}$. If $(KQ,\overline{Q})$ is henselian, then K^h is equal to $K^{h(Q)}$ where $(K^{h(Q)},Q)$ is the henselization of (K,Q). In general, $(K^hQ,\overline{Q})$ is the henselization of $(KQ,\overline{Q})$, and (K^h,Q) is a tame unramified extension of the henselization $(K^{h(Q)},Q)$ of (K,Q).*

If in addition $(L,P)|(K,P)$ is an h–finite extension, then

$$(LQ,\overline{Q})|(KQ,\overline{Q})$$

is an h–finite extension too and the extension $(L,Q)^r|(K,Q)^r$ of the ramification fields is finite.

Proof: The first assertion is seen as follows: Let $(KQ,\overline{Q})$ be henselian. The henselization $(K^{h(Q)},Q)$ of (K,Q) has residue field $K^{h(Q)}Q = KQ$. Now $(K^{h(Q)},Q)$ and $(K^{h(Q)}Q,\overline{Q})$ being henselian it follows that $(K^{h(Q)},P)$ is henselian too (cf. Lemma 2.15), hence it includes the henselization (K^h,P) of (K,P). On the other hand (K^h,P) being henselian yields that (K^h,Q) is henselian (again by Lemma 2.15), hence it includes the henselization $(K^{h(Q)},Q)$ of (K,Q). This shows $K^h = K^{h(Q)}$, as contended.

The second assertion can be deduced from the first assertion: Since (K^h,P) is henselian, $(K^hQ,\overline{Q})$ and (K^h,Q) are henselian too (again by Lemma 2.15), hence $(K^hQ,\overline{Q})$ includes the henselization $((KQ)^{h(\overline{Q})},\overline{Q})$ of $(KQ,\overline{Q})$, and (K^h,Q) includes the henselization $(K^{h(Q)},Q)$ of (K,Q). By a straightforward construction an unramified subextension $K_1|K^{h(Q)}$ of $K^h|K^{h(Q)}$ may be found such that $K_1Q = (KQ)^{h(\overline{Q})}$. Since $(KQ)^{h(\overline{Q})}|KQ$ is a separable extension, K_1 can be chosen such that $(K_1,Q)|(K^{h(Q)},Q)$ is a tame extension. From the first assertion we conclude that the henselization $K_1^h = K^h$ must be equal to the henselization $K_1^{h(Q)}$ whose residue field $(K_1^{h(Q)}Q,\overline{Q})$ is just the henselization of $(KQ,\overline{Q})$.

If $(L,P)|(K,P)$ is h–finite then $(L,P)^h|(K,P)^h$ is finite and so is the extension of the residue fields $(L^hQ,\overline{Q})|(K^hQ,\overline{Q})$. By what we have just proved, $(L^hQ,\overline{Q}) = (LQ,\overline{Q})^h$

and $(K^hQ,\overline{Q}) = (KQ,\overline{Q})^h$, hence $(LQ,\overline{Q})|(KQ,\overline{Q})$ is an h–finite extension. Again by what we have proved, $(L^{h(P)},Q)|(L,Q)^h$ and $(K^{h(P)},Q)|(K,Q)^h$ are tame extensions, hence $(L,Q)^r = (L^{h(P)},Q)^r$ and $(L,Q)^r = (L^{h(P)},Q)^r$ and since $L^{h(P)}|K^{h(P)}$ is finite, so is $(L,Q)^r|(K,Q)^r$. □

Now we are able to prove:

Lemma 2.17 *In the situation as described above, the following formula holds:*

$$\mathrm{d}((L,P)|(K,P)) = \mathrm{d}((L,Q)|(K,Q)) \cdot \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q})) \ . \tag{10}$$

In particular, $(L,P)|(K,P)$ is defectless if and only if $(L,Q)|(K,Q)$ and $(LQ,\overline{Q})|(KQ,\overline{Q})$ are defectless. This yields: (K,P) is a defectless field if and only if (K,Q) and $(KQ,\overline{Q})$ are defectless fields. Note that we have to use our most general definition of the defect because $(L,Q)|(K,Q)$ is r–finite but might not be h–finite.

Proof: From Lemma 2.16 we infer

$$(K^hQ,\overline{Q}) = (KQ,\overline{Q})^h \ \text{ and } \ (L^hQ,\overline{Q}) = (LQ,\overline{Q})^h \ ,$$

whence

$$\mathrm{d}((L^hQ,\overline{Q})|(K^hQ,\overline{Q})) = \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q})) \ .$$

Again from Lemma 2.16, we infer

$$(K^h,Q) \subset (K,Q)^r \ \text{ and } \ (L^h,Q) \subset (L,Q)^r \ ,$$

whence by Lemma 2.10,

$$\mathrm{d}((L^h,Q)|(K^h,Q)) = \mathrm{d}((L^h,Q)^r|(K^h,Q)^r) = \mathrm{d}((L,Q)^r|(K,Q)^r) = \mathrm{d}((L,Q)|(K,Q)) \ .$$

Now we compute:

$$\begin{aligned}
\mathrm{d}((L,P)|(K,P)) &= \mathrm{d}((L,P)^h|(K,P)^h) \\
&= \frac{[L^h : K^h]}{(v(L^h) : v(K^h)) \cdot [\overline{L^h} : \overline{K^h}]} \\
&= \frac{[L^h : K^h]}{(v_Q(L^h) : v_Q(K^h)) \cdot (v_{\overline{Q}}(L^hQ) : v_{\overline{Q}}(K^hQ)) \cdot [\overline{L^hQ} : \overline{K^hQ}]} \\
&= \frac{[L^h : K^h]}{(v_Q(L^h) : v_Q(K^h)) \cdot [L^hQ : K^hQ]} \\
&\quad \cdot \frac{[L^hQ : K^hQ]}{(v_{\overline{Q}}(L^hQ) : v_{\overline{Q}}(K^hQ)) \cdot [\overline{L^hQ} : \overline{K^hQ}]} \\
&= \mathrm{d}((L^h,Q)|(K^h,Q)) \cdot \mathrm{d}((L^hQ,\overline{Q})|(K^hQ,\overline{Q})) \\
&= \mathrm{d}((L,Q)|(K,Q)) \cdot \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q})) \ .
\end{aligned}$$

From this it follows immediately that if (K,Q) and $(KQ,\overline{Q})$ are defectless fields, then (K,P) is a defectless field too. Conversely, let (K,P) be a defectless field. Given a finite extension $(L,Q)|(K,Q)$, then $(L,P)|(K,P)$ is defectless by hypothesis, and $(L,Q)|(K,Q)$

is defectless by what we have just shown. Hence (K,Q) is a defectless field. Given a finite extension $(k,\overline{Q})|(KQ,\overline{Q})$, a straightforward construction produces a finite extension $(L,P)|(K,P)$ such that $LQ = k$. By hypothesis, $(L,P)|(K,P)$ is defectless. By what we have shown it follows that $(k,\overline{Q})|(KQ,\overline{Q})$ is defectless too. Hence also $(KQ,\overline{Q})$ is a defectless field. □

2.2 Some generalities about the transcendence defect and valuation transcendence bases.

Given an extension $L|K$ of valued fields of finite transcendence degree, we define the *transcendence defect* to be

$$\mathrm{trdef}(L|K) = \mathrm{trdeg}(L|K) - \mathrm{rr}(v(L)/v(K)) - \mathrm{trdeg}(\overline{L}|\overline{K})$$

where $\mathrm{rr}(v(L)/v(K))$ denotes the rational rank of the abelian factor group $v(L)/v(K)$, i.e. the cardinality of any maximal set of rationally independent values of $v(L)$ modulo $v(K)$ (such set we will call "a transcendence basis of $v(L)|v(K)$"). It follows from [BOU], chapter VI, §10.3, Theorem 1, that the transcendence defect is always a nonnegative integer (see also Lemma 2.20 below). If $\mathrm{trdef}(L|K) = 0$ then we say that $L|K$ *has no transcendence defect*. If $L|K$ has infinite transcendence degree, then we will say that $L|K$ has no transcendence defect if every finitely generated subextension $F|K$ of $L|K$ has no transcendence defect.

Consider two extensions $L|K$ and $M|L$, both extensions having finite transcendence degree. Compared with the defect, the transcendence defect shows an analogous behaviour; it satisfies the following formula:

$$\mathrm{trdef}(M|K) = \mathrm{trdef}(M|L) + \mathrm{trdef}(L|K)\ . \tag{11}$$

This follows immediately from the additivity of the transcendence degree and the rational rank. We conclude:

Lemma 2.18 *$M|K$ has no transcendence defect if and only if both $M|L$ and $L|K$ have no transcendence defect.*

A subset $\mathcal{T} = \{x_1,\dots,x_r,y_1,\dots,y_s\}$ of a valued extension field L of K will be called *algebraically valuation–independent* over K if the values $v(x_1),\dots,v(x_r)$ are positive and form a transcendence basis of $v(L)|v(K)$, and $y_1,...,y_s$ are elements of $\mathcal{O}_L^\times$ whose residues $\overline{y_1},\dots,\overline{y_s}$ form a transcendence basis of $\overline{L}|\overline{K}$. A transcendence basis $\mathcal{T}$ of $L|K$ will be called *valuation transcendence basis*, if it is algebraically valuation–independent over K. Note that our definitions given here are not direct analogues to the definitions of "linearly valuation–independent" and "valuation basis". We could have had defined that $\mathcal{T}$ should be called algebraically valuation–independent if the value of every polynomial f in $K[\mathcal{T}]$ is equal to the minimum of the values of the monomials in f; indeed, every $\mathcal{T}$ which is algebraically valuation–independent according to our definition, does also have this property; but our definition is a bit more restrictive since this form serves our purposes better. Nevertheless, it can be proved that whenever a set $\mathcal{T}$ satisfies the above condition on the value of polynomials $f \in K[\mathcal{T}]$, then there exists an algebraically valuation–independent set $\mathcal{T} \subset L$ such that $K(\mathcal{T}) = K(\mathcal{T}')$.

We note:

Lemma 2.19 *Let $L|K$ be a valued field extension and $\mathcal{T} \subset L$ an algebraically valuation–independent set over K. Then the value of every polynomial f in $K[\mathcal{T}]$ is equal to the minimum of the values of the monomials in f (we will always assume that in the representation of f, two different monomials never differ just by a constant factor from K). Consequently, the valuation on $K(\mathcal{T})$ is uniquely determined by its restriction to K and the values of the elements of $\mathcal{T}$. Furthermore, $\mathcal{T}$ is algebraically independent over K.*

Proof: Cf. [BOU], chapter VI, §10.3, Theorem 1. □

The connection between the transcendence defect and valuation transcendence bases is the following:

Lemma 2.20 *Let $L|K$ be a valued field extension of finite transcendence degree with valuation v and place $P = P_v$. If this extension admits a valuation transcendence basis*

$$\left.\begin{array}{l} \mathcal{T} = \{x_1, \ldots, x_r, y_1, \ldots, y_s\} \;\; \textit{where} \\ \textit{the positive values } v(x_1), \ldots, v(x_r) \textit{ form a transcendence basis of } v(L)|v(K) \\ \textit{the residues } \overline{y_1}, \ldots, \overline{y_s} \textit{ form a transcendence basis of } \overline{L}|\overline{K} \end{array}\right\} \qquad (12)$$

then $r = \mathrm{rr}(v(L)/v(K))$, $s = \mathrm{trdeg}(\overline{L}|\overline{K})$ and $\mathrm{trdeg}(L|K) = r + s$, hence the extension has no transcendence defect. Moreover, the following holds:

$$\begin{array}{rcl} v(K(\mathcal{T})) & = & v(K) \oplus \mathbb{Z}\, v(x_1) \oplus \ldots \oplus \mathbb{Z}\, v(x_r)\;, \\ \overline{K(\mathcal{T})} & = & \overline{K}(\overline{y_1}, \ldots, \overline{y_s}) = KP(\mathcal{T}P)\;, \end{array}$$

and if L is finitely generated over K then $v(L)$ is finitely generated over $v(K)$ and LP is finitely generated over KP.

Conversely, if the extension $L|K$ has no transcendence defect then it admits a valuation transcendence basis.

Proof: The first part holds by definition and [BOU], chapter VI, §10.3, Theorem 1. Note that $x_iP = 0$ for $1 \le i \le r$, hence $\mathcal{T}P \setminus \{0\} = \{y_1P, ..., y_sP\} = \{\overline{y_1}, ..., \overline{y_s}\}$. If L is finitely generated over K then $L|K(\mathcal{T})$ is a finite extension, which shows that $v(L)$ is finitely generated over $v(K)$ since $v(K(\mathcal{T}))$ is, and that $\overline{L}$ is finitely generated over $\overline{K}$ since $\overline{K(\mathcal{T})}$ is.

For the proof of the converse, we assume that the extension $L|K$ has no transcendence defect. We choose elements $x_1, \ldots, x_r, y_1, \ldots, y_s$ in L such that $v(x_1), \ldots, v(x_r)$ form a transcendence basis of $v(L)|v(K)$ and $\overline{y_1}, \ldots, \overline{y_s}$ form a transcendence basis of $\overline{L}|\overline{K}$. Then by Lemma 2.19, the system $x_1, \ldots, x_r, y_1, \ldots, y_s$ is algebraically independent over K. On the other hand, by hypothesis the transcendence degree of $L|K$ is equal to $r+s$ which shows that the chosen elements form a transcendence basis, hence by construction a valuation transcendence basis of $L|K$. □

The following corollary treats the case of an infinite valuation transcendence basis $\mathcal{T}$:

Corollary 2.21 *Let $L|K$ be a valued field extension and $\mathcal{T}$ a valuation transcendence basis of $L|K$ of the form*

$$\left.\begin{array}{l}\mathcal{T}=\{x_i,y_j\mid i\in I,\ j\in J\}\quad\textit{where}\\ \textit{the positive values } v(x_i),\ i\in I \textit{ form a transcendence basis of } v(L)|v(K)\\ \textit{the residues } \overline{y_j},\ j\in J \textit{ form a transcendence basis of } \overline{L}|\overline{K}\end{array}\right\}\qquad(13)$$

then $L|K$ has no transcendence defect, and the following holds:

$$\begin{array}{rcl} v(K(\mathcal{T})) &=& v(K)\oplus\bigoplus_{i\in I}\mathbb{Z}\,v(x_i)\;,\\ \overline{K(\mathcal{T})} &=& \overline{K}(\overline{y_j}\mid j\in J)=KP(\mathcal{T}P)\;.\end{array}$$

Proof: The second part is immediately deduced from the foregoing lemma by means of a transfinite induction over an enumeration of $\mathcal{T}$. The first assertion is seen as follows. If $K'|K$ is a finitely generated subextension of $L|K$, then there exist elements $z_1,...,z_n\in\mathcal{T}$ such that $K'(z_1,...,z_n)$ is algebraic over $K(z_1,...,z_n)$. Thus $z_1,...,z_n$ is a valuation transcendence basis of $K'(z_1,...,z_n)$ over K. By Lemma 2.20, it follows that the extension $K'(z_1,...,z_n)$ has no transcendence defect. By the additivity of the transcendence defect, we conclude that $K'|K$ has no transcendence defect. □

Now we consider an extension $L|K$ of finite transcendence degree and assume again that the valuation v resp. its associated place P is composite:

$$P=Q\overline{Q}\;.$$

Then $LQ|KQ$ has finite transcendence degree too, and the following formula holds:

$$\mathrm{trdef}((L,P)|(K,P))=\mathrm{trdef}((L,Q)|(K,Q))+\mathrm{trdef}((LQ,\overline{Q})|(KQ,\overline{Q}))\;.\qquad(14)$$

(The proof is straightforward.) This yields:

Lemma 2.22 $(L,P)|(K,P)$ *has no transcendence defect if and only if both extensions* $(L,Q)|(K,Q)$ *and* $(LQ,\overline{Q})|(KQ,\overline{Q})$ *have no transcendence defect.*

Now we will study the behaviour of valuation transcendence bases under decompositions of the place.

Lemma 2.23 *Let $K(\mathcal{T})|K$ be an extension of finite transcendence degree with valuation transcendence basis $\mathcal{T}=\{x_1,\dots,x_r,y_1,\dots,y_s\}$, where the values $v(x_1),\dots,v(x_r)$ form a transcendence basis of $v(K(\mathcal{T}))|v(K)$ and the residues $\overline{y_1},\dots,\overline{y_s}$ form a transcendence basis of $\overline{K(\mathcal{T})}\,|\,\overline{K}$. Then for every decomposition $P=P_1P_2P_3$, there exists a valuation transcendence basis*

$$\mathcal{T}^*=\{x_1^*,\dots,x_r^*,y_1,\dots,y_s\}\subset K(\mathcal{T})$$

of $(K(\mathcal{T}),P)|(K,P)$ which is also a valuation transcendence basis of the extensions $(K(\mathcal{T}),P_1)|(K,P_1)$ and of $(K(\mathcal{T}),P_1P_2)|(K,P_1P_2)$, such that $(K(\mathcal{T}),P)|(K(\mathcal{T}^*),P)$ is defectless and $\mathcal{T}^*$ has the following property:

$$\mathcal{T}':=\{x_i^*P_1,y_jP_1\mid 1\le i\le r,\ 1\le j\le s,\ x_i^*P_1\neq 0\}$$

is a valuation transcendence basis of $(K(\mathcal{T})P_1,P_2)|(KP_1,P_2)$ *and*

$$K(\mathcal{T}^*)P_1 = KP_1(\mathcal{T}') \;.$$

Here all elements y_jP_1 *are residue–transcendental with respect to* P_2*, and if* P_3 *is trivial, then all elements* x_i^* *with* $x_i^*P_1 \neq 0$ *are value–transcendental with respect to* P_2*.*

For any algebraic extension $L|K$*, the valuation transcendence basis* $\mathcal{T}^*$ *will satisfy in addition that* $(L(\mathcal{T}),P)|(L(\mathcal{T}^*),P)$ *is defectless and* $L(\mathcal{T}^*)P_1 = LP_1(\mathcal{T}')$*.*

Proof: Given any decomposition $P = P_1P_2P_3$, the elements $y_1,\ldots,y_s$ have algebraically independent residues under $P = P_1P_1P_3$, hence under P_1P_2 and P_1 too, as we infer from Lemma 2.19. Furthermore, the elements $y_1P_1,\ldots,y_sP_1$ have algebraically independent residues under P_2P_3 and hence under P_2, and the elements $y_1P_1P_2,\ldots,y_sP_1P_2$ have algebraically independent residues under P_3. So we only have to consider the elements x_i. We choose a transcendence basis $\alpha_1,\ldots,\alpha_\lambda$ of $v_{P_3}(K(\mathcal{T})P_1P_2)|v_{P_3}(KP_1P_2)$ (where $\lambda = 0$ if P_3 is trivial), a transcendence basis $\alpha_{\lambda+1},\ldots,\alpha_\mu$ of $v_{P_2}(K(\mathcal{T})P_1)|v_{P_2}(KP_1)$ (where $\mu = \lambda$ if P_2 is trivial), and a transcendence basis $\alpha_{\mu+1},\ldots,\alpha_\nu$ of $v_{P_1}(K(\mathcal{T}))|v_{P_1}(K)$ (where $\nu = \mu$ if P_1 is trivial). All appearing values may be chosen to be positive. Let $x_1^*,\ldots,x_\nu^*$ be monomials in $x_1,\ldots,x_r$ over K such that

$$\begin{aligned} v_{P_3}(x_i^*P_1P_2) &= \alpha_i \;\text{ for }\; 1\leq i\leq\lambda \\ v_{P_2}(x_i^*P_1) &= \alpha_i \;\text{ for }\; \lambda< i\leq\mu \\ v_{P_1}(x_i^*) &= \alpha_i \;\text{ for }\; \mu< i\leq\nu \;. \end{aligned}$$

Now the values $v(x_1^*),\ldots,v(x_\nu^*)$ form a transcendence basis of $v(K(\mathcal{T}))|v(K)$, hence $\nu = r$ and the elements $x_1^*,\ldots,x_r^*,y_1,\ldots,y_s$ form a valuation transcendence basis of $(K(\mathcal{T}),P)|(K,P)$ which we will call $\mathcal{T}^*$.

By construction, the values $v_{P_3}(x_1^*P_1P_2),\ldots,v_{P_3}(x_\lambda^*P_1P_2)$ form a transcendence basis of $v_{P_3}(K(\mathcal{T})P_1P_2)|v_{P_3}(KP_1P_2)$ and the P_3–residues of the elements $y_1P_1P_2,\ldots,y_sP_1P_2$ form a transcendence basis of $\overline{K(\mathcal{T})}\,|\,\overline{K}$. The extension $(K(\mathcal{T})P_1P_2,P_3)|(KP_1P_2,P_3))$ has no transcendence defect (as we know from Lemma 2.22) which shows that

$$x_1^*P_1P_2,\ldots,x_\lambda^*P_1P_2,\; y_1P_1P_2,\ldots,y_sP_1P_2$$

is a valuation transcendence basis of this extension. In particular, this shows that the P_2–residues of the elements $x_1^*P_1,\ldots,x_\lambda^*P_1,y_1P_1,\ldots,y_sP_1$ form a transcendence basis of $K(\mathcal{T})P_1P_2|KP_1P_2$. On the other hand, the values $v_{P_2}(x_{\lambda+1}^*P_1),\ldots,v_{P_2}(x_\mu^*P_1)$ form a transcendence basis of $v_{P_2}(K(\mathcal{T})P_1)|v_{P_2}(KP_1)$. Again from Lemma 2.22 we know that the extension $(K(\mathcal{T})P_1,P_2)|(KP_1,P_2))$ has no transcendence defect which shows that

$$x_1^*P_1,\ldots,x_\mu^*P_1,\; y_1P_1,\ldots,y_sP_1$$

is a valuation transcendence basis of this extension. In particular, this proves that the P_1–residues of the elements $x_1^*,\ldots,x_\mu^*,y_1,\ldots,y_s$ form a transcendence basis of $K(\mathcal{T})P_1|KP_1$. On the other hand, the values $v_{P_1}(x_{\mu+1}^*)$,$\ldots,v_{P_1}(x_\nu^*)$ form a transcendence basis of $v_{P_1}(K(\mathcal{T}))|v_{P_1}(K)$. Again from Lemma 2.22 we know that $(K(\mathcal{T}),P_1)|(K,P_1))$ has no transcendence defect which shows that $\mathcal{T}^*$ is a valuation transcendence basis of this extension. From Lemma 2.20 we infer

$$K(\mathcal{T}^*)P_1 = K(x_1^*P_1,\ldots,x_\mu^*P_1,y_1P_1,\ldots,y_sP_1)\;,$$

the latter being equal to $K(\mathcal{T}')$ because all elements x_i^*, $\mu+1 \le i \le \nu$ have positive v_{P_1}–values and consequently vanish under P_1. Hence $\mathcal{T}'$ is equal to the valuation transcendence basis of $(K(\mathcal{T})P_1, P_2)|(KP_1, P_2))$ that we have constructed above.

As we have shown, the P_1P_2–residues of $x_1^*, \ldots, x_\lambda^*, y_1, \ldots, y_s$ form a transcendence basis of $K(\mathcal{T})P_1P_2|KP_1P_2$. On the other hand, the $v_{P_1P_2}$–values of the elements $x_{\lambda+1}, \ldots, x_r$ form a transcendence basis of $v_{P_1P_2}(K(\mathcal{T}))|v_{P_1P_2}(K)$ and since $(K(\mathcal{T}), P_1P_2)|(K, P_1P_2)$ has no transcendence defect by virtue of Lemma 2.22, these facts prove that $\mathcal{T}^*$ is a valuation transcendence basis of $(K(\mathcal{T}), P_1P_2)|(K, P_1P_2)$.

If P_3 is trivial, then $\lambda = 0$ and the nonzero P_1–residues of $\mathcal{T}^*$ are just the elements x_i^*, $\lambda+1 = 1 \le i \le \mu$ which are all value–transcendental with respect to P_2.

If $L|K$ is algebraic, then $\mathcal{T}^*$ is also a valuation transcendence basis for $L(\mathcal{T}^*)|L$ and $L(\mathcal{T}^*)P_1 = LP_1(\mathcal{T}')$ can be derived as it was done for K in the place of L.

It remains to show that $(K(\mathcal{T}), P)|(K(\mathcal{T}^*), P)$ and $(L(\mathcal{T}), P)|(L(\mathcal{T}^*), P)$ are defectless extensions. The valuation v induces a homomorphism from the multiplicative group $\mathcal{G}$ generated by $K^\times$ and the elements $x_1, ..., x_r$ (as a subgroup of the multiplicative group of $K(\mathcal{T})$) onto $v(K(\mathcal{T}))$ and an isomorphism from $\mathcal{G}/K^\times$ onto $v(K(\mathcal{T}))/v(K)$. By construction (the elements x_i^* were chosen to be monomials in $x_1, \ldots, x_r$), the multiplicative group $\mathcal{H}$ generated by $K^\times$ and the elements $x_1^*, \ldots, x_r^*$ is a subgroup of $\mathcal{G}$, and induced by v, the subgroup $\mathcal{H}/K^\times$ of $\mathcal{G}/K^\times$ is isomorphic to $v(K(\mathcal{T}^*))/v(K)$. Consequently, the index $(\mathcal{G} : \mathcal{H})$ is equal to the index $v(K(\mathcal{T}))/v(K(\mathcal{T}^*))$. On the other hand, the index $(\mathcal{G} : \mathcal{H})$ cannot be smaller than the degree of the field extension $K(\mathcal{T})|K(\mathcal{T}^*)$ which is generated by the elements $x_1, ..., x_r$. Moreover, the residue fields of $K(\mathcal{T})$ and $K(\mathcal{T}^*)$ are both equal to $\overline{K}(\overline{y_1}, \ldots, \overline{y_s})$. Hence

$$[K(\mathcal{T}) : K(\mathcal{T}^*)] \le (\mathcal{G} : \mathcal{H}) = (v(K(\mathcal{T})) : v(K(\mathcal{T}^*))) \le [K(\mathcal{T}) : K(\mathcal{T}^*)]$$

showing that "=" holds everywhere and that the extension $K(\mathcal{T})|K(\mathcal{T}^*)$ is defectless, as asserted. With the same arguments it can be shown that $L(\mathcal{T})|L(\mathcal{T}^*)$ is defectless. This completes the proof of our lemma. □

Note that in general it is not possible to choose $\mathcal{T}^*$ such that $K(\mathcal{T}^*) = K(\mathcal{T})$. The obstruction is that an element may be value–transcendental with respect to a composite valuation $v \circ w$ over a field K while its value with respect to v may lie in the divisible hull of $v(K)$ without lying in $v(K)$ itself.

2.3 Approximation types and distances.

Let (L, v) be a valued field and R a subgroup of the additive group of L. Then $v(R)$ is an ordered subset of $v(L)$. (In many of our applications, R will be a field.) Note that we always exclude $\infty = v(0)$ from $v(R)$ and $v(L)$. A map

$$\mathbf{A}\colon v(R) \cup \{\infty\} \longrightarrow \mathcal{P}(R) \times \mathcal{P}(R)\,, \quad \alpha \mapsto (\mathbf{A}_\alpha, \mathbf{A}_\alpha^\circ)$$

from the value set of R into the product of the power set of R with itself, will be called *approximation type* over R if it satisfies $\mathbf{A}_\infty^\circ = \emptyset$ and the following conditions:

(at 0) $\mathbf{A}_\alpha^\circ \subset \mathbf{A}_\alpha \subset R$
(at 1) $\alpha > \beta \Longrightarrow \mathbf{A}_\alpha \subset \mathbf{A}_\beta^\circ$
(at 2) $a, b \in \mathbf{A}_\alpha \Longrightarrow v(a-b) \geq \alpha$
(at 2°) $a, b \in \mathbf{A}_\alpha^\circ \Longrightarrow v(a-b) > \alpha$
(at 3) $(v(a-b) \geq \alpha \wedge a \in \mathbf{A}_\alpha) \Longrightarrow b \in \mathbf{A}_\alpha$
(at 3°) $(v(a-b) > \alpha \wedge a \in \mathbf{A}_\alpha^\circ) \Longrightarrow b \in \mathbf{A}_\alpha^\circ$

for all elements $a, b \in R$ and all values $\alpha, \beta \in v(R) \cup \{\infty\}$. As immediate consequences of (at 0) and (at 1) we note:

(at 4) $\alpha \geq \beta \Longrightarrow \mathbf{A}_\alpha \subset \mathbf{A}_\beta$
(at 4°) $\alpha \geq \beta \Longrightarrow \mathbf{A}_\alpha^\circ \subset \mathbf{A}_\beta^\circ$

for all $\alpha, \beta \in v(R) \cup \{\infty\}$. $\mathbf{A}$ will be called *value–immediate approximation type*, if it satisfies in addition

(at 5_v) $\forall c \in R \ \exists \alpha \in v(R) \cup \{\infty\} : \ c \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$,

and *residue–immediate approximation type*, if it satisfies in addition

(at 5_r) $\forall \alpha \in v(R) : \ \mathbf{A}_\alpha \neq \emptyset \Longrightarrow \mathbf{A}_\alpha^\circ \neq \emptyset$.

$\mathbf{A}$ will be called *immediate approximation type*, if it is value–immediate and residue–immediate.

If we write

$$\begin{aligned} \mathcal{O}_\alpha &= \{a \in R \mid v(a) \geq \alpha\} \\ \mathcal{M}_\alpha &= \{a \in R \mid v(a) > \alpha\} \end{aligned}$$

for every $\alpha \in v(R) \cup \{\infty\}$, then every nonempty $\mathbf{A}_\alpha$ may just be presented as the coset

$$\mathbf{A}_\alpha = c_\alpha + \mathcal{O}_\alpha$$

for an arbitrary $c_\alpha \in \mathbf{A}_\alpha$ (this is the content of (at 2) and (at 3)), and every nonempty $\mathbf{A}_\alpha^\circ$ may be presented as the coset

$$\mathbf{A}_\alpha^\circ = c_\alpha^\circ + \mathcal{M}_\alpha$$

for an arbitrary $c_\alpha^\circ \in \mathbf{A}_\alpha^\circ$ (this is the content of (at 2°) and (at 3°)).

Now let $x \in L$. Then the map

$$v(R) \cup \{\infty\} \ni \alpha \mapsto (\mathrm{appr}(x, R)_\alpha, \mathrm{appr}(x, R)_\alpha^\circ)$$

where

$$\begin{aligned} \mathrm{appr}(x, R)_\alpha &= \{r \in R \mid v(x-r) \geq \alpha\} , \\ \mathrm{appr}(x, R)_\alpha^\circ &= \{r \in R \mid v(x-r) > \alpha\} , \end{aligned}$$

will be called the *approximation type of x over R* and will be denoted by

$$\mathrm{appr}(x, R) .$$

This map is indeed an approximation type in the already defined sense; the proof is straightforward and thus left to the reader.

Conversely, if $\mathbf{A}$ is an approximation type over R and there exists an element $x \in L$ such that $\mathbf{A} = \mathrm{appr}(x, R)$, then we say that x *realizes* $\mathbf{A}$ (in (L, v)). Note that $\mathbf{A}$ is realized by an element of R if and only if $\mathbf{A}_\infty \neq \emptyset$.

At this point, we should say some words about cuts in an ordered abelian group Γ. A *Dedekind cut* (or simply *cut*) in Γ is a partition (Λ, Λ') of Γ into two convex subsets Λ, Λ' where Λ is empty or an *initial segment* of Γ, i.e. if $\alpha \in \Lambda$ and $\Gamma \ni \beta \leq \alpha$, then $\beta \in \Lambda$. Similarly, if Λ' is nonempty, then it is a *final segment* of Γ, i.e. if $\alpha \in \Lambda'$ and $\Gamma \ni \beta \geq \alpha$, then $\beta \in \Lambda'$.

We say that the *cut* (Λ, Λ') *is realized* by the element $\delta \in \Gamma$ if $\Lambda \leq \delta \leq \Lambda'$; here "$\delta \geq \Lambda$" stands for "$\forall \gamma \in \Lambda: \ \delta \geq \gamma$" (correspondingly we will use the other relations $\leq, >, <$). If (Λ, Λ') and (Λ_1, Λ'_1) are two cuts then

$$(\Lambda, \Lambda') < (\Lambda_1, \Lambda'_1)$$

will indicate that $\Lambda_1 \setminus \Lambda \neq \emptyset$; if in this situation, (Λ, Λ'_1) and (Λ_1, Λ'_1) are realized by $\delta \in \Gamma$ and $\delta_1 \in \Gamma$ respectively, then it will follow that $\delta < \delta_1$. Note that every element $\gamma \in \Gamma$ may itself be interpreted as a cut by taking

$$\Lambda = \{\alpha \mid \alpha \leq \gamma\} \ ;$$

hence the above rules also determine the order between an element of Γ and an arbitrary cut.

A subset Υ of Γ determines a cut (Λ, Λ') of Γ by taking Λ to be the least initial segment containing Υ, or equivalently,

$$\Lambda' = \{\alpha \in \Gamma \mid \alpha > \Upsilon\} \ .$$

This cut will also be called the *supremum* of Υ, denoted by $\sup \Upsilon$.

In a canonical way, the cut (Λ, Λ') of Γ also induces a cut $(\Lambda_\Delta, \Lambda'_\Delta)$ in every ordered overgroup Δ of Γ, where Λ_Δ is taken to be the convex hull of Λ in Δ. In particular, the cut (Λ, Λ') of Γ determines a cut $\tilde{\Lambda}, \tilde{\Lambda}'$ in the divisible hull $\widetilde{\Gamma}$ of the value group Γ. If this cut is realized by an element δ of $\widetilde{\Gamma}$, then (Λ, Λ') is called a *rational cut* of Γ, and we will identify it with the element δ. Also in general, we will use "δ" to denote a cut; since every cut is realized in a suitable overgroup (e.g. if this overgroup is highly enough saturated), we may always view δ as an element of such overgroup.

The cut $\delta = (\Lambda, \Lambda')$ is called a *positive cut*, if $0 \in \Lambda$, and a *negative cut*, if $0 \in \Lambda'$. Given an integer i and an element $\gamma \in \Gamma$, we let $i \cdot \delta + \gamma$ be the cut determined by the least initial segment of Γ containing $i \cdot \Lambda + \gamma$ if i is nonnegative, resp. containing $i \cdot \Lambda' + \gamma$ if i is negative. Note that δ is positive if and only if $-\delta = (-1) \cdot \delta$ is negative.

The set

$$\Lambda(\mathbf{A}) = \{\alpha \in v(R) \mid \mathbf{A}_\alpha \neq \emptyset\}$$

will be called the *value set of* $\mathbf{A}$. Note that $\infty \notin \Lambda(\mathbf{A}) \subseteq v(R)$. It follows from (at 4) that $\Lambda(\mathbf{A})$ is an initial segment of $v(R)$, if it is nonempty. The *distance* $\mathrm{dist}(\mathbf{A})$ of $\mathbf{A}$ is the cut induced by $(\Lambda(\mathbf{A}), v(R) \setminus \Lambda(\mathbf{A}))$ in the divisible hull of the value group $v(\mathrm{Quot}(R))$. Note that $\mathrm{Quot}(R)$ is a subfield of L and that the divisible hull of $\mathrm{Quot}(R)$ is just the value group of the algebraic closure of $\mathrm{Quot}(R)$. If this cut is realized by an element δ of $v(\mathrm{Quot}(R))$, then we will call $\mathrm{dist}(\mathbf{A})$ a *rational distance*, and we will identify it with the element δ.

If $\mathbf{A} = \mathrm{appr}(x,R)$, then we will write

$$\mathrm{dist}(x,R)$$

in the place of "dist($\mathbf{A}$)". We say that the *distance is assumed* by an element of R if $\mathrm{dist}(\mathbf{A}) \in v(R) \cup \{\infty\}$ and $\mathbf{A}_\alpha \neq \emptyset$ for $\alpha = \mathrm{dist}(\mathbf{A})$. The *distance is finitely assumed* by an element of R if $\infty > \mathrm{dist}(\mathbf{A}) \in v(R)$ and $\mathbf{A}_\alpha \neq \emptyset$ for $\alpha = \mathrm{dist}(\mathbf{A})$. By writing

$$\mathrm{dist}(x,R) \asymp \mathrm{dist}(y,R)$$

we will express that both distances are equal and that moreover $\mathrm{appr}(x,R)$ is immediate if and only if $\mathrm{appr}(y,R)$ is immediate.

If $M \subset R$ is an arbitrary subset, then the cut

$$\sup\{\alpha \in v(R) \mid \mathbf{A}_\alpha \cap M \neq \emptyset\}$$

will be denoted by

$$\mathrm{dist}_R(x,M)\ ,$$

and we have

$$\mathrm{dist}_R(x,M) \leq \mathrm{dist}(x,R) = \mathrm{dist}_R(x,R) \leq \mathrm{dist}_L(x,R)\ .$$

Let $\mathbf{F}$ be a formula with one free variable z (not necessarily, but usually in the language of valued fields with constants from R); if $\mathbf{A}$ is an approximation type over R, then both expressions

$$c \nearrow \mathbf{A} \Longrightarrow \mathbf{F}(c)$$
$$\forall z \nearrow \mathbf{A} :\ \mathbf{F}(z)$$

will denote the assertion

$$\exists \alpha \in v(R)\ ,\ \alpha < \mathrm{dist}(\mathbf{A}),\ \forall \beta \geq \alpha\, \forall z \in \mathbf{A}_\beta :\ \mathbf{F}(z)\ .$$

In particular, this assertion includes the information that there exists an element $z \in R$ with $\mathbf{F}(z)$; this follows from the definition of the distance. If $\mathbf{A} = \mathrm{dist}(x,R)$, then we will also write

$$c \nearrow x\ ,\ \ \forall z \nearrow x$$

in the place of "$c \nearrow \mathbf{A}$" and "$\forall x \nearrow \mathbf{A}$".

Let $\mathbf{A}$ be an approximation type over a valued field K. Given an arbitrary polynomial $f(X) \in K[X]$, we say that $\mathbf{A}$ *fixes the value of* f if $v(f(c))$ is independent of $c \in K$ for $c \nearrow \mathbf{A}$. $\mathbf{A}$ is said to be a *transcendental approximation type* (over K) if $\mathbf{A}$ fixes the value of every polynomial $f(X) \in K[X]$. If $f(X)$ is a normed polynomial of minimal degree $\mathbf{d}$ such that $\mathbf{A}$ does not fix the value of f, then it will be called an *associated minimal polynomial* for $\mathbf{A}$, and $\mathbf{A}$ is said to be an *algebraic approximation type* of *degree* $\mathbf{d}$ (over K). Note that if there exists any polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, then there exists also a normed polynomial of the same degree having the same property (since this property is not lost by multiplication with nonzero constants from K). We take the degree of a transcendental approximation type to be $\mathbf{d} = \infty$. According to this terminology, an approximation type over K of degree $\mathbf{d}$ fixes the value of every polynomial $f \in K[X]$ with $\deg(f) < \mathbf{d}$. Note that an associated minimal polynomial f for $\mathbf{A}$ is always irreducible over K. Indeed, if $g, h \in K[X]$ are having degree $< \deg(f)$, then $\mathbf{A}$ fixes the value of g and h and thus also of $g \cdot h$. Since every polynomial $g \in K[X]$ of degree $\mathbf{d}$ whose value is not fixed by $\mathbf{A}$, is just a multiple $c \cdot f$ of an associated minimal polynomial f for $\mathbf{A}$ (with $c \in K^\times$), the irreducibility holds for every such polynomial too.

An approximation type of degree 1 is called *trivial*; it is of the form $\mathrm{appr}(c_0, K)$ with $c_0 \in K$, and every such approximation type is trivial (cf. Lemma 11.39).

We will give a detailed outline of the basic properties of approximation types and distances in the appendix.

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3 Function fields without transcendence defect over defectless ground fields.

In this section, our main goal is the proof of

Theorem 3.1 *Let $F|K$ be a function field without transcendence defect. If K is a defectless field then F is a defectless field.*

In view of Lemma 2.3, this is equivalent to:
Let $F|K$ be a henselian function field without transcendence defect. If K is a defectless field then F is a defectless field.

Once proved, this theorem has important consequences which we will describe now. We start with structure theorems for henselian function fields over defectless fields. For a function field $F|K$ we will denote by *degree of irrationality* the minimal degree of F over all possible transcendence bases $\mathcal{T}$:

$$[F:K]_{\mathrm{irr}} = \min_{\mathcal{T}}[F:K(\mathcal{T})]\;.$$

If in addition, $F|K$ is separable then we will denote by *degree of separable irrationality* the minimal degree of F over all possible separating transcendence bases $\mathcal{T}$:

$$[F:K]_{\mathrm{sep}} = \min_{\mathcal{T}}[F:K(\mathcal{T})]\;.$$

Theorem 3.2 *Let K be a defectless field and $F|K$ a henselian function field without transcendence defect. Then F is a finite defectless extension of a henselian rational function field F_0 . Moreover, F_0 can be chosen such that*

$$[F:F_0] = (v(F):v(K))_{\mathrm{tor}}\cdot[\overline{F}:\overline{K}]_{\mathrm{irr}} \tag{15}$$

where $(v(L):v(K))_{\mathrm{tor}}$ denotes the order of the torsion subgroup of $v(F)/v(K)$ and $[\overline{F}:\overline{K}]_{\mathrm{irr}}$ denotes the degree of irrationality of the function field $\overline{F}|\overline{K}$. Moreover,

$$[F:F_0] = \min_{\mathcal{T}}[F:K(\mathcal{T})^h] \tag{16}$$

where the minimum runs over all valuation transcendence bases of $F|K$.

Proof: Let

$$v(F)/v(K) = \Gamma\oplus\alpha_1\mathbb{Z}\oplus\ldots\oplus\alpha_r\mathbb{Z}$$

where Γ denotes the torsion subgroup of $v(F)/v(K)$ and $r\geq 0$. Choose $x_1,\ldots,x_r\in F$ such that

$$\alpha_i = v(x_i)+v(K)\,,\;1\leq i\leq r\;.$$

Furthermore, let $\overline{\mathcal{T}}\subset\overline{F}$ be a transcendence basis of $\overline{F}|\overline{K}$ with

$$[\overline{F}:\overline{K}(\overline{\mathcal{T}})] = [\overline{F}:\overline{K}]_{\mathrm{irr}}\;.$$

Choose $y_1,\ldots,y_s\in\mathcal{O}_F^\times$ $(s\geq 0)$ such that

$$\overline{\mathcal{T}} = \{\overline{y_1},\ldots,\overline{y_s}\}\;.$$

The set $\mathcal{T} = \{x_1, ..., x_r, y_1, ..., y_s\} \subset F$ is algebraically valuation independent by definition. Its cardinality is

$$r + s = \mathrm{rr}(v(F)/v(K)) + \mathrm{trdeg}(\overline{F}|\overline{K})$$

which is equal to $\mathrm{trdeg}(F|K)$ by our assumption that $F|K$ has no transcendence defect. Consequently, $\mathcal{T}$ is a valuation transcendence basis of $F|K$. By Theorem 3.1, $K(\mathcal{T})^h$ is a defectless field since by assumption, K is a defectless field. Since F is henselian finitely generated, it is a finite defectless extension of the henselian rational function field $F_0 = K(\mathcal{T})^h$ as asserted, and thus

$$[F : K(\mathcal{T})^h] = (v(F) : v(K(\mathcal{T})))[\overline{F} : \overline{K(\mathcal{T})}] \ . \tag{17}$$

From Lemma 2.20 we infer

$$\begin{aligned} v(K(\mathcal{T})) &= v(K) \oplus \mathbb{Z}v(x_1) \oplus ... \oplus \mathbb{Z}v(x_r) & (18)\\ \overline{K(\mathcal{T})} &= \overline{K}(\overline{y_1}, ..., \overline{y_s}) = \overline{K}(\overline{\mathcal{T}}) & (19) \end{aligned}$$

whence $v(F)/v(K(\mathcal{T})) \cong \Gamma$ and

$$\begin{aligned} (v(F) : v(K(\mathcal{T}))) &= (v(F) : v(K))_{\mathrm{tor}} \\ [\overline{F} : \overline{K(\mathcal{T})}] &= [\overline{F} : \overline{K}]_{\mathrm{irr}} \end{aligned}$$

which in view of (17) proves (15).
The last assertion of our theorem is seen as follows: For every valuation transcendence basis $\mathcal{T}' = \{x'_1, ..., x'_r, y'_1, ..., y'_s\}$ of the form (12) of $F|K$ we know by Lemma 2.20 that

$$\begin{aligned} v(K(\mathcal{T}')) &= v(K) \oplus \mathbb{Z}v(x'_1) \oplus ... \oplus \mathbb{Z}v(x'_r) \\ \overline{K(\mathcal{T}')} &= \overline{K}(\overline{y'_1}, ..., \overline{y'_s}) \end{aligned}$$

where $\overline{y'_1}, ..., \overline{y'_s}$ form a transcendence basis of $\overline{F}|\overline{K}$. Consequently, the factor group $v(F)/v(K(\mathcal{T}'))$ contains an isomorphic copy of the torsion subgroup of $v(F)/v(K)$ and we have

$$\begin{aligned} (v(F) : v(K(\mathcal{T}'))) &\geq (v(F) : v(K))_{\mathrm{tor}} \\ [\overline{F} : \overline{K(\mathcal{T}')}] &\geq [\overline{F} : \overline{K}]_{\mathrm{irr}} \ . \end{aligned}$$

F being henselian finitely generated over K, the extension $F|K(\mathcal{T}')^h$ is finite, and by Theorem 3.1 it is defectless. Hence

$$\begin{aligned} [F : K(\mathcal{T}')^h] &= (v(F) : v(K(\mathcal{T}')))[\overline{F} : \overline{K(\mathcal{T}')}] \\ &\geq (v(F) : v(K))_{\mathrm{tor}}[\overline{F} : \overline{K}]_{\mathrm{irr}} = [F : F_0] \end{aligned}$$

which proves (16) since by construction, $F_0 = K(\mathcal{T})^h$ where $\mathcal{T}$ is a valuation transcendence basis of $F|K$. □

Suitable conditions on the extensions of the value groups and the residue fields yield more structural information:

Theorem 3.3 *The situation being as in Theorem 3.2, put* $p = \text{char}(\overline{K})$ *and assume in addition that* $p = 0$ *or that* $v(F)/v(K)$ *has no torsion element of order* p *and* $\overline{F}|\overline{K}$ *is separable. Then there exists a valuation transcendence basis* $\mathcal{T}$ *of* $F|K$ *and an element* $a \in F$ *such that*

$$F = K(\mathcal{T})^h(a)$$

which is a tame extension of $K(\mathcal{T})^h$ *satisfying*

$$[F : K(\mathcal{T})^h] = [K(\mathcal{T}, a) : K(\mathcal{T})] = (v(F) : v(K))_{\text{tor}} \cdot [\overline{F} : \overline{K}]_{\text{sep}} \; . \tag{20}$$

If $v(K)$ *is pure in* $v(F)$ *and* $\overline{F}|\overline{K}$ *is regular then* K^h *is the exact constant field of* F*, i.e.*

$$K^h = \tilde{K} \cap F \; .$$

Conversely, if K^h *is the exact constant field of* F *then* $\overline{F}|\overline{K}$ *is regular.*

Proof: We choose elements $x_1, ..., x_r \in F$ as in the proof of Theorem 3.2. Furthermore, let $\overline{\mathcal{T}} \subset \overline{F}$ be a separating transcendence basis of $\overline{F}|\overline{K}$ with

$$[\overline{F} : \overline{K}(\overline{\mathcal{T}})] = [\overline{F} : \overline{K}]_{\text{sep}} \; .$$

Choose $y_1, ..., y_s \in \mathcal{O}_F^\times$ such that

$$\overline{\mathcal{T}} = \{\overline{y_1}, ..., \overline{y_s}\} \; .$$

As in the proof of Theorem 3.2 it is shown that the set

$$\mathcal{T} = \{x_1, ..., x_r, y_1, ..., y_s\} \subset F$$

is a valuation transcendence basis of $F|K$, that $F|K(\mathcal{T})^h$ is a finite defectless extension and that (18) and (19) hold. The latter shows that

$$(v(F) : v(K(\mathcal{T})))$$

is equal to the order of the torsion subgroup of $v(F)/v(K)$ and thus by hypothesis prime to p (if $p > 0$), and that $\overline{F}|\overline{K(\mathcal{T})}$ is separable. We conclude that $F|K(\mathcal{T})^h$ is a finite tame extension satisfying

$$[F : K(\mathcal{T})^h] = (v(F) : v(K))_{\text{tor}} \cdot [\overline{F} : \overline{K}]_{\text{sep}} \; . \tag{21}$$

Since $\overline{F}|\overline{K(\mathcal{T})}$ is a finite separable extension, Hensel's Lemma shows that there exists a finite separable extension F_1 of $K(\mathcal{T})$ within the henselian field F such that $\overline{F_1} = \overline{F}$ and $[F_1 : K(\mathcal{T})] = [\overline{F} : \overline{K(\mathcal{T})}]$. Since $v(F)/v(K(\mathcal{T}))$ is a finite group of order prime to p (if $p > 0$) and moreover $\overline{F_1} = \overline{F}$, Hensel's Lemma shows that there exists a finite separable extension F_2 of F_1 within the henselian field F such that $v(F_2) = v(F)$ and $[F_2 : F_1] = [v(F) : v(K(\mathcal{T}))]$. Since $F_2|K(\mathcal{T})$ is a separable extension we may write $F_2 = K(\mathcal{T}, a)$ for a suitable element $a \in F_2$, and we have

$$[K(\mathcal{T}, a) : K(\mathcal{T})] = (v(F) : v(K(\mathcal{T})))[\overline{F} : \overline{K(\mathcal{T})}] = [F : K(\mathcal{T})^h]$$

which together with (21) proves (20). By construction, $F|K(\mathcal{T})^h(a)$ is an immediate extension, and it is defectless by Theorem 3.1. Hence it is a trivial extension; this proves $F = K(\mathcal{T})^h(a)$.

The last two assertions of the theorem are seen as follows:
Let L be the exact constant field of F. Since F is henselian, L includes K^h. Since $L|K^h$ is algebraic, $v(L)/v(K)$ is a torsion group included in $v(F)/v(K)$, and $\overline{L}|\overline{K}$ is an algebraic extension included in $\overline{F}|\overline{K}$. By our assumption that $\overline{F}|\overline{K}$ is separable, Hensel's Lemma shows that $\overline{L}$ must be the exact constant field of $\overline{F}|\overline{K}$. Hence if K^h is the exact constant field of $F|K$, then $\overline{F}|\overline{K}$ is regular. Now assume that $\overline{F}|\overline{K}$ is regular and that $v(K)$ is pure in $v(F)$. This yields $\overline{L} = \overline{K}$ and $v(L) = v(K)$. Since K^h is defectless by hypothesis, $L|K^h$ is a defectless extension and since it is immediate, it must be trivial. This completes the proof of our theorem. □

The following theorem is a version of Theorem 3.3 for stronger assumptions on the value groups and residue fields. Note that these assumptions, even including regularity of $\overline{F}|\overline{K}$, are always fulfilled if $v(K) \prec_\exists v(F)$ and $\overline{K} \prec_\exists \overline{F}$ (cf. Lemma 8.3).

Theorem 3.4 *The situation being as in Theorem 3.2, assume in addition that $v(F)/v(K)$ has no torsion and that $\overline{F}|\overline{K}$ is separable. Then there exists a valuation transcendence basis $\mathcal{T}$ of $F|K$ and an element $a \in F$ such that*

$$F = K(\mathcal{T})^h(a)$$

which is a tame unramified extension of $K(\mathcal{T})^h$ satisfying

$$[F : K(\mathcal{T})^h] = [K(\mathcal{T}, a) : K(\mathcal{T})] = [\overline{K(\mathcal{T})}(\overline{a}) : \overline{K(\mathcal{T})}] = [\overline{F} : \overline{K}]_{\mathrm{sep}} .$$

Consequently, F is a henselian rational function field over K generated by a valuation transcendence basis if and only if $\overline{F}$ is a rational function field over $\overline{K}$. Moreover, $\overline{F}|\overline{K}$ is regular if and only if K^h is the exact constant field of $F|K$.

Proof: The theorem is an immediate consequence of Theorem 3.3 and its proof; we only have to add the information that the element $a \in F$ can be chosen as to satisfy $[K(\mathcal{T}, a) : K(\mathcal{T})] = [\overline{K(\mathcal{T})}(\overline{a}) : \overline{K(\mathcal{T})}]$. This can be done using Hensel's Lemma and the fact that $\overline{F}|\overline{K(\mathcal{T})}$ is a separable and consequently simple extension. □

A valued function field F having the properties as described in the last structure theorem will be called *henselian almost rational function field.*

If on the other hand the ground field K is not defectless, then Theorem 3.1 may be used to define a defect for the henselian function field $F|K$ and to put this defect into relation to the defect that appears in certain finite extensions of K. The details are given in section 5.

To derive model theoretic results from Theorem 3.1 we will use an embedding lemma whose algebraic part is the following:

Lemma 3.5 (Embedding Lemma I)
Let (K, v) be a defectless field (the valuation is allowed to be trivial), $F|K$ a henselian function field without transcendence defect and $(K^\star, v^\star)$ a henselian extension of (K, v). Assume that $v(F)/v(K)$ is torsion free and that $\overline{F}|\overline{K}$ is separable.
If $\rho: v(F) \longrightarrow v^\star(K^\star)$ is an embedding over $v(K)$ and $\sigma: \overline{F} \longrightarrow K^\star/v^\star$ is an embedding over $\overline{K}$, then there exists an embedding $\iota: (F, v) \longrightarrow (K^\star, v^\star)$ over (K, v) that respects ρ and σ, i.e. $v^\star(\iota(a)) = \rho(v(a))$ and $\iota(a)/v^\star = \sigma(\overline{a})$ for all $a \in F$.

Here, the embeddings of value group and residue field are understood to be monomorphisms of groups resp. fields. On the other hand, the notation "$\iota\colon (F,v) \longrightarrow (K^\star, v^\star)$" indicates an embedding of valued fields, i.e. a valuation preserving monomorphism:

$$\forall x \in F : \; x \in \mathcal{O}_F \Longleftrightarrow \iota x \in \mathcal{O}_{K^\star} \; .$$

The embedding lemma which is based on the above lemma and the model theoretic result derived from it are to be found in section 8. For the proof of Lemma 3.5, observe that in view of Theorem 3.4, it is an immediate consequence of the following

Lemma 3.6 (Embedding Lemma I′)
Let $F|K$ be a henselian function field admitting a valuation transcendence basis $\mathcal{T}$ such that $F|K(\mathcal{T})^h$ is a finite tame unramified extension. Let the assumptions on $(K^\star, v^\star)$ and on the embeddings of $v(F)$ and $\overline{F}$ be as in the preceding Embedding Lemma I. Then there exists an embedding $\iota\colon (F,v) \longrightarrow (K^\star, v^\star)$ over (K,v) that respects σ and τ.

Proof: First we will construct the embedding for $K(\mathcal{T})$ and then we will show how to prolongate it to F.

Let the given valuation transcendence basis $\mathcal{T}$ be of the form (12). We choose elements $x'_1, ..., x'_r \in K^\star$ such that $v^\star(x'_i) = \rho(v(x_i))$, $1 \leq i \leq r$. The values $v^\star(x'_1), ..., v^\star(x'_r)$ are rationally independent over $v(K)$ since the same holds for their foreimages $v(x_1), ..., v(x_r)$ and this property is preserved by every monomorphism over $v(K)$. Next, we choose elements $y'_1, ..., y'_s \in \mathcal{O}^\times_{K^\star}$ such that $y'_j/v^\star = \sigma(\overline{y_j})$, $1 \leq j \leq s$. The residues $y'_1/v^\star, ..., y'_s/v^\star$ are algebraically independent over $\overline{K}$ since the same holds for their foreimages $\overline{y_1}, ..., \overline{y_s}$ and this property is preserved by every monomorphism over $\overline{K}$. Consequently, $\mathcal{T}' = \{x'_1, ..., x'_r, y'_1, ..., y'_s\} \subset K^\star$ is an algebraically valuation–independent set over K. Recall that for every polynomial f in $K[\mathcal{T}]$, the value of f is equal to the value of (exactly) one of the monomials appearing in f (which is the unique monomial having the minimal value), cf. Lemma 2.19. The same is true for all polynomials in $K[\mathcal{T}']$. This shows that both sets $\mathcal{T}$ and $\mathcal{T}'$ are algebraically independent over K, so that the assignment

$$x_i \mapsto x'_i \; , \; y_j \mapsto y'_j \;\; 1 \leq i \leq r \, , \; 1 \leq j \leq s$$

induces an isomorphism $\iota : K(\mathcal{T}) \longrightarrow K(\mathcal{T}')$. Furthermore, it shows that this embedding of $K(\mathcal{T})$ into $K^\star$ respects the restriction of ρ to $v(K(\mathcal{T}))$ and the restriction of σ to $\overline{K(\mathcal{T})}$.

By the universal property of henselizations (cf. [RIB1], H 3), p. 176), ι extends to a valuation preserving embedding of $K(\mathcal{T})^h$ into $K^\star$ since by hypothesis, $K^\star$ is henselian. Since $K(\mathcal{T})^h|K(\mathcal{T})$ is immediate, this embedding trivially also respects the above mentioned restrictions of ρ and σ. Through this embedding, we will from now on identify $K(\mathcal{T})^h$ with its image in $K^\star$. To simplify notation, let us put $L = K(\mathcal{T})^h$.

Now we have to prolongate ι (which by our identification has become the identity) to an embedding of F into $K^\star$ (over L) which respects ρ and σ. This is done as follows:
By hypothesis, $F|L$ is finite, tame and unramified. Consequently, $\overline{F}|\overline{L}$ is a finite separable extension, generated by one element, say $\overline{a}$. Let $f \in \mathcal{O}_L[X]$ be monic such that its residue polynomial $\overline{f}$ is the minimal polynomial of $\overline{a}$ over $\overline{L}$; by hypothesis, $\overline{f}$ is separable. Hensel's Lemma shows that there exists exactly one root a of f in F having residue $\overline{a}$, and exactly one root a' of f in the henselian field $K^\star$ having residue $\sigma(\overline{a})$. The assignment

$$a \mapsto a'$$

induces an isomorphism $\iota : K(a) \longrightarrow K(a')$ which is valuation preserving since L is henselian. $F|L$ being unramified, $L(a)|L$ is unramified too. Thus ι respects ρ (which after the above identification is the identity). We have to show that ι also respects σ.

Let $n = [L(a) : L]$. Since the elements $1, \overline{a}, ..., \overline{a}^{n-1}$ are linearly independent, the basis $1, a, ..., a^{n-1}$ is a valuation basis of $L(a)|L$. Let $g(a) \in L[a]$ where $g \in L[X]$ is of degree $< n$; if the value of $g(a)$ is zero, then $g \in \mathcal{O}_L[X]$ and thus which shows $\overline{g(a)} = \overline{g}(\overline{a})$. In this case,

$$\overline{\iota(g(a))} = \overline{g(a')} = \overline{g}(\overline{a'}) = \overline{g}(\sigma\overline{a}) = \sigma(\overline{g}(\overline{a})) \ .$$

This proves that ι respects σ.
We have constructed an embedding of $L(a)$ into $K^\star$ which respects ρ and σ. But since $F|L$ is a finite tame and unramified extension, we have

$$[F : L] = [\overline{F} : \overline{L}] = [\overline{L(a)} : \overline{L}] = [L(a) : L]$$

which shows $F = L(a)$, and ι is the required embedding. □

We will now carry out a stepwise reduction of the proof of Theorem 3.1 in the subsections 3.1 to 3.6. After this reduction the proof will split into two parallel parts, one for fields of characteristic $p > 0$ (subsection 3.7) and one for fields of characteristic 0 with residue characteristic $p > 0$ (subsection 3.8). **Note that in case of $\mathrm{char}(\overline{K}) = 0$, Theorem 3.1 is a trivial consequence of the Lemma of Ostrowski (Lemma 2.4). So we will always assume in the following that**

$$p = \mathrm{char}(\overline{K}) > 0 \ .$$

3.1 Reduction to henselian rational function fields of transcendence degree 1.

To attack the problem of proving Theorem 3.1 we want to begin with the following reduction:

Lemma 3.7 *To prove Theorem 3.1, it suffices to prove*
(R) *Every henselian rational function field of transcendence degree 1 without transcendence defect over a defectless field is a defectless field.*

Proof: Assume that F is a henselian function field without transcendence defect over the defectless field K and that assertion **(R)** is true. Let $n = \mathrm{trdeg}(F|K)$.
If $n = 0$, then F is a finite extension of K and hence defectless by Lemma 2.9.
Let $n \geq 1$, and assume that Theorem 3.1 was already shown to be true for every henselian function field of transcendence degree $< n$. Choose any $x \in F$ which is transcendent over K. Let F_0 be a henselization of $K(x)$ inside the henselian field F. By Lemma 2.18, both extensions

$$F|F_0 \quad \text{and} \quad F_0|K$$

have no transcendence defect. Since Theorem 3.1 is assumed to be true for henselian function fields of transcendence degree 1 and thus for $F_0|K$, F_0 is a defectless field. By the induction hypothesis we know that Theorem 3.1 is true for every henselian function field of transcendence degree $n-1$, thus for $F|F_0$ showing that F is a defectless field. □

In the preceding proof, the transcendent element x may be chosen such that

– either its residue $\overline{x}$ is transcendent over $\overline{K}$,

– or its value $v(x)$ is rationally independent over $v(K)$.

This is possible since by hypothesis $F|K$ has no transcendence defect which means that $\overline{F}|\overline{K}$ is transcendent or $v(F)/v(K)$ has rational rank ≥ 1.
In the first case we call x *residue–transcendental*, in the second case *value–transcendental* over K. If one of these cases holds for x, we call x *valuation–transcendental*. Hence we can reformulate the lemma in the following way:

To prove Theorem 3.1, it suffices to prove:

(R1) *Every henselian rational function field of transcendence degree 1 with a valuation–transcendental generator over a defectless ground field is a defectless field.*

In the residue–transcendental case, the valuation on $K(x)$ is nothing else but the *functional valuation* or *Gauß valuation* associated to x, i.e. for a polynomial

$$f(x) = c_0 + c_1 x + \ldots + c_n x^n \in K[x]$$

we have

$$v(f(x)) = \min_{0\leq i\leq n} v(c_i)$$

and consequently

$$v(F) = v(K) \text{ and } \overline{F} = \overline{K}(\overline{x}) \ . \tag{22}$$

In the value–transcendental case, the valuation on $K(x)$ is uniquely determined by the *cut* which is induced by $v(x)$ in the divisible hull $\widetilde{v(K)} = v(\tilde{K})$ of the value group $v(K)$ (cf. subsection 2.3). Given a polynomial $f(x)$ like above we have

$$v(f(x)) = \min_{0\leq i\leq n} (v(c_i) + iv(x)) \ ,$$

and consequently

$$v(F) = v(K) \oplus \mathbb{Z}v(x) \text{ and } \overline{F} = \overline{K} \ . \tag{23}$$

3.2 Reduction to algebraically closed ground fields.

Now we want to reduce further to the case where, in addition, K is algebraically closed. Note that every algebraically closed valued field trivially is a defectless field.

Lemma 3.8 *To prove* **(R1)***, it suffices to prove*

(R2) *Every henselian rational function field of transcendence degree 1 with a valuation–transcendental generator over an algebraically closed ground field is a defectless field.*

To prove this lemma, we assume $K(x)^h$ to be a henselian rational function field over the defectless field K with a valuation–transcendental generator x as described in the last section. The assumption of the lemma implies that the henselian rational function field $\tilde{K}(x)^h$, which is a henselization of the rational function field $\tilde{K}(x)$, is a defectless field. From this we have to deduce that $K(x)^h$ is a defectless field. This deduction will be done now in a more general setting. For its description we need the following definition. Let $F|K$ be an extension of valued fields. We call $F|K$ *valuation–regular* if

– the residue field extension $\overline{F}|\overline{K}$ is regular

– the value factor group $v(F)/v(K)$ is torsionfree.

If K is a henselian defectless field then every valuation–regular extension $F|K$ is regular. This can be seen as follows. Since $\overline{K}$ is relatively algebraically closed in $\overline{F}$ and $v(F)/v(K)$ is torsionfree, the relative algebraic closure of K in F must be an immediate extension of K. Hence it is equal to K since K is defectless and thus admits no nontrivial immediate algebraic extensions. To show that F is linearly disjoint from $\tilde{K}$ over K, let $L|K$ be a finite extension. By assumption on K, $L|K$ is defectless, i.e. $[L:K] = [\overline{L}:\overline{K}]\cdot(v(L):v(K))$. Since $\overline{F}|\overline{K}$ is regular and $v(F)/v(K)$ is torsionfree, we have (for any fixed extension of the valuation v from F to $F.L$):

$$[\overline{F.L}:\overline{F}] \geq [\overline{F}.\overline{L}:\overline{F}] = [\overline{L}:\overline{K}]$$
$$(v(F.L):v(F)) \geq (v(F)+v(L):v(F)) = (v(L):v(K))\ ,$$

hence

$$\begin{aligned}[F.L:F] &\geq (v(F.L):v(F))\cdot[\overline{F.L}:\overline{F}]\\ &\geq (v(L):v(K))\cdot[\overline{L}:\overline{K}] = [L:K] \geq [F.L:F]\end{aligned}$$

showing that in all these inequalities, "=" holds everywhere and that there exists only one extension of the valuation v from F to $F.L$. In particular, we get

$$v(F.L) = v(F)+v(L) \text{ and } \overline{F.L} = \overline{F}.\overline{L}\ .$$

This will also hold if $L|K$ is an infinite extension since for every $\alpha \in v(F.L)$ there is already a finite extension $L_0|K$ such that $\alpha \in v(F.L_0)$, and a similar argument works for the residue fields. We have proved:

Lemma 3.9 *Let K be a henselian defectless field and $F|K$ a valuation–regular extension. Then the following holds: for every finite extension $L|K$ there exists a unique valuation v on $F.L$ extending the valuation v of F, and $F.L|F$ is defectless. Furthermore, every algebraic extension $L|K$ satisfies:*

$$v(F.L) = v(F)+v(L) \quad \textit{and} \quad \overline{F.L} = \overline{F}.\overline{L}\ . \tag{24}$$

On the other hand, it proves that F is linearly disjoint from every finite extension L over K. This yields the following

Lemma 3.10 *If K is a henselian defectless field and $F|K$ is a valuation–regular extension, then $F|K$ is regular.*

From (22) and (23) we see that $K(x)^h|K$ is valuation–regular if x is valuation–transcendental over K. Thus our Reduction Lemma 3.8 will follow from

Lemma 3.11 *Let F be a valuation–regular extension of the defectless field K. If $F.\tilde{K}$ is a defectless field, then F is also a defectless field.*

Proof: Assume that $F.\tilde{K}$ is a defectless field. To prove that F is a defectless field we may assume by Lemma 2.3 that F is henselian; indeed, $F|K$ is valuation–regular if and only if $F^h|K$ is valuation–regular, and $F.\tilde{K}$ is a defectless field if and only $F^h.\tilde{K}$ is a defectless field (this follows from Lemma 2.3 and $(F.\tilde{K})^h = (F^h.\tilde{K})^h$). We may also assume that K is henselian since in view of the fact that the henselization of K is an immediate extension of K, we may replace K by its henselization in F^h. Given a finite extension $E|F$, we want to show that $E|F$ is defectless. By our assumption on $F.\tilde{K}$ we know that $E.\tilde{K}|F.\tilde{K}$ is defectless, i.e.

$$[E.\tilde{K} : F.\tilde{K}] = (v(E.\tilde{K}) : v(F.\tilde{K})) \cdot [\overline{E.\tilde{K}} : \overline{F.\tilde{K}}]$$

since F and thus also $F.\tilde{K}$ is henselian. Now we choose a finite extension L of K as large as to guarantee

$$[E.L : F.L] = [E.\tilde{K} : F.\tilde{K}] \; ;$$

this equation then will also hold for every algebraic extension of L in the place of L. After a suitable finite enlargement of L, also the following equations will be satisfied:

$$v(E.L) + v(F.\tilde{K}) = v(E.\tilde{K})$$
$$\overline{E.L} \, . \, \overline{F.\tilde{K}} = \overline{E.\tilde{K}} \; ;$$

this is true because the extensions

$$v(E.\tilde{K}) \supset v(F.\tilde{K}) \quad \text{and} \quad \overline{E.\tilde{K}} \,|\, \overline{F.\tilde{K}}$$

are finite and thus generated by the values resp. residues of finitely many elements from $E.\tilde{K}$.

Using the preceding equations, we deduce

$$\begin{aligned} [E.L : F.L] &\geq (v(E.L) : v(F.L)) \cdot [\overline{E.L} : \overline{F.L}] \\ &\geq (v(E.L) + v(F.\tilde{K}) : v(F.L) + v(F.\tilde{K})) \cdot [\overline{E.L}.\overline{F.\tilde{K}} : \overline{F.L}.\overline{F.\tilde{K}}] \\ &= (v(E.\tilde{K}) : v(F.\tilde{K})) \cdot [\overline{E.\tilde{K}} : \overline{F.\tilde{K}}] = [E.\tilde{K} : F.\tilde{K}] = [E.L : F.L] \; , \end{aligned}$$

hence "=" holds everywhere showing that $E.L|F.L$ is a defectless extension. On the other hand, we know by Lemma 3.9 that $F.L|F$ is defectless. Consequently, $E.L|F$ and its subextension $E|F$ are defectless. This proves our lemma, thereby completing the proof of Lemma 3.8. □

3.3 Reduction to finite rank.

In this section we want to show that for the proof of **(R2)** we may assume in addition that the ground field K has finite rank. Again, we prove a more general lemma. For its formulation, we need a definition that generalizes the definition of "valuation–regular" given in the previous section. Two extensions $L|K$ and $N|K$ are called *valuation–disjoint* if

(a) $\overline{L}$ is linearly disjoint from $\overline{N}$ over $\overline{K}$,
(b) $v(L)$ is linearly disjoint from $v(N)$ over $v(K)$ (in the group–theoretical sense).

Note that an extension $F|K$ is valuation–regular if and only if it is valuation–disjoint from $\tilde{K}|K$.

Lemma 3.12 *Let K be a valued field and $E_1|F_1$ an extension of valued fields such that $K.E_1|K.F_1$ is h–finite. Assume further that there exists a subfield k of F_1 and K such that*

1. *$k.E_1|k.F_1$ is defectless,*
2. *$k.E_1|k.F_1$ is valuation–disjoint from $K.F_1|k.F_1$.*

Then these properties remain true if k is replaced by an arbitrary subfield K' of K containing k. In particular, $K.E_1|K.F_1$ is defectless, and

$$[(k.E_1)^h : (k.F_1)^h] = [(K'.E_1)^h : (K'.F_1)^h] = [(K.E_1)^h : (K.F_1)^h] \,.$$

Proof: Since $v(k.E_1)$ is linearly disjoint from $v(K.F_1)$ over $v(k.F_1)$ and $\overline{k.E_1}$ is linearly disjoint from $\overline{K.F_1}$ over $\overline{k.F_1}$, we find for every subfield K' of K which contains k:

$$\begin{aligned}(v(K'.E_1) : v(K'.F_1)) &\geq (v(k.E_1) + v(K'.F_1) : v(K'.F_1)) = (v(k.E_1) : v(k.F_1))\\ [\overline{K'.E_1} : \overline{K'.F_1}] &\geq [\overline{k.E_1}.\overline{K'.F_1} : \overline{K'.F_1}] = [\overline{k.E_1} : \overline{k.F_1}] \,.\end{aligned}$$

Using this and the hypothesis that $k.E_1|k.F_1$ is defectless, we deduce

$$\begin{aligned}[(k.E_1)^h : (k.F_1)^h] &\geq [(K'.E_1)^h : (K'.F_1)^h] \geq (v(K'.E_1) : v(K'.F_1)) \cdot [\overline{K'.E_1} : \overline{K'.F_1}]\\ &\geq (v(k.E_1) : v(k.F_1)) \cdot [\overline{k.E_1} : \overline{k.F_1}] = [(k.E_1)^h : (k.F_1)^h]\end{aligned}$$

showing that "=" holds everywhere. This proves that $K'.E_1|K'.F_1$ is defectless and that $[(K'.E_1)^h : (K'.F_1)^h] = [(k.E_1)^h : (k.F_1)^h]$. Moreover, it yields

$$\begin{aligned}v(K'.E_1) &= v(k.E_1) + v(K'.F_1)\\ \overline{K'.E_1} &= \overline{k.E_1}\,.\,\overline{K'.F_1}\,.\end{aligned}$$

On the other hand, $v(k.E_1) + v(K'.F_1)$ is linearly disjoint from $v(K.F_1)$ over $v(K'.F_1)$ as well as $\overline{k.E_1}\,.\,\overline{K'.F_1}$ is linearly disjoint from $\overline{K.F_1}$ over $\overline{K'.F_1}$. This proves that $K'.E_1$ is valuation–disjoint from $K.F_1$ over $K'.F_1$. □

With the help of this lemma, we achieve the next reduction step:

Corollary 3.13 *To prove* **(R2)***, it suffices to prove*

(R3) *Every henselian rational function field of transcendence degree 1 with a valuation–transcendental generator over an algebraically closed ground field of finite rank is a defectless field.*

Proof: Let $F = K(x)^h$ be a henselian rational function field with a valuation–transcendental generator x over the algebraically closed field K. Let E be a finite extension of F. Assuming **(R3)** we want to show that $E|F$ is defectless. Let k' be a finitely generated field and E_1 a finite extension of $k'(x)$ such that

1. $(K.E_1)^h = E$,
2. $[E_1^h : k'(x)^h] = [E : K(x)^h]$.

The second property will remain true for every subfield of K containing k', hence in particular for the algebraic closure k of k' that is contained in the algebraically closed field K. (k, v) has finite rank since k has finite transcendence degree over its prime field. By hypothesis **(R3)** and Lemma 2.3, the extension $k.E_1|k(x)$ is defectless. If we are able to show that $k.E_1$ is valuation–disjoint from $K(x)$ over $k(x)$, then it will follow from the previous lemma that the extension $K.E_1|K(x)$ is defectless. Since $(K.E_1)^h = E$ and $K(x)^h = F$, this yields that $E|F$ is defectless, as desired.

It remains now to prove that $k.E_1$ is valuation–disjoint from $K(x)$ over $k(x)$. We consider the following two cases:

<u>case 1:</u> x is value–transcendental over K. Then $v(k(x)) = v(k) \oplus \mathbb{Z}v(x)$, $v(K(x)) = v(K) \oplus \mathbb{Z}v(x)$ and $v(k.E_1) = v(k) \oplus \mathbb{Z}\frac{1}{n}v(x)$ for a suitable integer n since $v(k)$ is divisible. This shows that $v(k.E_1)$ is linearly disjoint from $v(K(x))$ over $v(k(x))$ since $v(x)$ is rationally independent over $v(K)$. Furthermore, $\overline{k(x)} = \overline{k} = \overline{k.E_1}$ since $\overline{k}$ is algebraically closed, showing that $\overline{k.E_1}$ is linearly disjoint from $\overline{K(x)} = \overline{K}$ over $\overline{k(x)}$.

<u>case 2:</u> x is residue–transcendental over K. Then $\overline{k(x)} = \overline{k}(\overline{x})$ where $\overline{k}$ is algebraically closed, showing that $\overline{k.E_1}$ being an algebraic extension of $\overline{k}(\overline{x})$ is linearly disjoint from $\overline{K(x)} = \overline{K}(\overline{x})$ over $\overline{k(x)}$ since $\overline{x}$ is transcendental over $\overline{K}$. Furthermore, $v(k(x)) = v(k)$, $v(K(x)) = v(K)$ and $v(k.E_1) = v(k)$ since $v(k)$ is divisible. This shows that $v(k.E_1)$ is linearly disjoint from $v(K(x))$ over $v(k(x))$.

This completes the proof of our corollary. □

3.4 Reduction to rank 1.

In the previous subsection, we have reduced the problem of proving Theorem 3.1 to the proof of the following assertion:

(R3) *Every henselian rational function field F of transcendence degree 1 with a valuation–transcendental generator over an algebraically closed ground field K of finite rank is a defectless field.*

Note that in this case we have $rk(K) \leq rk(F) \leq rk(K) + 1$. Now we want to reduce further to the case where the rank of F is 1.

Lemma 3.14 *To prove* **(R3)**, *it suffices to prove*

(R4) *Every henselian rational function field of transcendence degree 1 and rank 1 with a valuation–transcendental generator over an algebraically closed ground field is a defectless field.*

Proof: Let $F = K(x)^h$ satisfy the assumptions of **(R3)**. The assertion of **(R3)** is trivial in the case where $\mathrm{rk}(F)$ is zero since every trivially valued field is automatically a defectless field.

Assuming **(R4)** we want to show now that **(R3)** is true whenever $\infty > \mathrm{rk}(F) \geq 1$. This will be done by induction on $\mathrm{rk}(F)$, the case $\mathrm{rk}(F) = 1$ being covered by our hypothesis that **(R4)** is true. We assume now $n > 1$ and that **(R3)** is true whenever $\mathrm{rk}(F) < n$. If $\mathrm{rk}(F) = n$, then the place $P = P_v$ associated to v on F allows a decomposition

$$P = Q\overline{Q}$$

where Q and $\overline{Q}$ are both places of rank $< n$.

By Lemma 2.3, it suffices to prove that $K(x)$ is a defectless field. Consequently, we will consider the fields $(K(x), Q)$ and $(K(x)Q, \overline{Q})$. To begin with, we note that by hypothesis K and thus also KQ are algebraically closed fields. Now we will consider the following three cases:

<u>case 1:</u> x is residue–transcendental over K. It follows that also xQ is residue–transcendental over KQ, and in particular, it must be transcendental over KQ (according to Lemma 2.19). Hence x is a residue–transcendental generator of the rational function field $(K(x), Q)$ over (K, Q). Lemma 2.20 shows that xQ is a generator of the rational function field $(K(x)Q, \overline{Q}) = (KQ(xQ), \overline{Q})$ over the algebraically closed field KQ. By our induction hypothesis and Lemma 2.3, we know that $(K(x), Q)$ and $(K(x)Q, \overline{Q})$ are defectless fields.

<u>case 2:</u> x is value–transcendental over K, but $v_Q(x)$ is not rationally independent over $v_Q(K)$. Then there is an element $c \in K$ such that $v_Q(cx) = 0$; this is true since $v_Q(K)$, being the value group of an algebraically closed field, is divisible. As x is value–transcendental over (K, P), so is cx. Consequently, w.l.o.g. we may assume from the start that $c = 1$. It follows that xQ must be value–transcendental over $(KQ, \overline{Q})$, and in particular, it must be transcendental over KQ (according to 2.19). As in case 1 it is now shown that $(K(x), Q)$ and $(K(x)Q, \overline{Q}) = (KQ(xQ), \overline{Q})$ are defectless fields.

<u>case 3:</u> x is value–transcendental over K and $v_Q(x)$ is rationally independent over $v_Q(K)$. This yields that x is a value–transcendental generator of $(K(x), Q)$ over (K, Q), and from our induction hypothesis we obtain that $(K(x), Q)$ is a defectless field. Furthermore it follows by (23) that $K(x)Q$ equals the algebraically closed field KQ. In this case, $(K(x)Q, \overline{Q})$ is trivially a defectless field.

We have obtained in every case that $(K(x), Q)$ and $(K(x)Q, \overline{Q})$ are defectless fields. By virtue of Lemma 2.17 this implies that $(K(x), P)$ is a defectless field, proving our lemma and **(R3)** for $\mathrm{rk}(F) = n$. □

Note that if $\mathrm{rk}(F) = 1$ then K may very well be trivially valued. But this can only appear in the case where x is value–transcendental over K.

3.5 Reduction to extensions of degree p $(= \mathrm{char}(\overline{K}) > 0)$.

To prove **(R4)**, we have to prove that every finite extension $E|F$ of a henselian rational function field $F = K(x)^h$ satisfying the assumptions of **(R4)**, is defectless. But the structure of such extensions cannot be easily determined. Hence we would like to reduce this problem to the investigation of classes of extensions which are more easily describable. The key to this reduction is the following Lemma together with Lemma 3.17 that we will prove below. Recall that by our general assumption for the reduction steps, $p = \mathrm{char}(\overline{K}) > 0$.

Lemma 3.15 *For every finite extension $L|K$, the extension $L.K^r|K^r$ is a tower of normal extensions of degree p, the separable ones being Galois extensions. For every finite extension $L|K$, there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

Proof: We know from Lemma 2.10 that $K^{sep}|K^r$ is a p–extension. Then for every intermediate fields L_1, L_2 with $[L_1 : L_2] = p$, it follows that $L_1|L_2$ is normal since in general the degree of the normal hull of L_1 over L_2 must be a divisor of $p!$, whereas in our case it can only be equal to a power of p; this shows that it must be equal to p. The finite tame extension N of K^h is obtained by letting it be generated by all the finitely many elements of K^r that are necessary for the defining relations of the extensions forming the tower $L.K^r|K^r$. □

The lemma shows for our present case, that $E.K(x)^r|K(x)^r$ is a tower of normal extensions of degree p, the separable ones being Galois extensions and thus having the form of Artin–Schreier–extensions if $\mathrm{char}(K) = p > 0$. These two classes, the purely inseparable extensions of degree p and the Galois extensions of degree p, are the classes that we looked for. And indeed, in view of Lemma 2.11 it suffices to prove that $E.K(x)^r|K(x)^r$ is defectless. But unfortunately $K(x)^r$ is not any more of the form $K(x)^h$ to which we had reduced because it is so easy to handle. This problem is solved by the observation that given a finite extension $E|F$, it suffices already to take a finite tame extension $N|K(x)^h$ to achieve that $E.N|N$ decomposes into a tower as described above. And Lemma 3.17 will show that N is again (almost) of the form $K(x)^h$. The next problem appearing is the fact that we are dealing with a tower of extensions of degree p, not only with a single one. So we will use induction, but for that we have to know that the nice form of the field N that we will describe in Lemma 3.17 is maintained in every step of the induction, i.e. that the given extensions of degree p inherit this form. This will be shown in Lemma 3.19.

To begin with, we want to fix exactly what we meant by "(almost) of the form $K(x)^h$". In the sequel we have to deal with the following special form of henselian almost rational function fields:

$$F = K(x,y)^h \text{ where } \left\{ \begin{array}{l} K \text{ is algebraically closed} \\ x \text{ is valuation–transcendental over } K \\ F|K(x)^h \text{ is finite, tame and unramified} \\ y = 0 \text{ if } x \text{ is value–transcendental;} \\ \text{otherwise } \overline{F} = \overline{K}(\overline{x},\overline{y}) \text{ and} \\ [K(x,y) : K(x)] = [\overline{K}(\overline{x},\overline{y}) : \overline{K}(\overline{x})] \,. \end{array} \right\} \tag{25}$$

Altogether, the following considerations will show

Lemma 3.16 *To prove* **(R4)**, *it suffices to prove*

(R5): *Let F be of rank 1 and of the form (25). Then every Galois extension of degree p and every purely inseparable extension of degree p is defectless.*

We begin with the proof of the fact that finite tame extensions preserve the form (25).

Lemma 3.17 *Let K be an algebraically closed valued field.*
a) Let $F = K(x)^h$ be a henselian rational function field with a value–transcendental generator x over K. If $N|F$ is a tame extension of degree n, then N is a henselian rational function field $K(x')^h$ with a value–transcendental generator x' over K satisfying $(x')^n = x$.
b) Let $F = K(x,y)^h$ be a henselian almost rational function field of the form (25) where x is residue–transcendental over K. If $N|F$ is a tame extension of degree n, then N is a henselian almost rational function field $K(x,y')^h$ for a suitable $y' \in N$ with

$$[K(x,y') : K(x)] = [\overline{K}(\overline{x},\overline{y'}) : \overline{K}(\overline{x})] = n \cdot [\overline{K}(\overline{x},\overline{y}) : \overline{K}(\overline{x})] \,. \tag{26}$$

Proof: a) Since $N|F$ is tame, hence defectless, and since $\overline{K(x)} = \overline{K}$ is algebraically closed, being the residue field of an algebraically closed field, we have $\overline{N} = \overline{K(x)}$ and

$$n = (v(N) : v(K(x))) \cdot [\overline{N} : \overline{K(x)}] = (v(N) : v(K(x))) \;.$$

In particular, this shows $(p, n) = 1$. Since $v(K)$ is divisible, being the value group of an algebraically closed field, this yields $v(N) = v(K) \oplus \mathbb{Z}\frac{v(x)}{n}$. Knowing that $\frac{v(x)}{n} \in v(N)$ and that $\overline{N} = \overline{K(x)}$, we deduce from Hensel's Lemma that there exists an element x' in the henselian field N such that $(x')^n = x$, hence $v(x') = \frac{v(x)}{n}$. Now

$$n = [N : K(x)^h] \geq [K(x')^h : K(x)^h] \geq (v(K(x')) : v(K(x))) = n$$

shows $N = K(x')^h$. By construction, x' is value–transcendental over K.
b) Since $v(K(x)) = v(K)$ is divisible, we have $v(N) = v(K) = v(K(x, y))$. $N|F$ is tame by assumption, hence defectless, whence

$$n = (v(N) : v(K(x, y))) \cdot [\overline{N} : \overline{K(x, y)}] = [\overline{N} : \overline{K(x, y)}]$$

where $\overline{N}|\overline{K(x, y)}$ is separable. Since by hypothesis $\overline{K(x, y)}|\overline{K(x)}$ is also separable, the extension $\overline{N}|\overline{K(x)}$ is separable and thus admits a primitive element which we will call $\overline{\eta}$. Knowing that $\overline{\eta} \in \overline{N}$ and that $\overline{\eta}$ is separable over $\overline{K(x)}$, we deduce from Hensel's Lemma that there exists an element y' in the henselian field N such that $\overline{y'} = \overline{\eta}$. Since it is possible to choose a normed polynomial over $K(x)$ whose reduction is the minimal polynomial of $\overline{\eta}$ over $\overline{K(x)}$, we can even achieve that

$$[K(x, y') : K(x)] = [\overline{N} : \overline{K(x)}] \;, \tag{27}$$

hence

$$[K(x, y') : K(x)] = [\overline{N} : \overline{K(x, y)}] \cdot [\overline{K(x, y)} : \overline{K(x)}] = n \cdot [\overline{K(x, y)} : \overline{K(x)}] \;. \tag{28}$$

Now $\overline{K(x, y')} \supset \overline{K}(\overline{x}, \overline{\eta}) = \overline{N}$, and

$$n = [N : K(x, y)^h] \geq [K(x, y')^h : K(x, y)^h] \geq [\overline{K(x, y')} : \overline{K(x, y)}] \geq [\overline{N} : \overline{K(x, y)}] = n$$

shows that $N = K(x, y')^h$ and $\overline{N} = \overline{K}(\overline{x}, \overline{y'})$. Together with (27) and (28) this yields also equation (26). □

Corollary 3.18 *Assume that* **(R5)** *is true and let F be of rank 1 and of the form (25). Then F does not admit any nontrivial immediate algebraic extension.*

Proof: It suffices to prove that F does not admit any nontrivial finite immediate extension. Let $E|F$ be an arbitrary nontrivial extension. If it is tame and thus defectless, there is nothing to show. Otherwise we may choose by Lemma 3.15 a finite tame extension $N|F$ such that $E.N|N$ is a (nonempty) tower of normal extensions of degree p and hence possesses a normal subextension $E'|N$ of degree p. By the preceeding lemma, N is again of the form (25), and it has also rank 1 since it is an algebraic extension of F. From our hypothesis that **(R5)** is true it follows that $E'|N$ is defectless. Hence $E.N|N$ cannot be an immediate extension and in view of Corollary 2.12 this yields that $E|F$ cannot be an immediate extension. This proves our assertion. □

The next lemma shows how finite defectless purely wild extensions preserve the form (25). The foregoing corollary is a main ingredient of its proof.

Lemma 3.19 *Let K be an algebraically closed field and assume that* **(R5)** *is true.*
a) Let $F = K(x)^h$ be a henselian rational function field with a value–transcendental generator x over K. If $F'|F$ is a defectless extension of degree p, then it must be purely wild and F' is a henselian rational function field $K(x')^h$ of the same rank as F with a value–transcendental generator x' over K satisfying $p \cdot v(x') = v(x)$.
b) Let $F = K(x,y)^h$ be a henselian almost rational function field of the form (25) where x is residue–transcendental over K. If $F'|F$ is a purely wild defectless extension of degree p, then F' is a henselian almost rational function field $K(x',y')^h$ of the same rank as F with x' residue–transcendental over K and

$$[K(x',y') : K(x')] = [\overline{K}(\overline{x'},\overline{y'}) : \overline{K}(\overline{x'})] = [\overline{K}(\overline{x},\overline{y}) : \overline{K}(\overline{x})]$$

and

$$\overline{F'} = \overline{K(x',y')} = \overline{K}(\overline{x'},\overline{y'}) = \overline{K}(\overline{x},\overline{y})^{1/p}$$

with $\overline{x'} = \overline{x}^{1/p}$ and $\overline{y'} = \overline{y}^{1/p}$. Hence F' is a tame unramified extension of $K(x')^h$ and thus also of the form (25).

Note that the version of b) where $F'|F$ is tame, is already contained in part b) of Lemma 3.17.

Proof: a) Since $F'|F$ is defectless, and since $\overline{K(x)} = \overline{K}$ is algebraically closed (being the residue field of an algebraically closed field), we have $\overline{F'} = \overline{K(x)}$ and thus

$$p = (v(F') : v(K(x))) \cdot [\overline{F'} : \overline{K(x)}] = (v(F') : v(K(x))) \, .$$

Since $v(K)$ is divisible (being the value group of an algebraically closed field), this yields $v(F') = v(K) \oplus \mathbb{Z}\frac{v(x)}{p}$. We choose an element $x' \in F'$ such that $p \cdot v(x') = v(x)$, hence

$$v(K(x')) = v(K) \oplus \mathbb{Z}\frac{v(x)}{p} = v(F') \, .$$

On the other hand, $\overline{K(x')} = \overline{K} = \overline{F'}$. Now the henselian field F' contains the henselization $K(x')^h$, and we have just shown that $F'|K(x')^h$ is an immediate extension. But $K(x')^h$ satisfies the conditions of Corollary 3.18, and consequently it does not admit any proper immediate algebraic extension. This yields $F' = K(x')^h$. By construction, x' is value–transcendental over K.
b) Since $F'|F$ is defectless and since $v(K(x)) = v(K)$ is divisible, we have $v(F') = v(K(x)) = v(K(x,y))$ and

$$p = (v(F') : v(K(x,y))) \cdot [\overline{F'} : \overline{K(x,y)}] = [\overline{F'} : \overline{K(x,y)}]$$

where $\overline{F'} \,|\, \overline{K(x,y)}$ is purely inseparable since $F'|F$ is purely wild by assumption. $\overline{K}$ being algebraically closed, $\overline{K(x,y)} = \overline{K}(\overline{x},\overline{y})$ has p–degree 1 and hence we have

$$\overline{F'} = \overline{K}(\overline{x},\overline{y})^{1/p} = \overline{K}(\overline{x}^{1/p},\overline{y}^{1/p}) \, .$$

We choose elements $x',y' \in F'$ such that $\overline{x'} = \overline{x}^{1/p}$ and $\overline{y'} = \overline{y}^{1/p}$, hence $\overline{K(x',y')} = \overline{F'}$. On the other hand, $v(K(x',y')) = v(K) = v(F')$. Now the henselian field F' contains the henselization $K(x',y')^h$, and we have just shown that $F'|K(x',y')^h$ is an immediate extension. But $K(x',y')^h$ satisfies the conditions of Corollary 3.18, and consequently it does not admit any proper immediate algebraic extension. This yields $F' = K(x',y')^h$. From $\overline{x'} = \overline{x}^{1/p}$ and $\overline{y'} = \overline{y}^{1/p}$ it follows that $[\overline{K}(\overline{x'},\overline{y'}) : \overline{K}(\overline{x'})] = [\overline{K}(\overline{x},\overline{y}) : \overline{K}(\overline{x})]$.

By assumption, $K(x,y)^h$ is a tame extension of $K(x)^h$, thus the extension $\overline{K}(\overline{x},\overline{y})|\overline{K}(\overline{x})$ is separable. Consequently, $\overline{K}(\overline{x}^{1/p},\overline{y}^{1/p})|\overline{K}(\overline{x}^{1/p})$ is separable too. This shows that by Hensel's Lemma we may actually choose y' in the henselian field F' such that

$$[K(x',y') : K(x')] = [\overline{K}(\overline{x'},\overline{y'}) : \overline{K}(\overline{x'})]\ .$$

Moreover, $v(F') = v(K) = v(K(x'))$ shows that $F'|K(x')^h$ is unramified, and it is a tame extension since $\overline{F'}\,|\,\overline{K(x')}$ is separable, as we had just shown. Altogether, we have proved that F' is again of the form (25). □

Now we are ready for the

Proof of Lemma 3.16:
Let F satisfy the conditions of **(R4)**. Given an arbitrary finite extension $E|F$, we have to show that it is defectless. As explained already, by Lemma 2.11 it suffices to prove that $E.N|N$ is defectless, where $N|F$ is a finite tame extension such that $E.N|N$ is a tower of normal extensions of degree p. By Lemma 3.17, N satisfies the conditions of **(R5)** (note that N is of rank 1 because it is an algebraic extension of F). The proof that $E.N|N$ is defectless is now done by induction on the number of extensions appearing in the tower. If this number is zero, the assertion is trivial. Otherwise there exists a normal subextension $E'|N$ of $E.N|N$ of degree p. From **(R5)** it follows that this extension is defectless, and from Lemma 3.19 we infer that E' again satisfies the conditions of **(R5)** (again, its rank is 1 since it is an algebraic extension of N). By the induction hypothesis, $E.N|E'$ is also defectless since it has a smaller degree than $E.N|N$. Hence by Lemma 2.8, $E.N|N$ is defectless. □

3.6 Inseparably defectless fields. Reduction to Galois extensions of degree p.

We will now treat the case of a purely inseparable extension. We will do this under more general conditions than that of **(R5)**.

Lemma 3.20 *Let F be a subhenselian function field with valuation transcendence basis $\mathcal{T}$ over K and assume that $F^h|K(\mathcal{T})^h$ is a tame extension. If K is an inseparably defectless field of characteristic $p>0$, then F is an inseparably defectless field too.*

Proof: In view of Lemma 2.3 and Lemma 2.11, it suffices to show that $K(\mathcal{T})$ is an inseparably defectless field. Let $\mathcal{T}$ be of the form (12), cf. page 25.
Every finite purely inseparable extension L of $K(\mathcal{T})$ is contained in an extension $E = K'(\mathcal{T}^{1/p^e}) = K'(t^{1/p^e} \mid t \in \mathcal{T})$ for a suitable $e \in \mathbb{N}$ and some finite purely inseparable extension K' of K. Since $K'|K$ is algebraic, $\mathcal{T}^{1/p^e}$ will again be a valuation transcendence basis over K' which shows that

$$\begin{aligned} v\left(K'(\mathcal{T}^{1/p^e})\right) &= v(K') \oplus \mathbb{Z}v(x_1^{1/p^e}) \oplus \ldots \oplus \mathbb{Z}v(x_r^{1/p^e}) \\ &= v(K') \oplus \mathbb{Z}\frac{v(x_1)}{p^e} \oplus \ldots \oplus \mathbb{Z}\frac{v(x_r)}{p^e} \end{aligned}$$

and

$$\overline{K'(\mathcal{T}^{1/p^e})} = \overline{K'}(\overline{y_1^{1/p^e}}, \dots, \overline{y_s^{1/p^e}}) = \overline{K'}(\overline{y_1}^{1/p^e}, \dots, \overline{y_s}^{1/p^e}) \;,$$

whence

$$\begin{aligned} [E : K(\mathcal{T})] &= [K' : K] \cdot p^{e(r+s)} = (v(K') : v(K)) \cdot p^{er} \cdot [\overline{K'} : \overline{K}] \cdot p^{es} \\ &= (v(E) : v(K(\mathcal{T}))) \cdot \left[\overline{E} : \overline{K(\mathcal{T})}\right] \end{aligned}$$

since by hypothesis $K'|K$ is defectless. This equation shows that $E|K(\mathcal{T})$ and thus also its subextension $L|K(\mathcal{T})$ is defectless. (Note that every purely inseparable algebraic extension admits a unique prolongation of the valuation.) Since $L|K(\mathcal{T})$ was an arbitrary finite purely inseparable extension, we have shown that $K(\mathcal{T})$ is an inseparably defectless field. □

Having proved this theorem, we don't have to consider inseparable extensions any longer. Namely, we get the following reduction:

Corollary 3.21 *To prove* **(R5)***, it suffices to prove*

(R6): *Let F be of rank 1 and of the form (25). Then every Galois extension of degree p is defectless.*

The proof of **(R6)** will now be split into two different cases. First we will consider the case of K having the same characteristic $p > 0$ as its residue field $\overline{K}$. In this case a Galois extension of degree p is nothing else but an Artin–Schreier–extension. The second case will be the case of "mixed characteristic" where the field K has characteristic 0 whereas $\overline{K}$ has characteristic $p > 0$. The case $\mathrm{char}(K) = 0 = \mathrm{char}(\overline{K})$ doesn't appear here since in that case by the Lemma of Ostrowski, every field is a defectless field and therefore Theorem 3.1 is trivially true.

3.7 Galois extensions of degree p in characteristic p.

We want to prove **(R6)** in the case $\mathrm{char}(K) = \mathrm{char}(\overline{K}) = p > 0$. In this case, every Galois extension $E|F$ of degree p is an Artin–Schreier–extension:

$$E = F(\vartheta), \quad \wp(\vartheta) = \vartheta^p - \vartheta = a \in F \;. \tag{29}$$

For $c \in F$ we have

$$E = F(\vartheta + c), \quad \wp(\vartheta + c) = \vartheta^p + c^p - \vartheta - c = a + \wp(c) \in F \tag{30}$$

showing that all elements of the class $a + \wp(F)$ determine the same Artin–Schreier–extension. We will choose a suitable representative from which we are able to read off immediately that the extension is defectless. The following lemma shows that in the present case, $\mathcal{O}_F \subset \wp(F)$:

Lemma 3.22 *Assume F is a henselian field of arbitrary characteristic and $F(\vartheta)|F$ is an extension with $\vartheta^p - \vartheta = a \in F$. Then $\mathcal{M}_F \subset \wp(F)$ and consequently, $F(\vartheta)|F$ is trivial if $v(a) > 0$. If in addition $\overline{F}$ is closed under Artin–Schreier–extensions, then $\mathcal{O}_F \subset \wp(F)$; hence in this case $F(\vartheta)|F$ is trivial if $v(a) \geq 0$.*

Proof: If $a \in \mathcal{M}_F$, then the (separable) polynomial $Y^p - Y - \overline{a} = Y^p - Y$ always admits 0 as a simple root over $\overline{F}$. If $a \in \mathcal{O}_F$, then the (separable) polynomial $Y^p - Y - \overline{a}$ admits a simple root over $\overline{F}$ if $\overline{F}$ is closed under Artin–Schreier–extensions. By Hensel's Lemma, this shows $a \in \wp(F)$ in either case. $\square$

Corollary 3.23 *Assume F is a henselian field of characteristic $p > 0$ and $F(\vartheta)|F$ is a nontrivial purely wild extension with $\vartheta^p - \vartheta = a \in F$. Then* $\mathrm{dist}(\vartheta, F) \leq 0$ *and if* $\mathrm{dist}(\vartheta, F) = 0$*, then this distance is not assumed by an element of F.*

Proof: If $\mathrm{dist}(\vartheta, F) > 0$ or if if $\mathrm{dist}(\vartheta, F) = 0$ is assumed by an element of F then there exists an element $c \in F$ such that $v(\vartheta - c) \geq 0$ and consequently $v(a - c^p + c) = v((\vartheta - c)^p - (\vartheta - c)) \geq 0$. Then by the foregoing lemma, either $\vartheta - c \in F$, or $v(\vartheta - c) = 0$ and $\overline{F(\vartheta - c)}|\overline{F}$ is a separable Artin–Schreier–extension of degree p. In both cases, it follows that the extension $F(\vartheta)|F$ is tame since it is equally generated by the element $\vartheta - c$. $\square$

3.7. **A** We will first discuss the ramified case:

$$\left.\begin{array}{l} F = K(x)^h \text{ is of rank 1 and } K \text{ is algebraically closed} \\ \mathrm{char}(F) = p > 0 \\ x \text{ is value–transcendental over } K. \end{array}\right\} \tag{31}$$

In this case, $\overline{F} = \overline{K}$ (cf. (23) on page 39) and consequently, $\overline{F}$ is algebraically closed.

We will consider the ring

$$R = K[x, x^{-1}] \tag{32}$$

which consists of all finite *Laurent series* of the form

$$\varphi(x) = \sum_{i \in I} c_i x^i \ , \quad c_i \in K \ , \tag{33}$$

i.e. the index set $I \subset \mathbb{Z}$ is finite.

Lemma 3.24 *If $F = K(x)^h$ has arbitrary characteristic and is of rank 1 with value–transcendental generator x, the following holds:*

$$F = R + \mathcal{O}_F \ ,$$

i.e. R is dense in F. If in addition $\overline{K}$ is Artin–Schreier–closed, the following holds:

$$F = R + \wp(F) \ .$$

("Artin–Schreier–closed" shall indicate that every polynomial of the form $X^p - X - a$ admits a zero, where p is the characteristic of the field).

Proof: From the hypothesis that F is of rank 1 we deduce that $K(x)$ is dense in its henselization $K(x)^h$. Consequently,

$$F = K(x) + \mathcal{O}_F$$

and it suffices to prove that

$$K(x) \subset R + \mathcal{O}_F \ , \tag{34}$$

i.e. *R is dense in its quotient field* $K(x)$, or in other words: every element of $K(x)$ has distance ∞ from R. For this it is enough to show for every $0 \neq \varphi(x) \in R$:

$$\mathrm{dist}_{K(x)}(\frac{1}{\varphi(x)}, R) = \infty \, .$$

Using the notation of (33) we have

$$v(\varphi(x)) = \min_{i\in I} v(c_i x^i) = v(c_k x^k) \tag{35}$$

for a unique $k \in I$ since x is value–transcendental over K. We write

$$\frac{1}{\varphi(x)} = \frac{c_k^{-1}x^{-k}}{1-\psi(x)}$$

bearing in mind that $c_k x^k$ is a unit in R which yields

$$\psi(x) = 1 - c_k^{-1}x^{-k}\varphi(x) \in R$$

with $v(\psi(x)) > 0$. The latter shows that $(1-\psi)^{-1}$ is a limit of the sequence of partial sums

$$\sum_{\nu=0}^{\mu} \psi^\nu \in R \quad (\mu \in I\!N)$$

of the geometrical series. Our hypothesis that the rank of F is 1 implies

$$\mathrm{dist}_F(\frac{1}{1-\psi}, R) \geq \sup_{\nu\in I\!N} v(\psi^\nu) = \infty$$

which shows that $\mathrm{dist}_F(\frac{1}{\varphi}, R) = \infty$, as asserted. □

According to this lemma, under the assumptions of (31) all Artin–Schreier–extensions of $K(x)^h$ are already determined by elements from R. But we can do more:

Lemma 3.25 *If* $F = K(x)^h$ *is of rank 1 with value–transcendental generator* x *over a perfect Artin–Schreier–closed field* K *of characteristic* $p > 0$*, then for every* $a \in F$ *there exists a finite Laurent–series* $\varphi(x) \in R$ *such that*

$$a \equiv \varphi(x) \bmod \wp(F) \tag{36}$$

and, using the notation of (33),

$$\forall i \in I: \quad i \equiv 0 \bmod p \Rightarrow c_i = 0 \, .$$

Proof: If K is Artin–Schreier–closed, then the same holds for the residue field $\overline{K}$. Thus the first assertion follows directly from the previous lemma. To prove the second assertion, we will show how to replace a summand $c_{jp}x^{jp}$ of $\varphi(x)$ by a summand $c'_j x^j$. Then it is possible by a finite repetition of this procedure to replace a given finite Laurent series $\varphi(x) \in a + \wp(F)$ by a finite Laurent series in $a + \wp(F)$ that satisfies also the second assertion.

Firstly, c_0 may be omitted since $c_0 \in \wp(F)$ by our assumption that K is Artin–Schreier–closed. Secondly, let $c_i \neq 0$ for some $i = jp \in I$, $0 \neq j \in \mathbb{Z}$. Since K is assumed to be perfect, we have $c_i^{1/p} \in K$ and thus

$$(c_i x^i)^{1/p} = c_i^{1/p} x^j \in R\ .$$

Consequently,

$$\varphi(x) \equiv \varphi(x) - c_i x^i + c_i^{1/p} x^j \bmod \wp(R)$$

the latter being a Laurent series with i-th coefficient equal to zero. This is the required replacement procedure. □

Now we deduce the following normal form:

Lemma 3.26 *If F satisfies the assumptions of (31) then every Galois extension $E|F$ of degree p has the following form:*

$$E = F(\vartheta), \quad \vartheta^p - \vartheta = a = \sum_{i \in I} c_i x^i$$

with finite index set $I \subset \mathbb{Z} \setminus p\mathbb{Z}$ and

$$\forall i \in I:\ v(c_i x^i) < 0.$$

In particular, $v(\sum_{i\in I} c_i x^i) = \min_{i \in I} v(c_i x^i) < 0$ is not divisible by p in $v(F)$.

Proof: We know already that $E|F$ is of the form (29). Since K and thus also $\overline{K}$ are algebraically closed by hypothesis (31), we may assume in view of (30) that a is a finite Laurent series $\varphi(x)$ satisfying the assertions of the previous lemma. We write

$$\varphi(x) = \sum_{i \in I} c_i x^i$$

where $I \subset \mathbb{Z} \setminus p\mathbb{Z}$ is finite and $c_i \in K$. By Lemma 3.22 we know that $\mathcal{O}_F \subset \wp(F)$ since $\overline{F} = \overline{K}$ is algebraically closed, and in view of (30) we may thus assume that every monomial $c_i x^i$ appearing in $\varphi(x)$ has value < 0. Since the extension is assumed to be nontrivial, there must at least be one nonzero monomial. The last assertion holds by (35) and the fact that $v(c_i x^i)$ is divisible by p in $v(F) = v(K) \oplus \mathbb{Z} v(x)$ if and only if i is divisible by p. □

Corollary 3.27 *If F satisfies the assumptions of (31) then every Galois extension $E|F$ of degree p is purely wild and defectless. This proves* **(R6)** *in the ramified case (31).*

Proof: We assume that $E|F$ has the form as described in the foregoing lemma. Since $v(a) < 0$, from $a = \vartheta^p - \vartheta$ it follows $v(\vartheta) < 0$ and

$$pv(\vartheta) = v(\wp(\vartheta)) = v(a)$$

Consequently,

$$v(\vartheta) = v(a)/p \notin v(F)\ .$$

Since p is prime, this is only possible if

$$(v(E) : v(F)) = p = [E : F]$$

showing that $E|F$ is purely wild and defectless, as asserted. □

Note that for an extension $E|F$ which is already of the form (29) with a satisfying (36), the proof that $E|F$ is defectless does not require any more a condition on the rank of F.

3.7. **B** Now we will discuss the unramified case:

$$\left.\begin{array}{l} F = K(x,y)^h \ \text{ is of rank 1 and of the form (25)} \\ \mathrm{char}(F) = p > 0 \\ x \ \text{ is residue–transcendental over } K. \end{array}\right\} \qquad (37)$$

In this case, $v(F) = v(K)$ (cf. (22) on page 39) and consequently, $v(F)$ is p–divisible.

We recall the crucial properties of the ring R as defined in (32), which enabled us to prove **(R6)** in the ramified case (31).

(I) R contains K and its quotient field $\mathrm{Quot}(R)$ is dense in F.

(II.A) R admits a valuation basis $\mathcal{B} = \{u_j \mid j \in J\}$ over K, containing the element 1, such that the values $v(u_j)$, $j \in J$, form a basis of $v(F)$ over $v(K)$.

(III) The basis $\mathcal{B}$ is closed under p–th powers ($p = \mathrm{char}(\overline{K})$), i.e. for every element $u_j \in \mathcal{B}$, the p–th power u_j^p is also contained in $\mathcal{B}$.

If the characteristic of K is p then a K–basis $\mathcal{B}$ having the property (III) will be called *Frobenius–closed.*

For the unramified case we have to replace property (II.A) by the following property:

(II.B) R admits a valuation basis $\mathcal{B} = \{u_j \mid j \in J\}$ over K, containing the element 1, such that the residues $\overline{u_j}$, $j \in J$, form a basis of $\overline{F}|\overline{K}$.

Note that the important property of R to be dense in its quotient field follows from $K \subset R$ and property (II.A) resp. (II.B) if F has rank 1 and is of the form (25):

Lemma 3.28 *Let F be of rank 1, of the form (25) and of arbitrary characteristic. Then the properties* (I) *and* (II.A) *resp.* (II.B) *of R imply that R is dense in F. Under our assumption (37), this implies*

$$F = R + \wp(F)$$

and consequently, every Artin–Schreier–extension $E|F$ is of the form

$$E = F(\vartheta) \quad \text{with} \quad \vartheta^p - \vartheta = a \in R\,. \qquad (38)$$

Proof: The proof is an analogue to the proof of Lemma 3.24; we have to show that R is dense in its quotient field for which it suffices to show that $\mathrm{dist}_F(r^{-1}, R) = \infty$. Assume that there exists an element $s \in R^\times$ with $v(rs - 1) > 0$ and write

$$\frac{1}{r} = \frac{s}{1-(1-rs)}\,.$$

Note that $1 - rs \in R$ and proceed as in the proof of Lemma 3.24. It remains to show the existence of s. Now the condition $K \subset R$ which is part of property (I) together with property (II.A) resp. (II.B) implies that $v(R) = v(F)$ and $\overline{R} = \overline{F}$ which shows that $v(r)$

has an inverse in $v(R)$, say $v(s_1)$ for suitable $s_1 \in R$, and it shows that the residue of the element $rs_1 \in R \cap \mathcal{O}_F^\times$ has an inverse in $\overline{R}$, say $\overline{s_2}$ for suitable $s_2 \in R$. Then the element $s = s_1 s_2$ has the desired property since $v(rs_1s_2) = v(rs_1) = 0$ and $\overline{rs_1s_2} = 1$. □

From this lemma we derive the following normal form:

Lemma 3.29 *If F satisfies the assumptions of (37) and R has properties* (I) *and* (II.B) *then every Galois extension $E|F$ can be written in the form (38) with*

$$a = \sum_{i \in I} c_i u_i \;, \qquad c_i \in K,\; u_i \in \mathcal{B}$$

where $I \subset J$ is a finite index set, no element u_i is a p–th power in $\mathcal{B}$ if $c_i \neq 0$, and

$$\forall i \in I:\; v(c_i u_i) = v(c_i) \leq 0 \;.$$

Consequently,

$$v(a) = \min_{i \in I} v(c_i) \leq 0 \;,$$

$E|F$ being tame if $v(a) = 0$. Moreover, if $E|F$ is tame then a may be chosen such that $v(a) = 0$.

Proof: We know already that every Galois extension $E|F$ of degree p is of the form (29). According to Lemma 3.28 we assume that $E|F$ is of the form (38) and we write

$$a = \sum_{i \in I} c_i u_i \;, \qquad c_i \in K,\; u_i \in \mathcal{B}$$

where $I \subset J$ is a finite index set. Note that

$$v(a) = \min_{i \in I} v(c_i u_i) = \min_{i \in I} v(c_i) \tag{39}$$

since property (II.B) says that the elements u_i, $i \in I$, form a valuation basis of R over K, and that all of them have value 0.

We may assume that no element u_i is a p–th power of another element in $\mathcal{B}$ since otherwise we could use a replacement procedure similar to the one used in the proof of Lemma 3.25, to produce a sum that satisfies this condition and is equivalent to a modulo $\wp(F)$; note that if $u_i \neq 1$ then $u_i \in F \setminus K$ which shows that there exists an integer $\nu = \nu(u_i)$ such that $u_i \notin F^{p^\nu}$, and thus also an integer $\mu = \mu(u_i) \leq \nu$ such that $u_i \notin \mathcal{B}^{p^\mu}$. If $u_i = 1$, then $c_i u_i = c_i \in K$ may be omitted since $c_i \in \wp(F)$ by our hypothesis that K is algebraically closed.

Furthermore, we know from Lemma 3.22 that $\mathcal{M}_F \subset \wp(F)$, and in view of (30) we may thus assume that every monomial appearing in a has value ≤ 0. Consequently, we have:

$$v(a) = \min_{i \in I} v(c_i u_i) \leq 0 \;.$$

If $v(a) = 0$ then the residue polynomial $X^p - X - \overline{a}$ which is an Artin–Schreier polynomial, does not admit a zero in $\overline{F}$ since otherwise $E|F$ would be trivial by Hensel's Lemma, contrary to our assumption that its degree is p. Hence in this case, $\overline{E}|\overline{F}$ is a separable extension of the same degree p as $E|F$. This shows that $E|F$ is tame if $v(a) = 0$. On the other hand, if $E|F$ is a tame Galois extension then $\overline{E}|\overline{F}$ must be separable by definition, and since $v(F) = v(K)$ is divisible by our assumption (37) it follows that $[\overline{E} : \overline{F}] = [E : F] = p$. Consequently, $\overline{E}|\overline{F}$ is a Galois extension of degree p and thus an Artin–Schreier–extension generated by an Artin–Schreier–root of $\overline{a}$ for a suitable element $a \in \mathcal{O}_F^\times$. By Hensel's Lemma, E is then generated by an Artin–Schreier–root of a. This shows that we may choose a with $v(a) = 0$ if $E|F$ is a tame extension. □

As a further preparation we need the following two lemmata:

Lemma 3.30 *Independently of the characteristic of K,* (II.B) *and* (III) *imply that the basis $\overline{\mathcal{B}}$ of $\overline{F}|\overline{K}$ consisting of all $\overline{u_j}$, $j \in J$, is also Frobenius–closed. If $\overline{u_m} = \overline{u_n}^{\,p}$ then $u_m = u_n^p$.*

Proof: Since $\mathcal{B}$ is Frobenius–closed, every u_j^p is an element of $\mathcal{B}$. Hence $\overline{u_j}^{\,p} = \overline{u_j^p} \in \overline{\mathcal{B}}$ which shows that $\overline{\mathcal{B}}$ is Frobenius–closed. If $\overline{u_m} = \overline{u_n}^{\,p}$ then $v(u_m - u_n^p) > 0 = v(u_m)$ which is only possible if $u_m = u_n^p$ since $\mathcal{B}$ is assumed to be a valuation basis. □

Lemma 3.31 *Every Frobenius–closed basis z_j, $j \in J$, of an extension $k_2|k_1$ of fields of characteristic $p > 0$ has the following property: if the sum*

$$s = \sum_{i\in I} d_i z_i\,, \quad d_i \in k_1\,,\ I \subset J \text{ finite}$$

is a p–th power, then for every $i \in I$ with $d_i \neq 0$, the basis element z_i is a p–th power of another basis element.

Proof: Assume that

$$s = (\sum_{j\in J_0} d_j{}' z_j)^p\,, \quad d_j{}' \in k_1$$

where $J_0 \subset J$ is a finite index set. Then

$$\sum_{i\in I} d_i z_i = s = \sum_{j\in J_0} (d_j{}')^p z_j^p$$

where the elements z_j^p are also basis elements by hypothesis which shows that every z_i which appears on the left hand side (i.e. $d_i \neq 0$) equals a p–th power z_j^p appearing on the right hand side. □

Now we are ready to prove:

Lemma 3.32 *If in the unramified case (37) there exists a subring $R \subset F$ satisfying properties* (I), (II.B) *and* (III) *then every Galois extension $E|F$ of degree p is defectless.*

Proof: By Lemma 3.29 we may assume that $E|F$ has the form (38) and that a satifies the assertions as described in that lemma. Since tame extensions are defectless (by definition), we only have to deal with the case where $E|F$ is not tame; by Lemma 3.29 we may thus assume that

$$v(a) = \min_{i\in I} v(c_i) = v(c_k) < 0$$

for a suitable $k \in I$. This shows $v(\vartheta) < 0$, which yields

$$pv(\vartheta) = v(\wp(\vartheta)) = v(a) .$$

We put $b = a/c_k$ and $d_i = c_i/c_k$ so that $v(b) = 0$ and $v(d_i) \geq 0$ with $d_k = 1$. The element $\eta = \vartheta/c_k^{1/p}$ satisfies $v(\eta) = 0$ and

$$\eta^p - c_k^{(1-p)/p}\eta = b .$$

We have

$$v(c_k^{(1-p)/p}\eta) = \frac{1-p}{p}v(c_k) > 0 = v(\eta^p)$$

and consequently

$$\overline{\eta}^p = \overline{b} = \sum_{i\in J} \overline{d_i}\,\overline{u_i} . \tag{40}$$

According to Lemma 3.30, the basis $\overline{\mathcal{B}}$ of $\overline{F}|\overline{K}$ is also Frobenius–closed, hence by virtue of Lemma 3.31 the assumption $\overline{\eta} \in \overline{F}$ together with (40) would imply that every basis element appearing in the sum is a p–th power of another basis element. In particular, this would be the case for $\overline{u_k}$ since $d_k = 1$ makes sure that $\overline{u_k}$ appears in the sum. But according to Lemma 3.30 this would imply that also u_k is the p–th power of another basis element, contrary to our assumption. This shows

$$\overline{\eta} \notin \overline{F}$$

whence

$$[\overline{E} : \overline{F}] = p = [E : F]$$

which proves that $E|F$ is defectless. We note that under the assumption $v(a) < 0$,

$$\overline{E} = \overline{F}(\overline{\eta}) = \overline{F}^{1/p}$$

which is purely inseparable of degree p over $\overline{F}$. □

In the sequel we will show the existence of such a ring R in F. Since K and thus also $\overline{K}$ are algebraically closed, there exists a Frobenius–closed basis of $\overline{F}$ over $\overline{K}$, as we will show in subsection 3.9. We have to lift this basis to a Frobenius–closed basis of F over K.

Since K is algebraically closed, it contains a field of representatives for the residue field $\overline{K}$ which we identify with $\overline{K}$ so that we may write

$$\overline{K} \subset K . \tag{41}$$

Then the residue map on K induces the identity on $\overline{K}$. The embedding (41) can be prolongated to an embedding of $\overline{F}$ into F as follows:
By hypothesis (25) we have $F = K(x, y)^h$ with

$$[K(x, y) : K(x)] = [\overline{K}(\overline{x}, \overline{y}) : \overline{K}(\overline{x})] \tag{42}$$

where $\overline{K}(\overline{x}, \overline{y})|\overline{K}(\overline{x})$ is a separable extension. Let

$$f(x, y) = 0$$

be the irreducible equation for x, y over K, normed such that f has integral coefficients with $\overline{f}(X,Y) \neq 0$. Condition (42) means that $f(X,Y)$ and $\overline{f}(X,Y)$ have the same degree in Y; moreover we have

$$\frac{\partial \overline{f}(\overline{x},\overline{y})}{\partial \overline{y}} \neq 0 \tag{43}$$

because $\overline{x}$ is a separating transcendent element for $\overline{F}|\overline{K}$. By (41) we may view the polynomial $\overline{f}(x,Y)$ as a polynomial over $K(x) \subset F$; from (43) it follows by Hensel's Lemma that this polynomial has exactly one zero $y' \in F$ with $\overline{y'} = \overline{y}$. We have $K(x,y')^h \subset K(x,y)^h$. Again from (43) it follows that the polynomial $f(x,Y)$ has exactly one root in $K(x,y')^h$ whose residue is equal to $\overline{y'} = \overline{y}$. This root must be y, hence $y \in K(x,y')^h$ and we have shown

$$K(x,y')^h = K(x,y)^h = F\ . \tag{44}$$

The residue map induces on $\overline{K}$ the identity and an isomorphism

$$\overline{K}(x) \longrightarrow \overline{K}(\overline{x})$$

since both x and $\overline{x}$ are transcendental over $\overline{K}$. It leaves the coefficients of the irreducible polynomial $\overline{f}(X,Y)$ fixed and sends the zero (x,y') of $\overline{f}(X,Y)$ to the zero $(\overline{x},\overline{y})$, hence it induces an isomorphism

$$\overline{K}(x,y') \longrightarrow \overline{K}(\overline{x},\overline{y}) = \overline{F}\ .$$

By this isomorphism we identify

$$x = \overline{x}\ , \quad y' = \overline{y}$$

such that

$$\overline{F} \subset F\ , \quad F = (K.\overline{F})^h\ , \tag{45}$$

the latter being a consequence of (44).

By construction, K and $\overline{F}$ are linearly disjoint over $\overline{K}$. We form the subring generated by both fields in F:

$$R = K \otimes_{\overline{K}} \overline{F} \subset F\ .$$

From (45) it follows that F is the henselization of the quotient field of R. Since the rank of F is 1, the field Quot(R) is dense in its henselization, hence R satisfies property (I). Every $\overline{K}$–basis of $\overline{F}$ is at the same time a K–basis of R; in view of the fact that the residue map induces the identity on $\overline{F}$ this yields that this basis is a valuation basis and that R satisfies (II.B). If we choose, as we indicated above, a Frobenius–closed basis of $\overline{F}|\overline{K}$ then R together with this basis satisfies also property (III).

We summarize what we have proved:

Lemma 3.33 *In the unramified case (37) there exists an embedding of the residue field $\overline{F}$ into F respecting the residue map such that $\overline{K} = K \cap \overline{F}$, that K is linearly disjoint from $\overline{F}$ over $\overline{K}$ and that $F = (K.\overline{F})^h$. The ring $R = K \otimes_{\overline{K}} \overline{F} \subset F$ satisfies properties* (I), (II.B) *and* (III).

An application of Lemma 3.32 now proves

Corollary 3.34 *If F satisfies the assumptions of (37) then every Galois extension $E|F$ of degree p is defectless. This proves* **(R6)** *in the unramified case (37).*

Note that for an extension $E|F$ which is already of the form (38), the proof that $E|F$ is defectless does not require any more a condition on the rank of F.

3.8 Galois extensions of degree p in characteristic 0.

We want to prove **(R6)** in the case where $\mathrm{char}(K) = 0$. Every field having residue characteristic 0 is a defectless field by the Lemma of Ostrowski (cf. Lemma 2.4). Hence it only remains to prove our assertion under the additional assumption that F has residue characteristic $p > 0$. Recall that we assume F to be of rank 1 and of the form (25).

Since the algebraically closed field K contains all p–th roots of unity, Kummer theory shows that every Galois extension E of F of degree p is of the form

$$E = F(\vartheta),\ \vartheta^p = b \in F\ . \tag{46}$$

For every $d \in F^\times$ we have

$$E = F(d\vartheta),\ (d\vartheta)^p = d^p\vartheta^p = d^p b \in F \tag{47}$$

showing that all elements of the class $a \cdot (F^\times)^p$ determine the same extension (46). Now we distinguish three cases:

<u>case 1:</u> $v(b) \notin pv(F)$.
Then $(v(E) : v(F)) = p = [E : F]$ and the extension $E|F$ is purely wild and defectless;

<u>case 2:</u> $v(b) \in pv(F)$.
Then there exists an element $d \in F^\times$ such that $v(d^p b) = 0$ and $E = F(d\vartheta)$;

<u>case 2.1:</u> $\overline{d^p b} \notin \overline{F}^p$.
Then $[\overline{E} : \overline{F}] = p = [E : F]$, $\overline{E}|\overline{F}$ is purely inseparable and the extension $E|F$ is purely wild and defectless;

<u>case 2.2:</u> $\overline{d^p b} \in \overline{F}^p$.
Then there exists an element $d_1 \in \mathcal{O}_F^\times$ such that $\overline{d_1^p d^p a} = 1$ and $E = F(d_1 d\vartheta)$.

Consequently, for our proof that $E|F$ is defectless we may from now on assume $v(b) = 0$ and $\overline{b} = 1$, thus $b = 1 + a$ with $a \in \mathcal{M}_F$, hence $E|F$ is of the form

$$E = F(\vartheta),\ \vartheta^p = 1 + a \in F \ \text{ with } \ a \in \mathcal{M}_F\ . \tag{48}$$

Note that this implies $v(\vartheta) = 0$.

The algebraically closed field K contains an element C which satisfies the equation

$$C^{p-1} = -p\ .$$

(Let us mention that such C is a generator of the minimal extension of $\mathbb{Q}_p$ containing all p–th roots of unity.) Note that

$$C^p = -pC$$

and

$$v(C) = \frac{1}{p-1} v(p) > 0\ ,$$

hence $C \in \mathcal{M}_F$.

If we substitute $C\eta + 1$ for ϑ in (48), we get

$$\begin{aligned} 0 &= \vartheta^p - 1 - a = (C\eta + 1)^p - 1 - a \\ &= C^p\eta^p + pC\left(\sum_{i=2}^{p-1} C^{i-1}\frac{1}{p}\binom{p}{i}\eta^i\right) + pC\eta - a \\ &\equiv C^p\eta^p + pC\eta - a \mod pC\mathcal{M}_F \end{aligned}$$

because all coefficients $C^{i-1}\frac{1}{p}\left(\begin{array}{c}p\\i\end{array}\right)$ for $2\leq i\leq p-1$ are elements of $C\mathcal{O}_F\subset\mathcal{M}_F$ since $\left(\begin{array}{c}p\\i\end{array}\right)$ is divisible by p for such i. Dividing by C^p, we obtain

$$\eta^p-\eta-\frac{a}{C^p}\in\mathcal{M}_F\;.$$

We conclude that $v(a/C^p)\leq 0$ since otherwise Hensel's Lemma would show that η is an element of the henselian field F, contradicting our hypothesis that $F(\vartheta)|F$ is of degree p.

If $v(a/C^p)=0$, we conclude that $\overline{F}(\overline{\eta})|\overline{F}$ is an Artin–Schreier–extension (of degree p), since if it were a trivial extension, Hensel's Lemma would again yield a contradiction. In this case, $E|F$ is a tame defectless extension with

$$[\overline{E}:\overline{F}]=p=[E:F]\;.$$

Note that this case is only possible if $\overline{F}$ is not algebraically closed.

If $v(a/C^p)<0$, then $v(\eta)<0$ and $v(\eta^p)=pv(\eta)<v(\eta)$. Hence

$$v(\eta^p-\frac{a}{C^p})=v(\eta)>v(\eta^p)\;,$$

whence

$$v(\eta^p)=v(\frac{a}{C^p})$$

and consequently, the extension $E|F$ has the form

$$E=F(C\eta),\;v((C\eta)^p-a)>v(a)\;. \tag{49}$$

If we are now able to derive normal forms for a which are similar to those that we have obtained in the case of equal characteristic $\mathrm{char}(K)=p=\mathrm{char}(\overline{K})$, then we will immediately know from (49) that
$pv(C\eta)\in v(F)\setminus pv(F)$ in the ramified case, or
$\overline{c'C\eta}^{\,p}\in\overline{F}\setminus\overline{F}^p$ in the unramified case,
for a suitable element $c'\in F$ with $v(c')=-v(C\eta)$. This would prove for both cases that the extension $E|F$ is defectless.

In the sequel, we will derive such normal forms for a. Since in the case of fields of characteristic 0, we do not have the additivity of $x\mapsto x^p$ which we used in the case of F having characteristic $p>0$ (in particular for the replacement procedure in the proof of Lemma 3.25), we have to use here the following general lemmata:

Lemma 3.35 *If*

$$a=1+y+\sum_{j\in J}z_j^p$$

with $y,z_j\in\mathcal{M}_F$ and a finite index set J, and if

$$d=(1+\sum_{j\in J}z_j)^{-1}\;,$$

then

$$d\equiv 1\bmod\mathcal{M}_F\quad\textit{and}\quad d^pa\equiv 1+d^py\mod p\mathcal{M}_F\;.$$

Proof: Since all z_j are elements of $\mathcal{M}_F$ we have

$$a \equiv (1 + \sum_{j\in J} z_j)^p + y \mod p\mathcal{M}_F$$

which implies the assertion by virtue of $d \in \mathcal{O}_F$ and

$$\overline{d} = \overline{1 + \sum_{j\in J} z_j}^{-1} = 1^{-1} = 1 .$$

□

Lemma 3.36 *Let F be any henselian field of characteristic 0 with residue characteristic $p > 0$, and let $y, z, \tilde{z} \in \mathcal{M}_F$. Assume that F contains an element C with $C^{p-1} = -p$. Then*

$$1 + y - p\tilde{z} \in (1 + y + z) \cdot (F^\times)^p$$

if the following conditions hold:

1. $v(\tilde{z}^p - z) > \frac{p}{p-1} v(p)$,
2. $2v(\tilde{z}) > \frac{1}{p-1} v(p)$,
3. $v(z + p\tilde{z}) + v(y - p\tilde{z}) > \frac{p}{p-1} v(p)$.

Proof: We compute

$$\frac{1 + y + z}{1 + y - p\tilde{z}} = 1 + \frac{z + p\tilde{z}}{1 + y - p\tilde{z}} = 1 + z + p\tilde{z} + A$$

where $A \in F$ with $v(A) \geq v(z + p\tilde{z}) + v(y - p\tilde{z})$. Let

$$f(X) = X^p - (1 + z + p\tilde{z} + A) .$$

Putting $X = Y + 1 + \tilde{z}$, we get

$$\begin{aligned} f(X) &= (Y + 1 + \tilde{z})^p - (1 + z + p\tilde{z} + A) \\ &= (Y + 1)^p + \tilde{z}^p + p(Y + 1)^{p-1}\tilde{z} + p\tilde{z}^2 B - (1 + z + p\tilde{z} + A) \\ &= Y^p + 1 + pY + pY^2 D + \tilde{z}^p + p\tilde{z} + p\tilde{z}Y D' + p\tilde{z}^2 B - (1 + z + p\tilde{z} + A) \\ &= Y^p + pY + pY^2 D + \tilde{z}^p - z + p\tilde{z}Y D' + p\tilde{z}^2 B - A \end{aligned}$$

where $B, D, D' \in \mathcal{O}_F[Y]$. Putting $Y = CZ$ and

$$g(Z) := \frac{1}{C^p} f(X) = \frac{1}{C^p} f(CZ + 1 + \tilde{z}) ,$$

and using $C^{p-1} = -p$, we get

$$\begin{aligned} g(Z) &= \frac{1}{C^p}(C^p Z^p - C^p Z - C^{p+1} Z^2 D + \tilde{z}^p - z - C^p \tilde{z} Z D' - C^{p-1}\tilde{z}^2 B - A) \\ &= Z^p - Z - CZ^2 D + C^{-p}(\tilde{z}^p - z) - \tilde{z} Z D' - C^{-1}\tilde{z}^2 B - C^{-p} A \end{aligned}$$

with $B, D, D' \in \mathcal{O}_F[Z]$. The polynomial $g(Z)$ has integral coefficients and residue $\overline{Z}^p - \overline{Z}$, if the following conditions hold:

$$v(C) > 0\,,\ v(\tilde{z}^p - z) > pv(C)\,,\ v(\tilde{z}) > 0\,,$$
$$2v(\tilde{z}) > v(C)\,,\ v(z + p\tilde{z}) + v(y - p\tilde{z}) > pv(C)\ .$$

In view of $v(C) = v(p)/(p-1) > 0$, these conditions hold if and only if conditions *1.*, *2.*, *3.* hold. Note that $v(p) > 0$ since by hypothesis, $\mathrm{char}(\overline{F}) = p \neq 0 = \mathrm{char}(F)$. □

Putting $\tilde{z} = 0$ or $z = \tilde{z}^p$ respectively, one deduces from this lemma both parts of the following well known corollary:

Corollary 3.37 *Let the assumptions be as in Lemma 3.36. Then:*
a) $1 + y \in (1 + y + z) \cdot (F^\times)^p$ *if* $v(z) > \frac{p}{p-1}v(p)$.
b) $1 + y - p\tilde{z} \in (1 + y + \tilde{z}^p) \cdot (F^\times)^p$ *if* $v(y) \geq v(p)$ *and* $v(\tilde{z}^p) > v(p)$.

Corollary 3.38 *Let the assumptions be as in Lemma 3.36. Then:*
$1 + y \in (1 + y + z) \cdot (F^\times)^p$ *if* $1 + z \in (F^\times)^p$ *and* $v(yz) > \frac{p}{p-1}v(p)$.

Proof: $1 + y \in (1 + y + z)(F^\times)^p$ is true if the following quotient is an element of $(F^\times)^p$:

$$\frac{1 + y + z}{1 + y} = 1 + \frac{z}{1 + y} = 1 + z + A$$

where $A \in \mathcal{O}_F$ with $v(A) = v(yz)$, since $y, z \in \mathcal{M}_F$. By hypothesis,

$$v(A) = v(yz) > \frac{p}{p-1}v(p)\ .$$

Thus by part a) of Corollary 3.37 and by our assumption on $1 + z$,

$$1 + z + A \in (1 + z) \cdot (F^\times)^p = (F^\times)^p\ ,$$

as desired. □

3.8. **A** We will first discuss the ramified case:

$$\left.\begin{array}{l} F = K(x)^h \text{ is of rank 1 and } K \text{ is algebraically closed} \\ \mathrm{char}(F) = 0\,,\ \mathrm{char}(\overline{F}) = p > 0 \\ x \ \text{ is value–transcendental over } K. \end{array}\right\} \qquad (50)$$

In this case, $\overline{F} = \overline{K}$ (cf. (23) on page 39) and consequently, $\overline{F}$ is algebraically closed.

As in 3.7. A we will consider the subring

$$R = K[x, x^{-1}]$$

of F. Recall that every element of this ring is a finite sum of monomials, all of them having different values, whose minimum represents the value of the sum (cf. (35) on page 51). From

Lemma 3.24 we know that R is dense in F. In view of assumption (48) and part a) of Corollary 3.37, we may thus assume from now on that

$$E = F(\vartheta) \quad , \quad \vartheta^p = 1 + a = 1 + \sum_{i\in I} c_i x^i \in R \qquad (51)$$
$$\text{with } \; c_i x^i \in \mathcal{M}_F \; \text{ and } \; I \subset \mathbb{Z} \; \text{ finite.}$$

Note that since K is assumed to be algebraically closed, a monomial $c_i x^i$ is a p–th power in F if and only if $i \in p\mathbb{Z}$; indeed, if $i \notin p\mathbb{Z}$, then $v(c_i x^i) \notin v(K) \oplus p\mathbb{Z}v(x) = pv(F)$. Now we apply Lemma 3.35 to $1+a$, taking the elements z_j to be all monomials $c_{jp}x^{jp}$ with $jp \in I$ and $j \in \mathbb{Z}$. We find that we may replace a by $d^p y + A$ with $A \in p\mathcal{M}_F$, where y contains all monomials appearing in a that are not p–th powers, and d is chosen as in the lemma. Since $d - 1 \in \mathcal{M}_F$ and R is dense in F, we may write

$$d \equiv 1 + \sum_{i\in\tilde{I}} \tilde{c}_i x^i \mod p\mathcal{M}_F$$

where all monomials $\tilde{c}_i x^i$ are elements of $\mathcal{M}_F$. Consequently,

$$d^p \equiv 1 + \sum_{i\in\tilde{I}} \tilde{c}_i^p x^{ip} \mod p\mathcal{M}_F \; .$$

Since by our choice, y contains no monomials which are p–th powers, we may write

$$d^p y + A = \sum_{i\in I'} c'_i x^i + A'$$

where $A' \in p\mathcal{M}_F$ and $I' \subset \mathbb{Z} \setminus p\mathbb{Z}$ is a finite index set. Since R is dense in F, by virtue of part a) of Corollary 3.37 we may replace A' by a suitable element from R which is a finite sum of monomials which in view of (35) must have value $> v(p)$. We have found that from now on we may assume:

$$E = F(\vartheta),\ \vartheta^p = 1 + a = 1 + \sum_{i\in I} c_i x^i$$
$$\text{with} \left\{ \begin{array}{l} \text{finite } I \subset \mathbb{Z}\,, \\ c_i x^i \in \mathcal{M}_F\,, \\ c_i x^i \in p\mathcal{M}_F \;\text{ if }\; i \in p\mathbb{Z}\,. \end{array} \right\} \qquad (52)$$

Let us assume for the following that there are no summands $c_i x^i$ having value $\leq v(p)$. Then part b) of Corollary 3.37 shows that we may replace every monomial of the form $c_i x^i$ with $i \in p\mathbb{Z}$ by the monomial $-p\tilde{c}_i x^{i/p}$ where $\tilde{c}_i$ is any element of K with $\tilde{c}_i^p = c_i$. After an iterated application we may assume that I contains no multiples of p different from 0. To get rid of c_0, we use Corollary 3.38, where we put

$$z = c_0 \;\text{ and }\; y = \sum_{0\neq i\in I} c_i x^i \; .$$

Since K is algebraically closed, $1 + z = 1 + c_0$ is a p–th power in K and thus in F. By our assumption that all summands $c_i x^i$ have value $> v(p)$, we know that $v(y) > v(p)$ and that $v(z) = v(c_0) > v(p)$, hence $v(yz) > 2v(p) \geq \frac{p}{p-1}v(p)$. Now Corollary 3.38 shows that

$$1 + \sum_{0\neq i\in I} c_i x^i \in (1 + \sum_{i\in I} c_i x^i) \cdot (F^\times)^p \; .$$

Hence we may assume $I \cap p\mathbb{Z} = \emptyset$.

Finally, since we know by part a) of Corollary 3.37 that we may omit every monomial $c_i x^i$ with value $> \frac{p}{p-1} v(p)$ appearing in a, we have derived the following normal form for the case 2.2:

Lemma 3.39 *Assume that F satisfies (50) and $E|F$ is of the form (48). Then there exists $\vartheta \in F$ such that*

$$E = F(\vartheta),\ \vartheta^p = 1 + \sum_{i \in I} c_i x^i$$

with finite index set $I \subset \mathbb{Z}$ and

$$\forall i \in I:\ 0 < v(c_i x^i) \leq \frac{p}{p-1} v(p)\ \wedge\ (v(c_i x^i) \leq v(p) \Longrightarrow i \notin p\mathbb{Z})\ .$$

If there exists no $i \in I$ with $v(c_i x^i) \leq v(p)$, then it may in addition be assumed that $I \cap p\mathbb{Z} = \emptyset$. In any case, $v(\sum_{i \in I} c_i x^i) = \min_{i \in I} v(c_i x^i)$ is not divisible by p in $v(F)$.

Corollary 3.40 *If F satisfies the assumptions of (50), then every Galois extension of degree p is purely wild and defectless. This proves* **(R6)** *in the case (50).*

Proof: As we have discussed in the beginning, we may assume that the extension is given in the form (46). Furthermore the assertion was immediately proved in the cases 1 and 2.1 (actually, the latter cannot appear if F satisfies (50)). For the following we may thus assume case 2.2, i.e. $E|F$ is of the form (48). Consequently, we may assume that $E|F$ has the form as described in the preceding Lemma. From the discussion at the beginning of this subsection, we know that the extension $E|F$ must have the form (49), and in particular, $v(a)/p \in v(E)$. On the other hand, we know from the preceding lemma that the value of a is not divisible by p in $v(F)$. This yields

$$(v(E) : v(F)) = p = [E : F]$$

showing that $E|F$ is purely wild and defectless. □

3.8. **B** Now we will discuss the unramified case:

$$\left.\begin{array}{l} F = K(x,y)^h \text{ is of rank 1 and of the form (25)} \\ \mathrm{char}(F) = 0\,,\ \mathrm{char}(\overline{F}) = p > 0 \\ x \ \text{ is residue–transcendental over } K. \end{array}\right\} \quad (53)$$

In this case, $v(F) = v(K)$ (cf. (22) on page 39) and consequently, $v(F)$ is p–divisible.

Using Lemma 3.28, we derive the following normal form:

Lemma 3.41 *If F satisfies (53) and R has properties* (I) *and* (II.B)*, then for every Galois extension $E|F$ of the form (48), we may assume in addition:*

$$a = r^p \sum_{i \in I} c_i u_i + r' \in R\ , \qquad r, r' \in R,\ c_i \in K,\ u_i \in \mathcal{B}$$

where $r \in R$ has value 0, $r' \in p\mathcal{M}_F$, $I \subset J$ is a finite index set, no element u_i is a p–th power in $\mathcal{B}$, and

$$\forall i \in I: \ 0 < v(r^p c_i u_i) = v(c_i) \leq \frac{p}{p-1} v(p) \ .$$

If there exists no $i \in I$ with $v(c_i) \leq v(p)$, then it may in addition be assumed that $r = 1$ and $r' = 0$. In any case,

$$v(a) = v(\sum_{i \in I} c_i u_i) = \min_{i \in I} v(c_i u_i) = \min_{i \in I} v(c_i) \leq \frac{p}{p-1} v(p) \tag{54}$$

and if $v(c_k) = v(a)$ for $k \in I$ then $u_k \notin \mathcal{B}^p$. $E|F$ is tame if $v(a) = \frac{p}{p-1}v(p)$ holds. Conversely, if $E|F$ is tame then a may be chosen such that $v(a) = \frac{p}{p-1}v(p)$.

Proof: Let $E|F$ be given in the form (48). In view of Lemma 3.28 and part a) of Lemma 3.37, we may assume that $a \in R$ and we write

$$a = \sum_{i \in I'} c'_i u_i \ , \qquad c'_i \in K, \ u_i \in \mathcal{B}$$

where $I' \subset J$ is a finite index set. Since $a \in \mathcal{M}_F$ and $\mathcal{B}$ is a valuation basis of R over K, all monomials $c'_i u_i$ must be elements of $\mathcal{M}_F$ (cf. equation (39) on page 54).

First, we apply Lemma 3.35 to $1 + a$, taking the elements z_j to be all monomials $c'_j u_j$ with $u_j \in \mathcal{B}^p$. We find that we may replace a by $d^p y + A$, where y contains all monomials $c'_j u_j$ with $u_j \notin \mathcal{B}^p$ and d is chosen as in the lemma. We put $r' := A \in p\mathcal{M}_F$. Let $d_0 \in K$ be such that $v(d_0) = v(d)$ and put $r = d/d_0 \in R$; consequently, $v(r) = 0$. From now on we may assume

$$a = d^p y + A = r^p \sum_{i \in I} c_i u_i + r'$$

where no element u_i is a p–th power in $\mathcal{B}$. Moreover, we may assume that every c_i has value $\leq \frac{p}{p-1}v(p)$ since otherwise, the summand may be omitted according to part a) of Corollary 3.37. On the other hand, every c_i satisfies $v(c_i) \geq v(\sum c_i u_i)$ since $\mathcal{B}$ is a valuation basis of R over K by (II B), and this shows that $v(c_i) \geq v(a) > 0$; moreover, it proves (54).

For the case that there exists no $i \in I$ with $v(c_i) \leq v(p)$, the further assertions on the monomials $c_i u_i$ are derived like in the ramified case (50) (cf. page 62).

Now assume that $v(a) = \frac{p}{p-1}v(p)$. Then the substitution $X = CZ + 1$ where C is chosen as in Lemma 3.36, transforms the polynomial $X^p - (1 + a)$ into a polynomial in Z with residue polynomial $Z^p - Z - \overline{a/C^p}$ where $v(a/C^p) = 0$ (cf. the proof of Lemma 3.36). This is an Artin–Schreier polynomial which does not admit a zero in $\overline{F}$ since otherwise $E|F$ would be trivial by Hensel's Lemma, contrary to our assumption that its degree is p. Hence in this case, $\overline{E}|\overline{F}$ is a separable extension of the same degree p as $E|F$. This shows that $E|F$ is tame if $v(a) = \frac{p}{p-1}v(p)$.

Assume now that $E|F$ is a tame Galois extension. Then $\overline{E}|\overline{F}$ must be separable by definition, and since $v(F) = v(K)$ is divisible by our assumption (53), it follows that $[\overline{E} : \overline{F}] = [E : F] = p$. Consequently, $\overline{E}|\overline{F}$ is a Galois extension of degree p and thus an Artin–Schreier–extension generated by an Artin–Schreier-root of $\overline{y}$ for a suitable element $y \in \mathcal{O}_F^\times$. Now choose C as above and let $f(Z) \in F[Z]$ be the polynomial that satisfies

$$(CZ + 1)^p = C^p Z^p + C^p f(Z) - C^p Z + 1$$

(note that $C^p = -pC$); it follows that $C^p f(Z) \in pC^2\mathcal{O}_F[Z] = C^{p+1}\mathcal{O}_F[Z]$, hence

$$f(Z) \in \mathcal{M}_F[Z] \,.$$

By Hensel's Lemma, E is then generated over F by a root of

$$Z^p + f(Z) - Z - y \,.$$

Multiplying by C^p and transforming with $X = CZ + 1$ we get a polynomial

$$X^p - (1 + C^p y)$$

admitting a root in E which generates the extension $E|F$. Moreover, $v(y) = 0$ and $v(C^p y) = \frac{p}{p-1}v(p)$. This shows that we may choose $a = C^p y$ with $v(a) = \frac{p}{p-1}v(p)$ if $E|F$ is a tame extension. □

On the basis of this lemma, we prove:

Lemma 3.42 *If in the unramified case (53) there exists a subring $R \subset F$ satisfying properties* (I), (II.B) *and* (III)*, then every Galois extension $E|F$ of degree p is defectless.*

Proof: As we have discussed in the beginning, we may assume that the extension is given in the form (46). Furthermore the assertion was immediately proved in the cases 1 and 2.1 (actually, the former cannot appear if F satisfies (50)). For the following we may thus assume case 2.2, i.e. $E|F$ is of the form (48). Furthermore, we may assume that a satifies the assertions as described in Lemma 3.41. Since tame extensions are defectless (by definition), we only have to deal with the case where $E|F$ is not tame; by Lemma 3.41 we may thus assume that

$$v(a) = \min_{i\in I} v(c_i) = v(c_k) < \frac{p}{p-1}v(p) \tag{55}$$

for a suitable $k \in I$, i.e. $v(a/C^p) < 0$. The discussion at the beginning of this subsection has shown that in this case, the extension $E|F$ is of the form (49). Now $c_k^{1/p} \in K$ since K is algebraically closed by assumption, and $pv(c_k^{-1/p}C\eta) = v(a) - v(c_k) = 0$, i.e. $v(c_k^{-1/p}C\eta) = 0$. Moreover, we infer from (49) that

$$\overline{c_k^{-1/p}C\eta}^{\,p} = \overline{a/c_k} = \overline{\sum_{i\in I} d_i u_i}$$

where $d_i = c_i/c_k$ so that $v(\sum_{i\in I} d_i u_i) = 0$ and $v(d_i) \geq 0$ with $d_k = 1$. As in the proof of Lemma 3.32 it is now shown that

$$\overline{\eta} \notin \overline{F} \,,$$

whence

$$[\overline{E} : \overline{F}] = p = [E : F]$$

which proves that $E|F$ is defectless and that

$$\overline{E} = \overline{F}(\overline{\eta}) = \overline{F}^{1/p}$$

which is purely inseparable of degree p over $\overline{F}$. Note that Lemma 3.30 and Lemma 3.31 which are used for the proof, are independent of the characteristic of F. □

Now we will show the existence of a ring R in F that has properties (I), (II.B) and (III). Since K and thus also $\overline{K}$ are algebraically closed, there exists a Frobenius–closed basis $\overline{\mathcal{B}}$ of $\overline{F}$ over $\overline{K}$, as we will show in subsection 3.9. We have to lift this basis $\overline{\mathcal{B}}$ to a Frobenius–closed basis of F over K.

By assumption, the characteristic of K is zero and the characteristic of $\overline{K}$ is $p > 0$. Consequently, K contains $\mathbb{Q}$ and its valuation induces the p–adic valuation v_p on $\mathbb{Q}$. Since K is algebraically closed, it contains a valued overfield (K_0, v) of $(\mathbb{Q}, v)$ such that $\overline{K_0} = \overline{K}$ and $v(K_0) = v_p(\mathbb{Q}) = \mathbb{Z}v(p)$; this field can be constructed as follows:
Take $\mathcal{T}$ to be a set of foreimages for a transcendence basis $\overline{\mathcal{T}}$ of $\overline{K}|\mathbb{F}_p$; then $\mathcal{T}$ is a valuation–independent set and we have $\overline{\mathbb{Q}(\mathcal{T})} = \mathbb{F}_p(\overline{\mathcal{T}})$ and $v(\mathbb{Q}(\mathcal{T})) = v(\mathbb{Q})$ by Lemma 2.21. Now $\overline{K} \,|\, \overline{\mathbb{Q}(\mathcal{T})}$ is an algebraic extension which can be viewed as a transfinite tower of algebraic extensions, every successor being a finite extension of the predecessor and the index set of this tower being well–ordered. By induction on this well–ordering one can lift succesively all these finite extensions preserving the degree; this is possible since we are working in the algebraically closed field K. By this construction, we get a tower of finite extensions, starting from the field $\mathbb{Q}(\mathcal{T})$, and since all these have the same degree as the corresponding extension of their residue fields, all these extensions will have the same value group as $\mathbb{Q}(\mathcal{T})$ which is $\mathbb{Z}v(p)$. The union over this tower is the desired field K_0.

Now by Lemma 2.20,

$$\overline{K_0(x)} = \overline{K_0}(\overline{x}) = \overline{K}(\overline{x}) \; . \tag{56}$$

By hypothesis (53), we have $F = K(x,y)^h$ with

$$[K(x,y) : K(x)] = [\overline{K}(\overline{x},\overline{y}) : \overline{K}(\overline{x})] \tag{57}$$

where $\overline{K}(\overline{x},\overline{y})|\overline{K}(\overline{x})$ is a separable extension. Let

$$f(x,y) = 0$$

be the irreducible equation for x, y over K, normed such that f has integral coefficients with $\overline{f}(X,Y) \neq 0$. Condition (57) implies that $f(X,Y)$ and $\overline{f}(X,Y)$ have the same degree in Y; moreover we have

$$\frac{\partial \overline{f}(\overline{x},\overline{y})}{\partial \overline{y}} \neq 0 \tag{58}$$

because $\overline{x}$ is a separating transcendent element for $\overline{F} \,|\, \overline{K}$. By (56), we may choose a polynomial $g(X,Y) \in K_0[X,Y]$ with integral coefficients such that g has the same degree in Y as f and

$$\overline{g}(X,Y) = \overline{f}(X,Y) \; .$$

From (58) it follows by Hensel's Lemma that $g(x,Y)$ has exactly one zero $y' \in F$ with $\overline{y'} = \overline{y}$. We have $K(x,y')^h \subset K(x,y)^h$. Again from (58) it follows that the polynomial $f(x,Y)$ has exactly one root in $K(x,y')^h$ whose residue is equal to $\overline{y'} = \overline{y}$. This root must be y, hence $y \in K(x,y')^h$ and we have shown

$$K(x,y')^h = K(x,y)^h = F \; .$$

Hence we assume from now on that y is algebraic over $K_0(x)$ with

$$[K_0(x,y):K_0(x)] = [\overline{K_0}(\overline{x},\overline{y}):\overline{K_0}(\overline{x})] = [\overline{K}(\overline{x},\overline{y}):\overline{K}(\overline{x})] = [K(x,y):K(x)]\;.$$

In particular, this shows that the function field $F_0 = K_0(x,y)$ is linearly disjoint from K over K_0. Moreover, we have

$$\overline{F_0} = \overline{K_0}(\overline{x},\overline{y}) = \overline{F}$$

and

$$F = (K.K_0(x,y))^h = (K.F_0)^h \tag{59}$$

Now we lift the Frobenius–closed $\overline{K}$–basis $\overline{\mathcal{B}}$ of $\overline{F}$ to F_0 in the following way:
First we observe that every basis element $\overline{u}$ in $\overline{\mathcal{B}}$ which is not equal to 1, is an element of $\overline{F}\setminus\overline{K}$ which implies that there exists an integer $\nu = \nu(\overline{u})$ such that $\overline{u} \notin \overline{F}^{p^\nu}$. This shows that

$$\overline{\mathcal{B}} = \{1, \overline{u}^{p^n} \mid n \in I\!N \text{ and } \overline{u} \in \overline{\mathcal{B}} \setminus \overline{F}^p\}\;.$$

For every $\overline{u} \in \overline{\mathcal{B}} \setminus \overline{F}^p$ we choose an element $u \in F_0$ with residue $\overline{u}$. Let $\mathcal{B}'$ be the collection of all these elements u. Then

$$\mathcal{B} = \{1, u^{p^n} \mid n \in I\!N \text{ and } u \in \mathcal{B}'\}$$

is a linearly valuation–independent set and a set of representatives for $\overline{\mathcal{B}}$. Let

$$R_0 = K_0[\mathcal{B}]$$

be the subring of F_0 generated over K_0 by the elements from $\mathcal{B}$. Since

$$v(R_0) = v_p(\mathbb{Q}) = v(F_0) \quad\text{and}\quad \overline{R_0} = \overline{F} = \overline{F_0}$$

and since the value group $v_p(\mathbb{Q})$ is isomorphic to $\mathbb{Z}$, we conclude that R_0 is dense in F_0.

Recall that K and F_0 are linearly disjoint over K_0. We form the subring generated by K and R_0 in F:

$$R = K \otimes_{K_0} R_0 \subset F\;.$$

Since R_0 is dense in F_0 and K is of rank 1 by our assumption (53), the ring R is dense in the ring

$$R' = K \otimes_{K_0} F_0 \subset F\;.$$

From (59) it follows that F is the henselization of the quotient field of R'. Since the rank of F is 1, the field $\mathrm{Quot}(R')$ is dense in its henselization, and the fact that R is dense in R' implies that $\mathrm{Quot}(R)$ is dense in $\mathrm{Quot}(R')$; hence R satisfies property (I).
By construction, $\mathcal{B}$ is a valuation basis of R over K containing 1, closed under p–th powers and such that $\overline{\mathcal{B}}$ is a Frobenius–closed $\overline{K}$–basis of $\overline{F}$; hence R satisfies properties (II.B) and (III).

We summarize what we have proved:

Lemma 3.43 *In the unramified case (53) there exists a subfield K_0 of K and a function field F_0 linearly disjoint from K over K_0 such that $v(F_0) = v(K_0)$ is discrete, $\overline{K_0} = \overline{K}$ and $\overline{F_0} = \overline{F}$. $\mathcal{O}_{F_0}^\times$ contains a linearly valuation–independent set $\mathcal{B}$ including 1 and closed under p–th powers such that $\overline{\mathcal{B}}$ is a Frobenius–closed valuation basis of $\overline{F}$ over $\overline{K}$. The ring R_0 generated by $\mathcal{B}$ over K_0 is dense in F_0, and the ring $R = K \otimes_{K_0} R_0 \subset F$ satisfies properties* (I), (II.B) *and* (III).

An application of Lemma 3.42 now proves

Corollary 3.44 *If F satisfies the assumptions of (53), then every Galois extension $E|F$ of degree p is defectless. This proves* **(R6)** *in the unramified case (53).*

3.9 Frobenius–closed bases of algebraic function fields.

Let $F|K$ be an algebraic function field in one variable with constant field K of characteristic $p > 0$. Recall that a K–basis B of $F|K$ is called *Frobenius–closed* if $B^p \subset B$. Here B^p denotes the set of all p-th powers t^p with $t \in B$.

Lemma 3.45 *If F is an algebraic function field over an algebraically closed field K of arbitrary characteristic and q is an arbitrary natural number > 1, then there exists a basis of $F|K$ which is closed under q–th powers.*

Proof: If $F = K(x)$ is a rational function field, our lemma follows from the theorem on the partial fraction decomposition: every element $f \in F$ has a unique representation

$$f = c + \sum_{n>0} c_n x^n + \sum_{a\in K}\sum_{n>0} c_{a,n} \frac{1}{(x-a)^n}$$

where only finitely many of the coefficients $c, c_n, c_{a,n} \in K$ are nonzero. If we put

$$t_a = \frac{1}{x-a}\ ,\quad t_\infty = x$$

it follows that the elements

$$1,\ t_a^n \quad\text{with}\quad a \in K \cup \{\infty\},\ n \in I\!N$$

form a K–basis of F; this basis has the property that *every* power of a basis element is again a basis element.

For general function fields the theorem on the partial fraction decomposition remains true in the following modified form (according to Hasse). Let P_∞ be a fixed place of $F|K$ and R^∞ the ring of functions which are holomorphic in every $P \neq P^\infty$. It is well known that R^∞ is a Dedekind domain with F as its quotient field, and that the prime ideals of R^∞ correspond to the places $P \neq P_\infty$ of $F|K$. Hence by virtue of the approximation theorem for Dedekind domains (cf. [BOU], chapter VII, §2.4, proposition 2), there exists for every $P \neq P_\infty$ a function $t_P \in F$ such that

$$\begin{array}{rcl} v_P(t_P) & = & -1 \\ v_Q(t_P) & \geq & 0 \quad \text{for}\ \ Q \neq P, P_\infty\ . \end{array}$$

By construction, every t_P is the inverse of a uniformizing parameter for P. Every function $f \in F$ can be expanded P–adically with respect to such a uniformizing parameter, and the principal part that appears in this expansion has the form

$$h_P(f) = \sum_{n>0} c_{P,n} t_P^n\ ,$$

where only finitely many of the coefficients $c_{P,n} \in K$ are nonzero, namely $n \leq -v_P(f)$. By construction, t_P has only a single pole $\neq P_\infty$ and this pole is P; the same holds for $h_P(f)$ (if $h_P(f) \neq 0$). Consequently, the function

$$h = f - \sum_{P \neq P_\infty} h_P(f)$$

has no pole other than P_∞ and is thus an element of R^∞. We have shown that f has a unique representation

$$f = h + \sum_{P \neq P_\infty} \sum_{n>0} c_{P,n} t_P^n$$

with coefficients $c_{P,n} \in K$ and an element $h \in R^\infty$. This shows that the functions

$$t_P^n \text{ with } P \neq P_\infty, \; n \in I\!N$$

form a K–basis of F modulo R^∞ which has the property that every power of a basis element is again a basis element.

Now it remains to show that R^∞ admits a basis which is closed under q–th powers. An integer $n \in I\!N$ is called *pole number* of P_∞ if there exists a function $f \in R^\infty$ such that $v_{P_\infty}(f) = -n$. If we choose for every pole number n of P_∞ such a function $t_n \in R^\infty$, we get a K–basis

$$1, \; t_n \text{ with } n \in I\!N$$

of R^∞. To get a basis which is closed under q–th powers, we have to carry out our choice as follows:

Let $H_\infty \subseteq I\!N$ be the set of all pole numbers. H_∞ is closed under addition; in particular

$$qH_\infty \subset H_\infty \; .$$

For every $m \in H_\infty \setminus qH_\infty$ we choose an arbitrary element $t_m \in R^\infty$ with $v_{P_\infty}(t_m) = -m$. Every $n \in H_\infty$ can uniquely be written as

$$n = q^\nu m \text{ where } \nu \geq 0 \text{ and } m \in H_\infty \setminus qH_\infty \; .$$

Accordingly we put

$$t_n = t_m^{q^\nu}$$

which implies

$$v_{P_\infty}(t_n) = q^\nu \cdot v_{P_\infty}(t_m) = -q^\nu m = -n \; .$$

This construction produces a K–basis

$$1, \; t_m^{q^\nu} \text{ with } m \in H_\infty \setminus qH_\infty, \; \nu \geq 0$$

of R^∞ which is closed under q–th powers. □

For the generalization of this lemma to perfect ground fields of characteristic $p > 0$ we have to choose $q = p$:

Lemma 3.46 *If F is an algebraic function field over a perfect field K of characteristic $p > 0$ then there exists a Frobenius–closed basis for $F|K$.*

Proof: If K is not algebraically closed, we have to modify the proof of the previous lemma since not every place P of K has degree 1. (Such a modification is also necessary for the theorem on the partial fraction decomposition in $K(x)$ if K is not algebraically closed.) The modification reads as follows:

For every place P of $F|K$, let

$$d_P = [FP : K]$$

be the degree of P. For every $P \neq P_\infty$ we choose elements $u_{P,i} \in R^\infty$, $1 \leq i \leq d_P$, such that their residues $u_{P,1}P, ..., u_{P,d_P}P$ form a K–basis of FP. We note that for every $\nu \geq 0$, the p^ν-th powers $u_{P,i}^{p^\nu}$ of these elements have the same property: their P–residues also form a K–basis of FP since K is perfect.
We write every $n \in I\!N$ in the form

$$n = p^\nu m \;\text{ with }\; m \in I\!N,\; (p, m) = 1,\; \nu \geq 0$$

and observe that the elements

$$u_{P,i}^{p^\nu} t_P^n \;\text{ with }\; P \neq P_\infty,\; n \in I\!N,\; 1 \leq i \leq d_P$$

form a Frobenius–closed K–basis of F modulo R^∞.

It remains to construct a Frobenius–closed K–basis of R^∞. This is done as follows:

We consider the vector spaces of the multiples of the divisors nP_∞ in the sense of the Riemann–Roch Theorem:

$$L_n = L(nP_\infty) = \{x \in F \mid v_{P_\infty}(x) \geq -n \text{ and } v_P(x) \geq 0 \text{ for } P \neq P_\infty\} \; .$$

We have $L_0 = K$ and

$$R^\infty = \bigcup_{n \in I\!N} L_n \; .$$

For

$$d_{\infty,n} = \dim L_n / L_{n-1}$$

we have $d_{\infty,n} \geq 0$ and

$$d_{\infty,n} \leq [FP_\infty : K] = d_\infty \; .$$

(By the Riemann – Roch Theorem, equality holds for large enough n.)
Now for $n = 1, 2, ...$ we will choose succesively basis elements $t_{n,i} \in L_n$ modulo L_{n-1}. Then the elements

$$1,\; t_{n,i} \;\text{ with }\; n \in I\!N,\; 1 \leq i \leq d_{\infty,n}$$

form a K–basis of R^∞. To obtain that this basis is Frobenius–closed, we organize our choice as follows:

If $n = pm$, the p–th powers $t_{m,i}^p \in L_n$ are linearly independent modulo $L_{p(m-1)}$ and even modulo $L_{pm-1} = L_{n-1}$. This fact follows from our hypothesis that K is perfect: the existence of nonzero elements $c_i \in K$ with $\sum c_i t_{m,i}^p \in L_{p(m-1)}$ would yield $\sum c_i^{1/p} t_{m,i} \in L_k$ for some $k < m$, a contradiction. In our choice of the elements $t_{n,i}$ we are thus free to take all the elements $t_{m,i}^p$ and to extend this set to a basis of L_n modulo L_{n-1} by arbitrary further elements, if necessary (for n large enough, the elements $t_{m,i}^p$ will already form such a basis). This procedure guarantees that the p–th power of every basis element $t_{m,i}$ is again a basis element, namely equal to $t_{pm,j}$ for suitable j. Hence a basis constructed in this way will be Frobenius–closed. □

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4 The relation of immediate Artin–Schreier–extensions and immediate purely inseparable extensions in characteristic $p > 0$.

In this section, we want to classify immediate Artin–Schreier–extensions according to the question whether they are in some sense similar to immediate purely inseparable extensions or not. We study the relation between such immediate extensions, and the results will enable us to break up the property "defectless field" into similar properties which only deal with more special classes of extensions (such as "separably defectless field"). At the end of the first subsection, we will derive some persistence results which show that the considered properties are inherited by certain algebraic extensions. By an application to an earlier result which was proved in section 3, we obtain Theorem 4.16 which is an analogue of Theorem 3.1 for inseparably defectless fields.

In the second subsection, we will use these results to establish two important characterizations of algebraically complete fields whose applications are to be found in section 5 and section 10.

4.1 Classification and general results.

We will consider the following situation:

$L|K$ an immediate Artin–Schreier–extension of henselian fields of characteristic $p > 0$

ϑ an Artin–Schreier–generator of $L|K$

$a = \wp(\vartheta) = \vartheta^p - \vartheta \in K$

$\delta = \mathrm{dist}(\vartheta, K)$ the distance of ϑ from K.

Since $L|K$ is immediate, we know by Theorem 11.27 that $\delta > v(\vartheta)$. ϑ' is another Artin–Schreier–generator of $L|K$ if and only if

$$\vartheta' = n\vartheta + u \quad \text{with } u \in K \text{ and } 1 \leq n \leq p-1. \tag{60}$$

As a consequence, the distance δ does not depend on the choice of the Artin–Schreier–generator and is thus an invariant of the extension $L|K$. Assume that there exists an Artin–Schreier–generator $\vartheta' = \vartheta - c$ with $v(\wp(\vartheta')) = v(\vartheta') \geq 0$; if in this case, there exists an Artin–Schreier–root $\wp^{-1}(\overline{\wp(\vartheta')})$ in $\overline{K}$, then $K(\vartheta') = K$ by Hensel's Lemma; otherwise $\overline{K(\vartheta')}$ is an Artin–Schreier–extension of $\overline{K}$ implying that $L|K$ is a tame extension and thus defectless. Since we assume $L|K$ to be nontrivial and immediate, we deduce from Corollary 3.23 that

$$\forall c \in K : v(\vartheta - c) < 0 \tag{61}$$

which shows that δ is a negative cut.

We will now distinguish two types of immediate Artin–Schreier–extensions. To this end, we consider δ as a cut in the value group $v(\tilde{K})$ which is equal to the divisible hull

$\widetilde{v(K)}$ of the value group $v(K)$. This is done in the way as described in section 2.3, page 30: if $\delta = (\Lambda, \Lambda')$ in $v(\tilde{K})$, then Λ is the convex hull of

$$\Lambda(\mathrm{appr}(\vartheta, K)) = \{v(\vartheta - c) \mid c \in K\}$$

in $v(\tilde{K})$. Note that Λ admits no maximal element since $L|K$ is assumed to be immediate, hence by virtue of Theorem 11.27 it follows that $\mathrm{appr}(\vartheta, K)$ is immediate. We consider the following interval in $v(\tilde{K})$:

$$U_\delta = \{\alpha \in v(\tilde{K}) \mid \delta \leq \alpha \leq -\delta\} \ .$$

Here $-\delta$ denotes the cut $(-\Lambda', -\Lambda)$ derived from $\delta = (\Lambda, \Lambda')$, where $-\Lambda = \{-\alpha \mid \alpha \in \Lambda\}$. If U_δ is a group (hence a convex subgroup of $v(K)$), we call the immediate Artin–Schreier–extension $L|K$ *independent*. If U_δ is not a group, we call $L|K$ *dependent*. Note that the associated ideal

$$I_\delta = \{c \in K \mid v(c) > -\delta\}$$

is a prime ideal of $\mathcal{O}_K$ if and only if U_δ is a convex subgroup of $v(K)$. If I_δ is a prime ideal, there exists a coarsening v_δ of v such that $v_\delta(\tilde{K}) \cong v(\tilde{K})/U_\delta$. Furthermore, Lemma 11.73 shows that U_δ is a group if and only if $|\delta|$ is a *distinguished* cut and U_δ coincides with the *invariance subgroup* $\mathcal{I}(\delta, v(K))$ (cf. section 11 for definitions and further details). Note that for the following it will be crucial that we took the cut δ as a cut in the divisible hull $\widetilde{v(K)} = v(\tilde{K})$ of the value group $v(K)$; indeed, the interval $[\delta, -\delta]$ in $v(K)$ may be a group whereas as an interval in $v(\tilde{K})$ it may not.

To start with, we observe the following

Lemma 4.1 *U_δ is a group if and only if*

$$\forall \alpha \in v(\tilde{K}) : \alpha < \delta \Leftrightarrow p\alpha < \delta$$

which fact we express in short terms by writing

$$\delta = p\delta \ .$$

Note that $\alpha < \delta \Rightarrow p\alpha < \delta$ follows already from $\delta \leq 0$.

Proof: Let U_δ be a group. Then $\alpha \geq \delta$, i.e. $\alpha \in U_\delta$ implies $p\alpha \in U_\delta$, i.e. $p\alpha \geq \delta$. For the converse, assume that U_δ is not a group. Then there exist elements $\alpha, \beta \in U_\delta$ such that

$$\delta \leq \alpha \leq \beta < 0 \ \text{ and } \ \alpha + \beta < \delta.$$

This yields

$$p\alpha \leq 2\alpha \leq \alpha + \beta < \delta$$

whereas $\alpha \geq \delta$. □

We have chosen the name "dependent" since dependent immediate Artin–Schreier–extensions depend on immediate purely inseparable extensions in a way that we will describe in the following lemma.

Lemma 4.2 *Assume that $L|K$ is dependent, hence in particular $\delta < 0$. Then there exists an immediate purely inseparable extension $K(\vartheta^*)|K$ of degree p; moreover we can choose ϑ^* such that it satisfies the same approximation type as $\vartheta - u$ over K for some $u \in K$ and has the same distance as $\vartheta - u$ and ϑ from K:*

$$\begin{gathered}\mathrm{appr}(\vartheta^*, K) = \mathrm{appr}(\vartheta - u, K) \\ \mathrm{dist}(\vartheta^*, K) \asymp \mathrm{dist}(\vartheta - u, K) \asymp \mathrm{dist}(\vartheta, K)\ .\end{gathered}$$

Proof: For $u \in K$, we consider the Artin–Schreier–generator $\vartheta - u$ and put

$$a_u = \wp(\vartheta - u) = a - \wp(u) \in K.$$

We try to find u such that

$$\vartheta^* = (a_u)^{\frac{1}{p}}$$

satisfies the same approximation type as $\vartheta - u$, i.e.

$$\forall c \in K : v(\vartheta^* - c) = v(\vartheta - u - c). \tag{62}$$

If this is true then we see that $\vartheta^* \notin K$ since otherwise we would get a contradiction by putting $c = \vartheta^*$, hence $K(\vartheta^*)$ is of degree p over K; on the other hand, we conclude that

$$\mathrm{dist}(\vartheta^*, K) \asymp \mathrm{dist}(\vartheta - u, K) \asymp \mathrm{dist}(\vartheta, K) = \delta$$

and in view of Lemma 11.83, that $L^*|K$ is an immediate extension.
According to the definition of ϑ, we have:

$$\vartheta^* - c = (a_u)^{\frac{1}{p}} - c = \wp(\vartheta - u)^{\frac{1}{p}} - c = \vartheta - u - c \;-\; (\vartheta - u)^{\frac{1}{p}}.$$

Hence (62) will be satisfied if

$$v(\vartheta - u - c) < v((\vartheta - u)^{\frac{1}{p}}) = \frac{v(\vartheta - u)}{p}$$

for all $c \in K$, which is equivalent to

$$\delta \leq \frac{v(\vartheta - u)}{p}. \tag{63}$$

By definition of δ we have $v(\vartheta - u) < \delta$. To show the existence of $u \in K$ satisfying (63) we use that U_δ by hypothesis is not a group. In view of Lemma 4.1 we choose an element $\alpha \in v(\tilde{K})$ such that

$$p\alpha < \delta \leq \alpha\ . \tag{64}$$

Then, by definition of δ, there exists $u \in K$ with $p\alpha < v(\vartheta - u) < \delta$, and dividing by p we deduce (63). □

Corollary 4.3 *If K admits no proper immediate purely inseparable extensions, then K admits no immediate dependent Artin–Schreier–extensions.*

The converse of this corollary is not true: every separable–algebraically closed valued field K of characteristic $p > 0$ which is not algebraically closed is a counterexample, since its value group is divisible and its residue field is algebraically closed and hence the proper purely inseparable extension $\tilde{K}|K$ is immediate. But a closer look shows that the irreversibility stems only from immediate purely inseparable extensions which lie in the completion of K:

Lemma 4.4 *Assume that K admits an immediate purely inseparable extension $K(\vartheta^*)|K$ of degree p such that*

$$\delta^* = \mathrm{dist}(\vartheta^*, K) < \infty$$

(which means that there exists an element $v(K)$ which is an upper bound for $v(\vartheta^* - c)$, $c \in K$). Then K admits an immediate dependent Artin–Schreier–extension $K(\tilde{\vartheta})|K$. More precisely, given any $\beta \in v(K)$ such that

$$\frac{p-1}{p}\beta + \frac{1}{p}v(\vartheta^*) \geq \delta^*,$$

and if $b \in K$ has value $v(b) = \beta$, then the Artin–Schreier–generator $\tilde{\vartheta}$ can be chosen as to satisfy

$$\begin{aligned} v(\tilde{\vartheta}) &= v(\vartheta^*) - \beta \\ \mathrm{appr}(\tilde{\vartheta}, K) &= \mathrm{appr}(\vartheta^*/b, K) \\ \mathrm{dist}(\tilde{\vartheta}, K) &\asymp \mathrm{dist}(\vartheta^*, K) - \beta. \end{aligned}$$

All immediate Artin–Schreier–extensions obtained in this way are dependent.

Proof: Let Θ be a root of the polynomial

$$Y^p - b^{p-1}Y - \vartheta^{*p} \in K[Y].$$

By Corollary 11.46 we know that

$$\mathrm{appr}(\Theta, K) = \mathrm{appr}(\vartheta^*, K)\ ,$$

and in view of Lemma 11.83 we conclude that $K(\Theta)|K$ is a nontrivial immediate extension. Putting $Y = bZ$ we find that $\tilde{\vartheta} = \Theta/b$ is a root of

$$Z^p - Z - (\frac{\vartheta^*}{b})^p \tag{65}$$

and hence an Artin–Schreier–generator of $K(\Theta)|K$. By hypothesis we have

$$\frac{p-1}{p}\beta + \frac{1}{p}v(\vartheta^*) \geq \delta^* > v(\vartheta^*) \tag{66}$$

from which we deduce

$$\beta > v(\vartheta^*)$$

showing that

$$v\left(\left(\frac{\vartheta^*}{b}\right)^p\right) < 0$$

and hence that

$$v(\tilde{\vartheta}) = v(\frac{\vartheta^*}{b}) = v(\vartheta^*) - \beta$$

since $\tilde{\vartheta}$ is a root of the polynomial (65). As an immediate consequence we get

$$\mathrm{dist}(\tilde{\vartheta}, K) \asymp \mathrm{dist}(\vartheta^*, K) - \beta.$$

It remains to show that the cut $\tilde{\delta} = \mathrm{dist}(\vartheta^*, K) - \beta$ cannot induce a convex subgroup $U_{\tilde{\delta}}$ of $v(K)$. To see this, we rewrite (66) in the form

$$\begin{aligned} \tfrac{1}{p}(v(\vartheta^*) - \beta) &\geq \delta^* - \beta \\ p \cdot \tfrac{1}{p}(v(\vartheta^*) - \beta) = v(\vartheta^*) - \beta &< \delta^* - \beta \end{aligned}$$

which shows that U_δ is not a group. □

For the proof of the preceding Lemma, we have transformed an immediate purely inseparable extension which was not contained in the completion, into an immediate separable extension. On the other hand, an immediate purely inseparable extension with a generator ϑ^* in the completion of K, i.e.

$$\mathrm{dist}(\vartheta^*, K) = \infty \;, \tag{67}$$

cannot be transformed into any immediate separable extension with a generator of the same approximation type since the above equation implies that any element having the same approximation type over K as ϑ^* can be mapped to ϑ^* by a valuation preserving isomorphism over K. Moreover it is well known that K is separable–algebraically closed in its completion since K is assumed to be henselian.

If K admits any immediate purely inseparable extension that does not lie in the completion of K, then K satisfies the hypothesis of the preceding lemma. Indeed, if $\eta \in \sqrt{K} \setminus K^c$ such that $K(\eta)|K$ is an immediate extension, we may assume that $\eta^p \in K^c$ (otherwise replace η by a suitable p^ν–th power). We have $\mathrm{dist}(\eta, K) < \infty$, hence

$$\mathrm{dist}_K(\eta^p, K^p) < \infty = \mathrm{dist}(\eta^p, K)$$

which shows that there is some $d \in K$ with $v(\eta - d^{1/p}) > \mathrm{dist}(\eta, K)$. By Lemma 11.26, this implies

$$\mathrm{appr}(d^{1/p}, K) = \mathrm{appr}(\eta, K) \;,$$

hence by Lemma 11.46, $K(d^{1/p})|K$ is an immediate extension not contained in K^c. This yields:

Corollary 4.5 *Assume K does not admit any immediate dependent Artin–Schreier–extension. Then every immediate purely inseparable extension lies in the completion of K. If K is Artin–Schreier–closed, then the perfect hull of K lies in the completion of K. In particular, if K is separable–algebraically closed, then $\tilde{K}$ lies in the completion of K.*

The following lemma shows that the condition of the preceeding corollary is inherited by the completion of K:

Lemma 4.6 *If K^c admits an immediate dependent Artin–Schreier–extension, then K admits an immediate dependent Artin–Schreier–extension with an Artin–Schreier–generator of the same distance.*

Proof: Let $Y^p - Y - a$ be the minimal polynomial of a dependent Artin–Schreier–extension of K^c and let ϑ be a root of it. Since $\mathrm{dist}(\vartheta, K^c) < \infty$, we may choose an element $a^* \in K$ such that $v(a - a^*) > p \cdot \mathrm{dist}(\vartheta, K^c)$. Then by Lemma 11.83, $Y^p - Y - a^*$ is the minimal polynomial of an immediate Artin–Schreier–extension and if ϑ^* is a root of it, then $\mathrm{dist}(\vartheta^*, K) \asymp \mathrm{dist}(\vartheta, K^c)$. In particular, the extension $K(\vartheta^*)|K$ is also dependent. □

An immediate consequence of this lemma is:

Corollary 4.7 *If K does not admit any immediate dependent Artin–Schreier–extension, then K^c does not admit any immediate purely inseparable extension. In particular, this holds if K is separable–algebraically maximal.*

None of the Artin–Schreier–extensions constructed in the last lemma can be independent. This follows from a more general fact that at the same time justifies our choice of the notion "independent":

Lemma 4.8 *If $L|K$ is an independent Artin–Schreier–extension, then there exists no immediate purely inseparable extension $K(\vartheta^*)|K$ such that*

$$\mathrm{appr}(\vartheta, K) = \mathrm{appr}(\vartheta^*, K) \ . \tag{68}$$

This can be expressed by writing

$$\mathrm{dist}(\vartheta, K) = \mathrm{dist}(\vartheta, \sqrt{K}) \ .$$

Proof: To deduce a contradiction, assume the contrary and choose $\nu \geq 1$ such that $\vartheta^{*p^\nu} \in K$. From $\mathrm{appr}(\vartheta, K) = \mathrm{appr}(\vartheta^*, K)$ we deduce

$$v(\vartheta - \vartheta^*) \geq \mathrm{dist}(\vartheta, K) =: \delta \ .$$

By hypothesis we know that U_δ is a group, hence

$$\delta = p\delta$$

in view of Lemma 4.1. From $\vartheta^p = \vartheta + a$ we compute

$$\vartheta^{p^\nu} = \vartheta + a' \ \text{ where } \ a' = a + \ldots + a^{p^{\nu-1}} \in K \ .$$

Putting all these equations together we get

$$\delta = \mathrm{dist}(\vartheta, K) > v(\vartheta - (\vartheta^{*p^\nu} - a')) = v(\vartheta^{p^\nu} - \vartheta^{*p^\nu}) = p^\nu v(\vartheta - \vartheta^*) \geq p^\nu \delta = \delta \ ,$$

the desired contradiction. □

For later use, we need the following

Lemma 4.9 *If $K_0 \subset K_1 \subset K_2$ is an extension of henselian fields of characteristic $p > 0$ such that $K_1|K_0$ is finite and purely inseparable and $K_2|K_1$ is an independent immediate Artin–Schreier–extension, then there exists an Artin–Schreier–extension $L|K_0$ such that $K_2 = K_1.L$, and every such extension $L|K_0$ is an independent immediate Artin–Schreier–extension.*

Proof: Let η be an Artin–Schreier–generator of $K_2|K_1$ and choose $\nu \geq 1$ such that

$$K_1^{p^\nu} \subset K_0 \; .$$

Then

$$\wp(\eta^{p^\nu}) = (\wp(\eta))^{p^\nu} \in K_0 \; ,$$

hence

$$K_0(\eta^{p^\nu})|K_0$$

is an Artin–Schreier–extension: it is nontrivial since $K_0(\eta)|K_0$ is not purely inseparable. Computing degrees we see that $K_2 = K_1.K_0(\eta^{p^\nu})$.

Now let $L|K_0$ be any such Artin–Schreier–extension. Let ϑ be an Artin–Schreier–generator of $L|K_0$ and hence of $K_2|K_1$ too. We choose ν as above and compute:

$$\mathrm{dist}(\vartheta^{p^\nu}, K_1) = \mathrm{dist}(\vartheta, K_1)$$

since ϑ^{p^ν} is also an Artin–Schreier–generator of $K_2|K_1$;

$$\mathrm{dist}(\vartheta, K_1) = \delta = p\delta = \mathrm{dist}(\vartheta^{p^\nu}, K_1^{p^\nu})$$

in view of Lemma 4.1 since U_δ is a group by hypothesis;

$$\mathrm{dist}(\vartheta^{p^\nu}, K_1^{p^\nu}) \leq \mathrm{dist}(\vartheta^{p^\nu}, K_0) \leq \mathrm{dist}(\vartheta^{p^\nu}, K_1)$$

because $K_1^{p^\nu} \subset K_0 \subset K_1$. Putting these three equations together, we deduce

$$\mathrm{dist}(\vartheta^{p^\nu}, K_0) = \mathrm{dist}(\vartheta^{p^\nu}, K_1) = \mathrm{dist}(\vartheta, K_1)$$

showing that $L|K_0$ is an independent immediate Artin–Schreier–extension like $K_2|K_1$. □

Another property of independent immediate Artin–Schreier–extensions is their persistence under maximal immediate extensions, in the following sense:

Lemma 4.10 *If K admits an immediate Artin–Schreier–extension $L|K$ with Artin–Schreier–generator ϑ of distance $\delta = 0$, then every algebraically maximal immediate extension (and in particular every maximal immediate extension) M of K contains also an immediate Artin–Schreier–extension with an Artin–Schreier–generator ϑ^* of distance 0 and of the same approximation type* $\mathrm{appr}(\vartheta^*, K) = \mathrm{appr}(\vartheta, K)$.

Proof: If $L \subset M$, there is nothing to show. Assume the contrary. Then $L.M|M$ is also an Artin–Schreier–extension with Artin–Schreier–generator ϑ. Since M is algebraically maximal, the extension $L.M|M$ is defectless and thus admits a valuation basis, according to Lemma 2.7. From Lemma 11.8 we infer the existence of an element $u \in M$ satisfying

$$v(\vartheta - u) \geq \mathrm{dist}(\vartheta, M) \; .$$

On the other hand, $K \subset M$ implies

$$\mathrm{dist}(\vartheta, M) \geq \mathrm{dist}(\vartheta, K) = \delta$$

showing that $v(\vartheta - u) \geq 0$. We put

$$a_u = \wp(\vartheta - u) = a - \wp(u) \in M$$

and note that $v(a_u) \geq 0$. Since $M|K$ is immediate, there exists $b \in K$ having the same residue class as a_u:

$$\overline{b} = \overline{a_u} .$$

We consider the Artin–Schreier–polynomial $\wp(X) - b \in K[X]$. By construction, after reduction to $\overline{M}[X]$ we have:

$$\wp(X) - \overline{b} = \wp(X) - \overline{a_u} .$$

Since $\wp(X) - a_u$ has the zero $\vartheta - u$ in the henselian field $L.M$, by Hensel's Lemma it follows that also $\wp(X) - b$ has a zero $\eta \in L.M$ and that we may assume

$$\overline{\eta} = \overline{\vartheta - u} . \tag{69}$$

We put

$$\vartheta^* = \vartheta - \eta .$$

To show that $\vartheta^* \in M$ we consider the automorphism σ of $L|K$ defined by

$$\sigma(\vartheta) - \vartheta = 1$$

which generates the Galois group of $L|K$ and admits a unique prolongation to a generating automorphism of $L.M|M$. $u \in M$ implies

$$\sigma(\vartheta - u) - (\vartheta - u) = \sigma(\vartheta) - \vartheta = 1 .$$

By construction, also η satisfies an Artin – Schreier – equation over M, thus $\sigma(\eta) - \eta = n$ for a unique $n \in \mathbb{Z}/p$. Equation (69) shows

$$\sigma(\eta) - \eta = \sigma(\vartheta - u) - (\vartheta - u) = 1 .$$

This yields

$$\sigma(\vartheta^*) - \vartheta^* = \sigma(\vartheta) - \vartheta - (\sigma(\eta) - \eta) = 0, \text{ hence } \vartheta^* \in M .$$

Moreover, since $v(\eta) \geq 0$ we have for every $c \in K$:

$$v(\vartheta^* - c) = v(\vartheta - c - \eta) = \min(v(\vartheta - c) , v(\eta)) = v(\vartheta - c) < 0$$

proving that $\mathrm{appr}(\vartheta^*, K) = \mathrm{appr}(\vartheta, K)$. In particular, this shows that $\vartheta^* \notin K$ so that $K(\vartheta^*)|K$ is a proper immediate Artin–Schreier–extension. □

From this lemma, we deduce the following

Corollary 4.11 *If there exists a maximal immediate extension in which K is separable–algebraically closed, then K admits no independent immediate Artin–Schreier–extension with distance $\delta = 0$.*

The converse of this corollary is not true. For example, let k be an algebraically closed valued field of rank 1 and $k(x)$ an immediate extension of k, and let K be a henselization of $k(x)$; note that $k(x)$ is dense in K, i.e. $K = k(x) + \mathcal{M}_K$ and in view of $\mathcal{M}_K \subset \wp(K)$ (cf. Lemma 3.22), we have $K = k(x) + \wp(K)$. To show that K admits no independent immediate Artin–Schreier–extensions, let L be any Artin–Schreier–extension of K; since $v(K) = v(k)$ is divisible and $\overline{K} = \overline{k}$ is algebraically closed, $L|K$ must be immediate. We have shown above that $K = k(x) + \wp(K)$, thus $L|K$ admits an Artin–Schreier–generator ϑ such that $\wp(\vartheta) \in k(x)$. Again from Lemma 3.22, we know that $v(\wp(\vartheta)) < 0$. Since for every element a in $k(x) \setminus k$, there is a $\nu \geq 1$ such that $a \notin k(x)^{p^\nu}$, we may moreover assume that $a = \wp(\vartheta) \in k(x) \setminus k(x)^p$; otherwise we could replace ϑ by the Artin–Schreier–generator $\vartheta - a^{\frac{1}{p}}$ which satisfies

$$(\vartheta - a^{\frac{1}{p}})^p - (\vartheta - a^{\frac{1}{p}}) = a^{\frac{1}{p}}) \ .$$

Now we consider the element $\vartheta - a^{\frac{1}{p}}$ in $\tilde{K}$. We compute

$$v((\vartheta - a^{\frac{1}{p}})^p - (\vartheta - a^{\frac{1}{p}})) = v(a^{\frac{1}{p}}) = \frac{1}{p}v(a) = v(\vartheta) < 0 \ ,$$

hence

$$v(\vartheta - a^{\frac{1}{p}}) = \frac{1}{p}v(\vartheta) > v(\vartheta)$$

which is only possible if

$$\mathrm{appr}(\vartheta, K) = \mathrm{appr}(a^{\frac{1}{p}}, K)$$

showing that $L|K$ is dependent.

We have proved that K does not admit independent Artin–Schreier–extensions. On the other hand, it admits proper Artin–Schreier–extensions, and any of these is immediate. But all of them are included in every maximal immediate extension of K. This is true because the fact that $\overline{K}$ is algebraically closed and $v(K)$ is divisible implies that every maximal immediate extension of K is algebraically closed and therefore contains $\tilde{K}$. We have shown that K is not separable–algebraically closed in any of its maximal immediate extensions whereas it doesn't admit independent immediate Artin–Schreier–extensions.

In this connection, note that every Artin–Schreier–extension of a valued field K, whether immediate or not, becomes an independent immediate extension over the perfect hull $\sqrt{K}$ of K.

In the sequel we will consider independent immediate Artin–Schreier–extensions $L|K$ with distance $\delta < 0$. In this case, U_δ is a proper subgroup of $v(\tilde{K})$, and we let v_δ denote the coarsening of v with

$$v_\delta(\tilde{K}) \cong v(\tilde{K})/U_\delta \ .$$

By definition of δ and U_δ, we see that

$$\begin{aligned} &\forall c \in K : v_\delta(\vartheta - c) < 0 \ , \\ &\sup_{c \in K} v_\delta(\vartheta - c) = 0 \ . \end{aligned}$$

This means that also $(L, v_\delta)|(K, v_\delta)$ is an immediate extension and that the v_δ–distance $\mathrm{dist}_{v_\delta}(\vartheta, K)$ is 0. Hence $(L, v_\delta)|(K, v_\delta)$ is covered by the case that was treated in Lemma 4.10. From this it follows

Lemma 4.12 *Assume that for every coarsening w of v, there exists an immediate algebraically complete extension (M_w, w) of (K, w) such that K is separable–algebraically closed in M_w. Then K admits no independent immediate Artin–Schreier–extensions.*

Lemma 4.13 *The condition of Lemma 4.12 is satisfied if K is a finite extension of a separable–algebraically maximal field K_0 such that $K|K_0$ is defectless.*

Proof: Let w be any coarsening of v. Since (K_0, v) is separable–algebraically maximal, the same is true for (K_0, w) since every finite separable immediate extension of (K_0, w) would also be immediate for the finer valuation v. Now let (N_w, w) be a maximal immediate extension of (K_0, w). (K_0, w) being separable–algebraically maximal, K_0 is separable–algebraically closed in N_w. Hence K_0 itself satisfies the condition of Lemma 4.12. By this, our lemma becomes a consequence of the following more general lemma. □

Lemma 4.14 *Assume that for every coarsening w of v, K_0 admits an immediate algebraically complete extension $(N_w, w)|(K_0, w)$ such that K_0 is relatively algebraically closed (resp. separable–algebraically closed) in N_w. If $K|K_0$ is finite and defectless, then for every coarsening w of v, $(M_w, w) = (N_w.K, w)$ is an immediate algebraically complete extension of (K, w) such that K is relatively algebraically closed (resp. separable–algebraically closed) in M_w.*

Proof: Since $(K, v)|(K_0, v)$ is defectless by hypothesis, the same is true for the extension $(K, w)|(K_0, w)$ by Lemma 2.17. We note that (K_0, w) is henselian since it is assumed to be separable–algebraically closed in the henselian field (N_w, w). So we may apply Lemma 2.13: since $(N_w, w)|(K_0, w)$ is immediate and $(K, w)|(K_0, w)$ is defectless, $(N_w.K, w)|(K, w)$ is immediate and N_w is linearly disjoint from K over K_0. The latter shows that K is relatively algebraically closed (resp. separable–algebraically closed) in N_w. On the other hand, $(M_w, w) = (N_w.K, w)$ is algebraically complete being a finite extension of an algebraically complete field. □

A property that is shifted through every finite extension (even if it is not defectless), is the property of being inseparably defectless:

Lemma 4.15 *Every finite extension K of an inseparably defectless field K_0 of characteristic $p > 0$ is inseparably defectless too.*

Proof: From Corollary 2.9 it follows that every finite purely inseparable extension of an inseparably defectless field is again an inseparably defectless field. Thus it remains to show the lemma in case $K|K_0$ is separable. We fix a prolongation of v to $\tilde{K}$ and consider the ramification fields K_0^r and K^r of K_0 and K with respect to that prolongation. By Lemma 2.11, we know that K_0 is inseparably defectless if and only if K_0^r is inseparably defectless, and the same holds for K. By Lemma 2.10, we have $K^r = K.K_0^r$. The same lemma shows that $K_0^{sep}|K_0^r$ is a p–extension, hence $K^r|K_0^r$ can be viewed as a tower of Artin–Schreier–extensions (cf. Lemma 3.15). Replacing K_0 and K by their ramification fields we may assume from the start that they are henselian and that $K|K_0$ is a tower of Artin–Schreier–extensions. Let $L|K_0$ be an Artin–Schreier–extension contained in $K|K_0$. We want to show that L is inseparably defectless since then by induction, it follows that K is inseparably defectless. Since $\sqrt{L} = L.\sqrt{K_0}$, it suffices to show for every finite purely inseparable extension $K_1|K_0$ (which itself is defectless by hypothesis), that $K_2 = K_1.L$ is a defectless extension of L. This follows immediately if $K_2|K_1$ and thus $K_2|K_0$ are defectless. Now assume that $K_2|K_1$ is immediate. K_1 is an inseparably defectless field being a finite purely inseparable extension of the inseparably defectless field K_0. In particular, this yields that K_1 admits no immediate purely inseparable extension and hence by virtue of Lemma 4.2, no dependent immediate Artin–Schreier–extension. $K_2|K_1$ is thus an independent immediate Artin–Schreier–extension. An application of Lemma 4.9 shows that $L|K_0$ is immediate. Since L and K_1 are linearly disjoint over K_0, we have

$$[K_2 : L] = [K_1.L : L] = [K_1 : K_0] \;,$$

and we compute

$$\begin{aligned} \mathrm{d}(K_2|K_0) &= \mathrm{d}(K_2|K_1)\cdot \mathrm{d}(K_1|K_0) = [K_2|K_1]\cdot 1 = \frac{[K_2 : K_0]}{[K_1 : K_0]} = \frac{[K_2 : K_0]}{[K_2 : L]} \\ &= \frac{[K_2 : L]\cdot [L : K_0]}{[K_2 : L]} = [L : K_0] = \mathrm{d}(L : K_0) \end{aligned}$$

whence

$$\mathrm{d}(K_2|L) = \frac{\mathrm{d}(K_2|K_0)}{\mathrm{d}(L : K_0)} = 1$$

showing that $K_2|L$ is defectless, as desired. □

As a corollary to this lemma and Lemma 3.20, we get:

Theorem 4.16 *Let $F|K$ be a henselian function field without transcendence defect. If K is inseparably defectless then F is inseparably defectless.*

4.2 Characterizations of defectless fields.

Our results on immediate Artin–Schreier–extensions enable us to prove the following useful characterization of the property "defectless field":

Theorem 4.17 *Let K be a separable–algebraically maximal field of characteristic $p > 0$. If in addition K is inseparably defectless, then K is algebraically complete.*

Proof: We note that K is henselian since it is separable–algebraically maximal. Let $L|K$ be a finite extension. We want to show that $L|K$ is defectless. Since any subextension of a defectless extension is defectless too, we may assume w.l.o.g. that $L|K$ is normal. Hence there exists an intermediate field K_1 such that $L|K_1$ is separable and $K_1|K$ is purely inseparable. By hypothesis, we know that $K_1|K$ is defectless. It remains to prove that $L|K_1$ is defectless.

Let K' be a finite tame extension of K_1 such that $L.K'|K'$ is a tower of Artin–Schreier–extensions, and put $L' = L.K'$. By Lemma 2.11, $L|K_1$ is defectless if and only if $L'|K'$ is defectless. Since $K_1|K$ is defectless and $K'|K_1$ is tame and hence defectless, both extensions being finite, $K'|K$ is finite and defectless. Using Lemma 4.15 we conclude that K' is inseparably defectless too and therefore does not admit immediate purely inseparable extensions, showing by virtue of Lemma 4.2 that every immediate Artin–Schreier–extension of K' must be independent. Moreover, from Lemma 4.13 we infer that K' satisfies the hypothesis of Lemma 4.12 which shows that K' admits no independent immediate Artin–Schreier–extensions. Consequently, given an Artin–Schreier–extension $K''|K'$ contained in $L'|K'$, this extension must be defectless. In view of Lemma 4.15, Lemma 4.14 and Lemma 4.12, K'' will again be inseparably defectless and will not admit any independent immediate Artin–Schreier–extension. By induction, we conclude that all Artin–Schreier–extensions that $L'|K'$ consists of are defectless, hence $L'|K'$ and thus $L|K_1$ and $L|K$ are defectless, as asserted. □

Conversely, every defectless field is immediately seen to be separable–algebraically maximal and inseparably defectless. Note that, if $[\overline{K} : \overline{K}^p]$ and $(v(K) : pv(K))$ are finite (and $\mathrm{char}K = p > 0$), we can replace "inseparably defectless" by "every immediate extension is separable" or by

$$[K : K^p] = [\overline{K} : \overline{K}^p] \cdot (v(K) : pv(K))$$

since it was proved by Delon that in this case, all the three properties are equivalent (see [DEL1], Proposition 1.43).

The reader may have noticed that one could use Theorem 4.17 to organize our reduction steps in section 3 in quite a different way; in particular, one could dispense with Lemma 3.19 showing that the structure of a henselian almost rational function field is preserved under defectless extensions of degree p. But since Theorem 4.17 works only in the case $\mathrm{char}(K) = p > 0$ and since we wanted to prove the case of "mixed characteristic" simultaneously together with the case of "equal characteristic", we had to choose the way as developped in the previous section.

Now we will give a second characterization for algebraically complete fields. For this we need the following lemma.

Lemma 4.18 *Let $L|K$ be a finite defectless purely inseparable extension and let $a_1, ..., a_m$ be elements of L. Then for every value $\alpha \in v(K)$, there exists a defectless separable extension $L'|K$ having the same degree as $L|K$ such that $v(L') = v(L)$, $\overline{L'} = \overline{L}$ and L contains elements $a'_1, ..., a'_m$ with $v(a_i - a'_i) > \alpha$, $1 \leq i \leq m$.*
If among the elements a_i there is a valuation basis of $L|K$, say $a_1, ..., a_n$, and if α is chosen to be greater than $\max_{1\leq i\leq n}(v(a_i))$, then the elements $a'_1, ..., a'_n$ will form a valuation basis of $L'|K$.

Proof: The extension $L|K$ is a tower of purely inseparable defectless extensions of degree p. According to this we may choose a special valuation basis of $L|K$ as follows:

$$S_i := \{1, s_i, s_i^2, ..., s_i^{p-1}\}$$

is a valuation basis of the i–th extension such that
— either $v(s_i) > 0$ and $0, v(s_i), ..., (p-1)v(s_i)$ is a basis of $v(K(s_1, ..., s_i))|v(K(s_1, ..., s_{i-1}))$,
— or $v(s_i) = 0$ and $1, \overline{s_i}, ..., \overline{s_i}^{p-1}$ is a basis of $\overline{K(s_1, ..., s_i)}|\overline{K(s_1, ..., s_{i-1})}$.

If we are given m elements $a_1, ..., a_m$ in $K(s_1)$ and a value $\alpha \in v(K)$, then we may write

$$a_i = \sum_{j=0}^{p-1} s_1^j c_{ij} \text{ with } c_{ij} \in K \; .$$

We put

$$\gamma_1 := \max_{i,j}(v(s_1), \alpha - v(c_{ij})) \ .$$

If we choose r_1 to be a root of the polynomial

$$Y^p - d_1 Y - s_1^p$$

where we choose $d_1 \in K$ with value $v(d_1)$ as large as to guarantee

$$v(s_1 - r_1) > \gamma_1 \ ,$$

whence $v(r_1) = v(s_1) \geq 0$ and

$$\forall j,\ 0 < j < p :\ v(s_1^j - r_1^j) > \gamma_1 \ ,$$

then every

$$a_i' := \sum_{j=0}^{p-1} r_1^j c_{ij}$$

satisfies $v(a_i - a_i') > \beta$ and moreover, $1, r_1, ..., r_1^{p-1}$ is a valuation basis for $K(r_1)|K$ since the r_1^j have the same values resp. residues as the s_1^j. By construction, $[K(r_1) : K] = p$ and $v(K(r_1)) = v(K(s_1))$, $\overline{K(r_1)} = \overline{K(s_1)}$.

Now we have to use induction. Assume that whenever given m elements $a_1, ..., a_m$ in $K(s_1, ..., s_{k-1})$ and a value $\alpha \in v(K)$, then we can find $r_1, ..., r_{k-1} \in \tilde{K}$ and elements $a_1', ..., a_m' \in K(r_1, ..., r_{k-1})$ such that $\forall i : v(a_i - a_i') > \alpha$ and that the extension $K(r_1, ..., r_k)|K$ is separable and defectless. Now let m elements $a_1, ..., a_m$ in $K(s_1, ..., s_k)$ and a value $\alpha \in v(K)$ be given. We may write

$$a_i = \sum_{j=0}^{p-1} s_k^j c_{ij} \ \text{ with } c_{ij} \in K(s_1, ..., s_{k-1}) \ .$$

We put

$$\gamma_k := \max_{i,j}(v(s_k), \alpha - v(c_{ij}))$$

(where $1 \leq i \leq m$ and $0 \leq j \leq p-1$). Let us now choose a suitable separable defectless extension $K(r_1, ..., r_{k-1})$ of K and elements $c_{ij}', \tilde{r}_k \in K(r_1, ..., r_{k-1})$ with

$$v(c_{ij} - c_{ij}') > \max_{i,j}(\alpha, v(c_{i,j}))$$

and

$$v(s_k^p - \tilde{r}_k) > p \cdot \gamma_k \ .$$

Such extension and elements exist by our induction hypothesis. Note that by our choice, $v(c_{i,j}') = v(c_{i,j})$ for all i, j. If we choose r_k to be a root of the polynomial

$$Y^p - d_k Y - \tilde{r}_k$$

where we choose $d_k \in K$ with value $v(d_k)$ as large as to guarantee

$$v(r_k^p - \tilde{r}_k) > p \cdot \gamma_k \ ,$$

then we will have

$$v(s_k - r_k) \geq \frac{1}{p} \min(v(s_k^p - \tilde{r}_k), v(r_k^p - \tilde{r}_k)) > \gamma_k$$

and consequently, $v(r_k) = v(s_k) \geq 0$ and

$$\forall j,\ 0 < j < p:\ v(s_k^j - r_k^j) > \gamma_k\ ,$$

which shows firstly that $1, r_k, ..., r_k^{p-1}$ is a valuation basis for

$$K(r_1, ..., r_k)|K(r_1, ..., r_{k-1})\ .$$

Secondly, every

$$a'_i := \sum_{j=0}^{p-1} r_k^j c'_{ij}$$

satisfies

$$\begin{aligned} v(a_i - a'_i) &= v\left(\sum_{j=0}^{p-1} s_k^j c_{ij} - \sum_{j=0}^{p-1} r_k^j c'_{ij}\right) \\ &\geq \min\left(v\left(\sum_{j=0}^{p-1} s_k^j (c_{ij} - c'_{i,j})\right), v\left(\sum_{j=0}^{p-1} (r_k^j - s_k^j) c'_{ij}\right)\right) > \alpha\ . \end{aligned}$$

By construction,

$$[K(r_1, ..., r_k) : K(r_1, ..., r_{k-1})] = p$$

and

$$v(K(r_1, ..., r_k)) = v(K(s_1, ..., s_k))\ ,\ \ \overline{K(r_1, ..., r_k)} = \overline{K(s_1, ..., s_k)}\ .$$

By induction, this completes the proof of the first part of our lemma.

Assume now that $a_1, ..., a_n$ is a valuation basis of $L|K$ and that

$$\alpha \geq \max_{1 \leq i \leq n} (v(a_i))\ .$$

This implies

$$\forall i:\ v(a'_i - a_i) > v(a_i) = v(a'_i)$$

and consequently, for every $c_1, ..., c_n \in K$:

$$\begin{aligned} v(a'_1 c_1 + ... + a'_n c_n) &= v(a_1 c_1 + ... + a_n c_n + \sum_{j=1}^{n} (a'_j - a_j) c_j) \\ &= \min_{1 \leq i \leq n} (v(a_i c_i), v(\sum_{j=1}^{n} (a'_j - a_j) c_j)) = \min_{1 \leq i \leq n} (v(a_i c_i)) = \min_{1 \leq i \leq n} (v(a'_i c_i)) \end{aligned}$$

which shows that the elements $a'_1, ..., a'_n \in L'$ are valuation–independent over K, and since $[L' : K] = [L : K] = n$, this yields that they form a valuation basis of $L'|K$. □

Now we are ready to prove

Theorem 4.19 *Let K be a separable–algebraically complete field of characteristic $p > 0$. If in addition $K^c|K$ is separable, then K is algebraically complete.*

Proof: According to Theorem 4.17, it suffices to show that K is inseparably defectless. Assume that this is not the case. Since every purely inseparable extension is a tower of extensions of degree p, it follows that there exists a finite defectless purely inseparable extension $L|K$ and elements $a, b \in \sqrt{K}$ such that $b^p = a \in L$ and the extension $L(b)|L$ is immediate. Let $t_1, ..., t_n$ be a valuation basis of $L|K$. By our hypothesis that $K^c|K$ is separable, we know that

$$[K^c.L(b) : K^c.L] = p \; .$$

Since $L|K$ is finite, thus $K^c.L = L^c$, this shows that the distance of b from L^c is not ∞:

$$\delta := \mathrm{dist}(b, L) < \infty \; . \qquad (70)$$

Note that $L = t_1K \oplus ... \oplus t_nK$. We show: if $t'_1, ..., t'_n \in \tilde{K}$ are other valuation–independent elements over K such that

$$\forall i: \; v(t_i - t'_i) \geq \delta - v(b) + v(t_i) \qquad (71)$$

then

$$\mathrm{dist}(b, L) = \mathrm{dist}_L(b, t_1K \oplus ... \oplus t_nK) = \mathrm{dist}_L(b, t'_1K \oplus ... \oplus t'_nK) \; . \qquad (72)$$

Indeed, for any $c_1, ..., c_n \in K$ such that $\forall i : v(t_ic_i) \geq v(b)$, we have

$$\begin{aligned} v(b - \sum_{i=1}^{n} t'_ic_i) &= v(b - \sum_{i=1}^{n} t_ic_i + \sum_{i=1}^{n}(t_i - t'_i)c_i) \\ &= \min(v(b - \sum_{i=1}^{n} t_ic_i), v(\sum_{i=1}^{n}(t_i - t'_i)c_i)) = v(b - \sum_{i=1}^{n} t_ic_i)) < \delta \end{aligned}$$

since $\forall i : v((t_i - t'_i)c_i) \geq \delta - v(b) + v(t_i) + v(b) - v(t_i) = \delta$. This yields (72).

If we take $\alpha > \max_i(p\delta, \delta - v(b) + v(t_i))$, the previous lemma shows the existence of a defectless separable extension $L'|K$ which admits a valuation basis $t'_1, ..., t'_n$ satisfying condition (71), and that in addition there is an element $a' \in L'$ such that $v(a - a') \geq p\delta$. The latter condition yields

$$v(b - (a')^{1/p}) \geq \delta$$

which by Lemma 11.26 implies

$$\delta = \mathrm{dist}(b, L) = \mathrm{dist}(b, L') = \mathrm{dist}((a')^{1/p}, L')$$

by virtue of (72) and

$$L' = t'_1K \oplus ... \oplus t'_nK \; .$$

If we now take b' to be a root of the polynomial

$$Y^p - dY - a'$$

where we choose d with value $v(d)$ as large as to satisfy

$$v(b' - (a')^{1/p}) \geq \delta \; ,$$

then we get an immediate separable extension $L'(b')|L'$ with

$$\mathrm{dist}(b', L') = \mathrm{dist}((a')^{1/p}, L') = \delta\ .$$

Altogether, we have constructed a finite separable extension $L'(b')|K$ which is not defectless in contradiction to our assumption on K. Hence we have shown K to be inseparably defectless and thus algebraically complete, as asserted. □

Corollary 4.20 *Let K be a henselian field of characteristic $p > 0$. If K is a separable–algebraically complete field, then K^c is an algebraically complete field, and vice versa.*

Proof: K^c is henselian like K; cf. the proof of Lemma 5.12. By virtue of the preceding Theorem, K^c is separable–algebraically complete if and only if it is algebraically complete. Thus it suffices to prove that K^c is separable–algebraically complete if and only if K is separable–algebraically complete.

Assume that K^c is separable–algebraically complete. Let $L|K$ be an arbitrary finite separable extension. By Lemma 11.70, the henselian field K is separable–algebraically closed in K^c. Consequently, every finite separable extension of K is linearly disjoint from K^c over K which shows that $[L.K^c : K^c] = [L : K]$. By hypothesis, $L.K^c|K^c$ is defectless. On the other hand, $L.K^c = L^c$ is the completion of L and thus an immediate extension of L. Consequently,

$$[L : K] = [L^c : K^c] = (v(L^c) : v(K^c)) \cdot [\overline{L^c} : \overline{K^c}] = (v(L) : v(K)) \cdot [\overline{L} : \overline{K}]$$

showing that $L|K$ is defectless. Since $L|K$ was an arbitrary finite separable extension we have shown that K is separable–algebraically complete.

Now assume that K^c is not separable–algebraically complete. Then there exists a finite separable extension $L_1|K^c$ with nontrivial defect. By Lemma 3.15, there exists a finite tame extension of $N|K^c$ such that $N.L_1|N$ is a tower of Galois extensions of degree p which are Artin–Schreier–extensions. Since $\mathrm{d}(N.L_1|N) = \mathrm{d}(L_1|K^c) > 1$, at least one of them must be immediate. Hence putting N and some suitable defectless Artin–Schreier–extensions of the tower together, we obtain a finite separable defectless extension $L|K^c$ and the existence of an element $a \in L$ such that the Artin–Schreier–extension with Artin–Schreier–generator ϑ satisfying $\vartheta^p - \vartheta = a$, is immediate. By Lemma 3.22,

$$\mathrm{dist}(\vartheta, L) \leq 0 < \infty \quad \text{and} \quad v(a) < 0\ .$$

Let $t_1, ..., t_n$ be a valuation basis of $L|K^c$. Let $L = K(b)$ and let $f(X) \in K^c[X]$ be the minimal polynomial of b over K^c. We write

$$a = g(b)/h(b) \quad \text{and} \quad t_i = g_i(b)/h_i(b)$$
$$\text{with} \quad g(X), h(X), g_i(X), h_i(X) \in K^c[X]\ .$$

Replacing the coefficients of f, g, h, g_i, h_i by sufficiently close elements from K, we get polynomials $f^*, g^*, h^*, g_i^*, h_i^*$, and taking b^* to be a root of f^* we get $a^* := g^*(b^*)/h^*(b^*) \in K(b^*)$ and $t_i^* := g_i^*(b^*)/h_i^*(b^*) \in K(b^*)$, and we may assume

$$v(a - a^*) \geq 0\ , \quad v(t_i - t_i^*) \geq v(t_i) - v(\vartheta) > v(t_i)\ ,$$

and that $K(b^*)|K$ is separable of the same degree as $K^c(b)|K^c$. In particular, this yields that the elements $t_1^*, ..., t_n^*$ form a valuation basis of $K(b^*)|K$. Now we take ϑ^* to be a root of the polynomial

$$Y^p - Y - a^* .$$

Since $v(a - a^*) \geq 0$ and $(\vartheta - \vartheta^*)^p - (\vartheta - \vartheta^*) = a - a^*$, we have $v(\vartheta - \vartheta^*) \geq 0 \geq \mathrm{dist}(\vartheta, L)$ which by Lemma 11.26 implies

$$0 \geq \mathrm{dist}(\vartheta, L) = \mathrm{dist}(\vartheta^*, L) .$$

Now $L = t_1 K^c \oplus ... \oplus t_n K^c$, and as in the proof of the last lemma one can show that

$$\mathrm{dist}(\vartheta^*, L) = \mathrm{dist}_L(\vartheta^*, t_1 K^c \oplus ... \oplus t_n K^c) = \mathrm{dist}_L(\vartheta^*, t_1^* K^c \oplus ... \oplus t_n^* K^c) .$$

Furthermore, we have

$$0 \geq \mathrm{dist}_L(\vartheta^*, t_1^* K^c \oplus ... \oplus t_n^* K^c) \geq \mathrm{dist}_L(\vartheta^*, t_1^* K \oplus ... \oplus t_n^* K) = \mathrm{dist}(\vartheta^*, K(b^*)) .$$

We will show equality here. Indeed, for every choice of elements $c_1, ..., c_n \in K^c$, we can find elements $c_1^*, ..., c_n^* \in K$ with $v(t_i^*(c_i - c_i^*)) > 0$ which consequently satisfy

$$v(\vartheta^* - \sum_i t_i^* c_i^*) = v(\vartheta^* - \sum_i t_i^* c_i + \sum_i (t_i^*(c_i - c_i^*)) = v(\vartheta^* - \sum_i t_i^* c_i) < 0 .$$

This gives the asserted equality. We have now proved that the extension $K(b^*, \vartheta^*)|K(b^*)$ is immediate and thus the whole separable extension $K(b^*, \vartheta^*)|K$ is not defectless. Hence K is not separable–algebraically complete if K^c is not separable–algebraically complete. □

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5 Defects of function fields without transcendence defect over arbitrary ground fields.

In this section we will consider function fields without transcendence defect over their ground fields, i.e. function fields $F|K$ admitting a valuation transcendence basis. Using Theorem 3.1 we will define the defect for such function fields and show how this defect corresponds to extensions of the ground field and their defect. Furthermore, we will define and investigate two other notions of "defect" which are not as strong as the ordinary one but nevertheless can carry specific information. We will define a completion defect for finite extensions as well as for function fields without transcendence defect. In particular, we will use it to give a characterization of separably defectless fields.

5.1 Definition and basic properties of the defect of subhenselian function fields without transcendence defect.

Our results on function fields without transcendence defect over defectless fields enable us to give also a simple definition for the defect of function fields without transcendence defect over arbitrary ground fields. We are considering the following situation:

K a valued field,
F a subhenselian function field over K, i.e. F^h is the henselization of a valued function field over K,
v the valuation of K and F.

For every transcendence basis $\mathcal{T}$ of F over K, we consider

$$\mathrm{d}(F|K(\mathcal{T}))\ .$$

We define

$$\mathrm{d}(F|K) := \sup_{\mathcal{T}} \mathrm{d}(F|K(\mathcal{T}))$$

and we will show in the following that this supremum is a finite number whenever $F|K$ has no transcendence defect, in which case it is equal to $\mathrm{d}(F|K(\mathcal{T}))$ for every valuation transcendence basis $\mathcal{T}$ of $F|K$.

Lemma 5.1 *For every (not necessarily finite) valuation transcendence basis $\mathcal{T}$ over K, $K(\mathcal{T})|K$ and $K(\mathcal{T})^h|K^h$ are regular and valuation–regular extensions.*

Proof: As a purely transcendental extension, $K(\mathcal{T})|K$ is regular. Corollary 2.21 shows that $K(\mathcal{T})|K$ is valuation–regular. From this, the same follows for $K(\mathcal{T})^h|K^h$ since $K^h|K$ and $K(\mathcal{T})^h|K(\mathcal{T})$ are immediate extensions. Finally, by Lemma 3.9 it now follows that $K(\mathcal{T})^h|K^h$ is regular. □

Lemma 5.2 *For every h–finite extension $L|K$ and every valuation transcendence basis $\mathcal{T}$,*

$$\mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L|K)\ .$$

Proof: Let $\mathcal{T} = \{x_1, ..., x_r, y_1, ..., y_s\}$ where $v(x_1), ..., v(x_r)$ are rationally independent over $v(K)$ and $\overline{y_1}, \ldots, \overline{y_s}$ are algebraically independent over $\overline{K}$. Then by Lemma 2.20,

$$\begin{aligned} v(K(\mathcal{T})) &= v(K) \oplus \mathbb{Z}v(x_1) \oplus \ldots \oplus \mathbb{Z}v(x_r), \\ v(L(\mathcal{T})) &= v(L) \oplus \mathbb{Z}v(x_1) \oplus \ldots \oplus \mathbb{Z}v(x_r), \\ \overline{K(\mathcal{T})} &= \overline{K}\,(\overline{y_1}, ..., \overline{y_s})\,, \\ \overline{L(\mathcal{T})} &= \overline{L}\,(\overline{y_1}, ..., \overline{y_s})\ . \end{aligned}$$

This shows

$$\begin{aligned} \left[\overline{L(\mathcal{T})} : \overline{K(\mathcal{T})}\right] &= \left[\overline{L} : \overline{K}\right] \\ (v(L(\mathcal{T})) : v(K(\mathcal{T})) &= (v(L) : v(K)) \end{aligned}$$

and thus, in view of

$$[L(\mathcal{T})^h : K(\mathcal{T})^h] = [L^h.K(\mathcal{T})^h : K(\mathcal{T})^h] = [L^h : K^h]\ ,$$

which is a consequence of the foregoing lemma, we have

$$\mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L|K)$$

as contended. □

Lemma 5.3 *For every valuation transcendence basis of $F|K$ there exists a finite extension $K_{\mathcal{T}}$ of K such that for every algebraic extension L of K containing $K_{\mathcal{T}}$ the following holds:*

1. *the extension $L.F|L(\mathcal{T})$ is defectless*
2. $\mathrm{d}(L.F|K(\mathcal{T})) = \mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L|K)$.

If K is henselian then $K_{\mathcal{T}}$ may be chosen to be a finite purely wild extension of K. If char$(K) = p > 0$, then $K_{\mathcal{T}}$ may be chosen to be a finite immediate separable extension of a finite purely inseparable extension of K.

Proof: For any finite extension L of K for which $L.F|L(\mathcal{T})$ is defectless we compute

$$\begin{aligned} \mathrm{d}(L.F|K(\mathcal{T})) &= \mathrm{d}(L.F|L(\mathcal{T})) \cdot \mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) \\ &= \mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L|K) \end{aligned}$$

where the last equation holds by Lemma 5.2. Hence we may restrict our attention to the fulfillment of assertion 1.

Furthermore, we will show that w.l.o.g. we may replace F and K by their henselizations. Indeed, by definition we have for every valuation transcendence basis $\mathcal{T}$:

$$\mathrm{d}(F|K(\mathcal{T})) = \mathrm{d}(F^h|(K(\mathcal{T}))^h) = \mathrm{d}((F^h)^h|(K^h(\mathcal{T}))^h) = \mathrm{d}(F^h|K^h(\mathcal{T}))\ .$$

Now if $K_{\mathcal{T}}$ is a finite purely wild extension of K^h such that $L.F^h|L(\mathcal{T})$ is defectless for any finite extension L of $K_{\mathcal{T}}$, then we may choose a finite extension $K'_{\mathcal{T}}$ of K such that $(K'_{\mathcal{T}})^h = K'_{\mathcal{T}}.K^h = K_{\mathcal{T}}$ and hence for every finite extension L' of $K'_{\mathcal{T}}$ we get:

$$\begin{aligned} \mathrm{d}(L'.F|L'(\mathcal{T})) &= \mathrm{d}((L'.F)^h|(L'(\mathcal{T}))^h) \\ &= \mathrm{d}((L'^h.F^h)^h|(L'^h(\mathcal{T}))^h) \\ &= \mathrm{d}(L'^h.F^h|L'^h(\mathcal{T})) = 1 \end{aligned}$$

since L'^h is a finite extension of $(K'_{\mathcal{T}})^h = K_{\mathcal{T}}$. Thus we may assume from the start that F and K are henselian.

The extension $\tilde{K}.F|\tilde{K}(\mathcal{T})^h$ is defectless by Theorem 3.1. We will show now that there exists a finite extension $K_{\mathcal{T}}$ such that for every finite extension L of K containing $K_{\mathcal{T}}$ the following holds:

1. $[L.F : (L(\mathcal{T}))^h] = \left[\tilde{K}.F : (\tilde{K}(\mathcal{T}))^h\right]$,

2. $(v(L.F) : v(L(\mathcal{T}))) = (v(\tilde{K}.F) : v(\tilde{K}(\mathcal{T})))$,

3. $\left[\overline{L.F} : \overline{L(\mathcal{T})}\right] = \left[\overline{\tilde{K}.F} : \overline{\tilde{K}(\mathcal{T})}\right]$.

$\tilde{K}.F|\tilde{K}(\mathcal{T})^h$ being defectless, these three conditions yield immediately that $L.F|L(\mathcal{T})^h$ is defectless too. To prove the existence of $K_{\mathcal{T}}$, we observe first that there is a finite extension K_1 of K such that

$$\begin{aligned}[K_1.F : (K_1(\mathcal{T}))^h] &= [K_1.F : K_1.(K(\mathcal{T}))^h] \\ &= [\tilde{K}.F : \tilde{K}.(K(\mathcal{T}))^h] \\ &= [\tilde{K}.F : (\tilde{K}(\mathcal{T}))^h]\,,\end{aligned}$$

and this remains true if we replace K_1 by any algebraic extension since such a replacement cannot increase the degree.

Secondly, we note that $v(\tilde{K})$ is trivially pure in $v(\tilde{K}.F)$. Since $\tilde{K}.F$ is a henselian function field without transcendence defect over $\tilde{K}$, we deduce

$$\begin{aligned}v(\tilde{K}.F) &= v(\tilde{K}) \oplus \mathbb{Z}v(z_1) \oplus \ldots \oplus \mathbb{Z}v(z_r) \\ v(\tilde{K}(\mathcal{T})) &= v(\tilde{K}) \oplus \mathbb{Z}n_1 v(z_1) \oplus \ldots \oplus \mathbb{Z}n_r v(z_r)\end{aligned}$$

for suitable elements $z_1, \ldots, z_r \in \tilde{K}.F$. Let K_2 be a finite extension of K_1 such that $z_1, \ldots, z_r \in K_2.F$. For any algebraic extension L of K containing K_2 this implies

$$\begin{aligned}v(L.F) &\supseteq v(L) \oplus \mathbb{Z}v(z_1) \oplus \ldots \oplus \mathbb{Z}v(z_r) \\ v(L(\mathcal{T})) &= v(L) \oplus \mathbb{Z}n_1 v(z_1) \oplus \ldots \oplus \mathbb{Z}n_r v(z_r)\end{aligned}$$

showing that

$$(v(L.F) : v(L(\mathcal{T}))) \geq n_1 \cdot \ldots \cdot n_r = (v(\tilde{K}.F) : v(\tilde{K}(\mathcal{T})))\,. \tag{73}$$

Furthermore, we observe that $\overline{\tilde{K}.F}$ is a finite extension of $\overline{\tilde{K}(\mathcal{T})}$ which in turn equals the rational function field

$$\overline{\tilde{K}}\left(\overline{y_1}, \ldots, \overline{y_s}\right)\ ,\ y_1, \ldots, y_s \in \mathcal{T}\ .$$

Hence

$$\overline{\tilde{K}.F} = \overline{\tilde{K}}\left(\overline{y_1}, \ldots, \overline{y_s}, \overline{a_1}, \ldots, \overline{a_m}\right)$$

for suitable elements $a_1, \ldots, a_m \in \tilde{K}.F$. Let $f_1(X), \ldots, f_m(X)$ be the minimal polynomials of $a_1, \ldots, a_m$ over $\tilde{K}(\mathcal{T})^h$ and take the elements $c_1, \ldots, c_k \in \tilde{K}(\mathcal{T})^h$ to be their coefficients. Let $K_{\mathcal{T}}$ be a finite extension of K_2 such that $a_1, \ldots, a_m \in F.K_{\mathcal{T}}$ and $c_1, \ldots, c_k \in K_{\mathcal{T}}(\mathcal{T})$. Then for every algebraic extension L of K containing $K_{\mathcal{T}}$,

$$\left[\overline{L.F} : \overline{L(\mathcal{T})}\right] \geq \left[\overline{L.F} : \overline{L}(\overline{y_1}, \ldots, \overline{y_s})\right] = \left[\overline{\tilde{K}.F} : \overline{\tilde{K}(\mathcal{T})}\right]\ . \tag{74}$$

Putting equations (73) and (74) together, we get

$$\begin{aligned}\left[\tilde{K}.F:\tilde{K}(\mathcal{T})^h\right] &= [L.F:L(\mathcal{T})^h] \geq \left[\overline{L.F}:\overline{L(\mathcal{T})}\right]\cdot(v(L.F):v(L(\mathcal{T})))\\ &\geq [\overline{\tilde{K}.F}:\overline{\tilde{K}(\mathcal{T})}]\cdot(v(\tilde{K}.F):v(\tilde{K}(\mathcal{T}))) = [\tilde{K}.F:\tilde{K}(\mathcal{T})^h]\;,\end{aligned}$$

hence "=" must hold here as well as in inequalities (73) and (74) for every algebraic extension L of $K_{\mathcal{T}}$.

It remains to show that $K_{\mathcal{T}}$ may be chosen satisfying the additional conditions stated at the end of the lemma. Assume K to be henselian. Since we may replace $K_{\mathcal{T}}$ by a finite extension of $K_{\mathcal{T}}$, we may assume w.l.o.g. that there exists an intermediate field K' of $K_{\mathcal{T}}|K$ such that $K_{\mathcal{T}}|K'$ is a tame and $K'|K$ is a purely wild extension. We want to show that we may replace $K_{\mathcal{T}}$ by K'. To this end we only have to show that $L'.F|L'(\mathcal{T})$ is defectless for every algebraic extension L' of K'. By what we have shown already,

$$\mathrm{d}((K_{\mathcal{T}}.\,L').F|(K_{\mathcal{T}}.\,L')(\mathcal{T})) = 1 \tag{75}$$

since $K_{\mathcal{T}}.\,L'$ is an algebraic extension of $K_{\mathcal{T}}$. On the other hand, the extension $K_{\mathcal{T}}.\,L'|L'$ is finite and tame like $K_{\mathcal{T}}|K'$, hence it is defectless. Using Lemma 5.2 we deduce

$$\mathrm{d}((K_{\mathcal{T}}.\,L')(\mathcal{T})|L'(\mathcal{T})) = \mathrm{d}(K_{\mathcal{T}}.\,L'|L') = 1\;. \tag{76}$$

Putting equations (75) and (76) together we see that $(K_{\mathcal{T}}.\,L').F|L'(\mathcal{T})$ is defectless. Thus the same holds for the subextension $L'.F|L'(\mathcal{T})$, as asserted.

We have shown that if K is henselian then $K_{\mathcal{T}}$ may be chosen to be a purely wild extension of K. Now any maximal algebraic purely wild extension contains the perfect hull $\sqrt{K}$ of K, and if $\mathrm{char}(K) = p > 0$ then it is just a maximal immediate algebraic extension of $\sqrt{K}$. In the general case, in view of what we have shown at the beginning of our proof, we may choose $K_{\mathcal{T}}$ such that $K_{\mathcal{T}}.K^h$ is a finite purely wild extension of K^h, hence contained in a maximal immediate algebraic extension of $\sqrt{K^h} = \sqrt{K}^{\,h}$. Since $\sqrt{K}^{\,h}$ is an immediate extension of $\sqrt{K}$ we may express this by saying that $K_{\mathcal{T}}$ is contained in a maximal immediate algebraic extension of $\sqrt{K}$. This completes our proof. □

After this preparation, we are able to prove the finiteness of $\mathrm{d}(F|K)$ and its independence of the choice of the valuation transcendence basis $\mathcal{T}$:

Theorem 5.4 *Let $F|K$ be a subhenselian function field without transcendence defect. Then for every valuation transcendence basis $\mathcal{T}$ of $F|K$,*

$$\mathrm{d}(F|K) = \mathrm{d}(F|K(\mathcal{T})) < \infty\;. \tag{77}$$

Moreover, there exists a finite extension K' of K such that for every algebraic extension L of K containing K' we have

1. *for every valuation transcendence basis $\mathcal{T}$ of $F|K$, the extension $L.F|L(\mathcal{T})$ is defectless,*

2. $\mathrm{d}(F|K) = \dfrac{\mathrm{d}(L|K)}{\mathrm{d}(L.F|F)} = \max\limits_{N|K \text{ finite}} \dfrac{\mathrm{d}(N|K)}{\mathrm{d}(N.F|F)}\;.$

If K is henselian then K' may be chosen to be a finite purely wild extension of K. If $char(K) = p > 0$, then K' may be chosen to be a finite immediate separable extension of a finite purely inseparable extension of K.

Proof: Let $\mathcal{T}_0$ be any transcendence basis of $F|K$. Then by additivity of the transcendence defect, the transcendence defect of $K(\mathcal{T}_0)|K$ is zero. Hence $K(\mathcal{T}_0)$ admits a valuation transcendence basis $\mathcal{T}$ over K. We compute

$$\mathrm{d}(F|K(\mathcal{T}_0)) \leq \mathrm{d}(F|K(\mathcal{T}_0)) \cdot \mathrm{d}(K(\mathcal{T}_0)|K(\mathcal{T})) = \mathrm{d}(F|K(\mathcal{T}))$$

showing that

$$\mathrm{d}(F|K) = \sup_{\mathcal{T}} \mathrm{d}(F|K(\mathcal{T}))$$

where $\mathcal{T}$ runs over valuation transcendence bases only. Since F is a subhenselian function field, every $\mathrm{d}(F|K(\mathcal{T}))$ is a finite number. It remains to show that for any two valuation transcendence bases $\mathcal{T}_1$ and $\mathcal{T}_2$,

$$\mathrm{d}(F|K(\mathcal{T}_1)) = \mathrm{d}(F|K(\mathcal{T}_2)) \ .$$

We choose finite extensions $K_{\mathcal{T}_1}$ and $K_{\mathcal{T}_2}$ according to Lemma 5.3. Putting $L_0 = K_{\mathcal{T}_1}.K_{\mathcal{T}_2}$ we get by Lemma 5.3:

$$\mathrm{d}(L_0.F|K(\mathcal{T}_1)) = \mathrm{d}(L_0|K) = \mathrm{d}(L_0.F|K(\mathcal{T}_2))$$

and from this we deduce

$$\mathrm{d}(F|K(\mathcal{T}_1)) = \frac{\mathrm{d}(L_0.F|K(\mathcal{T}_1))}{\mathrm{d}(L_0.F|F)} = \frac{\mathrm{d}(L_0.F|K(\mathcal{T}_2))}{\mathrm{d}(L_0.F|F)} = \mathrm{d}(F|K(\mathcal{T}_2)) \ .$$

This proves (77).

Furthermore, using Lemma 5.2, for any finite extension L of K containing $K_{\mathcal{T}_1}$ we observe the following:

$$\begin{aligned} \mathrm{d}(L(\mathcal{T}_2)|K(\mathcal{T}_2)) &= \mathrm{d}(L|K) = \mathrm{d}(L.F|K(\mathcal{T}_1)) = \mathrm{d}(L.F|F) \cdot \mathrm{d}(F|K(\mathcal{T}_1)) \\ &= \mathrm{d}(L.F|F) \cdot \mathrm{d}(F|K(\mathcal{T}_2)) = \mathrm{d}(L.F|K(\mathcal{T}_2)) \end{aligned}$$

showing that

$$\mathrm{d}(L.F|L(\mathcal{T}_2)) = \mathrm{d}(L.F|K(\mathcal{T}_2))/\mathrm{d}(L(\mathcal{T}_2)|K(\mathcal{T}_2)) = 1 \ .$$

Hence every algebraic extension L of $K_{\mathcal{T}_1}$ satisfies assertion 1 and also the first part of assertion 2, because

$$\mathrm{d}(F|K) = \mathrm{d}(F|K(\mathcal{T})) = \frac{\mathrm{d}(L.F|K(\mathcal{T}))}{\mathrm{d}(L.F|F)} = \frac{\mathrm{d}(L|K)}{\mathrm{d}(L.F|F)}$$

where the last equation holds by Lemma 5.3. The second part of assertion 2 follows from

$$\begin{aligned} \mathrm{d}(N|K) &= \mathrm{d}(N(\mathcal{T})|K(\mathcal{T})) \\ &\leq \mathrm{d}(N.F|N(\mathcal{T})) \cdot \mathrm{d}(N(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(N.F|K(\mathcal{T})) \\ &= \mathrm{d}(N.F|F) \cdot \mathrm{d}(F|K(\mathcal{T})) = \mathrm{d}(N.F|F) \cdot \mathrm{d}(F|K) \ . \end{aligned}$$

We have actually shown that K' may be taken to be equal to the field $K_{\mathcal{T}}$ arising from Lemma 5.3 for any valuation transcendence basis $\mathcal{T}$. This shows that K' may be chosen as to satisfy the same additional conditions, as asserted in the theorem. □

Corollary 5.5 *Let E and F be subhenselian function fields over K. If $E|F$ is algebraic and $F|K$ has no transcendence defect, then $E|F$ is h–finite and the following multiplicativity holds for the defect:*

$$\mathrm{d}(E|K) = \mathrm{d}(E|F) \cdot \mathrm{d}(F|K) \; .$$

Proof: Taking any valuation transcendence basis $\mathcal{T}$ of $F|K$ (which is also a valuation transcendence basis of $E|K$ since $E|F$ is algebraic), we compute

$$\mathrm{d}(E|K) = \mathrm{d}(E|K(\mathcal{T})) = \mathrm{d}(E|F) \cdot \mathrm{d}(F|K(\mathcal{T})) = \mathrm{d}(E|F) \cdot \mathrm{d}(F|K)$$

using Theorem 5.4. □

Corollary 5.6 *For every subhenselian function field F without transcendence defect over K there exists a finite extension K' of K such that*

$$\mathrm{d}(K'.\,F|K) = \mathrm{d}(K'|K) \; .$$

Proof: Choosing K' according to Theorem 5.4, applying assertion 2. of Theorem 5.4 for $L = K'$ and using the preceding corollary, we get

$$\mathrm{d}(K'|K) = \mathrm{d}(K'.F|F) \cdot \mathrm{d}(F|K) = \mathrm{d}(K'.F|K) \; .$$

□

The following theorem is an immediate consequence of the two preceding corollaries:

Theorem 5.7 *If K is a defectless field then for any subhenselian function field F without transcendence defect over K, $\mathrm{d}(F|K)$ is trivial.*

On the other hand we know by Theorem 3.1 that any subhenselian function field F without transcendence defect over a defectless field K is itself a defectless field. Note that the foregoing theorem can also be proved by an application of Theorem 3.1 to $K(\mathcal{T})$ which shows that $\mathrm{d}(F|K(\mathcal{T}) = 1$ for every valuation transcendence basis $\mathcal{T}$ of $F|K$.

As further consequences of Theorem 5.4 we get the following corollary for function fields:

Corollary 5.8 *Let F be a valued function field without transcendence defect over K and let P be the place associated to v. Assume that $P = Q\overline{Q}$. Then FQ is also a function field without transcendence defect over KQ, and the following holds:*

$$\mathrm{d}((F,P)|(K,P)) = \mathrm{d}((F,Q)|(K,Q)) \cdot \mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q})) \; .$$

Proof: According to Lemma 2.23 we may choose a valuation transcendence basis $\mathcal{T}$ of $(F,P)|(K,P)$ such that it is also a valuation transcendence basis for $(F,Q)|(K,Q)$ and that the nonzero elements of the set $\{tQ|t \in \mathcal{T}\}$ form a valuation transcendence basis $\mathcal{T}'$ of $(FQ,\overline{Q})|(KQ,\overline{Q})$. Hence (F,Q) has no transcendence defect over (K,Q), and by Lemma 2.20, FQ is finitely generated over KQ. Hence $(FQ,\overline{Q})$ is a valued function field without transcendence defect over $(KQ,\overline{Q})$. Now using Theorem 5.4 and Lemma 2.17 we compute:

$$\begin{aligned}
\mathrm{d}((F,P)|(K,P)) &= \mathrm{d}((F,P)|(K(\mathcal{T}),P)) \\
&= \mathrm{d}((F,Q)|(K(\mathcal{T}),Q)) \cdot \mathrm{d}((FQ,\overline{Q})|(K(\mathcal{T})Q,\overline{Q})) \\
&= \mathrm{d}((F,Q)|(K(\mathcal{T}),Q)) \cdot \mathrm{d}((FQ,\overline{Q})|(KQ(\mathcal{T}'),\overline{Q})) \\
&= \mathrm{d}((F,Q)|(K,Q)) \cdot \mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q}))\ .
\end{aligned}$$

□

For the next corollary we need an additional lemma:

Lemma 5.9 *Let K be a valued field and $E|F$ an extension of valued fields such that $K.E|K.F$ is h–finite. Then there exists a finitely generated subfield k of K such that for every subfield K_0 of K containing k, the following holds:*

1. $[(K_0.E)^h : (K_0.F)^h] = [(K.E)^h : (K.F)^h]$,
2. $(v(K_0.E) : v(K_0.F)) \geq (v(K.E) : v(K.F))$,
3. $\left[\overline{K_0.E} : \overline{K_0.F}\right] \geq \left[\overline{K.E} : \overline{K.F}\right]$,
4. $\mathrm{d}(K_0.E|K_0.F) \leq \mathrm{d}(K.E|K.F)$.

Proof: Since $[(K.E)^h : (K.F)^h]$ is finite, $(v(K.E) : v(K.F))$ and $\left[\overline{K.E} : \overline{K.F}\right]$ are finite too. Hence there exist $\beta_1,\ldots,\beta_r \in v(K.E)$ such that

$$v(K.E) = v(K.F) + \mathbb{Z}\beta_1 + \ldots + \mathbb{Z}\beta_r\ ,$$

and there exist $\overline{b_1},\ldots,\overline{b_s} \in \overline{K.E}$ such that

$$\overline{K.E} = \overline{K.F}(\overline{b_1},\ldots,\overline{b_s})\ .$$

Whenever K_0 is such that

$$\beta_1,\ldots,\beta_r \in v(K_0.E) \tag{78}$$

and

$$\overline{b_1},\ldots,\overline{b_s} \in \overline{K_0.F}\ , \tag{79}$$

then

$$(v(K_0.E) : v(K_0.F)) \geq (v(K.E) : v(K.F)) \tag{80}$$
$$\left[\overline{K_0.E} : \overline{K_0.F}\right] \geq \left[\overline{K.E} : \overline{K.F}\right]\ , \tag{81}$$

the left sides not necessarily being finite.

Now we choose a finitely generated subfield k of K such that every subfield K_0 of K containing k satisfies

$$[(K_0.E)^h : (K_0.F)^h] = [(K.E)^h : (K.F)^h] \tag{82}$$

as well as (78) and (79), hence also (80) and (81), where the left sides now have to be finite. Then by (80), (81) and (82),

$$\begin{aligned} \mathrm{d}(K_0.E|K_0.F) &= \frac{[(K_0.E)^h : (K_0.F)^h]}{(v(K_0.E) : v(K_0.F)) \cdot \left[\overline{K_0.E} : \overline{K_0.F}\right]} \\ &\leq \frac{[(K.E)^h : (K.F)^h]}{(v(K.E) : v(K.F)) \cdot \left[\overline{K.E} : \overline{K.F}\right]} = \mathrm{d}(K.E|K.F) \;. \end{aligned}$$

□

Corollary 5.10 *Let E be a valued function field without transcendence defect over K. Then there exists a finitely generated field K_0 and a function field E_0 without transcendence defect over K_0 such that*

1. $E = K.E_0$,
2. $\mathrm{d}(E|K) \geq \mathrm{d}(E_0|K_0)$
3. *if E^{const} and E_0^{const} are the exact constant fields of E resp. E_0, then $E^{const} = K.E_0^{const}$ and $[E^{const} : K] = [E_0^{const} : K_0]$.*

Given a valuation transcendence basis $\mathcal{T}$ of $E|K$, $E_0|K_0$ may be chosen to have the same valuation transcendence basis $\mathcal{T}$ and to satisfy

1. $[E^h : K(\mathcal{T})^h] = [E_0^h : K_0(\mathcal{T})^h]$,
2. $(v(E_0) : v(K_0(\mathcal{T}))) \geq (v(E) : v(K(\mathcal{T})))$,
3. $\left[\overline{E_0} : \overline{K_0(\mathcal{T})}\right] \geq \left[\overline{E} : \overline{K(\mathcal{T})}\right]$.

Proof: Let $\mathcal{T}$ be a valuation transcendence basis of $E|K$ and $a_1, \ldots, a_n$ be elements of E such that $E = K(\mathcal{T}, a_1, \ldots, a_n)$. Let K_1 be a finitely generated subfield of K such that

$$K_1(\mathcal{T}, a_1, \ldots, a_n)|K_1(\mathcal{T})$$

is algebraic and hence finite. Put

$$F_1 = K_1(\mathcal{T}) \;\text{ and }\; E_1 = K_1(\mathcal{T}, a_1 \ldots, a_n) \;.$$

Then $K.F_1 = K(\mathcal{T})$ and $K.E_1 = E$. In addition, we may choose K_1 as large as to satisfy $E^{const} = K.E_1^{const}$ and $[E^{const} : K] = [E_1^{const} : K_1]$, where E_1^{const} denotes the exact constant field of E_1 over K_1, and that in addition $E_1|E_1^{const}$ is linearly disjoint from $K.E_1^{const}|E_1^{const}$. Now we choose a finitely generated subfield k of K according to Lemma 5.9. Let K_0 be an arbitrary finitely generated subfield K containing k and K_1. Then

$$\begin{aligned} [(K_0.E_1)^h : (K_0.F_1)^h] &= [(K.E_1)^h : (K.F_1)^h] \\ (v(K_0.E_1) : v(K_0.F_1)) &\geq (v(K.E_1) : v(K.F_1)) \\ \left[\overline{K_0.E_1} : \overline{K_0.F_1}\right] &\geq \left[\overline{K.E_1} : \overline{K.F_1}\right] \\ \mathrm{d}(K_0.E_1|K_0.F_1) &\leq \mathrm{d}(K.E_1|K.F_1) \;. \end{aligned}$$

Putting $E_0 = K_0.E_1$ and $F_0 = K_0.F_1 = K_0(\mathcal{T})$ we see that $\mathcal{T}$ is a valuation transcendence basis of $E_0|K_0$ and $F_0|K_0$, and the following holds:

$$\begin{aligned}
&E = K.E_1 = K.E_0 \ , \\
&[E_0^h : K_0(\mathcal{T})^h] = [(K_0.E_1)^h : (K_0.F_1)^h] = [(K.E_1)^h : (K.F_1)^h] = [E^h : K(\mathcal{T})^h] \ , \\
&(v(E_0) : v(K_0(\mathcal{T}))) \geq (v(K.E_1) : v(K.F_1)) = (v(E) : v(K(\mathcal{T}))) \ , \\
&\left[\overline{E_0} : \overline{K_0(\mathcal{T})}\right] \geq \left[\overline{K.E_1} : \overline{K.F_1}\right] = \left[\overline{E} : \overline{K(\mathcal{T})}\right] \ .
\end{aligned}$$

and in view of Theorem 5.4:

$$\begin{aligned}
\mathrm{d}(E_0|K_0) &= \mathrm{d}(E_0|K_0(\mathcal{T})) = \mathrm{d}(K_0.E_1|K_0.F_1) \\
&\leq \mathrm{d}(K.E_1|K.F_1) = \mathrm{d}(E|K(\mathcal{T})) = \mathrm{d}(E|K) \ .
\end{aligned}$$

Moreover $K_0.E_1^{const}$ is algebraically closed in $E_0 = K_0.E_1$ since E_1^{const} is algebraically closed in E_1 and the extension $E_1|E_1^{const}$ was assumed to be linearly disjoint from $K.E_1^{const}|E_1^{const}$ and thus from $K_0.E_1^{const}|E_1^{const}$. Hence $E_0^{const} = K_0.E_1^{const}$; together with

$$[E_1^{const} : K_1] \leq [K_0.E_0^{const} : K_0] \leq [K.E_0^{const} : K] = [E^{const} : K] = [E_1^{const} : K_1]$$

it yields

$$E^{const} = K.E_0^{const} \quad \text{and} \quad [E^{const} : K] = [E_0^{const} : K_0] \ .$$

This completes the proof of our corollary. □

5.2 Definition and properties of completion defect and defect quotient, and their relation to the ordinary defect.

In this subsection, we will define and investigate the *completion defect* and the *defect quotient*, the quotient of henselian defect and completion defect. For every h–finite extension $L|K$ we define the completion defect by

$$\mathrm{d_m}\,(L|K) := \mathrm{d}((L^h)^c|(K^h)^c)$$

and the defect quotient by

$$\mathrm{d_c}\,(L|K) := \frac{\mathrm{d}(L|K)}{\mathrm{d_m}\,(L|K)} \ ,$$

hence by definition

$$\mathrm{d}(L|K) = \mathrm{d_m}\,(L|K) \cdot \mathrm{d_c}\,(L|K) \ . \tag{83}$$

An h–finite extension $L|K$ is called *c–defectless* if $\mathrm{d_m}\,(L|K) = 1$, and it is called *q–defectless* if $\mathrm{d_c}\,(L|K) = 1$. Accordingly, a valued field K is called *c–defectless* if every h–finite (or equivalently, every finite) extension $L|K$ is c–defectless, and *q–defectless* if every h–finite (or equivalently, every finite) extension $L|K$ is q–defectless. Thus for h–finite extensions of q–defectless fields, the completion defect equals the ordinary defect.

For subhenselian function fields F without transcendence defect over K we define the completion defect and the defect quotient by

$$\begin{aligned}
\mathrm{d_m}\,(F|K) &= \sup_{\mathcal{T}} \mathrm{d_m}\,(F|K(\mathcal{T})) \\
\mathrm{d_c}\,(F|K) &= \sup_{\mathcal{T}} \mathrm{d_c}\,(F|K(\mathcal{T}))
\end{aligned}$$

where the supremum is taken over all transcendence bases of $F|K$.

The following observation is immediate:

Lemma 5.11 *Every h–finite extension $L|K$ satisfies:*

$$\begin{aligned} \mathrm{d_m}\,(L|K) &= \mathrm{d_m}\,(L^h|K^h) \\ \mathrm{d_c}\,(L|K) &= \mathrm{d_c}\,(L^h|K^h)\;. \end{aligned}$$

Hence K is an c–defectless resp. q–defectless field if and only if its henselization K^h is an c–defectless resp. q–defectless field.

A similar assertion for subhenselian function fields over K will be shown later. To shorten our formulas, we define for any valued field K:

$$K^{hc} := (K^h)^c\;.$$

The correspondence $K \mapsto (K^h)^c = K^{hc}$ may look a bit weird, but at least it has the nice property to be idempotent:

Lemma 5.12 *The completion of a henselian field is henselian too. Consequently,*

$$(K^{hc})^{hc} = K^{hc}\;.$$

Proof: Since for any field L we know that $(L^c)^c = L^c$, it suffices to show that $(K^{hc})^h = K^{hc}$ or equivalently, that the completion of any henselian field is again henselian. This can be shown using the fact that the zeros of a polynomial depend continuously on the coefficients of that polynomial, cf. Theorem 4.5 of [PZ], p. 329. Let $f \in \mathcal{O}_{K^{hc}}[X]$ a polynomial satisfying the hypothesis of Hensel's Lemma, i.e. $\overline{f}$ should admit a simple zero $\overline{a}$. For every $\alpha \in v(K)$, we may choose a polynomial $f_\alpha \in K^h[X]$ such that all coefficients of $f - f_\alpha$ all have value $\geq \alpha$, i.e. the greater α is chosen, the better the coefficients of f are approximated by the coefficients of f_α. For every $\alpha > 0$, the polynomial $\overline{f_\alpha}$ will admit $\overline{a}$ as a simple zero and will thus have a root a_α in the henselian field K^h with residue $\overline{a_\alpha} = \overline{a}$. Now for $\alpha \to \infty$, these roots will converge to a root of f also having residue $\overline{a}$. □

completion defect and defect quotient have the following properties:

Lemma 5.13 *Let $L|K$ be an h–finite extension. Then*

$$\mathrm{d}(L|K) \geq \mathrm{d_m}\,(L|K) \;\text{ and }\; \mathrm{d}(L|K) \geq \mathrm{d_c}\,(L|K)\;, \tag{84}$$

and $\mathrm{d_m}\,(L|K)$, $\mathrm{d_c}\,(L|K)$ are integers dividing $\mathrm{d}(L|K)$ and hence are powers of p. The completion defect and the defect quotient are multiplicative: if M is an h–finite extension of L, then

$$\begin{aligned} \mathrm{d_m}\,(M|K) &= \mathrm{d_m}\,(M|L)\cdot \mathrm{d_m}\,(L|K) \\ \mathrm{d_c}\,(M|K) &= \mathrm{d_c}\,(M|L)\cdot \mathrm{d_c}\,(L|K)\;. \end{aligned}$$

Any h–finite separable extension $L|K$ is q–defectless. A finite purely inseparable extension $L|K$ satisfies

$$\mathrm{d_m}\,(L|K) = \mathrm{d}(L^c|K^c) \tag{85}$$

$$\mathrm{d_c}\,(L|K) = \frac{[L:K]}{[L^c:K^c]} \tag{86}$$

and it is q–defectless if and only if it is linearly disjoint from the completion of K.

Proof: Since finite extensions of complete resp. henselian fields are again complete resp. henselian, for every finite extension $L|K$ we get $L^c = L.K^c$, and for every h–finite extension $L|K$ we get $L^h = L.K^h$, and $L^{hc} = L.K^{hc}$. From this and the multiplicativity of extension degree, ramification index and inertia degree we obtain the multiplicativity of the completion defect and the defect quotient. Furthermore K^{hc} contains the henselization K^h of K and L^{hc} contains the henselization L^h of L. In view of

$$\begin{array}{rcl} v(K^{hc}) = v(K^h) = v(K) & , & v(L^{hc}) = v(L^h) = v(L) \;, \\ \overline{K^{hc}} = \overline{K^h} = \overline{K} & , & \overline{L^{hc}} = \overline{L^h} = \overline{L} \end{array}$$

and the fact that $[L^{hc} : K^{hc}] = [L.K^{hc} : K^{hc}] \leq [L.K^h : K^h] = [L^h : K^h]$, we deduce

$$\begin{aligned} \mathrm{d_m}\,(L|K) &= \mathrm{d}(L^{hc}|K^{hc}) = \frac{[L^{hc} : K^{hc}]}{(v(L^{hc})) : v(K^{hc}) \cdot \left[\overline{L^{hc}} : \overline{K^{hc}}\right]} \\ &\leq \frac{[L^h : K^h]}{v(L^h) : v(K^h)) \cdot \left[\overline{L^h} : \overline{K^h}\right]} = \mathrm{d}(L^h|K^h) = \mathrm{d}(L|K) \;. \end{aligned}$$

Since on the other hand, $\mathrm{d_m}\,(L|K)$ is the ordinary defect of the extension $L^{hc}|K^{hc}$, it is a power of p and consequently a divisor of $\mathrm{d}(L|K)$. This yields that

$$\mathrm{d_c}\,(L|K) = \frac{\mathrm{d}(L|K)}{\mathrm{d_m}\,(L|K)}$$

is also an integer dividing $\mathrm{d}(L|K)$ and a power of p. In the above inequality, equality holds if and only if

$$[L.K^{hc} : K^{hc}] = [L.K^h : K^h] \tag{87}$$

expressing the property of $L.K^h$ to be linearly disjoint from K^{hc} over K^h. Since the henselian field K^h is relatively separable–algebraically closed in its completion, for every finite separable extension $L|K$ equation (87) holds, proving the fact that every such extension is q–defectless.

Now let $L|K$ be a finite purely inseparable extension. Assume that we are able to show

$$[L.K^{hc} : K^{hc}] = [L.K^c : K^c] \;. \tag{88}$$

Then it follows by virtue of $L^c = L.K^c$:

$$\mathrm{d_m}\,(L|K) = \mathrm{d}(L.K^{hc}|K^{hc}) = \mathrm{d}(L^c|K^c) \;,$$

which proves (85). Using this and the fact that $[L^h : K^h] = [L.K^h : K^h] = [L : K]$ (since $K^h|K$ is separable), we deduce

$$\mathrm{d_c}\,(L|K) = \mathrm{d}(L|K)/\mathrm{d_m}\,(L|K) = [L^h : K^h]/[L^c : K^c] = [L : K]/[L^c : K^c] \;,$$

which proves (86). Hence $L|K$ is c–defectless if and only if

$$[L : K] = [L^c : K^c]$$

which expresses the property of L to be linearly disjoint from K^c over K. This will prove the last assertion.

It remains to show equation (88) which claims that $L.K^c$ is linearly disjoint from K^{hc} over K^c. Assume that this does not hold. Then there exists an intermediate field N between L and K and an element $a \in L \setminus N$, $a^p \in N$, such that

$$a \notin N.K^c \text{ but } a \in N.K^{hc} \;.$$

Since $a \notin N.K^c = N^c$, the distance $\mathrm{dist}(a,N)$ of a to N must be finite. Since $a \in N.K^{hc} = (N.K^h)^c = (N^h)^c = N^{hc}$, there exists an element $b \in N^h$ such that $v(a-b) > \mathrm{dist}(a,N)$, hence $\mathrm{appr}(a,N) = \mathrm{appr}(b,N)$. a being an element of $\tilde{N}$, Lemma 11.92 proves that this is contradictory, which completes our proof. □

Corollary 5.14 *Let $L|K$ be a finite extension. Then*

$$\mathrm{d_C}\,(L|K) = \frac{[L:K]_{\mathrm{insep}}}{[L^c:K^c]_{\mathrm{insep}}} \;.$$

In particular, if $L|K$ is c–defectless and immediate, then $L|K$ is purely inseparable and L is included in the completion of K.

Proof: Let $L_s|K$ be the maximal separable subextension of $L|K$; then $L|L_s$ is purely inseparable. By Lemma 5.13, $\mathrm{d_C}\,(L_s|K) = 1$ and

$$\mathrm{d_C}\,(L|K) = \mathrm{d_C}\,(L|L_s)\cdot \mathrm{d_C}\,(L_s|K) = \mathrm{d_C}\,(L|L_s) = \frac{[L:L_s]}{[L^c:L_s^c]} \;.$$

On the other hand, $[L:L_s] = [L:K]_{\mathrm{insep}}$. Furthermore, $L^c = L.K^c$ and $L_s^c = L_s.K^c$ shows that $L^c|L_s^c$ is purely inseparable and $L_s^c|K^c$ is separable and consequently, $[L^c:L_s^c] = [L^c:K^c]_{\mathrm{insep}}$. This proves the first assertion of our lemma.

Now let $L|K$ be c–defectless and immediate, i.e. $\mathrm{d_m}\,(L|K) = 1$ which in view of (83) yields

$$[L:K] = \mathrm{d}(L|K) = \mathrm{d_C}\,(L|K) \;.$$

By what we have proved above, this implies

$$[L:K] = \frac{[L:K]_{\mathrm{insep}}}{[L^c:K^c]_{\mathrm{insep}}} \;.$$

This shows $[L:K] = [L:K]_{\mathrm{insep}}$, i.e. $L|K$ is purely inseparable. Moreover, it shows that $[L^c:K^c] = 1$, i.e. $L^c = K^c$ which is only possible if L is included in the completion of K, as contended. □

Using Lemma 5.13, we obtain the following important characterization:

Lemma 5.15 *K is q–defectless if and only if its completion is a separable extension. In particular, every complete field is q–defectless.*

Proof: K is q–defectless if and only if every finite extension $L|K$ is q–defectless. In view of the multiplicativity of the defect quotient and the fact that every finite extension is contained in a finite normal extension, it follows that K is q–defectless if and only if every finite normal extension $L|K$ is q–defectless. Again by multiplicativity, and by the fact that a normal extension $L|K$ admits an intermediate field N such that $N|K$ is purely inseparable and $L|N$ is separable and thus q–defectless, it follows that K is q–defectless if and only if every finite purely inseparable extension $L|K$ is q–defectless. This in turn is the case if and only if every finite purely inseparable extension $L|K$ is linearly disjoint from K^c, or in other words: if and only if $K^c|K$ is separable. □

From the multiplicativity of the defect quotient and the completion defect as stated in Lemma 5.13, one derives:

Lemma 5.16 *Let $L|K$ be an h–finite extension. Then K is a q–defectless field if and only if $L|K$ is q–defectless and L is a q–defectless field. The same holds for "c–defectless" instead of "q–defectless".*

From equation (86) of Lemma 5.13 one derives:

Lemma 5.17 *Let $(L,P)|(K,P)$ be a finite extension and $P = Q\overline{Q}$ a decomposition of P with nontrivial Q. Then*

$$\mathrm{d_c}\,((L,P)|(K,P)) = \mathrm{d_c}\,((L,Q)|(K,Q)) \tag{89}$$
$$\mathrm{d_m}\,((L,P)|(K,P)) = \mathrm{d_m}\,((L,Q)|(K,Q)) \cdot \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q}))\;. \tag{90}$$

Proof: To prove the equation for the defect quotient we use the assertion of Lemma 5.13 that every separable h–finite extension is q–defectless. Consequently, by the multiplicativity of the defect quotient, we may assume that $L|K$ is a finite purely inseparable extension and we may use equation (86) to compute:

$$\begin{aligned}\mathrm{d_c}\,((L,P)|(K,P)) &= [L:K]/[L^{c(P)}:K^{c(P)}]\\ &= [L:K]/[L^{c(Q)}:K^{c(Q)}] = \mathrm{d_c}\,((L,Q)|(K,Q))\;,\end{aligned}$$

using the hypothesis that Q is nontrivial which yields the equality of the completion under P and Q: $L^{c(P)} = L^{c(Q)}$ and $K^{c(P)} = K^{c(Q)}$. This proves equation (89). Using this result, we get

$$\begin{aligned}\mathrm{d_m}\,((L,P)|(K,P)) &= \frac{\mathrm{d}((L,P)|(K,P))}{\mathrm{d_c}\,((L,P)|(K,P))}\\ &= \frac{\mathrm{d}((L,Q)|(K,Q)) \cdot \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q}))}{\mathrm{d_c}\,((L,Q)|(K,Q))}\\ &= \mathrm{d_m}\,((L,Q)|(K,Q)) \cdot \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q}))\end{aligned}$$

proving equation (90). □

Another important property of completion defect and defect quotient is the following:

Lemma 5.18 *Let $L|K$ be a finite extension of henselian fields and assume that the rank of (K,P) has no last element (i.e. v admits no nontrivial coarsest coarsening). Then there is a decomposition $P = Q\overline{Q}$ with nontrivial Q and nontrivial $\overline{Q}$ such that*

$$\mathrm{d_c}\left((L,P)|(K,P)\right) = \mathrm{d_c}\left((L,Q)|(K,Q)\right) = \mathrm{d}((L,Q)|(K,Q)) \tag{91}$$

$$\mathrm{d_m}\left((L,P)|(K,P)\right) = \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q})) \tag{92}$$

For a separable extension $L|K$ this means

$$\mathrm{d}((L,P)|(K,P)) = \mathrm{d}((LQ,\overline{Q})|(KQ,\overline{Q}))\ . \tag{93}$$

Proof: Firstly we note that equation (92) follows from equation (91) by virtue of equation (83) and Lemma 2.17. So we will only consider equation (91) in the sequel. Moreover, the first equation of (91) is already stated in Lemma 5.17. Secondly, we prove: if K_1 is an intermediate field of $L|K$ and there are places Q_1 and Q_2 such that

$$\mathrm{d_c}\left((K_1,P)|(K,P)\right) = \mathrm{d_c}\left((K_1,Q_1)|(K,Q_1)\right) = \mathrm{d}((K_1,Q_1)|(K,Q_1))$$
$$\mathrm{d_c}\left((L,P)|(K_1,P)\right) = \mathrm{d_c}\left((L,Q_2)|(K_1,Q_2)\right) = \mathrm{d}((L,Q_2)|(K_1,Q_2))$$

then equation (91) holds if we choose Q to be the coarser one of the places Q_1 and Q_2. But this becomes an immediate consequence of the multiplicativity of the defects if only we can show: if equations of the type (91) hold for a place Q, then they also hold for any nontrivial coarsening Q' of Q. For the first "=" in equation (91) this follows from equation (89) of Lemma 5.17. Having proved

$$\begin{aligned}\mathrm{d}((L,Q)|(K,Q)) &= \mathrm{d_c}\left((L,Q)|(K,Q)\right) = \mathrm{d_c}\left((L,Q')|(K,Q')\right)\\ &\leq \mathrm{d}((L,Q')|(K,Q')) \leq \mathrm{d}((L,Q)|(K,Q))\ ,\end{aligned}$$

one derives

$$\mathrm{d}((L,Q)|(K,Q)) = \mathrm{d}((L,Q')|(K,Q'))$$

showing that the whole equation (91) holds for Q' too.

By what we have shown, it suffices now to prove our Lemma in the following two cases:

Case 1: $L = K(a)$ is a separable extension of K. Let $f(X) \in K[X]$ be the minimal polynomial of a over K and let c_i, $0 \leq i \leq n$ be the coefficients of f. Then by our hypothesis on the rank of P there exists a nontrivial coarsening Q of P such that Q is trivial on $k(c_0,\ldots,c_n)$ where k denotes the prime field of K. This shows that fQ is a separable polynomial over KQ of the same degree as f; moreover it is irreducible since if it were then the same would follow for f by Hensels Lemma ((K,Q) being henselian by our hypothesis on (K,P) and Lemma 2.15). Hence in this case, $[L:K] = [LQ:KQ]$ and consequently $\mathrm{d}((L,Q)|(K,Q)) = 1$ which proves equation (93). This implies equations (91) and (92) since by Lemma 5.13, every separable extension is q–defectless.

Case 2: $L = K(a)$ is a purely inseparable extension of degree p.
If $\mathrm{d}((L,P)|(K,P)) = \mathrm{d_c}\left((L,P)|(K,P)\right) = p$, then for any coarsening Q of P, equation (91) is fulfilled since by equation (89) of Lemma 5.17 we have

$$\mathrm{d_c}\left((L,Q)|(K,Q)\right) = \mathrm{d_c}\left((L,P)|(K,P)\right) = p$$

and consequently,

$$\mathrm{d}((L,Q)|(K,Q)) = p = \mathrm{d_C}\,((L,Q)|(K,Q))$$

since $\mathrm{d}((L,Q)|(K,Q))$ is greater or equal to $\mathrm{d_C}\,((L,Q)|(K,Q))$ but can't exceed $[L:K]=p$. The existence of a nontrivial coarsening of P is guaranteed by the hypothesis on the rank of P.
If on the other hand, $\mathrm{d_C}\,((L,Q)|(K,Q)) = 1$, then a cannot be an element of K^c and thus there is an element $\alpha \in v_P(K)$ such that $\forall b \in K:\ v_P(a-b) < \alpha$. By our hypothesis on the rank of P there exists a coarsening Q of P such that $v_Q(a-b)=0$, hence $aQ \neq bQ$ for all $b \in K$ (this is satisfied if the coarsening corresponds to a convex subgroup of $v_P(K)$ which includes α). This shows $aQ \notin KQ$ and thus $[LQ:KQ]=p$ which yields that $\mathrm{d}((L,Q)|(K,Q)) = 1 = \mathrm{d_C}\,((L,Q)|(K,Q))$. This completes the proof of our Lemma. □

Now we turn to the investigation of these defects for function fields. For the proof of the next theorem, we need the following auxiliary result:

Lemma 5.19 *Let $K(\mathcal{T})|K$ be an extension of valued fields with valuation transcendence basis $\mathcal{T}$. Then for every element $b \in K(\mathcal{T}) \setminus K$, there exist elements $c_1, c_2 \in K$ such that $c_1 b - c_2$ is valuation–transcendental.*

Proof: Let $b = f/g$ with $f, g \in K[\mathcal{T}]$. By Lemma 2.19, the value of the polynomials f, g is equal to the minimum of the values of the monomials in f resp. g, and these monomials are uniquely determined; we will call them f_0 and g_0. If f_0 differs from g_0 just by a constant factor from K which we will call c_2, then we put $h = f - c_2 g$ and observe that the monomial h_0 of least value in h will not any more differ from g_0 by a constant from K. If $f_0 \notin K g_0$, then we put $c_2 = 0$, $h = f$ and $h_0 = f_0$. Note that $h \neq 0$ and thus $h_0 \neq 0$ since by hypothesis, $f/g \notin K$. We have

$$b - c_2 = \frac{f}{g} - c_2 = \frac{h}{g} \quad \text{with} \quad v(\frac{h}{g}) = v(\frac{h_0}{g_0})\ ,$$

and we know that in the quotient h_0/g_0, at least one element of $\mathcal{T}$ appears with a nonzero (integer) exponent. If at least one of these appearing elements from $\mathcal{T}$ is value–transcendental, then h_0/g_0 and thus also $b - c_2$ is value–transcendental over K. In the remaining case, we write

$$\frac{h_0}{g_0} = c \cdot y_1^{e_1} \cdot \ldots \cdot y_s^{e_s}\ ,\quad e_1, \ldots, e_s \in \mathbb{Z}\ ,$$

where $c \in K$, and $y_1, \ldots, y_s$ are different residue–transcendental elements from $\mathcal{T}$. Since the residues $\overline{y_1}, \ldots, \overline{y_s}$ are algebraically independent over $\overline{K}$, this shows that h_0/cg_0 and thus also h/cg are residue–transcendental over K. Putting $c_1 = c^{-1}$ and replacing c_2 by c_2/c, we obtain that $c_1 b - c_2$ is residue–transcendental over K. □

Theorem 5.20 *Let F be a subhenselian function field without transcendence defect over K. Assume that K is a q–defectless field or that $v(K)$ is not cofinal in $v(F)$. In both cases, F is a q–defectless field.*

Proof: Let $\mathcal{T}$ be a valuation transcendence basis of $F|K$. In view of Lemma 5.16 we have only to show that $K(\mathcal{T})$ is a q–defectless field (hence we may assume $F = K(\mathcal{T})$). For this we have to show that the completion of F is a separable extension.

In the first case, let us assume that K is a q–defectless field and that $v(K)$ is cofinal in $v(F)$. Then the completion F^c of F contains the completion K^c of K. By our hypothesis on K and Lemma 5.15, $\sqrt{K}$ is linearly disjoint from K^c over K. We want to show now that $\sqrt{K}$ is even linearly disjoint from F^c over K for which fact we still have to prove that $\sqrt{K}.K^c$ is linearly disjoint from F^c over K^c.

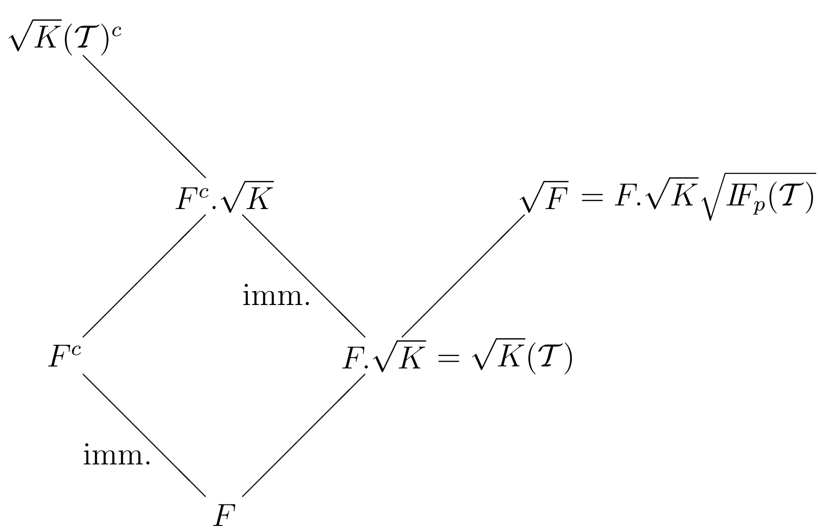
Assume the contrary. Then there exist two finite purely inseparable extensions $N \subset L$ of K and an element $a \in L \setminus N$ such that $a \in N.F^c \setminus N.K^c$. Since $a \notin N.K^c = N^c$, the distance $\mathrm{dist}(a, N)$ of a to N must be finite. Now $N.F^c = (N.F)^c$, hence there exists an element $b \in N.F = N(\mathcal{T})$ such that $v(a - b) > \mathrm{dist}(a, N)$. But according to the preceding Lemma, for every element $b \in N(\mathcal{T})$ there exist elements $c_1, c_2 \in N$ such that $c_1 b - c_2$ is valuation–transcendental. a being algebraic over N, this yields

$$v((c_1a - c_2) - (c_1b - c_2)) = \min(v(c_1a - c_2), v(c_1b - c_2)) \leq v(c_1a - c_2)$$

and consequently

$$\begin{aligned} v(a-b) &= v(c_1a - c_1b) - v(c_1) = v((c_1a - c_2) - (c_1b - c_2)) - v(c_1) \\ &\leq v(c_1a - c_2) - v(c_1) = v(a - c_2/c_1) \leq \mathrm{dist}(a, N)\ , \end{aligned}$$

a contradiction. We have shown that $\sqrt{K}$ is linearly disjoint from F^c over K. Conse-



quently, $F.\sqrt{K}$ is linearly disjoint from F^c over F. On the other hand, $\sqrt{F} = \sqrt{K(\mathcal{T})} = F.\sqrt{K}.\sqrt{\mathbb{F}_p(\mathcal{T})}$ is linearly disjoint from $F^c.\sqrt{K}$ over $F.\sqrt{K}$ since any finite extension of $F.\sqrt{K} = \sqrt{K}(\mathcal{T})$ within $F.\sqrt{K}.\sqrt{\mathbb{F}_p(\mathcal{T})}$ is defectless by Lemma 3.20, whereas the extension $F^c.\sqrt{K}|F.\sqrt{K}$ is immediate since $F^c.\sqrt{K}$ is included in $\sqrt{K}(\mathcal{T})^c$. Putting both results together, we see that $\sqrt{F}$ is linearly disjoint from F^c over F. Hence by Lemma 5.15, F is q–defectless. This completes our proof in the first case.

In the remaining case, $v(K)$ is not cofinal in $v(F)$, i.e. the convex hull of $v(K)$ in $v(F)$ is a proper convex subgroup of $v(F)$. Consequently, there exists a nontrivial coarsening w

of the valuation v on F which is trivial on K. Trivially, (K,w) is a defectless field, and so is (F,w) according to Theorem 3.1 since by Lemma 2.22 it is a function field without transcendence defect over (K,w). Thus any finite purely inseparable extension is defectless and thereby linearly disjoint from the w–completion $F^{c(w)}$ of F since this is an immediate extension of F. On the other hand, the topology induced by v equals the topology induced by any nontrivial coarsening of v, whence $F^{c(w)} = F^c$. Consequently, $\sqrt{F}$ is linearly disjoint from F^c. By virtue of Lemma 5.15, this completes our proof. □

On the basis of this theorem we are able to prove the following lemma:

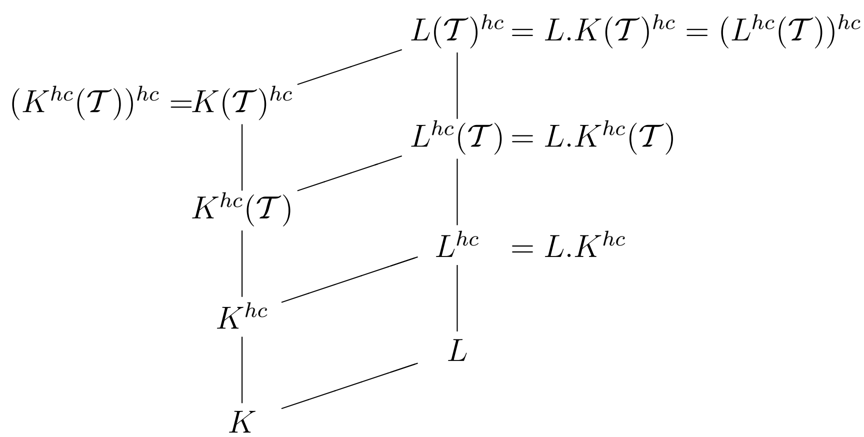
Lemma 5.21 *Let $K(\mathcal{T})|K$ be an extension of valued fields with valuation transcendence basis $\mathcal{T}$. Let L be a finite extension of K. If $v(K)$ is cofinal in $v(K(\mathcal{T}))$, then*

$$\begin{aligned} \mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T})) &= \mathrm{d_m}\,(L|K) \\ \mathrm{d_c}\,(L(\mathcal{T})|K(\mathcal{T})) &= \mathrm{d_c}\,(L|K)\,. \end{aligned}$$

If $v(K)$ is not cofinal in $v(K(\mathcal{T}))$, then

$$\mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d}(L|K)$$
$$\mathrm{d_c}\,(L(\mathcal{T})|K(\mathcal{T})) = 1\,.$$

Proof: If $v(K)$ is not cofinal in $v(K(\mathcal{T}))$, the assertion follows from Theorem 5.20, Lemma 5.2 and equation (83). Let us assume now that $v(K)$ is cofinal in $v(K(\mathcal{T}))$. In view of Lemma 5.2 and equation (83), it suffices to prove the first equality.



By definition,

$$\mathrm{d_m}\,(L|K) = \mathrm{d}(L^{hc}|K^{hc})\,. \tag{94}$$

Using $L^{hc}(\mathcal{T}) = (L.K^{hc})(\mathcal{T}) = L.(K^{hc}(\mathcal{T}))$, from Lemma 5.2 we infer

$$\mathrm{d}(L^{hc}|K^{hc}) = \mathrm{d}(L^{hc}(\mathcal{T})|K^{hc}(\mathcal{T}))\,. \tag{95}$$

The complete field K^{hc} is q–defectless by Lemma 5.15, hence by the preceding theorem, $K^{hc}(\mathcal{T})$ is q–defectless too. Consequently,

$$\mathrm{d}(L^{hc}(\mathcal{T})|K^{hc}(\mathcal{T})) = \mathrm{d_m}\,(L^{hc}(\mathcal{T})|K^{hc}(\mathcal{T}))\,. \tag{96}$$

By definition,

$$\mathrm{d_m}\,(L^{hc}(\mathcal{T})|K^{hc}(\mathcal{T})) = \mathrm{d}((L^{hc}(\mathcal{T}))^{hc}|(K^{hc}(\mathcal{T}))^{hc})\;. \tag{97}$$

From Lemma 5.12 we infer

$$(L^{hc}(\mathcal{T}))^{hc} = L(\mathcal{T})^{hc} \text{ and } (K^{hc}(\mathcal{T}))^{hc} = K(\mathcal{T})^{hc}\;.$$

This yields

$$\mathrm{d}((L^{hc}(\mathcal{T}))^{hc}|(K^{hc}(\mathcal{T}))^{hc}) = \mathrm{d}(L(\mathcal{T})^{hc}|K(\mathcal{T})^{hc})\;. \tag{98}$$

Again by definition,

$$\mathrm{d}(L(\mathcal{T})^{hc}|K(\mathcal{T})^{hc}) = \mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T}))\;. \tag{99}$$

Putting equations (94) – (99) together we obtain

$$\mathrm{d_m}\,(L|K) = \mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T}))\;,$$

as asserted. □

With the help of this lemma one proves the following theorem which is the analogue of Theorem 5.4 for completion defect and defect quotient.

Theorem 5.22 *Let F be a subhenselian function field without transcendence defect over K. Then for every valuation transcendence basis $\mathcal{T}$ of $F|K$,*

$$\mathrm{d_m}\,(F|K) = \mathrm{d_m}\,(F|K(\mathcal{T})) < \infty \tag{100}$$

$$\mathrm{d_c}\,(F|K) = \mathrm{d_c}\,(F|K(\mathcal{T})) < \infty\;. \tag{101}$$

This yields

$$\mathrm{d}(F|K) = \mathrm{d_m}\,(F|K)\cdot \mathrm{d_c}\,(F|K)\;. \tag{102}$$

Moreover, assume that $v(K)$ is cofinal in $v(F)$. Then there exists a finite extension K' of K such that for every finite extension L of K containing K' we have

$$\mathrm{d_m}\,(F|K) = \mathrm{d_m}\,(L|K)/\mathrm{d_m}\,(L.F|F) \tag{103}$$

$$= \max_{N|K \text{ finite}} \mathrm{d_m}\,(N|K)/\mathrm{d_m}\,(N.F|F) \tag{104}$$

$$\mathrm{d_c}\,(F|K) = \mathrm{d_c}\,(L|K)/\mathrm{d_c}\,(L.F|F) \tag{105}$$

$$= \max_{N|K \text{ finite}} \mathrm{d_c}\,(N|K)/\mathrm{d_c}\,(N.F|F)\;. \tag{106}$$

K' may be chosen as to satisfy the additional conditions stated in Theorem 5.4.

Proof: As it was done for the defect in the proof of Theorem 5.4, one shows that

$$\mathrm{d_m}\,(F|K) = \sup_{\mathcal{T}} \mathrm{d_m}\,(F|K(\mathcal{T}))$$

$$\mathrm{d_c}\,(F|K) = \sup_{\mathcal{T}} \mathrm{d_c}\,(F|K(\mathcal{T}))$$

where the supremum is only taken over all valuation transcendence bases of $F|K$. For the proof of equation (100) it suffices now to show that $\mathrm{d_m}\,(F|K(\mathcal{T}))$ is equal to a certain fixed number for any valuation transcendence basis $\mathcal{T}$, and for the proof of equation (101) we will show the same for the defect quotient.

If $v(K)$ is not cofinal in $v(F)$, then by virtue of Theorem 5.20, $K(\mathcal{T})$ is q–defectless and thus

$$\begin{array}{c} \mathrm{d_m}\,(F|K(\mathcal{T})) = \mathrm{d}(F|K(\mathcal{T})) = \mathrm{d}(F|K) \\ \mathrm{d_c}\,(F|K(\mathcal{T})) = 1 \end{array} .$$

If $v(K)$ is cofinal in $v(F)$, then $K(\mathcal{T})^{hc}$ contains K^{hc}. From this we deduce, using the fact that $K^{hc}(\mathcal{T})$ is q–defectless by Theorem 5.20:

$$\begin{aligned} \mathrm{d_m}\,(F|K(\mathcal{T})) &= \mathrm{d}(F^{hc}|K(\mathcal{T})^{hc}) = \mathrm{d}((F.K^{hc})^{hc}|(K^{hc}(\mathcal{T}))^{hc}) \\ &= \mathrm{d_m}\,(F.K^{hc}|K^{hc}(\mathcal{T})) = \mathrm{d}(F.K^{hc}|K^{hc}(\mathcal{T})) = \mathrm{d}(F.K^{hc}|K^{hc})\;. \end{aligned}$$

For the defect quotient, this implies

$$\mathrm{d_c}\,(F|K(\mathcal{T})) = \frac{\mathrm{d}(F|K(\mathcal{T}))}{\mathrm{d_m}\,(F|K(\mathcal{T}))} = \frac{\mathrm{d}(F|K)}{\mathrm{d}(F.K^{hc}|K^{hc})}\;.$$

This completes the proof of equations (100) and (101). Now equation (102) follows from equation (77) of Theorem 5.4 together with equations (83), (100) and (101).

For the remainder of the proof, we will assume that $v(K)$ is cofinal in $v(F)$. Let K' be as in Theorem 5.4 (hence it can be choosen as to satisfy the conditions stated there) and let L be a finite extension of K containing K'. Choosing any valuation transcendence basis $\mathcal{T}$ of $F|K$, by Theorem 5.4 we know that

$$\mathrm{d}(L.F|L(\mathcal{T})) = 1\;,$$

whence

$$\mathrm{d_m}\,(L.F|L(\mathcal{T})) = 1\;. \tag{107}$$

Using this equation and (100) as well as the multiplicativity of the completion defect, we deduce

$$\begin{aligned} \mathrm{d_m}\,(L.F|F)\cdot \mathrm{d_m}\,(F|K) &= \mathrm{d_m}\,(L.F|F)\cdot \mathrm{d_m}\,(F|K(\mathcal{T})) = \mathrm{d_m}\,(L.F|K(\mathcal{T})) \\ &= \mathrm{d_m}\,(L.F|L(\mathcal{T}))\cdot \mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T})) = \mathrm{d_m}\,(L(\mathcal{T})|K(\mathcal{T}))\;. \end{aligned}$$

In view of Lemma 5.21, $v(K)$ being cofinal in $v(F)$ by assumption, this yields equation (103). With regard to (102) and (83), equation (105) follows from equation (103) and the corresponding equation for the defect given in Theorem 5.4.

Equations (104) and (106) are shown as it was done for the defect in the proof of Theorem 5.4. □

As immediate consequences we get:

Corollary 5.23 *Assume F to be a subhenselian function field without transcendence defect over a q–defectless field K. Then $\mathrm{d_c}\,(F|K)$ is trivial.*

Proof: Let $\mathcal{T}$ be a valuation transcendence basis of $F|K$. From Theorem 5.20 we infer that $K(\mathcal{T})$ is a q–defectless field, hence in view of Theorem 5.22,

$$\mathrm{d_c}\,(F|K) = \mathrm{d_c}\,(F|K(\mathcal{T})) = 1\;.$$

□

Corollary 5.24 *Every subhenselian function field F without transcendence defect over K satisfies*

$$\begin{aligned}\mathrm{d_m}\,(F|K) &= \mathrm{d_m}\,(F^h|K) = \mathrm{d_m}\,(F^h|K^h)\\ \mathrm{d_c}\,(F|K) &= \mathrm{d_c}\,(F^h|K) = \mathrm{d_c}\,(F^h|K^h)\;.\end{aligned}$$

Proof: Any valuation transcendence basis $\mathcal{T}$ of $F|K$ is also a valuation transcendence basis of $F^h|K$ and of $F^h|K^h$. Hence with regard to Lemma 5.11,

$$\begin{aligned}\mathrm{d_m}\,(F|K) &= \mathrm{d_m}\,(F|K(\mathcal{T})) = \mathrm{d_m}\,(F^h|K(\mathcal{T})^h) = \mathrm{d_m}\,(F^h|(K^h(\mathcal{T}))^h)\\ &= \mathrm{d_m}\,((F^h)^h|(K^h(\mathcal{T}))^h) = \mathrm{d_m}\,(F^h|K^h(\mathcal{T})) = \mathrm{d_m}\,(F^h|K^h)\end{aligned}$$

and

$$\mathrm{d_m}\,(F^h|K(\mathcal{T})^h) = \mathrm{d_m}\,((F^h)^h|K(\mathcal{T})^h) = \mathrm{d_m}\,(F^h|K(\mathcal{T})) = \mathrm{d_m}\,(F^h|K)\;.$$

The assertions for the defect quotient are shown similarly. □

Corollary 5.25 *Let E and F be subhenselian function fields over K. If $E|F$ is algebraic and $F|K$ has no transcendence defect, then $E|F$ is h–finite and the following multiplicativity holds for the completion defect and defect quotient:*

$$\begin{aligned}\mathrm{d_m}\,(E|K) &= \mathrm{d_m}\,(E|F)\cdot\mathrm{d_m}\,(F|K)\\ \mathrm{d_c}\,(E|K) &= \mathrm{d_c}\,(E|F)\cdot\mathrm{d_c}\,(F|K)\;.\end{aligned}$$

Proof: First, we prove that $E|F$ is h–finite. E and F being subhenselian function fields, E^h and F^h are the henselizations of valued function fields E_0 and F_0 over K. Since $E|F$ is algebraic, also $E^h|F^h$ and $E_0.F_0|F_0$ are algebraic. As $E_0.F_0$ is also a function field over K, $E_0.F_0|F_0$ is finite and the same holds for $(E_0.F_0)^h|F_0^h$. But $E^h = E_0^h$ contains F and thus also F_0, whence $(E_0.F_0)^h = E^h$. By our choice of F_0 we have $F_0^h = F^h$, and thus we have proved that $E^h|F^h$ is finite. Taking any valuation transcendence basis $\mathcal{T}$ of $F|K$ (which is also a valuation transcendence basis of $E|K$ since $E|F$ is algebraic), we compute

$$\mathrm{d_m}\,(E|K) = \mathrm{d_m}\,(E|K(\mathcal{T})) = \mathrm{d_m}\,(E|F)\cdot\mathrm{d_m}\,(F|K(\mathcal{T})) = \mathrm{d_m}\,(E|F)\cdot\mathrm{d_m}\,(F|K)$$

using Theorem 5.22 and the multiplicativity of the completion defect. The proof for the defect quotient is similar. □

Corollary 5.26 *Let F be a subhenselian function field without transcendence defect over K. If $v(K)$ is cofinal in $v(F)$ then there exists a finite extension K' of K such that*

$$\begin{aligned}\mathrm{d_m}\,(K'.F|K) &= \mathrm{d_m}\,(K'|K)\\ \mathrm{d_c}\,(K'.F|K) &= \mathrm{d_c}\,(K'|K)\;.\end{aligned}$$

Proof: We take K' as in Theorem 5.22 and apply Corollary 5.25 to equation (103) and equation (105), where we set $L = K'$. □

The following theorem is a consequence of the two preceeding corollaries. It is the counterpart of Corollary 5.23.

Theorem 5.27 *Let F be a subhenselian function field without transcendence defect over K and let $v(K)$ be cofinal in $v(F)$. If K is an c–defectless field then* $\mathrm{d_m}\,(F|K)$ *is trivial and F is an c–defectless field.*

Proof: Let K be c–defectless and K' according to Corollary 5.26 such that

$$\mathrm{d_m}\,(K'.F|K) = \mathrm{d_m}\,(K'|K) = 1\ .$$

By Corollary 5.25, putting $E = K'.F$ we get $\mathrm{d_m}\,(F|K) = 1$. On the other hand, if F' is an arbitrary finite extension of F, then it is also a subhenselian function field without transcendence defect over K and consequently, like F it satisfies $\mathrm{d_m}\,(F'|K) = 1$. By Corollary 5.25, we conclude

$$\mathrm{d_m}\,(F'|F) = 1\ .$$

This shows that F is an c–defectless field. □

The following theorem identifies the class of c–defectless fields with the class of separably defectless fields:

Theorem 5.28 *A valued field K is an c–defectless field if and only if it is a separably defectless field.*

Proof: Let K be an c–defectless field. By Lemma 5.13, we know that every h–finite separable extension of K is q–defectless, i.e. its completion defect equals the ordinary defect. Thus every finite separable extension of K is defectless and consequently, K is a separably defectless field.

For the converse, assume that K is separably defectless. Then by Lemma 2.3, also its henselization is separably defectless, i.e. it is separable–algebraically complete. Now Corollary 4.20 shows that K^{hc} is algebraically complete. By virtue of the definition of the completion defect, this shows K to be c–defectless. □

The following theorem is a corollary to the preceding theorem and Theorem 5.27:

Theorem 5.29 *Let $F|K$ be a subhenselian function field without transcendence defect. If K is separably defectless and $v(K)$ is cofinal in $v(F)$ then F is separably defectless.*

From this theorem we derive the following structure theorems:

Theorem 5.30 *Let K be a separably defectless field and $F|K$ a subhenselian function field without transcendence defect. Assume that $v(K)$ is cofinal in $v(F)$. Then F is a finite c–defectless extension of a henselian rational function field F_0 . Moreover, F_0 can be chosen such that*

$$[F : F_0]_{\mathrm{sep}} = (v(L) : v(K))_{\mathrm{tor}} \cdot [\overline{F} : \overline{K}]_{\mathrm{irr}}\ .$$

Proof: The valuation transcendence basis $\mathcal{T}$ is constructed as in the proof of Theorem 3.2. By Theorem 5.28, K is an c–defectless field and by Theorem 5.27, $K(\mathcal{T})$ is an c–defectless field. Hence F is a finite c–defectless extension of the henselian rational function field $K(\mathcal{T})^h$. □

Theorem 5.31 *The situation being as in Theorem 5.30, assume in addition that $v(L)/v(K)$ has no torsion element of order p and that $\overline{L}|\overline{K}$ is separable. Then there exists a valuation transcendence basis $\mathcal{T}$ of $F|K$ and an element $a \in F$ such that F lies in the completion of $K(\mathcal{T})^h(a)$ which is a tame extension of $K(\mathcal{T})^h$ satisfying*

$$[K(\mathcal{T})^h(a) : K(\mathcal{T})^h] = [K(\mathcal{T})(a) : K(\mathcal{T})] = (v(F) : v(K))_{\mathrm{tor}} \cdot [\overline{F} : \overline{K}]_{\mathrm{sep}} \,.$$

Proof: The valuation transcendence basis $\mathcal{T}$ and the extension field $F_2 = K(\mathcal{T}, a)$ of $K(\mathcal{T})$ are constructed as in the proof of Theorem 3.3. Consequently, the extension $K(\mathcal{T}, a)^h|K(\mathcal{T})^h$ has the same properties. As in the proof of Theorem 5.30 it is shown that $K(\mathcal{T})$ is an c–defectless field. Hence F is a finite c–defectless extension of the henselian rational function field $K(\mathcal{T})^h$ and thus also of $K(\mathcal{T}, a)^h$. But by the construction of the latter field, the extension $F|K(\mathcal{T}, a)^h$ is immediate. But according to Corollary 5.14, an c–defectless immediate extension of a field must lie in its completion. □

Theorem 5.32 *The situation being as in Theorem 5.30, assume in addition that $v(L)/v(K)$ has no torsion and that $\overline{L}|\overline{K}$ is regular. Then there exists a valuation transcendence basis $\mathcal{T}$ of $F|K$ and an element $a \in F$ such that F lies in the completion of $K(\mathcal{T})^h(a)$ which is a tame unramified extension of $K(\mathcal{T})^h$ satisfying*

$$[K(\mathcal{T})^h(a) : K(\mathcal{T})^h] = [K(\mathcal{T})(a) : K(\mathcal{T})] = [\overline{K(\mathcal{T})}(\overline{a}) : \overline{K(\mathcal{T})}] = [\overline{F} : \overline{K}]_{\mathrm{sep}} \,.$$

Consequently, F lies in the completion of a henselian rational function field generated by a valuation transcendence basis over K if and only if $\overline{F}$ is a rational function field over $\overline{K}$.

Proof: The theorem can be easily deduced from Theorem 5.31; cf. the proof of Theorem 3.4. □

For the remainder of this section, we will consider the question to which extent Corollary 5.8 and Corollary 5.10 carry over to the completion defect.

Corollary 5.33 *Let F be a valued function field without transcendence defect over K and let P be the place associated to v. Assume that $P = Q\overline{Q}$ such that Q is nontrivial on F. Then the following holds:*

$$\mathrm{d_c}\,((F,P)|(K,P)) = \mathrm{d_c}\,((F,Q)|(K,Q))\,, \tag{108}$$
$$\mathrm{d_m}\,((F,P)|(K,P)) = \mathrm{d_m}\,((F,Q)|(K,Q)) \cdot \mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q}))\,. \tag{109}$$

In particular, if $\mathrm{d_m}\,((F,P)|(K,P))$ is trivial, then $\mathrm{d_m}\,((F,Q)|(K,Q))$, $\mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q}))$ and $\mathrm{d_m}\,((FQ,\overline{Q})|(KQ,\overline{Q}))$ are trivial too.

Proof: According to Lemma 2.23, we choose a valuation transcendence basis $\mathcal{T}$ of $(F,P)|(K,P)$ that is at the same time a valuation transcendence basis of $(F,Q)|(K,Q)$. Since Q is assumed to be nontrivial, we may apply equation (89) of Lemma 5.17 to compute

$$\begin{aligned}\mathrm{d_c}\,((F,P)|(K,P)) &= \mathrm{d_c}\,((F,P)|(K(\mathcal{T}),P))\\ &= \mathrm{d_c}\,((F,Q)|(K(\mathcal{T}),Q)) = \mathrm{d_c}\,((F,Q)|(K,Q))\;.\end{aligned}$$

This proves assertion (108). Using this result, we get

$$\begin{aligned}\mathrm{d_m}\,((F,P)|(K,P)) &= \frac{\mathrm{d}((F,P)|(K,P))}{\mathrm{d_c}\,((F,P)|(K,P))}\\ &= \frac{\mathrm{d}((F,Q)|(K,Q))\cdot \mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q}))}{\mathrm{d_c}\,((F,Q)|(K,Q))}\\ &= \mathrm{d_m}\,((F,Q)|(K,Q))\cdot \mathrm{d}((FQ,\overline{Q})|(KQ,\overline{Q}))\end{aligned}$$

by virtue of equation (102) of Theorem 5.22 and Corollary 5.8. □

This corollary shows clearly that the ground field K may be c–defectless while even a rational function field $F = K(x)$ may not. But this case can only arise if $v(K)$ is not cofinal in $v(F)$.

Corollary 5.34 *Let E be a valued function field without transcendence defect over K. Then there exists a finitely generated field K_0 and a function field E_0 without transcendence defect over K_0 satisfying all conditions of Corollary 5.10 and in addition:*

$$\mathrm{d_m}\,(E|K) \geq \mathrm{d_m}\,(E_0|K_0)\;.$$

Proof: Choose K_0 and E_0 according to Corollary 5.10. Assume that $v(E_0)$ is not cofinal in $v(E)$. Then the place P associated to v can be decomposed $P = Q\overline{Q}$ such that Q is nontrivial on E but trivial on E_0. We identify E_0 with E_0Q. Applying Corollary 5.10 to EQ and KQ instead of E and K, we see that K_0 and E_0 (with Q trivial on K_0 and F_0) can actually be chosen as large as to satisfy not only the required conditions, but also

$$\mathrm{d}((EQ,\overline{Q})|(KQ,\overline{Q})) \geq \mathrm{d}((E_0Q,\overline{Q})|(K_0Q,\overline{Q})) = \mathrm{d}((E_0,P)|(K_0,P))\;,$$

whence

$$\mathrm{d_m}\,((E,P)|(K,P)) \geq \mathrm{d}((EQ,\overline{Q})|(KQ,\overline{Q})) \geq \mathrm{d}((E_0,P)|(K_0,P)) \geq \mathrm{d_m}\,((E_0,P)|(K_0,P))$$

by virtue of Corollary 5.33.

Now assume that $v(E_0)$ is cofinal in $v(E)$. Then for a given valuation transcendence basis $\mathcal{T}$, the fields E_0 and K_0 may be replaced by larger fields that not only satisfy the conditions of Corollary 5.10 but also

$$\begin{aligned}[E:K(\mathcal{T})]_{\mathrm{insep}} &= [E_0:K_0(\mathcal{T})]_{\mathrm{insep}}\\ [E^c:K(\mathcal{T})^c]_{\mathrm{insep}} &= [E_0^c:K_0(\mathcal{T})^c]_{\mathrm{insep}}\;.\end{aligned}$$

In view of Lemma 5.14 and equation (101) of Theorem 5.22, this yields

$$\begin{aligned}\mathrm{d_c}\,(E|K) &= \mathrm{d_c}\,(E|K(\mathcal{T})) = \frac{[E:K(\mathcal{T})]_{\mathrm{insep}}}{[E^c:K(\mathcal{T})^c]_{\mathrm{insep}}} = \frac{[E_0:K_0(\mathcal{T})]_{\mathrm{insep}}}{[E_0^c:K_0(\mathcal{T})^c]_{\mathrm{insep}}}\\ &= \mathrm{d_c}\,(E_0|K_0(\mathcal{T})) = \mathrm{d_c}\,(E_0|K_0)\;.\end{aligned}$$

Hence

$$\mathrm{d_m}\,(E|K) = \frac{\mathrm{d}(E|K)}{\mathrm{d_c}\,(E|K)} \geq \frac{\mathrm{d}(E_0|K_0)}{\mathrm{d_c}\,(E_0|K_0)} = \mathrm{d_m}\,(E_0|K_0)\ .$$

□

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6 Tame extensions, tame and separably tame fields.

In this section we will state some important properties of tame and separably tame fields that we will use in the subsequent sections. Moreover, we will introduce a valuation theoretical property that characterizes the Galois groups of tame Galois extensions.

Let K be a valued field with $p = \mathrm{char}(\overline{K})$ ($p = 1$ if $\mathrm{char}(\overline{K}) = 0$). K is said to be a *tame field* if it is henselian and $\tilde{K}|K$ is a tame extension, and a *separably tame field*, if it is henselian and $K^{sep}|K$ is a tame extension ($\tilde{K}$ resp. K^{sep} denote the algebraic closure resp. the separable–algebraic closure of K). Here, an algebraic extension of a henselian field is called *tame* if every finite subextension $L|K$ satisfies

(a) the ramification index $(v(L) : v(K))$ is relatively prime to p,
(b) the residue field extension $\overline{L}|\overline{K}$ is separable,
(c) the extension $L|K$ is defectless.

An arbitrary extension $L|K$ is called *purely wild* if

(a') $(v(L) : v(K))$ is a p–torsion–group,
(b') $\overline{L}|\overline{K}$ is purely inseparable algebraic.

(Cf. the definitions on p. 46 of [KPR].)

For a henselian field K, let us recall the following facts stated in [KPR]:

Every subextension of a purely wild extension is purely wild. Every purely inseparable extension is purely wild. (Cf. the remarks on p. 47 in [KPR].)
An algebraic extension of K is purely wild if and only if it is K–linearly disjoint from K^r. (Cf. Lemma 4.2 on p. 47 in [KPR].)

The *ramification field* K^r (= the ramification field of the extension $K^{sep}|K$; see [END2] for its definition) is the maximal tame algebraic extension of K. Consequently, a Galois extension of a henselian field is tame iff its ramification group is trivial. If $\mathrm{char}(\overline{K}) = 0$ then $K^r = \tilde{K}$ and every algebraic extension of K is tame. On the other hand, in [PAN1], [PAN2], M. Pank proved the following important theorem:

Theorem 6.1 (M. Pank)
Let K be a henselian field with residue characteristic $p > 0$. There exist algebraic field complements W of K^r over K, i.e. $K^r.W = \tilde{K}$ and W is linearly disjoint from K^r over K. These complements W can be characterized as the maximal algebraic purely wild extensions of K. They are unique up to K–isomorphism if there does not exist any finite tame extension of K whose degree is divisible by p. Moreover, $v(W)$ is the p–divisible hull of $v(K)$, and $\overline{W}$ is the perfect hull of $\overline{K}$.

For the proof and further information on purely wild extensions, cf. [KPR].

6.1 Tame fields.

Tame fields can be characterized as follows:

Lemma 6.2 *The following assertions are equivalent:*
1) K is tame
2) Every algebraic purely wild extension of K is trivial
3) K is algebraically maximal and closed under every purely wild extension by p–th roots

4) K is algebraically maximal, $v(K)$ is p–divisible and $\overline{K}$ is perfect.

In particular, a valued field K of characteristic $p > 0$ is tame if and only if it is algebraically maximal and perfect. Consequently, given a valued field (K, v) of positive characteristic p, any maximal immediate algebraic extension (W, v) of $(\sqrt{K}, v)$ is a tame field having the p–divisible hull of $v(K)$ as value group and the perfect hull of $\overline{K}$ as residue field.

Every tame field is algebraically complete (hence for perfect valued fields of positive characteristic, "algebraically maximal" and "algebraically complete" are equivalent).

Proof: Let K be a tame field, i.e. $K^r = \tilde{K}$. Then by Pank's Theorem, every maximal algebraic purely wild extension of K is trivial. This proves 1) $\Longrightarrow$ 2).
If K has no algebraic purely wild extension then in particular it has no purely wild extension by p–th roots. Since moreover every finite tame extension is defectless (by definition), K is a defectless field and since it is assumed to be henselian, it is algebraically complete and thus algebraically maximal. This proves 2) $\Longrightarrow$ 3).
Assume now that K is an algebraically maximal field closed under purely wild extensions by p–th roots. Let a be an arbitrary element of K. Assume that $v(a)$ is not divisible by p in $v(K)$; then the extension $K(b)|K$ generated by an element $b \in \tilde{K}$ with $b^p = a$ satisfies $(v(K(b) : v(K))) = p = [K(b) : K]$ and is thus purely wild contrary to our assumption on K. Assume that $v(a) = 0$ and that $\overline{a}$ has no p–th root in $\overline{K}$; then the extension $K(b)|K$ generated as above satisfies $[\overline{K(b)} : \overline{K}] = p = [K(b) : K]$ and is again purely wild contrary to our assumption. By this, we have shown that $v(K)$ is p–divisible and $\overline{K}$ is perfect. This proves 3) $\Longrightarrow$ 4).
Assume now that K is an algebraically maximal (and thus henselian) field such that $v(K)$ is p–divisible and $\overline{K}$ is perfect. The latter condition yields (by definition) that every algebraic purely wild extension of K is immediate. But since K is assumed to be algebraically maximal every such extension must be trivial. This shows that K has no nontrivial algebraic purely wild extension at all and thus it follows from Pank's Theorem that $\tilde{K} = K^r$. This proves 4) $\Longrightarrow$ 1).
The second part of our lemma follows from the first part since if K has positive characteristic then every extension by p–th roots is purely inseparable and thus purely wild. Finally, the last assertions follow already from our proof. □

For the model theory of tame fields, the following corollary gives a basic information:

Corollary 6.3 *The property of being a tame field of fixed residue characteristic is recursively first order axiomatizable.*

Proof: If the residue characteristic is fixed to 0 then "tame" is equivalent to "henselian" which is recursively first order definable by an axiom scheme that just expresses Hensel's Lemma. If the residue characteristic is fixed to be a positive prime then the assertion is seen as follows:
By the foregoing lemma, a valued field of positive residue characteristic is tame if and only if it is an algebraically maximal field having p–divisible value group and perfect residue field. A valued field K has p–divisible value group if and only if it satisfies the following first order axiom:

$$\forall x \in K\, \exists y \in K :\ v(xy^p) = 0\ .$$

Furthermore, K has perfect residue field if and only if it satisfies the following first order axiom:

$$\forall x \in K\ \exists y \in K:\ (v(x) = 0 \Longrightarrow v(xy^p - 1) > 0)\ .$$

Finally, the property of being algebraically maximal is axiomatizable by a recursive scheme of first order axioms as it is shown in [DEL1], p. 15, Proposition 1.17, and p. 16, Corollaire 1.18. □

Let be given an abelian group Γ and a field k of positive characteristic p which are elementarily equivalent to the value group resp. the residue field of some tame field of positive characteristic (hence by Lemma 6.2, Γ is p–divisible and k is perfect). The next corollary will show that it is rather easy to construct tame fields with value group Γ and residue field k which moreover satisfy that its cardinality does not exceed the maximum of the cardinalities of value group and residue field.

Corollary 6.4 *Let p be a prime number, Γ a p–divisible ordered abelian group and k a perfect field of characteristic p. Then there exists a tame field K of characteristic p having Γ as its value group and k as its residue field such that $K|\mathbb{F}_p$ admits a valuation transcendence basis and the cardinality of K is equal to the maximum of the cardinalities of Γ and k.*

Proof: Given Γ and k both of cardinality at most κ, we proceed as follows:

Take α_i, $i \in I$ to be a maximal set of rationally independent values in Γ, and z_j, $j \in J$ to be a transcendence basis of $k|\mathbb{F}_p$. Let $\mathcal{T} = \{x_i, y_j \mid i \in I\,, j \in J\}$ be a set of algebraically independent elements over $\mathbb{F}_p$ and let $K_0 = \mathbb{F}_p(\mathcal{T})$ be valued such that

$$\forall i \in I:\ v(x_i) = \alpha_i \ \text{ and } \ \forall j \in J:\ \overline{y_j} = z_j\ .$$

Note that by Lemma 2.19, the valuation v is uniquely determined by the above assignments. Now take (K_1, v) to be a maximal immediate algebraic extension of the perfect hull $(\sqrt{K_0}, v)$. By Lemma 6.2, (K_1, v) is a tame field with the p–divisible hull of $v(K_0)$ as value group and the perfect hull of $\overline{K_0}$ as residue field. But since by hypothesis, Γ is p–divisible which shows $v(K_1) \subset \Gamma$. Similarly, k is perfect by hypothesis which shows $\overline{K_1} \subset k$. Now $\Gamma/v(K_1)$ is a torsion group without p–torsion, and $k|\overline{K_1}$ is a separable–algebraic extension. Hence there exists a tame algebraic extension (K, v) of (K_1, v) having value group Γ and residue field k. We have $K^r = K_1^r = \tilde{K_1} = \tilde{K}$ which shows that (K, v) is a tame field like (K_1, v). By construction, $K|\mathbb{F}_p$ has $\mathcal{T}$ as valuation transcendence basis.

Finally, note that the cardinality of $\mathcal{T}$ does not exceed the cardinality of Γ and k; this shows

$$|K| = \max\{\aleph_0, |\mathcal{T}|\} \leq \max\{|\Gamma|, |k|\}$$

since K is an infinite subfield of the algebraic closure of $\mathbb{F}_p(\mathcal{T})$. The inequality on the right hand side holds since at least $|\Gamma|$ is infinite. On the other hand, the cardinality of a valued field cannot be smaller than the cardinalities of its value group and its residue field which shows that the above inequality is indeed an equality. □

Note that Lemma 6.2 also yields that every tame field is perfect and that for a perfect field of positive characteristic the properties "algebraically maximal" and "algebraically complete" are equivalent. This is true since every such field is tame and thus defectless if it is algebraically maximal. Since it is also henselian if it is algebraically maximal, this implies that it is algebraically complete if it is algebraically maximal.

Every valued field (K,v) admits algebraic extensions which are tame fields and minimal in the sense that no proper subextensions are tame fields. These are exactly the maximal purely wild algebraic extensions of K, as one concludes from Lemma 6.2 together with Theorem 6.1. They have the p–divisible hull of $v(K)$ as value group and the perfect hull of $\overline{K}$ as residue field. Note that in general they are not unique up to isomorphism; cf. the appendix of [KPR]. Here it remains to note an additional property of these extensions concerning the "side conditions" as they appear in the hypothesis of Ax–Kochen–Ershov–principles:

Lemma 6.5 *If (K,v) is a tame field and $(L,v)|(K,v)$ an extension with $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, then every maximal purely wild algebraic extension (W,v) of (L,v) is a tame field satisfying $v(K) \prec_\exists v(W)$ and $\overline{K} \prec_\exists \overline{W}$.*

Proof: As mentioned above, $v(W)$ is the p–divisible hull $\frac{1}{p^\infty}v(L)$ of $v(L)$ and $\overline{W}$ is the perfect hull $\sqrt{\overline{L}}$ of $\overline{L}$. So we only have to prove that $v(K)$ (which is itself p–divisible by Lemma 6.2) is existentially closed in $\frac{1}{p^\infty}v(L)$ and that $\overline{K}$ (which is itself perfect by Lemma 6.2) is existentially closed in the perfect hull $\sqrt{\overline{L}}$ of $\overline{L}$.

By Lemma 8.1, the hypothesis $v(K) \prec_\exists v(L)$ implies that $v(L)$ is embeddable over $v(K)$ into every $|v(L)|^+$–saturated elementary extension of $v(K)$. Such an elementary extension is p–divisible like $v(K)$, hence the embedding can be extended to an embedding of $\frac{1}{p^\infty}v(L)$ which by Lemma 8.1 shows $v(K) \prec_\exists \frac{1}{p^\infty}v(L)$.

Again by Lemma 8.1, the hypothesis $\overline{K} \prec_\exists \overline{L}$ implies that $\overline{L}$ is embeddable over $\overline{K}$ into every $|\overline{L}|^+$–saturated elementary extension of $\overline{K}$. Such an elementary extension is perfect like $\overline{K}$, hence the embedding can be extended to an embedding of $\sqrt{\overline{L}}$ which by Lemma 8.1 shows $\overline{K} \prec_\exists \sqrt{\overline{L}}$. □

Now we will prove an important lemma on tame fields that we will need in several instances.

Lemma 6.6 *Let L be a tame field and $K \subset L$ a relatively algebraically closed subfield. If in addition $\overline{L}|\overline{K}$ is an algebraic extension, then K is also a tame field and moreover, $v(K)$ is pure in $v(L)$ and $\overline{K} = \overline{L}$.*

Proof: The following short and elegant version of the proof was given by F. Pop. Since L is tame, it is henselian and perfect and since K is relatively algebraically closed in L, it is henselian and perfect too. Assume that $K_1|K$ is a finite purely wild extension; in view of Lemma 6.2, we have to show that it is trivial. The degree $[K_1 : K]$ is a power of p, say p^m. Since K is perfect, $L|K$ and $K_1|K$ are separable extensions and since K is relatively algebraically closed in L, we know that L and K_1 are linearly disjoint over K, thus K_1 is relatively algebraically closed in $K_1.L$ and

$$[K_1.L : L] = [K_1 : K] = p^m .$$

Since L is assumed to be a tame field, the extension $K_1.L|L$ must be tame and this can only be the case if

$$\overline{K_1.L} \,|\, \overline{L}$$

is a separable extension of degree p^m. On the other hand, $\overline{K_1.L} \,|\, \overline{K_1}$ is an algebraic extension since by hypothesis, $\overline{L} \,|\, \overline{K}$ and thus also $\overline{K_1.L} \,|\, \overline{K}$ are algebraic extensions. Furthermore, $K_1.L$ being a henselian field and K_1 being relatively algebraically closed in $K_1.L$, Hensel's Lemma shows that

$$\overline{K_1.L} \,|\, \overline{K_1}$$

must be purely inseparable. This yields that

$$\begin{aligned} p^m &= [\overline{K_1.L} : \overline{L}]_{\rm sep} \leq [\overline{K_1.L} : \overline{K}]_{\rm sep} = [\overline{K_1.L} : \overline{K_1}]_{\rm sep} \cdot [\overline{K_1} : \overline{K}]_{\rm sep} \\ &= [\overline{K_1} : \overline{K}]_{\rm sep} \leq [\overline{K_1} : \overline{K}] \leq [K_1 : K] = p^m \;, \end{aligned}$$

showing that

$$\overline{K_1} \,|\, \overline{K}$$

is separable of degree p^m. Since $K_1|K$ was assumed to be purely wild, we have $p^m = 1$ and the extension $K_1|K$ is trivial.

We have now shown that K is a tame field; hence by Lemma 6.2, $v(K)$ is p–divisible and $\overline{K}$ is perfect. Since $\overline{L}|\overline{K}$ is assumed to be algebraic, it must be separable–algebraic. But L being henselian, the hypothesis that K is relatively algebraically closed in L yields by Hensel's Lemma that $\overline{L} = \overline{K}$. Furthermore, $v(L)/v(K)$ has no p–torsion; thus in view of $\overline{L} = \overline{K}$ and Hensel's Lemma, the hypothesis that K is relatively algebraically closed in L yields that $v(L)/v(K)$ has no torsion at all. □

The same lemma holds for separably tame fields, as stated in Lemma 6.17 below. The following corollaries will show some nice properties of the class of tame fields. They also possess generalizations to separably tame fields, see Corollary 6.18 below.

Corollary 6.7 *For every valued function field F with given transcendence basis $\mathcal{T}$ over a tame field K, there exists a tame subfield K_0 of K of finite rank with $\overline{K_0} = \overline{K}$ and $v(K_0)$ pure in $v(K)$, and furthermore a function field F_0 with transcendence basis $\mathcal{T}$ over K_0 such that*

$$F = K.F_0 \tag{110}$$

and

$$[F_0 : K_0(\mathcal{T})] = [F : K(\mathcal{T})] \;. \tag{111}$$

Proof: It is well known that there exists a finitely generated subfield K_1 of K admitting a finite extension F_1 of $K_1(\mathcal{T})$ such that (110) and (111) hold for K_1 and F_1 in the place of K_0 and F_0. As a finitely generated field, K_1 has finite rank. Now let y_j, $j \in J$, be a system of elements in K such that the residues $\overline{y_j}$, $j \in J$, form a transcendence basis of $\overline{K}$ over $\overline{K_1}$. According to Lemma 2.21, the field $K_1(y_j|j \in J)$ has residue field $\overline{K_1}(\overline{y_j}|j \in J)$ and the same value group as K_1, hence it is again a field of finite rank. Let K_0 be the relative closure of this field within K. Since by construction, $\overline{K}|\overline{K_1}(\overline{y_j}|j \in J)$ and thus also $\overline{K}|\overline{K_0}$ are algebraic, we may infer from the preceding lemma that K_0 is a tame field with $\overline{K_0} = \overline{K}$ and $v(K_0)$ pure in $v(K)$. As an algebraic extension of a field of finite rank it is itself of finite rank. Finally, the function field $F_0 = K_0.F_1$ over K_0 satisfies assertions (110) and (111). □

Corollary 6.8 *For every extension $L|K$ with L a tame field, there exists a tame intermediate field L_0 such that the extension $L_0|K$ has no transcendence defect and the extension $L|L_0$ is immediate.*

Proof: Take $\mathcal{T}$ to be a maximal set of algebraically valuation–independent elements over K in L. With this choice, $v(L)/v(K(\mathcal{T}))$ is a torsion group and $\overline{L}|\overline{K(\mathcal{T})}$ is algebraic. Let L_0 be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by Lemma 6.6, we have that L_0 is a tame field, that $\overline{L} = \overline{L_0}$ and that $v(L_0)$ is pure in $v(L)$ and thus $v(L_0) = v(L)$ which shows that the extension $L|L_0$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $L_0|K$ by construction which shows that according to Lemma 2.21, this extension has no transcendence defect. □

Now we will state two lemmata which we will actually not use further since we will deduce a general embedding lemma in section 8 that will serve us to prove our model theoretic results on tame fields. Nevertheless, in view of the fact that section 8 will only cover the case of tame fields of positive characteristic, the case of tame fields of characteristic 0 needing a different approach which we have to postpone to a subsequent paper, it is certainly interesting to see how far we can use the above presented results.

Lemma 6.9 *Let $\mathcal{K}$ be an elementary class of tame fields given by the axiom*
1) "(K,v) is a tame field"
and optional axioms
2) on the characteristic of K,
3) on the value group of K,
4) on the residue field of K.
Then every extension L of a field $K \in \mathcal{K}$ such that $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, admits an extension $L' \in \mathcal{K}$ such that $v(K) \prec v(L')$ and $\overline{K} \prec \overline{L'}$.

Proof: Since $v(K) \prec_\exists v(L)$, in view of Lemma 8.1 we may embed $v(L)$ into a $|v(L)|^+$–saturated elementary extension $v(K)^*$ of $v(K)$. Since $\overline{K} \prec_\exists \overline{L}$, in view of Lemma 8.1 we may embed $\overline{L}$ into a $|\overline{L}|^+$–saturated elementary extension $\overline{K}^*$ of $\overline{K}$. Let (L'',v) be an arbitrary extension of (L,v) having value group $v(K)^*$ and residue field $\overline{K}^*$. Furthermore, we take L' to be a maximal immediate algebraic extension of L''. Then $v(L') = v(L'') = v(K)^*$ is a p–divisible group like $v(K)$, and $\overline{L'} = \overline{L''} = \overline{K}^*$ is a perfect field like $\overline{K}$. Thus by Lemma 6.2, L' is a tame field because it is algebraically maximal by construction. Since $\mathrm{char}(L') = \mathrm{char}(K)$, $\overline{K} \prec \overline{K}^* = \overline{L'}$, $v(K) \prec v(K)^* = v(L')$ and $K \in \mathcal{K}$, we have $L' \in \mathcal{K}$ by the special form of our axioms for $\mathcal{K}$. □

The notion of Ax–Kochen–Ershov–class which is used in the following corollary, will be introduced in section 8.

Corollary 6.10 *Let $\mathcal{K}$ be an elementary class of tame fields as in the preceding lemma. If every field in this class is existentially closed in every immediate function field of transcendence degree 1, then $\mathcal{K}$ is an Ax–Kochen–Ershov–class.*

Proof: Let L be an extension of the tame field K such that $\overline{K} \prec_\exists \overline{L}$ and $v(K) \prec_\exists v(L)$. We have to show $(K,v) \prec_\exists (L,v)$. By virtue of the preceding corollary we may w.l.o.g. assume from the start that $L \in \mathcal{K}$ and thus that L is also tame.

Let L_0 be the tame intermediate field of the extension $L|K$ which is described in corollary 6.8. Then $L_0 \in \mathcal{K}$ since $\mathrm{char}(L_0) = \mathrm{char}(L)$, $v(L_0) = v(L)$, $\overline{L_0} = \overline{L}$ and $L \in \mathcal{K}$. By Theorem 8.4, $v(K) \prec_\exists v(L) = v(L_0)$ and $\overline{K} \prec_\exists \overline{L} = \overline{L_0}$ implies $(K, v) \prec_\exists (L_0, v)$.

It remains to show $(L_0, v) \prec_\exists (L, v)$. For this we only have to show that (L_0, v) is existentially closed in every finitely generated subextension $(F, v) \subset (L, v)$. Let $\{x_1, ..., x_n\}$ be a transcendence basis of $F|K$ and L_i be the relative algebraic closure of $L_0(x_1, ..., x_i)$ within L. Note that $L|L_0$ and thus every extension $L|L_i$ is immediate. By Lemma 6.6, this shows that every L_i is a tame field and since $L \in \mathcal{K}$, that every L_i is a member of $\mathcal{K}$. L_n contains F, so if we are able to show that $(L_0, v) \prec_\exists (L_n, v)$, our lemma will be proved.

Now the transcendence degree of every extension $L_{i+1}|L_i$, $0 \leq i < n$, is equal to 1. Hence it remains to show that for every immediate extension $L|K$ of transcendence degree 1 of fields $K, L \in \mathcal{K}$, we have $(K, v) \prec_\exists (L, v)$. This is true if (K, v) is existentially closed in every finitely generated subextension $(F, v) \subset (L, v)$. But F being an immediate function field of transcendence degree 1 over K, this follows from the hypothesis that (K, v) is existentially closed in every such function field. □

6.2 Separably tame fields.

For the following, we will make no restriction on the characteristic of K. However, if $\mathrm{char}(K) = 0$, then "separably tame" is the same as "tame", and some results are already proved above; note that in this case, the perfect closure $\sqrt{K}$ of K has to be interpreted to be just K.

Recall that a valued field (K, v) is called *separably defectless*, if every finite separable extension is defectless; it will be called *separable–algebraically complete*, if in addition it is henselian. (K, v) is called *separable–algebraically maximal*, if it does not admit proper immediate separable–algebraic extensions. Since the henselization of (K, v) is an immediate separable–algebraic extension of (K, v), a separable–algebraically maximal field (K, v) will coincide with its henselization and thus be henselian. Note that "separable–algebraically complete" implies "separable–algebraically maximal ".

Since every finite separable–algebraic extension of a separably tame field is tame and thus defectless, a separably tame field is always separable–algebraically complete. The converse is not true; it needs additional assumptions on the value group and the residue field. Under the assumptions that we are going to use frequently, the converse will even hold for "separable–algebraically maximal" in the place of "separable–algebraically complete". Before proving this, we need a lemma which essentially makes use of a theorem of M. Pank.

Lemma 6.11 *Let (K, v) be a henselian field. (K, v) is defectless if and only if every finite purely wild extension of (K, v) is defectless. Similarly, (K, v) is separably defectless if and only if every finite separable purely wild extension of (K, v) is defectless.*

Proof: By a theorem of M. Pank (cf. [KPR], Theorem 2.1 and Theorem 4.3, or [PAN1]), there exists a field complement W of the (absolute) ramification field K^r over K in the separable–algebraic closure K^{sep} (hence $\sqrt{W}$ is a field complement of K^r in $\tilde{K}$). Consequently, given any finite extension (resp. finite separable extension) $(L', v)|(K, v)$, there is a finite extension (resp. finite separable extension) $(L, v)|(L', v)$, a finite tame extension

$N|K$ and a finite (resp. finite separable) purely wild subextension $W_0|K$ of $W|K$ such that $L = N.W_0$. If $(L,v)|(K,v)$ is defectless, then also $(L',v)|(K,v)$; hence (K,v) is defectless (resp. separably defectless), if and only if every such extension $(N.W_0,v)|(K,v)$ is defectless. By Lemma 2.11,

$$\mathrm{d}\,(L|N) = \mathrm{d}\,(W_0|K)\ ,$$

and since every tame extension is defectless by definition,

$$\mathrm{d}\,(L|K) = \mathrm{d}\,(L|N)$$

in view of the multiplicativity of the defect. Hence $(L,v)|(K,v)$ is defectless if and only if $(W_0,v)|(K,v)$ is defectless; this yields our assertion. □

Lemma 6.12 *Let (K,v) be a separable–algebraically maximal field with p–divisible value group and perfect residue field. Then (K,v) is separably tame.*

Proof: For every finite purely wild extension, the ramification index and the inertia degree are powers of p. Consequently, under our assumptions on the value group and the residue field, every purely wild algebraic extension of the henselian field (K,v) is immediate. Since by assumption, (K,v) does not admit immediate separable–algebraic extensions, every purely wild separable–algebraic extension is thus trivial. This shows by the foregoing lemma that every finite separable–algebraic extension $(L,v) \mid (K,v)$ is defectless, and by our condition on the value group and the residue field, p does not divide $(v(L) : v(K))$ and $\overline{L}|\overline{K}$ is separable; this proves that $K^{sep}|K$ is a tame extension. □

The conditions of the foregoing lemma are characteristical for separably tame fields:

Lemma 6.13 *Let (K,v) be a separably tame field. Then the value group of (K,v) is p–divisible, and the residue field is perfect. Moreover, the perfect hull $\sqrt{K}$ of K (equipped with the unique prolongation of v) lies in the completion of (K,v).*

Proof: In the case of $\mathrm{char}(K) = 0$, the assertion is included in Lemma 6.2. Let us now assume $\mathrm{char}(K) = p > 0$.

Every Artin–Schreier–extension of (K,v) is separable; consequently, for the assertion on the value group and the residue field it suffices to show that every henselian field (K,v) which is closed under purely wild Artin–Schreier–extensions, has p–divisible value group and perfect residue field. To deduce a contradiction, assume that there exists $\alpha \in v(K)$ not divisible by p. W.l.o.g. we may assume that $\alpha < 0$, and we may choose $a \in K$ with $v(a) = \alpha$. We consider the Artin–Schreier–polynomial $X^p - X - a$. Any root b of this polynomial will have value α/p, hence $K(b)$ will admit a unique prolongation of v, and this will satisfy $(v(K(b)) : v(K)) = p$. The extension $(K(b),v)|(K,v)$ will consequently be purely wild, in contradiction to our assumption.

Similarly, assume that there exists $a \in K$ such that $\overline{a}$ does not admit a p–th root in $\overline{K}$. We choose some $c \in K$ with $v(c) < 0$ and consider the Artin–Schreier–polynomial $X^p - X - ac^p$. Any root b of this polynomial will satisfy $\left(\overline{b/c}\right)^p = \overline{a}$, hence $K(b)$ will admit a unique prolongation of v, and this will induce a purely inseparable residue field extension $\overline{K(b)}|\overline{K}$ of degree p. The extension $(K(b),v)|(K,v)$ will consequently be purely wild, in contradiction to our assumption.

The value group of the perfect hull $(\sqrt{K}, v)$ is the p–divisible hull of $v(K)$, and the residue field of $(\sqrt{K}, v)$ is the perfect hull of $\overline{K}$. By what we have just shown, it follows that the extension $(\sqrt{K}, v)|(K, v)$ is immediate. The fact that it lies in the completion of (K, v) follows from the first assertion of Corollary 4.5 since immediate Artin–Schreier–extensions are purely wild. □

Lemma 6.14 *(K, v) is a separably tame field if and only if $(\sqrt{K}, v)$ is a tame field. Consequently, if $(\sqrt{K}, v)$ is a tame field, then the extension $(\sqrt{K}, v)|(K, v)$ is immediate.*

Proof: $(\sqrt{K}, v)|(K, v)$ is a purely wild algebraic extension contained in every maximal purely wild algebraic extension. Consequently, if $(\sqrt{K}, v)$ admits no purely wild algebraic extensions at all, then $(\sqrt{K}, v)$ is the unique maximal purely wild extension of (K, v), and thus it must be a field complement for K^r over K in $\tilde{K}$ (see the proof of Lemma 6.11); this yields that $K^r = K^{sep}$, i.e. $K^{sep}|K$ is a tame extension by Proposition 4.1 of [KPR], and thus (K, v) is a separably tame field. Conversely, if (K, v) is a separably tame field, then there is no separable purely wild extension of (K, v) which shows that $(\sqrt{K}, v)$ admits no purely wild extensions at all. So $\sqrt{K}$ is a field complement for K^r over $\sqrt{K}$ in $\tilde{K}$ which shows that $K^r = \tilde{K}$; hence $\tilde{K}|\sqrt{K}$ is a tame extension and thus $\sqrt{K}$ a tame field. This proves our first assertion.

If $(\sqrt{K}, v)$ is a tame field, then by what we have proved, (K, v) is a separably tame field and it follows from Lemma 6.13 that the extension $(\sqrt{K}, v)|(K, v)$ is immediate. □

Lemma 6.15 *Every immediate algebraic extension of a separably tame field (K, v) is purely inseparable and included in the completion of (K, v). In particular, every algebraic approximation type over K has distance ∞.*

Proof: Recall that a separably tame field (K, v) is henselian. Every immediate algebraic extension of (K, v) is purely wild and thus purely inseparable since (K, v) is assumed to admit no separable–algebraic purely wild extensions. Then such extension lies in the perfect hull of (K, v) which by Lemma 6.13 lies in the completion of (K, v). Since every algebraic approximation type over K induces an immediate algebraic extension of (K, v) by an element a whose distance $\operatorname{dist}(a, K)$ coincides with the distance of the approximation type (cf. Theorem 11.52), such approximation type must have distance ∞. □

The following lemma describes the behaviour of separably tame fields under a decomposition of their place.

Lemma 6.16 *Let (K, v) be a separably tame field and let P be the place associated to v. Assume $P = P_1P_2P_3$ where P_1 is a coarsening of P and P_2 is nontrivial. (P_3 may be trivial.) Then (KP_1, P_2) is a separably tame field. If also P_1 is nontrivial, then (KP_1, P_2) is a tame field.*

Proof: By Lemma 6.13, $v(K)$ is p–divisible. The same is then true for $v_{P_2}(KP_1)$. We will also show that the residue field KP_1P_2 is perfect. Indeed, assume that this were not the case. It is then possible to construct a separable extension of degree p of (K, P_1P_2) which adjoins a p–th root to the residue field KP_1P_2; in the case of $\mathrm{char}(K) = p > 0$, the extension is chosen to be an Artin–Schreier–extension as in the proof of Lemma 6.13. Since already this residue field extension is purely inseparable, the induced extension of the residue field

$\overline{K} = KP_1P_2P_3$ cannot be separable of degree p. This shows that the constructed separable extension is a purely wild extension of (K,v) contrary to our assumption on (K,v).

(KP_1,P_2) is a separable–algebraically complete field since every finite separable defect extension of this field could be lifted to some separable defect extension of (K,P) (of the same degree, but possibly bigger defect if P_3 is nontrivial). From Lemma 6.12 we may now infer that (KP_1,P_2) is a separably tame field. If in addition P_1 is nontrivial, one shows as above that the residue field KP_1 is perfect. Then it follows from Lemma 6.14 that (KP_1,P_2) is a tame field. □

The following is an analogue of the very useful Lemma 6.6.

Lemma 6.17 *Let (K,v) be a separably tame field and $k \subset K$ a relatively algebraically closed subfield of K. If the residue field extension $\overline{K}|\overline{k}$ is algebraic, then (k,v) is also a separably tame field.*

Proof: Since k is relatively algebraically closed in K it follows that also $\sqrt{k}$ is relatively algebraically closed in $\sqrt{K}$. (K,v) being a separably tame field, $(\sqrt{K},v)$ is a tame field according to Lemma 6.14. From this lemma we also know that $\overline{K} = \overline{\sqrt{K}} = \sqrt{\overline{K}}$ and $v(K) = v(\sqrt{K})$. Our assumption on $\overline{K}|\overline{k}$ yields that the extension $\overline{\sqrt{K}}|\overline{\sqrt{k}}$ is algebraic. From Lemma 6.6 we may now infer that $(\sqrt{k},v)$ is a tame field with $\overline{\sqrt{k}} = \overline{\sqrt{K}} = \overline{K}$ and $v(\sqrt{k})$ pure in $v(\sqrt{K}) = v(K)$. Again by Lemma 6.14, (k,v) is thus a separably tame field with $\overline{k} = \overline{\sqrt{k}} = \overline{K}$ and $v(k) = v(\sqrt{k})$ pure in $v(K)$. □

Corollary 6.18 *Corollary 6.7 also holds for separably tame fields in the place of tame fields. More precisely, if $F|K$ is a separable extension, then F_0 and K_0 may be chosen such that $F_0|K_0$ (and thus also $F_0^h|K_0$) is a separable extension. Moreover, if $v(K)$ is cofinal in $v(F)$ then it may also be assumed that $v(K_0)$ is cofinal in $v(F_0)$.*

Proof: Since the proof of Corollary 6.7 only involves Lemma 6.6, it can be generalized using Lemma 6.17. The first additional assertion is clear since F is finitely generated over K. The second additional assertion is seen as follows. If $v(F)$ admits a biggest proper convex subgroup, then let K_0 contain a nonzero element whose value does not lie in this subgroup. If $v(F)$ and thus also $v(K)$ does not admit a biggest proper convex subgroup, then first choose F_0 and K_0 as in the (generalized) proof of Lemma 6.7; since F_0 has finite rank, there exists some element in K whose value does not lie in the convex hull of $v(F_0)$ in $v(F)$, and adding this element to K_0 and F_0 will make $v(K_0)$ cofinal in $v(F_0)$. □

Corollary 6.19 *Corollary 6.8 also holds for separably tame fields in the place of tame fields.*

6.3 Valuation independence of Galois groups.

Given a Galois extension $(L,v)|(K,v)$ of degree n of henselian fields, then its Galois group $\mathrm{Gal}(L|K) = \{\sigma_1,\ldots,\sigma_n\}$ will be called *valuation independent*, if for every choice of elements $d_1,\ldots,d_n \in \tilde{L}$ there exists an element $d_0 \in L$ such that (for the unique extension of the valuation v from L to $\tilde{L}$):

$$v\left(\sum_{i=1}^{n} \sigma_i(d_0)\cdot d_i\right) = \min_{1\leq i\leq n} v(\sigma_i(d_0)\cdot d_i)\;. \tag{112}$$

We will show in this section that a Galois extension of a henselian field is tame iff it has a valuation independent Galois group. Note that we could give the above definition and the following results also for extensions which are not Galois, replacing automorphisms by embeddings; however, the normal hull of an algebraic extension $L|K$ of a henselian field K is a tame extension of K if and only if $L|K$ is a tame extension, so there is no loss of generality in restricting our scope to Galois extensions.

Lemma 6.20 *Every finite tame unramified Galois extension has a valuation independent Galois group.*

Proof: Let $(L,v)|(K,v)$ be a finite tame unramified Galois extension and $\{\sigma_1,\ldots,\sigma_n\}$ its Galois group; it induces the Galois group of the residue field extension $\overline{L}|\overline{K}$ through

$$\forall x\in\mathcal{O}_L:\ \overline{\sigma_i}(\overline{a}) = \overline{\sigma_i(a)}\;.$$

Since the degree of $\overline{L}|\overline{K}$ is also equal to n, the automorphisms $\overline{\sigma_i}\in\mathrm{Gal}(\overline{L}|\overline{K})$ are all distinct; thus they are distinct characters from $\overline{L}^\times$ into $\left(\overline{\tilde{L}}\right)^\times$. By a well known theorem of Artin we know that these characters are linearly independent in $\overline{\tilde{L}}$. This means: for given elements $d_1,\ldots,d_n$ of $\mathcal{O}_{\tilde{L}}$, not all in $\mathcal{M}_{\tilde{L}}$, there is some $d_0\in\mathcal{O}_L^\times$ such that

$$\overline{\sum_{1\leq i\leq n}\sigma_i(d_0)\cdot d_i} = \sum_{1\leq i\leq n}\overline{\sigma_i}(\overline{d_0})\cdot\overline{d_i}\neq 0\;. \tag{113}$$

Now let be given arbitrary elements $d_1,\ldots,d_n\in\tilde{L}$. Choose some d_{i_0} of minimal value among them. Then (112) holds for $d_1,\ldots,d_n$ if and only if it holds for $d_{i0}^{-1}d_1,\ldots,d_{i0}^{-1}d_n$. Hence we may assume w.l.o.g. that $d_1,\ldots,d_n$ are elements of $\mathcal{O}_{\tilde{L}}$, not all in $\mathcal{M}_{\tilde{L}}$. We choose some $d_0\in L$ such that (113) holds, and we compute, using the fact that $v(\sigma_i(d_0)) = v(d_0)$ since K is henselian,

$$v\left(\sum_{1\leq i\leq n}\sigma_i(d_0)\cdot d_i\right) = 0 = \min_{1\leq i\leq n} v(\sigma_i(d_0)\cdot d_i)\;;$$

this proves (112). □

Lemma 6.21 *Every tame Galois extension* $(K(b),v)|(K,v)$ *with*

$$((v(K)+\mathbb{Z}v(b)):v(K)) = [K(b):K] =: n \quad\textit{and}\quad b^n\in K$$

has a valuation independent Galois group.

Proof: We may write $\mathrm{Gal}(L|K)=\{\sigma_1,\dots,\sigma_n\}$ with $\sigma_i(b)=b\zeta^i$ where ζ is a primitive n–th root of unity. So $\sigma_i(b)/b=\zeta^i$, and since the characteristic of $\overline{L}$ does not divide n,

$$\overline{\sigma_i(b)/b}=\overline{\zeta^i}=\overline{\zeta}^i$$

where $\overline{\zeta}$ is also a primitive n–th root of unity in $\overline{L}$. The characters χ_i from the group of n–th roots of unity into $\left(\overline{\tilde{L}}\right)^\times$ where χ_i sends ζ to $\overline{\zeta}^i$ are all distinct ($1\leq i\leq n$), hence they are linearly independent in $\overline{\tilde{L}}$ by the theorem of Artin cited in the foregoing proof. This means: given elements $d_1,\dots,d_n$ of $\mathcal{O}_{\tilde{L}}$, not all in $\mathcal{M}_{\tilde{L}}$, there is some j with $1\leq j\leq n$ such that

$$\overline{\sum_{1\leq i\leq n}\zeta^{ij}\cdot d_i}=\sum_{1\leq i\leq n}\overline{\zeta^j}^{\,i}\cdot\overline{d_i}\neq 0\,. \tag{114}$$

Now let be given arbitrary elements $d_1,\dots,d_n\in L$ be given. As shown in the foregoing proof, we may assume w.l.o.g. that $d_1,\dots,d_n$ are elements of $\mathcal{O}_{\tilde{L}}$, not all in $\mathcal{M}_{\tilde{L}}$. We choose some $j\in\{1,\dots,n\}$ such that (114) holds, and we compute, using the fact that $v(\sigma_i(b^j))=v(b)$ since K is henselian,

$$\begin{aligned}v\left(\sum_{1\leq i\leq n}\sigma_i(b^j)\cdot d_i\right) &= v\left(\sum_{1\leq i\leq n}\frac{\sigma_i(b^j)}{b^j}\cdot d_i\right)+v(b^j)=v\left(\sum_{1\leq i\leq n}\zeta^{ij}\cdot d_i\right)+v(b^j)\\ &= v(b^j)=\min_{1\leq i\leq n}v(\sigma_i(b^j)\cdot d_i)\,;\end{aligned}$$

this proves (112) for $d_0=b^j$. □

Lemma 6.22 *Let $(N,v)|(K,v)$ a finite tame Galois extension of henselian fields, and let $L|K$ a Galois subextension of $N|K$. If $\mathrm{Gal}(L|K)$ and $\mathrm{Gal}(N|L)$ are valuation independent, then $\mathrm{Gal}(N|K)$ is valuation independent too.*

Proof: Let $m=[L:K]$ and $n=[N:L]$. We choose

$$\sigma_i,\tau_j\in\mathrm{Gal}(N|K)\,,\;\;1\leq i\leq m\,,\;1\leq j\leq n\,,$$

such that

$$\begin{aligned}\mathrm{Gal}(L|K) &= \{\sigma_i|_L\mid 1\leq i\leq m\}\\ \mathrm{Gal}(N|L) &= \{\tau_j\mid 1\leq j\leq n\}\,.\end{aligned}$$

Then

$$\mathrm{Gal}(N|K)=\{\sigma_i\tau_j\mid 1\leq i\leq m\,,\,1\leq j\leq n\}\,.$$

Note that since K and L are henselian, the automorphisms σ_i,τ_j do not change the value of elements when applied to them. Now we have to show that for any choice of elements $d_{ij}\in\tilde{N}$ there exists some $d_0\in N$ such that

$$v\left(\sum_{i,j}\sigma_i\tau_j(d_0)\cdot d_{ij}\right)=\min_{i,j}v(\sigma_i\tau_j(d_0)\cdot d_{ij})\,,$$

where the sum and the minimum run over $1 \leq i \leq m$, $1 \leq j \leq n$. Let i_0, j_0 be such that $d_{i_0 j_0}$ is of minimal value among the d_{ij}. By the assumption of the lemma, we may choose some $d'_0 \in N$ such that

$$v\left(\sum_j \tau_j(d'_0) \cdot \sigma_{i_0}^{-1}(d_{i_0 j})\right) = \min_j v(\tau_j(d'_0) \cdot \sigma_{i_0}^{-1}(d_{i_0 j})) = v(d'_0) + v(d_{i_0 j_0}) \,.$$

For $i \neq i_0$ we have:

$$v\left(\sum_j \tau_j(d'_0) \cdot \sigma_i^{-1}(d_{ij})\right) \geq \min_j v(\tau_j(d'_0) \cdot \sigma_i^{-1}(d_{ij})) \geq v(d'_0) + v(d_{i_0 j_0}) \,.$$

Again by the assumption, we may choose some $d''_0 \in L$ such that

$$v\left(\sum_i \sigma_i(d''_0) \cdot \sum_j \sigma_i \tau_j(d'_0) \cdot d_{ij}\right) = \min_i v\left(\sigma_i(d''_0) \cdot \sum_j \sigma_i \tau_j(d'_0) \cdot d_{ij}\right) \,. \qquad (115)$$

Since $d''_0 \in L$ and thus $\tau_j(d''_0) = d''_0$ for every j, the left hand side of (115) is equal to

$$v\left(\sum_{i,j} \sigma_i(d''_0) \cdot \sigma_i \tau_j(d'_0) \cdot d_{ij}\right) = v\left(\sum_{i,j} \sigma_i \tau_j(d''_0 d'_0) \cdot d_{ij}\right) \,.$$

The right hand side of (115) is equal to

$$\begin{aligned}
\min_i v\left(\sigma_i(d''_0) \cdot \sigma_i(\sum_j \tau_j(d'_0) \cdot \sigma_i^{-1}(d_{ij}))\right) &= \min_i v\left(d''_0 \cdot \sum_j \tau_j(d'_0) \cdot \sigma_i^{-1}(d_{ij})\right) \\
&= v(d''_0) + v\left(\sum_j \tau_j(d'_0) \cdot \sigma_{i_0}^{-1}(d_{i_0 j})\right) \\
&= v(d''_0) + v(d'_0) + v(d_{i_0 j_0}) \\
&= \min_{i,j} v(\sigma_i \tau_j(d''_0 d'_0) \cdot d_{ij}) \,.
\end{aligned}$$

Hence $d_0 = d''_0 d'_0 \in N$ is the required element. □

Theorem 6.23 *Every finite tame Galois extension has a valuation independent Galois group.*

Proof: Let $(N, v)|(K, v)$ be a finite tame Galois extension. Let (L, v) be the inertia field of this extension. Then $(L, v)|(K, v)$ is a Galois extension of the same form as required in the assumption of Lemma 6.20, hence it has a valuation independent Galois group. By the foregoing lemma, it now suffices to prove that the (purely ramified) Galois extension $(N, v)|(K, v)$ has a valuation independent Galois group. If this extension is not trivial, then it admits a subextension of the form as required in the assumption of Lemma 6.21, which consequently has a valuation independent Galois group. In view of the foregoing lemma, our assertion now follows by induction on the degree of the extension. □

To show that tame Galois extensions are characterized by valuation independent Galois groups, it remains to prove the converse of the foregoing lemma. This is actually very easy.

Theorem 6.24 *A Galois extension of a henselian field is tame if and only if its Galois group is valuation independent.*

Proof: If the extension $(L,v)|(K,v)$ is not tame then by our remark following the definition of "tame" (cf. page 112), its ramification group is not trivial, i.e. its Galois group G contains an isomorphism σ such that

$$\forall x \in L:\ v(\sigma x - x) > v(x) = v(\sigma x)\ .$$

Assume w.l.o.g. that the elements of G are numbered such that $\sigma_1 = \sigma$ and $\sigma_2 = id$. Choosing $d_1 = 1$, $d_2 = -1$ and $d_i = 0$ for $i > 2$, we thus have

$$\forall x \in L:\ v\left(\sum_{i=1}^{n} \sigma_i(x) \cdot d_i\right) v(\sigma x - x) > \min_{1\le i\le n} v(\sigma_i(x) \cdot d_i)$$

which shows that G is not valuation independent. □

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7 Henselian rationality of immediate henselian function fields.

In this section we will show:

Theorem 7.1 *Let K be a tame field and F an immediate function field over K. If its transcendence degree over K is 1, then F^h is a henselian rational function field over K^h. In the general case of transcendence degree ≥ 1, given any immediate extension N of F which is a tame field (such an extension does always exist), there is a finite immediate extension F_1 of F within N such that F_1^h is a henselian rational function field over K.*

Actually, we will prove the more general

Theorem 7.2 *Let K be a separably tame field and F a separable immediate function field over K. If its transcendence degree over K is 1, then F^h is a henselian rational function field over K. In the general case of transcendence degree ≥ 1, given any immediate extension N of F which is a tame field (such an extension does always exist), there is a finite separable immediate extension F_1 of F within N such that F_1^h is a henselian rational function field over K.*

Indeed, the latter theorem implies the former, since by Lemma 6.2 a tame field is always perfect, and for $\mathrm{char}(K) = 0$ both theorems coincide. Note that in case of $\mathrm{char}(K) = p > 0$, Theorem 7.1 can also be formulated as follows:
Let K be a defectless perfect field and F an immediate function field over K. If its transcendence degree over K is 1, then F^h is a henselian rational function field over K^h. In the general case of transcendence degree ≥ 1, given any immediate extension N of F which is a tame field (such an extension does always exist), there is a finite immediate extension F_1 of F within N such that F_1^h is a henselian rational function field over K^h. This version follows from Theorem 7.1 since by Lemma 6.2, the henselization of a defectless perfect field is a tame field.

Let $\mathrm{trdeg}(F|K) = 1$. If $x \in F^h$ is transcendental over K, then the finite extension $F^h|K(x)^h$ is immediate and thus purely wild. If $\mathrm{char}(\overline{K}) = 0$, it is consequently trivial which shows that Theorem 7.2 is trivial in this case; we will from now on always assume $\mathrm{char}(\overline{K}) = p > 0$. Our consideration also shows that for the proof of our assertion it suffices to look at minimal purely wild algebraic extensions of immediate henselian rational function fields $K(x)^h$, i.e. purely wild algebraic extensions which do not admit proper nontrivial subextensions. In case of $\mathrm{char}(K) = p > 0$ (the "equal characteristic case"), such extensions have the following special structure:

Lemma 7.3 (F. Pop)
Let L be a valued field of positive characteristic. Every minimal purely wild algebraic extension of L^h is generated by a root of a minimal polynomial

$$\mathcal{A}(Y) - a$$

where $\mathcal{A}(Y) \in L[Y]$ is an additive polynomial and $a \in L^h$. If moreover L is an immediate extension of a valued field K, then we may even assume $\mathcal{A}(Y) \in K[Y]$.

Proof: A proof is given in [POP]. □

We will use this lemma for the proof of Theorem 7.2 in the equal characteristic case. But in case of $\mathrm{char}(K) = 0$ with $\mathrm{char}(\overline{K}) = p > 0$ (the "mixed characteristic case"), we do not know of an analogue to this lemma. Only if K is algebraically closed, we can say that the minimal purely wild extensions are Galois extensions of degree $p = \mathrm{char}(\overline{K})$, cf. Lemma 7.22 below. In the mixed characteristic case we will hence use a different approach. We will first prove Theorem 7.1 for algebraically closed K. Then we will prove a "pull down principle" for henselian rationality through tame extensions which yields the desired general form of Theorem 7.1 since for a tame field, the extension $\tilde{K}|K$ is tame by definition. Note that we could have used the pull down principle also in the equal characteristic case, but our aim was also to prove normal form results for minimal purely wild extensions in the most general setting.

At this point, we insert a short discussion of a very special sort of immediate function fields over henselian fields. Here, stronger assumptions on the function field enable us to assume less structure for the ground field. We need the following lemma (cf. Corollary 7.6 of [PZ]):

Lemma 7.4 *Every algebraic extension of a henselian field within its completion is purely inseparable.*

Using this lemma we prove:

Theorem 7.5 *Let (K, v) be a henselian field of arbitrary characteristic. If the valued function field $(F, v)|(K, v)$ is a separable subextension of the completion of (K, v), then F^h is a henselian rational function field over (K, v), generated by every separating transcendence basis of $F|K$.*

Proof: Let $\mathcal{T}$ be a separating transcendence basis of $F|K$. Then (F, v) lies in the completion of $(K(\mathcal{T}), v)$ since it lies in the completion of (K, v). Consequently, also F^h lies in the completion of $K(\mathcal{T})^h$ (which is equal to K^c since the completion of a henselian field is again henselian: cf. Lemma 5.12). But $F^h|K(\mathcal{T})^h$ is a finite separable extension; by Lemma 7.4 it must be trivial. □

7.1 The case of separably tame ground fields of equal characteristic.

In this subsection, all fields shall have characteristic $p > 0$.

Let us assume the setting as described in the above Lemma of Pop. In order to determine a suitable normal form for the minimal polynomial $\mathcal{A}(Y) - a$, let us first look at the additive polynomial $\mathcal{A}$.

Lemma 7.6 *Let $\mathcal{A}' \in K[Y]$ be a separable additive polynomial of degree p^e. Then there are elements $c_1, c_2 \in K$ such that the additive polynomial $\mathcal{A}(Y) = c_1 \cdot \mathcal{A}'(c_2 \cdot Y)$ is of the form*

$$\mathcal{A}(Y) = k_e Y^{p^e} + k_{e-1} Y^{p^{e-1}} + \ldots + k_1 Y^p + Y \in \mathcal{M}_K[Y^p] + Y \,. \tag{116}$$

If c is an element of a henselian overfield L of K with $v(c) \geq 0$, then the polynomial

$$\mathcal{A}(Y) - c \in \mathcal{O}_L$$

has a zero in L with residue $\overline{c}$ by Hensel's Lemma; in other words,

$$\mathcal{O}_L \subset \mathcal{A}(L) \; .$$

Proof: Let k_0' be the coefficient of Y in $\mathcal{A}'$; it is nonzero by our hypothesis on $\mathcal{A}'$. We put $c_1 = (k_0' c_2)^{-1}$. Any choice of c_2 with a high enough value will now yield the asserted form of $\mathcal{A}$. If $v(c) \geq 0$, then $\overline{\mathcal{A}(Y) - c} = \overline{Y} - \overline{c}$ has $\overline{c} \in \overline{L}$ as a simple zero. By Hensel's Lemma this shows that $\mathcal{A}(Y) - c$ admits a zero over the henselian field L having residue $\overline{c}$. □

As a consequence of the two preceding lemmata we get:

Corollary 7.7 *Let $L|K$ be an immediate extension of valued fields and let L' be a nontrivial minimal purely wild separable–algebraic extension of L^h. Then L' is generated over L^h by a root of a minimal polynomial of the form $\mathcal{A}(Y) - a$ where $\mathcal{A}(Y)$ is an additive polynomial of the form (116), $p^e = \deg(\mathcal{A}) = [L' : L^h]$, and where*

$$a \in L^h \text{ with } v(a) < 0 \; . \tag{117}$$

For any element

$$a' \in a + \mathcal{A}(L^h) + \mathcal{O}_{L^h} = a + \mathcal{A}(L^h)$$

there is a root of the polynomial

$$\mathcal{A}(Y) - a'$$

which also generates L' over L^h.

The last assertion follows by $\mathcal{O}_{L^h} \subset \mathcal{A}(L^h)$ and the additivity of the polynomial $\mathcal{A}(Y)$; it yields the question whether there are also normal forms for the element a which suit our purposes well. We will answer this question in the special case $L = K(x)$. First we need:

Lemma 7.8 *If the rank of K is 1 and $K(x)|K$ is immediate, then $K[x]$ is dense in $K(x)^h$.*

Proof: Since any valued field of rank 1 is dense in its henselization, it suffices to show that $K[x]$ is dense in $K(x)$. For this we only have to show that for every $f(x) \in K[x]$ and every $\alpha \in v(K)$ there exists an element $g(x) \in K[x]$ such that $v(g(x) - 1/f(x)) > \alpha$. Since $K(x)|K$ is immediate there is an element $c \in K$ satisfying $v(c - f(x)) > v(f(x)) = v(c)$ which yields $v(1 - f(x)/c) > 0$. By our hypothesis on the rank which actually says that the value group $v(K)$ is archimedian, there exists $j \in \mathbb{N}$ such that $j \cdot v(1 - f(x)/c) > \alpha + v(c)$. Now we put $h = 1 - f(x)/c \in K[x]$ and compute

$$\begin{aligned} v\left(\frac{1}{f(x)} - c^{-1}\sum_{i=0}^{j-1} h^i\right) &= v\left(\frac{1}{c(1-h)} - c^{-1}\sum_{i=0}^{j-1} h^i\right) \\ &= v(c^{-1}h^j) = j \cdot v(1 - f(x)/c) - v(c) > \alpha \; . \end{aligned}$$

The sum being an element of $K[x]$, this proves our lemma. □

As an immediate consequence we have:

Corollary 7.9 *If $\mathcal{A}(Y)$ is a separable additive polynomial of the form (116) and if $a \in K(x)^h$ where K has rank 1 and $K(x)|K$ is an immediate extension, then there exists a polynomial*

$$f(x) \in K[x] \cap (a + \mathcal{O}_{K(x)^h}) \subset K[x] \cap (a + \mathcal{A}(K(x)^h)) .$$

Any extension of $K(x)^h$ generated by a root of the polynomial $\mathcal{A}(Y) - a$ will then also be generated by a root of the polynomial

$$\mathcal{A}(Y) - f(x) .$$

Adding the hypothesis that appr (x, K) is transcendental, we will now derive a stronger normal form for $f(x)$. For the following lemma, we do not need the hypothesis that $\mathbf{A}$ be separable.

Lemma 7.10 *Let the situation be as in Corollary 7.9, with* $\mathrm{appr}(x, K)$ *transcendental (and consequently x transcendental over K by virtue of Corollary 11.48). Let $\deg \mathcal{A} = p^e$ and $K[x] \cap (a + \mathcal{A}(K(x)^h)) \neq \emptyset$. Then there exists a finite purely inseparable extension $K'|K$ and a polynomial*

$$h(z) \in K'[x] \cap (f(x) + \mathcal{A}(K[x])) \tag{118}$$

where

$$\left.\begin{array}{l} h(z) = a_n z^n + \ldots + a_1 z + a_0 \in K'[z] \text{ where} \\ z = (x-c)/d\,, \text{ with } v(z) = 0\,,\ c \in K \text{ and } 0 \neq d \in K \\ \exists m \in I\!N\,,\ 0 \leq m < e\,,\ \forall i > 0:\ (\,i \neq p^m \wedge a_i \neq 0\,) \Longrightarrow v(a_{p^m}) < v(a_i) < 0\,. \end{array}\right\} \tag{119}$$

Note that $K[x] = K[z]$.

Proof: Let $\deg(f) = n$. We consider the Taylor expansion for an arbitrary x_0:

$$f(x) = \sum_{i=0}^{n} f_i(x_0)(x - x_0)^i$$

where f_i denotes the i–th formal derivative of f. For any i which is divisible by p^e, say $i = p^e r$, the summand $f_i(x_0)(x - x_0)^i$ in $f(x)$ is equivalent to

$$-\sum_{j=0}^{e-1} k_j (k_e^{-1} f_i(x_0))^{p^{j-e}} (x - x_0)^{p^j r}$$

modulo $\mathcal{A}(K_1[x])$, where

$$K_1 = K\left((k_e^{-1} f_i(x_0))^{p^{-e}} \right) .$$

By a repeated application of this procedure we find that modulo $\mathcal{A}(K'[x])$ where $K'|K$ is a finite purely inseparable extension, $f(x)$ is equivalent to the polynomial

$$f(x_0) + \sum_i{}' \sum_{\mu=0}^{e-1} \left(\sum_{\nu}{}^{(i)} k_{\mu,\nu} f_{ip^\nu}(x_0)^{p^{\mu-\nu}} \right) (x - x_0)^{ip^\mu} \tag{120}$$

where:

1. $\sum_i{}'$ denotes the sum over all $i \leq n$ with $(p, i) = 1$,

2. $\sum_\nu^{(i)}$ denotes the sum over all $\nu \geq e$ with $ip^\nu \leq n$,

3. $k_{\mu,\nu} \in K'$ are determined by the coefficients k_i of $\mathcal{A}$.

For big enough $\lambda \in I\!N$, every term

$$\left(\sum_\nu{}^{(i)} k_{\mu,\nu} f_{ip^\nu}(x_0)^{p^{\mu-\nu}} \right)^{p^\lambda} \tag{121}$$

is a polynomial in $K[x_0]$. Since $\mathrm{appr}(x,K)$ is assumed to be transcendental, we may choose

$$\alpha_0 \in \{v(x-x_0) | x_0 \in K\}$$

such that for all $x_0 \in K$ with $v(x-x_0) > \alpha_0$ the values of (121) are fixed for every i and μ and thus we also have for all $x_0 \in K$, $v(x-x_0) > \alpha_0$:

$$v\left(\sum_\nu{}^{(i)} k_{\mu,\nu} f_{ip^\nu}(x_0)^{p^{\mu-\nu}} \right) = \beta_{i,\mu} \tag{122}$$

where $\beta_{i,\mu}$ are elements of the p–divisible hull of $v(K)$. Since the set $\{v(x-x_0)|x_0 \in K\}$ has no greatest element ($K(x)|K$ being an immediate extension by hypothesis), we may choose x_0 with $v(x-x_0) > \alpha_0$ such that all values

$$\beta_{i,\mu} + i \cdot p^\mu \cdot v(x-x_0)$$

are different. Having chosen x_0, we choose $d_1 \in K$ such that $v(d_1) = v(x-x_0)$ and put

$$y = \frac{x-x_0}{d_1} \;,$$

hence $v(y) = 0$. The polynomial that we have derived from (120) we will now write as a polynomial in y:

$$g(y) = f(x_0) + \sum_i{}' \sum_{\mu=0}^{e-1} \left(d_1^{ip^\mu} \cdot \sum_\nu{}^{(i)} k_{\mu,\nu} f_{ip^\nu}(x_0)^{p^{\mu-\nu}} \right) y^{ip^\mu} \;. \tag{123}$$

In this polynomial all the coefficients of y^{ip^m} are equal to zero for $m \geq e$. We consider the Taylor expansion

$$g(y) = g(y_0) + \sum_{i=1}^{n} g_i(y_0)(y-y_0)^i \;.$$

Since $\mathrm{appr}(x,K)$ is assumed to be transcendental, the same is true for $\mathrm{appr}\,(y,K)$, and by virtue of Lemma 11.32 there exists an element

$$\beta_0 \in \{v(y-y_0)|y_0 \in K\} \;,\; \beta_0 \geq 0$$

and an integer $m \geq 0$ such that for all $y_0 \in K$ with $v(y-y_0) > \beta_0$ the following holds for every $i \geq 1$ with $n \geq i \neq p^m$:

$$v(g_{p^m}(y_0)(y-y_0)^{p^m}) < v(g_i(y_0)(y-y_0)^i) \;. \tag{124}$$

By virtue of $v(y) = 0$, Corollary 11.35 shows

$$m < e\,.$$

Now we choose an element $y_0 \in K$ such that $v(y - y_0) > \beta_0$ and an element $d_2 \in K$ with $v(d_2) = v(y - y_0)$, and we put

$$z = \frac{y - y_0}{d_2} = \frac{x - x_0 - d_1 y_0}{d_1 d_2}$$

so that $v(z) = 0$. Moreover we may choose y_0 such that

$$v(g_i(y_0)d_2^i) \neq 0$$

for $1 \leq i \leq n$. If $v(g_i(y_0)d_2^i) > 0$ for a certain i, then $g_i(y_0)(y - y_0)^i \in \mathcal{A}(K'[y])$. Consequently, modulo $\mathcal{A}(K'[y])$, $g(y)$ and thus also $f(x)$ is equivalent to a polynomial

$$h(z) = a_n z^n + \ldots + a_0 \in K'[x]$$

where

$$a_i = \begin{cases} g_i(y_0)d_2^i & \text{if } v(g_i(y_0)d_2^i) < 0 \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$. We put

$$c = x_0 + d_1 y_0 \in K \text{ and } d = d_1 d_2 \in K$$

so that $z = (x - c)/d$.

To show the remaining assertion of our lemma, let us assume that $h(z) \notin K'$ since otherwise it is trivially fulfilled. Now for the integer m with $m < e$ that we have determined above, $h(z) \notin K'$ implies $a_{p^m} \neq 0$ by virtue of (124), and by the definition of the coefficients a_i we have:

$$\forall i > 0:\; i \neq p^m \wedge a_i \neq 0 \Longrightarrow v(a_{p^m}) < v(a_i) < 0\,.$$

□

Next, we add the hypothesis that $(K(x), v)$ is an immediate extension of the defectless ground field (K, v) or that $(K(x), v)$ is a separable immediate extension of the separably tame ground field (K, v). Under both hypotheses, the crucial condition that $\mathrm{appr}\,(x, K)$ be transcendental is always fulfilled:

Lemma 7.11 *Let $(K(x), v)$ be an immediate extension of the henselian defectless field (K, v) or a separable immediate extension of the separably tame field (K, v). (The separability condition is trivially fulfilled if x is transcendental over K.) Then* $\mathrm{appr}\,(x, K)$ *is transcendental, i.e. it fixes the value of every polynomial over K. More generally, every immediate approximation type over (K, v) of degree > 1 is transcendental over (K, v) if it is realized in an arbitrary separable extension.*

Proof: Assume that $(K(x), v)$ is an immediate extension of the defectless field (K, v). Then also $(K(x), v)^h|(K, v)$ is immediate. Since (K, v) is henselian defectless, it is in particular algebraically maximal, which shows that $\mathrm{appr}\,(x, K)$ is transcendental. Now assume that $(K(x),v)$ is a separable immediate extension of the separably tame field (K,v) (in this case, K is already henselian). By Theorem 11.27, $\mathrm{appr}\,(x,K)$ is an immediate approximation type. In order to derive a contradiction, let us assume that $\mathrm{appr}\,(x,K)$ is algebraic. Then by Theorem 11.52, there is an immediate algebraic extension

$$(K(b),v)|(K,v)\;,\quad \mathrm{appr}\,(x,K)=\mathrm{appr}\,(b,K)\;.$$

By Lemma 6.15, $K(b)|K$ is purely inseparable and $b\in K^c$. By virtue of Lemma 11.26, $\mathrm{appr}\,(x,K)=\mathrm{appr}\,(b,K)$ implies

$$v(x-c)\geq \mathrm{dist}\,(b,K)=\infty\;,$$

hence $x=b$. This contradicts our assumption that $K(x)|K$ is separable.

Note that if it is assumed from the start that $\mathrm{appr}\,(x,K)$ is immediate, then the condition that $(K(x),v)|(K,v)$ is immediate may be omitted; this gives our second assertion. □

In the following, we will use the relative approximation degree, denoted by $\mathbf{h}_K$; for its definition and properties cf. sections 11.2 and 11.3. Note that we may use the definitions and results described there since by Lemma 7.11, $\mathrm{appr}\,(x,K)$ is transcendental. With this notation, assertion 119 of Lemma 7.10 implies $\mathbf{h}_K(z:h(z))\leq p^m<p^e$.

Corollary 7.12 *Let the situation be as in Lemma 7.10. Then there is a finite purely inseparable extension $K'|K$ such that*

$$h(z)\in K[z]\cap(f(x)+\mathcal{A}(K'(x)^h))$$

with

$$\mathbf{h}_K(z:h(z))<p^e\;.$$

If (K,v) is perfect or separably tame, we may even assume $K'=K$.

Proof: We only have to prove the last assertion for (K,v) separably tame. By Lemma 6.13, (K,v) lies dense in $(\sqrt{K},v)$ and thus also in (K',v). Consequently, $K[x]$ lies dense in $K'[x]$, and since the completion of a henselian field is henselian (cf. Lemma 5.12), $K(x)^h$ lies dense in $K'(x)^h$. In view of Corollary 7.7, the latter implies that $\mathcal{A}(K(x)^h)$ lies dense in $\mathcal{A}(K'(x)^h)$. Hence there exist elements $h_0(z)\in K[z]=K[x]$ and $b_0\in K(x)^h$ such that

$$v(h_0-h)>0\ \text{ and }\ v((h-f)-\mathcal{A}(b_0))>0\;,$$

whence

$$v((h_0-f)-\mathcal{A}(b_0))>0\;.$$

By Lemma 7.6 and the additivity of $\mathcal{A}$, this yields

$$h_0\in K[z]\cap(f+\mathcal{A}(K(x)^h))\;.$$

Explicitly, $h_0(z)$ may be obtained by replacing the coefficients a_i appearing in $h(z)$ by elements of K which are sufficiently close. Consequently, the same conditions on the values of the coefficients may be assumed for $h_0(z)$. From Lemma 11.34 we may thus infer $\mathbf{h}_K(z:h_0(z))<p^e$ (since by hypothesis, $\mathrm{appr}\,(x,K)$ is transcendental which yields the same for $\mathrm{appr}\,(z,K)$ so that condition (174) is satisfied).

It follows from Corollary 7.7 where we now take L to be $K(z)$, that $\mathcal{A}(Y)-h(z)$ and $\mathcal{A}(Y)-h_0(z)$ will generate the same extension; so we may indeed replace h by h_0. □

This corollary together with Corollary 7.7 now gives:

Corollary 7.13 *If $K(x)$ is an immediate extension of the tame field K of rank 1 resp. a separable immediate extension of the separably tame field K of rank 1, then every minimal purely wild separable–algebraic extension of $K(x)^h$ is generated by a root of a minimal polynomial*

$$\mathcal{A}(Y) - f(x)$$

where the additive polynomial $\mathcal{A}(Y)$ is of the form (116) and $f(x) \in K[x]$ satisfies $\mathbf{h}_K(x : f(x)) < p^e$.

If (K,v) is just a defectless field, then the above still holds if (K,v) is replaced by the henselization of a suitable finite purely inseparable extension.

Proof: Since we want to replace the polynomial $f(x)$ as given in Corollary 7.7 by the polynomial $h(z)$ which appears in the foregoing corollary, it remains to show

$$\mathbf{h}_K(z : h(z)) = \mathbf{h}_K(x : h(z)) .$$

But this follows from Lemma 11.61 and the trivial fact that $\mathbf{h}_K(x : z) = 1$ (since z is linear in x). □

The normal form that we have derived for $f(x)$ allows us to prove the following first assertion on the structure of henselian function fields over defectless fields of positive characteristic:

Lemma 7.14 *Let F be an immediate function field over the defectless field K or a separable immediate function field over the separably tame field K. Assume that its transcendence degree over K is 1 and that its rank is 1. Then for a suitable finite purely inseparable extension K' of K, $(F.K')^h = F^h.K'$ is a henselian rational function field over K'^h. Under the second hypothesis, we may even assume $K'^h = K' = K$.*

Proof: If K is defectless, then by Corollary 2.14, the immediate extension $F^h|K$ is separable, and for every $K'|K$ algebraic, $F^h.K'|K'$ remains separable. So under both hypotheses, we may choose a separating transcendental element x of $F^h.K'|K'$, and we will choose x and a finite purely inseparable extension $K'|K$ such that the degree $[F^h.K' : K'(x)^h]$ is minimal; under the second hypothesis, we leave $K' = K$ fixed. We will show that $[F^h.K' : K'(x)^h] = 1$ which yields the assertion of our lemma.

Assume the contrary. Then the extension $F^h.K'|K'(x)^h$ is a nontrivial finite separable extension and purely wild since it is immediate, hence it contains a nontrivial separable minimal purely wild extension $E^h|K'(x)^h$. According to the previous lemma, this extension is generated by a root b of a minimal polynomial $\mathcal{A}(Y) - f(x)$ with $\mathbf{h}_{K''^h}(x : f(x)) < p^e$ and $\mathcal{A}(Y) \in K'^h[Y]$, where K'' is a finite purely inseparable extension of K and equal to K under the second hypothesis. By Lemma 11.58 (which we may apply in view of Lemma 7.11),

$$[K''(x)^h : K''(f(x))^h] < p^e = \deg \mathcal{A}(Y) = [E^h : K'(x)^h] = [E^h.K'' : K''(x)^h] ,$$

which in view of $f(x) = \mathcal{A}(b) \in K'^h(b)$ shows that

$$\begin{aligned} [E^h.K'' : K''(b)^h] &= [K''(x)^h(b) : K''(b)^h] = [K''(x)^h(b) : K''(f(x))^h(b)] \\ &\leq [K''(x)^h : K''(f(x))^h] < p^e \leq [E^h.K'' : K''(x)^h] . \end{aligned}$$

If the extension $K''(x)^h|K''(f(x))^h$ happens to be inseparable, then we may use Corollary 7.7 to replace $f(x)$ by $f(x)+cx$ for some $c\in K^\times$ with $v(cx)>0$ so that x will become a simple root of f which forces the extension $K''(x)^h|K''(f(x))^h$ to be separable. Consequently, the same will be true for $K''(x,b)^h|K''(b)^h$ which proves that b is a separating element of $F^h.K''|K''$. We have thereby deduced a contradiction to the minimum assumption on x and K'. This completes the proof of our lemma. □

The next step towards the desired structure theorem is the following

Lemma 7.15 *Let F be an immediate function field over the defectless field K or a separable immediate function field over the separably tame field K. Assume that its transcendence degree over K is 1 and that its rank is finite. Then for a suitable finite purely inseparable extension K' of K, $(F.K')^h = F^h.K'$ is a henselian rational function field over K'^h. Under the second hypothesis, we may even assume $K'^h = K' = K$.*

Proof: Let P be the place that is associated to the valuation on F^h. Since F has finite rank and $F|K$ is an immediate extension of transcendence degree 1, there exist places P_1, P_2, P_3 where P_1 and P_3 may be trivial and P_2 has rank 1, such that $P=P_1P_2P_3$ and

$$\begin{aligned}\mathrm{trdeg}(F^hP_1|K^hP_1) &= 1\;,\\ \mathrm{trdeg}(F^hP_1P_2|K^hP_1P_2) &= 0\;.\end{aligned}$$

By Lemma 2.15 and Lemma 2.17 the hypothesis that (K^h,P) is a henselian defectless field yields that also $(K^hP_1P_2,P_3)$ is a henselian defectless field, and by Lemma 6.16, the hypothesis that (K,P) is a separably tame field yields that $(K^hP_1P_2,P_3)$ is a tame field. Since $(F^h,P)|(K^h,P)$ is immediate, it follows from Lemma 2.2 that $v_{P_1}(F^h)=v_{P_1}(K^h)$, $v_{P_2}(F^hP_1)=v_{P_2}(K^hP_1)$, and that the algebraic extension $(F^hP_1P_2,P_3)|(K^hP_1P_2,P_3)$ is immediate; consequently, this extension must be trivial. This yields that also

$$(F^hP_1,P_2)|(K^hP_1,P_2)$$

is an immediate extension. This extension has transcendence degree 1, and

$$(F.K^hP_1,P_2)|(K^hP_1,P_2)$$

is finitely generated by Lemma 2.20. By Lemma 2.15 and Lemma 2.17 the hypothesis that (K^h,P) is a henselian defectless field yields that also (K^hP_1,P_2) is a henselian defectless field, and by Lemma 6.16, the hypothesis that (K,P) is a separably tame field yields that also (K^hP_1,P_2) is a separably tame field. Since the field $(F^hP_1P_2,P_3)=(K^hP_1P_2,P_3)$ is already henselian, it follows by virtue of Lemma 2.16 that (F^hP_1,P_2) is the henselization of $(F.K^hP_1,P_2)$. We have shown that (F^hP_1,P_2) is an immediate henselian function field of rank 1 and of transcendence degree 1 over the henselian defectless resp. separably tame field (K^hP_1,P_2). By Lemma 7.14 there is a finite purely inseparable extension k of K^hP_1 such that $F^hP_1.k$ is a henselian rational function field over k, with $k=K^hP_1$ under the second hypothesis; we may write $F^hP_1.k = k(xP_1)^{h(P_2)}$ for a suitable $x\in F^h$ which is consequently transcendental over K. If P_1 is trivial, there is nothing more to show. Otherwise, we choose a finite purely inseparable extension K' of K such that $K'^hP_1=k$ and $[K':K]=[k:K^hP_1]$ from which it follows that $(F^h.K')P_1 = F^hP_1.k$; we put

$K' = K$ under the second hypothesis. Furthermore, $K'(x)P_1 = K'P_1(xP_1) = k(xP_1)$, and again from Lemma 2.16 we get $k(xP_1)^{h(P_2)} = K'(x)^h P_1$. Altogether, we have

$$(F^h.K')P_1 = K'(x)^h P_1 \ .$$

On the other hand, $(F^h.K',P)|(K^h.K',P)$ is immediate by virtue of Lemma 2.13 since the extension $(F^h,P)|(K^h,P)$ is immediate and the extension $(K'^h,P)|(K^h,P)$ is defectless. We deduce that also $(F^h.K',P)|(K'(x)^h,P)$ is immediate, hence

$$v_P(F^h.K') = v_P(K'(x)^h)$$

and consequently

$$v_{P_1}(F^h.K') = v_{P_1}(K'(x)^h) \ .$$

We have shown that $(F^h.K',P_1)|(K'(x)^h,P_1)$ is an immediate extension. Moreover we know that it is algebraic and that $(K'(x)^h,P_1)$ is henselian like $(K'(x)^h,P)$, by virtue of Lemma 2.15. If we are able to show that this extension is defectless, then it will follow that it must be trivial, or in other words that $F^h.K'$ is a henselian rational function field over K'^h proving our lemma.

Assume first that K is a defectless field. As a finite extension of (K,P), the field (K',P) is also defectless, by virtue of Corollary 2.9. The same holds for (K',P_1) by Lemma 2.17. Since xP_1 is transcendental over $K'P_1$, the field $(K'(x),P_1)^h$ is defectless by Theorem 3.1. By Lemma 2.16, the extension $(K'(x)^h,P_1)|(K'(x),P_1)^h$ is tame. By virtue of Lemma 2.11 it follows that the field $(K'(x)^h,P_1)$ is a defectless field and thus $(F^h.K',P_1)|(K'(x)^h,P_1)$ is a defectless extension, as desired.

Assume now that K is a separably tame field and $F|K$ separable (and thus also $F^h|K$ separable); now we have $K' = K$. By the lemma that will follow this proof, we may have chosen x to be a separating element of $F^h|K$, hence we may assume that the extension $(F^h,P_1)|(K(x)^h,P_1)$ is separable. By Lemma 6.16, (K,P_1) is a separably tame field like (K,P), hence it is separably defectless. In view of $v_{P_1}(F^h) = v_{P_1}(K^h)$, it follows from Theorem 5.29 that also $(K(x)^h,P_1)$ is a separably defectless field which yields that the separable extension $(F^h,P_1)|(K(x)^h,P_1)$ defectless, as desired. □

Lemma 7.16 *Let $L|K$ be a separable extension of transcendence degree 1 and let Q be a nontrivial place on L. Then for every $x \in L$ there exists a separating element y of $L|K$ which satisfies $v_Q(y) = v_Q(x)$ and if $v_Q(x) = 0$, also $yQ = xQ$.*

Proof: Let us choose any separating element z of $L|K$ and an element $w \in L^\times$ which satisfies $v_Q(w) > v_Q(x)$ and $v_Q(zw) > v_Q(x)$. Note that $L|K(x,z,w)$ and $K(x,z,w)|K$ are separable extensions. But

$$K(x,z,w) = K(x, x+wz, x+w)$$

with

$$xQ = (x+wz)Q = (x+w)Q \ .$$

On the other hand, at least one of the elements x, $x+wz$, $x+w$ must be a separating element for the separable extension $K(x,x+wz,x+z)|K$; we define y to be that element. Then y is also a separating element for $L|K$, and it satisfies our assertion. □

Now we are able to prove our structure theorem on immediate henselian function fields over separably tame fields of positive characteristic.

Proof of Theorem 7.2 for separably tame fields of positive characteristic: Let us first consider the case $\mathrm{trdeg}(F|K) = 1$. According to Corollary 6.18 there exists a separably tame subfield K_0 of K of finite rank and a function field F_0 of transcendence degree 1 over K_0 with $\overline{K_0} = \overline{K}$ and $v(K_0)$ pure in $v(K)$ such that $F = F_0.K$ and that $v(K_0)$ is cofinal in $v(F_0)$; since $F|K$ is assumed to be separable, then we may also assume $F_0|K_0$ to be separable. If we are able to show

$$F_0^h = K_0(x)^h \tag{125}$$

for some $x \in F_0^h$ then it will follow

$$F^h = (F_0.K)^h = (F_0^h.K)^h = (K_0(x)^h.K)^h = K(x)^h$$

and the first assertion of our theorem will be proved.

We distinguish the following two cases:

<u>Case 1:</u> $F_0|K_0$ is not immediate. Since

$$\overline{K_0} = \overline{K} = \overline{F} \supseteq \overline{F_0} \supseteq \overline{K_0}\ ,$$

equality holds everywhere; in particular we have $\overline{K_0} = \overline{F_0}$ and thus $v(K_0) \neq v(F_0)$. Since $v(K_0)$ is pure in $v(F_0)$ and F_0 is finitely generated of transcendence degree 1 over K_0, it follows

$$v(F_0) = v(K_0) \oplus \mathbb{Z}v(x) = v(K_0(x))$$

for a suitable $x \in F_0$; by the foregoing lemma, it may be chosen to be a separating element of $F_0^h|K$. $(K_0(x), v)$ is separably defectless by Theorem 5.29. Hence the immediate separable–algebraic extension $F_0^h|K_0(x)^h$ must be trivial which proves our assertion (125) in the first case.

<u>Case 2:</u> $F_0|K_0$ is immediate. Then (125) follows from Lemma 7.15 and the fact that K_0 is separably tame.

The second assertion of our theorem follows from the first by induction on the transcendence degree of $F|K$. Assume $n \geq 1$ and that the assertion is proved for every transcendence degree $\leq n$. Let $\{t_1, ..., t_n, t\}$ be a separating transcendence basis of $F|K$.

We show that there is always a separable immediate extension of F which is separably tame. Let N be a maximal immediate separable–algebraic extension of F. Since K is a separably tame field by hypothesis and $F|K$ is immediate, the value group $v(F) = v(K)$ is p–divisible and the residue field $\overline{F} = \overline{K}$ is perfect by virtue of Lemma 6.13. By Lemma 6.12 it follows that N is a separably tame field.

Assume that there is given a separable immediate extension N of F which is a separably tame field, $N|F$ not necessarily being algebraic. We denote by N' the relative algebraic closure of F within N. By Lemma 6.17, N' is also a separably tame field, and it is an immediate separable–algebraic extension of F. Now we take L to be the relative algebraic closure of $K(t_1, ..., t_n)$ within N'. Since $N'|K$ and thus $N'|K(t_1, ..., t_n)$ are immediate, it follows from Lemma 6.17 that L is also a separably tame field and that $F.L$ is a separable immediate function field of transcendence degree 1 over L. By the first assertion of our theorem, $(F.L)^h = L(x)^h$ for some $x \in (F.L)^h$. But there exists a subfield

$L_0 \subset L$ finitely generated over K such that $x \in (F.L_0)^h$ and $(F.L_0)^h = L_0(x)^h$ and the same will be true for every overfield of L_0 within L. L_0 being an immediate function field of transcendence degree n over K, it admits by hypothesis a finite extension L_1 within L such that $L_1^h = K(x_1, ..., x_n)^h$ for suitable elements $x_1, ..., x_n \in L_1^h$. For $F_1 = F.L_1$ it follows

$$F_1^h = (F.L_1)^h = L_1(x)^h = (L_1^h(x))^h = (K(x_1, ..., x_n)^h(x))^h = K(x_1, ..., x_n)^h$$

which shows that F_1^h is a henselian rational function field over K. On the other hand, $F_1|F$ is a finitely generated subextension of the immediate separable–algebraic extension $N'|F$, hence $F_1|F$ is finite, separable and immediate. Finally, F_1 is also a subfield of N. This completes the proof of our theorem. □

7.2 A pull down principle for henselian rationality through tame extensions.

Let $(L, v)|(K, v)$ be a finite tame extension of algebraically maximal fields of rank 1, and let $(F, v)|(K, v)$ be an immediate function field of transcendence degree 1 and F not contained in the completion K^c of K. We consider the following question:
Assume that $F^h.L|L$ is a henselian rational function field, does this imply the same for $F^h|K$?

Firstly, we observe that w.l.o.g. we may assume that the extension $L|K$ is a Galois extension; indeed, the normal hull of a tame extension is a tame extension too, and if $F^h.L|L$ is henselian rational, then the same holds for $F^h.L'|L'$ for every extension $L'|L$. From now on we assume that $L|K$ is a finite tame Galois extension and that $F^h.L = L(x)^h$. Note that $F.L|L$ is an immediate henselian function field like $F|K$. Moreover, $(L(x), v)|(L, v)$ satisfies hypotheses (186) and (187) of section 11.3 (with K replaced by L):

Lemma 7.17 *$F \not\subset K^c$ implies $F.L \not\subset L^c$, hence* $\mathrm{dist}\,(x, L) < \infty$.

Proof: Since $F \not\subset K^c$, there exists some $z \in F$ with $\mathrm{dist}\,(z, K) < \infty$. $L|K$ is a finite tame extension by assumption, hence it admits a valuation basis $b_1, \ldots, b_n$, $b_1 = 1$; since $F|K$ is immediate, it remains a valuation basis of $F.L|F$. For every element

$$a = \sum c_i b_i \in L\;, \quad c_i \in K\;,$$

we compute

$$v(z - a) = v\left(z - c_1 - \sum_{i>1} c_i b_i\right) = \min_{i>1}\{v(z - c_1), v(c_i b_i)\} \le v(z - c_1)\;.$$

This shows

$$\mathrm{dist}\,(z, L) = \mathrm{dist}\,(z, K)$$

and consequently, $F.L \not\subset L^c$, as asserted.

Furthermore, $\mathrm{dist}\,(x, L) = \infty$ would imply $L(x) \subset L^c$; since the rank of (K, v) is 1 by assumption, the same is true for $(L(x), v)$ and $L(x)$ is thus dense in $L(x)^h$, and we would get $F^h.L = L(x)^h \subset L^c$, a contradiction. □

Lemma 7.18 *If there exists an element $y \in F^h$ such that $L(y)^h = L(x)^h$, then $F^h = K(y)^h$.*

Proof: Since y must be transcendental over K, we have that $F^h|K(y)^h$ is a finite extension. It must be purely wild since $F^h|K$ and thus also $F^h|K(y)^h$ are immediate extensions. On the other hand, $F^h.L = L(x)^h = L(y)^h = K(y)^h.L$ is a tame extension of $K(y)^h$ induced by the tame extension $L|K$. But the purely wild extension $F^h|K(y)^h$ is a subextension of the tame extension $F^h.L|K(y)^h$ which shows that $F^h|K(y)^h$ must be trivial. Hence $F^h = K(y)^h$, as asserted. □

Since $L|K$ is a finite tame Galois extension, the extension $F^h.L|F^h$ is a finite tame Galois extension too. It is of the same degree as $L|K$, since $L|K$ admits a valuation basis and $F^h|K$ is immediate by assumption. With $n = [L : K]$, we write

$$\mathrm{Gal}(F^h.L|F^h) = \{\rho_i \mid 1 \leq i \leq n\} \; .$$

Then $\mathrm{Gal}(L|K) = \{\rho_i|_L \mid 1 \leq i \leq n\}$.

Furthermore, we have assumed the rank of (L, v) to be 1, hence by Lemma 11.55, every element in $L(x)^h \setminus L$ satisfies condition (188). In the next lemma, we will consider the conjugates $\rho_i(x)$.

Lemma 7.19 *For every i, $1 \leq i \leq n$, we have*

$$L(x)^h = L(\rho_i(x))^h \; ,$$

hence $\mathbf{h}_K(x : \rho_i(x)) = 1$ by Corollary 11.62.

Proof: Since $\rho_i \in \mathrm{Gal}(F^h.L|F^h)$ and $\rho_i|_L \in \mathrm{Gal}(L|K)$, we may compute

$$L(x)^h = F^h.L = \rho_i(F^h.L) = \rho_i(L(x)^h) = (\rho_i(L(x)))^h = (\rho_i(L)(\rho_i(x)))^h = L(\rho_i(x))^h \; .$$

□

The following lemma and theorem make essential use of the valuation independence of Galois groups of tame Galois extensions. Let Tr denote the trace.

Lemma 7.20 *There is an element $d_0 \in L$ such that*

$$\mathbf{h}_K(x : \mathrm{Tr}_{F^h.L|F^h}(d_0 x)) = 1 \; .$$

Proof: For $1 \leq i \leq n$, let d_i be an approximation coefficient of $\rho_i(x)$ in x. By Theorem 6.23, we may choose an element $d_0 \in L$ such that (112) holds. Then for $k_i = \rho_i(d_0)$, the hypothesis (191) of Lemma 11.67 holds, so we may infer from this lemma that

$$\begin{aligned} \mathbf{h}_K\left(x : \mathrm{Tr}_{F^h.L|F^h}(d_0 x)\right) &= \mathbf{h}_K\left(x : \sum_i \rho_i(d_0 x)\right) \\ &= \mathbf{h}_K\left(x : \sum_i \rho_i(d_0) \cdot \rho_i(x)\right) = 1 \; . \end{aligned}$$

□

Now we are able to derive the main theorem of this subsection:

Theorem 7.21 *Let (K,v) be an algebraically maximal field of rank 1, and let (F,v) be an immediate function field of transcendence degree 1 over (K,v), with $F \not\subset K^c$. If $F^h.L$ is a henselian rational function field over L for some tame extension $(L,v)|(K,v)$, then F^h is a henselian rational function field over K.*

Proof: If $(F^h.L,v)$ is a henselian rational function field $L(x)^h$ for some tame extension L of K, then $F^h.L'$ equals $L'(x)^h$ already for a suitable finite subextension $L'|K$ of $L|K$. Hence we may assume from the start that $L|K$ is finite, and as discussed in the beginning of this subsection, we may assume that it is Galois, so we may assume the situation as it was treated above. Now the foregoing lemma shows that there is some $d_0 \in L$ such that for $y := \mathrm{Tr}_{F^h.L|F^h}(d_0 x) \in F^h$ we have $\mathbf{h}_K(x:y) = 1$. By virtue of Corollary 11.64, $L(y)^h = L(x)^h$. From Lemma 7.18, we may now infer that F^h is henselian rational over K, as asserted. □

7.3 The case of tame ground fields of mixed characteristic.

In this subsection, we will first prove that every immediate henselian function field of transcendence degree 1 over an algebraically closed ground field of rank 1 and mixed characteristic is henselian rational. In this setting, the minimal purely wild algebraic extensions in question have the following special structure:

Lemma 7.22 *Let L be a henselian field of mixed characteristic with divisible value group and algebraically closed residue field. Then every purely wild algebraic extension W of L is a tower of Galois extensions of degree p. If L contains all p–th roots of unity, such extensions of degree p are generated by p–th roots of suitable elements.*

Proof: From our conditions on the value group and the residue field, it follows by virtue of the Lemma of Ostrowski that $\tilde{L}|L$ is a p–extension. Then for every intermediate fields L_1, L_2 with $[L_1 : L_2] = p$, it follows that $L_1|L_2$ is normal since in general the degree of the normal hull of L_1 over L_2 must be a divisor of $p!$, whereas in our case it can only be equal to a power of p; this shows that it must be equal to p. The second assertion of our lemma follows from Kummer theory. □

In the case of $L = K(x)^h$ an immediate function field over an algebraically closed ground field K of characteristic 0, a minimal (purely wild) algebraic extension is thus a Galois extension $L(\vartheta)|L$ given by

$$\vartheta^p = a \in K(x)^h \; .$$

In the following we will determine a suitable normal form for the element a in the case of (K,v) having rank 1. For every $d \in L^\times$ we have

$$L(\vartheta) = L(d\vartheta),\ (d\vartheta)^p = d^p\vartheta^p = d^p a \in L \tag{126}$$

showing that all elements of the class $a \cdot (L^\times)^p$ determine the same extension. Since the extension $L(\vartheta)|L$ is immediate and $v(L) = v(K)$ is divisible, there exists an element

$d_1 \in L^\times$ such that $v(d_1^p a) = 0$; moreover, since $\overline{L} = \overline{K}$ is algebraically closed, there exists an element $d_2 \in L^\times$ such that $\overline{d_1^p d_2^p a} = 1$. Consequently, we may from now on assume $v(a) = 0$ and $\overline{a} = 1$, thus $a = 1 + b$ with $b \in \mathcal{M}_L$ (= the valuation ideal of L), hence the extension is of the form

$$L(\vartheta)|L,\ \vartheta^p = 1 + b \in L \ \text{ with } \ b \in \mathcal{M}_L\ . \tag{127}$$

Note that this implies $v(\vartheta) = 0$.

The algebraically closed field K contains an element $C \in \mathcal{M}_K$ which satisfies

$$C^{p-1} = -p \ \text{ and } \ v(C) = \frac{1}{p-1} v(p) > 0 \tag{128}$$

(cf. Section 3.8; we will also use the lemmata that we have proved there).

As an immediate consequence of Lemma 7.8 and part a) of Corollary 3.37, we have:

Corollary 7.23 *If the rank of K is 1 and $K(x)|K$ is immediate, then the extension (127) will also be generated by a root of the polynomial*

$$X^p - (1 + f(x)) \tag{129}$$

for some suitable polynomial $f(x) \in \mathcal{M}_{K(x)} \cap K[x]$.

Let $f(x) = d_n x^n + \ldots + d_0$. We consider the Taylor expansion for an arbitrary $c \in K$:

$$f(x) = \sum_{i=1}^{n} f_i(c)(x - c)^i \tag{130}$$

where f_i denotes the i–th formal derivative of f. Since K is algebraically closed, it is algebraically maximal, and thus in view of Corollary 11.53, for $c \nearrow x$ the values of $f_i(c)$ will be fixed. Consequently, for $c \nearrow x$ the values of all monomials $f_i(c)(x - c)^i$ will be different (cf. Corollary 11.32) and thus bigger or equal to the value of $f(x) \in \mathcal{M}_{K(x)}$, hence $f_i(c)(x - c)^i \in \mathcal{M}_{K(x)}$. Since $K(x) = K(x - c)$, we may replace x by $x - c$ and d_i by $f_i(c)$, and we may thus assume from the start that all nonzero monomials $d_i x^i$ appearing in $f(x)$ have different values, all of them bigger or equal to $v(f(x))$. We may even assume the following:

$$\text{if } i = p^t \text{ and } j = p^t r \text{ with } t \geq 0,\ r > 0 \text{ and } d_j \neq 0, \text{ then } \ v(d_i x^i) < v(d_j x^j) \tag{131}$$

(cf. Lemma 11.31). Furthermore, $K(x)|K$ being immediate, there is some $d \in K$ with $v(dx) = 0$; since $K(x) = K(dx)$, we may replace x by dx and d_i by $d_i d^{-i}$, and we may thus assume from the start $v(x) = 0$ and that all nonzero monomials $d_i x^i$ appearing in $f(x)$ satisfy $v(d_i x^i) = v(d_i)$.

Lemma 7.24 *Let K be algebraically closed. In the situation of Corollary 7.23, and with the assumptions of the foregoing remark, the extension (127) will then also be generated by a root of the polynomial*

$$X^p - (1 + h(z)) \tag{132}$$

for a suitable polynomial $h(z) \in K[x]$ with

$$\left.\begin{array}{l} h(z) = a_n z^n + \ldots + a_1 z + a_0 \in \mathcal{M}_K[z] \text{ where} \\ z = (x-c)/d\,,\ \text{with } v(z)=0\,,\ c \in K \text{ and } 0 \neq d \in K, \\ \text{and where } a_1 \text{ is the unique coefficient of least value.} \end{array}\right\} \qquad (133)$$

Then

$$\mathbf{h}_K(x : h(z)) = 1\ .$$

Moreover, we may assume

$$\forall i \geq 1 :\ v(a_i) \leq v(p) \Longrightarrow (i,p) = 1\ .$$

If $v(a_1) > v(p)$, we may even assume that there is no nonzero coefficient a_i with $p|i$ or $v(a_i) > \frac{p}{p-1} v(p)$, and that all nonzero coefficients have different value.

Proof: We may assume that $f(x)$ is as in the remark preceding this lemma. For every i divisible by p, choosing $d_i^{1/p}$ to be any p–th root of d_i in the algebraically closed field K, we have

$$d_i^{1/p} x^{i/p} \in \mathcal{M}_K[x] \subset \mathcal{M}_{K(x)}\ . \qquad (134)$$

We put

$$d = (1 + \sum_{p|i} d_i^{1/p} x^{i/p})^{-1} \in K(x)^\times\ ,$$

and by Lemma 3.35 we find that the extension (127) will also be generated by a root of a polynomial

$$X^p - (1 + \tilde{f}(x))$$

where $\tilde{f}(x)$ is a polynomial in x with

$$\tilde{f}(x) \equiv (1 + d^p \cdot \sum_{(p,j)=1} d_j x^j) \mod p\mathcal{M}_K[x]\ .$$

Using the representation of $d \equiv 1$ modulo $\mathcal{M}_{K(x)}$ as a geometrical series together with our assumption that the rank of (K,v) is 1, we find that d is equivalent modulo $p\mathcal{M}_{K(x)}$ to a polynomial in x whose nonconstant monomials are elements of $\mathcal{M}_K[x]$. This implies that d^p is equivalent modulo $p\mathcal{M}_{K(x)}$ to a polynomial in x^p whose nonconstant monomials are again elements of $\mathcal{M}_K[x]$. At this point we take C to be as in equation (128) and introduce a polynomial $g(x)$ which is equivalent to $\tilde{f}(x)$ modulo $C^p\mathcal{M}_{K(x)}$. From part a) of Corollary 3.37 we infer that the extension (127) will also be generated by a root of the polynomial

$$X^p - (1 + g(x))\ .$$

Let $g(x) = c_m x^m + \ldots + c_0$; here, g may be chosen such that all monomials $c_i x^i$ are already appearing in $\tilde{f}$; consequently, every c_i with $p|i$ and $v(c_i x^i) = v(c_i) \leq v(p)$ is equal to zero.

Suppose that there are still monomials in $g(x)$ of value $\leq v(p)$. Since by assumption, all nonzero monomials $d_j x^j$ have different values, there is also a unique monomial $d_{j_0} x^{j_0}$ among those with $(p,j) = 1$; by (131) we have $j_0 = 1$. Note that $d^p \equiv 1$ modulo $\mathcal{M}_{K(x)}$; consequently, $c_1 x = d_1 x$ is the monomial of minimal value in $g(x)$, concluding our proof in this case.

Now let us assume that all monomials in $g(x)$ have value $> v(p)$. Changing notation, we will talk again of $f(x)$ instead of $g(x)$ just to show that from now on, the proof is somewhat parallel to the proof of Lemma 7.10. We consider the Taylor expansion for an arbitrary x_0:

$$f(x) = \sum_{i=1}^{n} f_i(x_0)(x - x_0)^i \tag{135}$$

where f_i denotes the i–th formal derivative of f. For $v(x_0) \geq 0$ (which holds for $x_0 \nearrow x$), every monomial $f_i(x_0)(x - x_0)^i$ has value $> v(p)$. For $p|i$, the analogue of (134) will now be:

$$f_i(x_0)^{1/p}(x - x_0)^{i/p} \in \mathcal{M}_K[x - x_0] \ ,$$

in view of part b) of Corollary 3.37, we may then replace the summand $f_i(x_0)(x - x_0)^i$ for $i > 0$ by

$$-p \cdot f_i(x_0)^{1/p}(x - x_0)^{i/p} \in p\mathcal{M}_K[x - x_0]$$

without changing the extension. Moreover, by Corollary 3.38 the constant coefficient $f_0(x_0)$ may be omitted since $1+f(x_0)$ admits a p–th root in the algebraically closed field K. Using the first procedure repeatedly if necessary, we arrive at a polynomial

$$\sum_i{}' \left(\sum_j{}^{(i)} p^{(j)} \cdot f_{ip^j}(x_0)^{p^{-j}} \right) (x - x_0)^i \tag{136}$$

where:

1. $\sum_i{}'$ denotes the sum over all $i \leq \deg(f)$ with $(p, i) = 1$,
2. $\sum_j{}^{(i)}$ denotes the sum over all $j \geq 0$ with $ip^j \leq \deg(f)$,
3. $p^{(j)}$ are elements of K such that $p^{(0)} = 1$ and $(-p^{(j)}/p)^p = p^{(j-1)}$; hence $v(p^{(j)}) \geq v(p)$ for all $j \geq 1$.
4. $.^{p^{-j}}$ denotes an arbitrary p^j–th roots of the indicated element.

Let us write the i–th coefficient of (136) as

$$c_i(x_0) := \sum_j{}^{(i)} p^{(j)} \cdot f_{ip^j}(x_0)^{p^{-j}} \ ;$$

this is not a polynomial in x_0. Nevertheless, we may proceed as follows. Since every summand $p^{(j)} \cdot f_{ip^j}(x_0)^{p^{-j}}$ in $c_i(x_0)$ has value $> v(p)$, a straightforward induction on the natural number μ shows

$$c_i(x_0)^{p^\mu} \equiv \sum_j{}^{(i)} (p^{(j)})^{p^\mu} \cdot f_{ip^j}(x_0)^{p^{\mu-j}} \mod C^{p^{\mu+1}-1}\mathcal{M} \ .$$

But for large enough μ, the latter is a polynomial in x_0 whose value is consequently fixed for $x_0 \nearrow x$. This shows: if the value of $c_i(x_0)$ is not fixed for $x_0 \nearrow x$, then it must be bigger than

$$\frac{1}{p^\mu} v(C^{p^{\mu+1}-1}) = v(C^p) - \frac{1}{p^\mu} v(C)$$

for every μ. In this case, we have for $x_0 \nearrow x$ that

$$c_i(x_0)(x - x_0)^i \in C^p\mathcal{M}_{K(x)} ,$$

(since also $x - x_0$ will have a value > 0), and by part a) of Lemma 3.37, we may omit this summand from the polynomial (136). So we may assume that the values of all $c_i(x_0)$ are fixed (and that there is at least one $c_i(x_0)$ of value $\leq \frac{p}{p-1}v(p)$ since the extension is assumed to be nontrivial). Hence there exists an integer $i_0 \geq 1$ not divisible by p such that for $x_0 \nearrow x$, the summand

$$c_{i_0}(x_0) \cdot (x - x_0)^{i_0}$$

of (136) is the unique summand having minimal value, and all summands have different values. Having chosen such an x_0, we choose $d_1 \in K$ such that $v(d_1) = v(x - x_0)$ and put

$$y = \frac{x - x_0}{d_1} ,$$

hence $v(y) = 0$. The polynomial that we have derived from (136) will now be written as a polynomial in y:

$$g(y) = \sum_i d_1^i c_i(x_0) y .$$

Proceeding further just as in the proof of Lemma 7.10, we arive at a polynomial $h(z)$; note that in the present case, $a_1 z$ has minimal value among all nonconstant monomials since the i–th coefficient of $g(y)$ is zero whenever $p|i$. As above, the constant monomial a_0 may be deleted by Corollary 3.38. It follows from Lemma 11.34 that $\mathbf{h}_K(z : h(z)) = 1$, and $\mathbf{h}_K(x : h(z)) = 1$ follows from Lemma 11.61 and the trivial fact that $\mathbf{h}_K(x : z) = 1$ (since z is linear in x).

It remains to show that $h(z)$ may even be chosen such that there is no nonzero coefficient a_i with $p|i$. To this end, we apply the above replacement procedure once more to $h(z)$. If

$$a_i z^i \in C^p\mathcal{M}_{K(x)} ,$$

then by part a) of Lemma 3.37, we may omit this summand from the polynomial $h(z)$. So we may assume that the above replacement procedure is only applied to monomials $a_i z^i$ of value $\leq \frac{p}{p-1}v(p)$; for those, we have

$$v(a_i z^i) \leq v(-p \cdot (a_i)^{1/p} z^{i/p}) ,$$

(where equality holds if and only if $v(a_i z^i) = \frac{p}{p-1}v(p)$). Moreover, the procedure is only applied to monomials $a_i z^i$ of value $> v(a_1 z)$; this shows that the monomial $a_1 z$ will remain the unique monomial of least value in the newly obtained polynomial. □

The normal form that we have derived, allows us to prove the announced assertion on the structure of henselian function fields over algebraically closed valued fields of mixed characteristic:

Lemma 7.25 *Let K be an algebraically closed valued field of mixed characteristic and F an immediate function field of transcendence degree 1 over K. Assume that its rank is 1. Then F^h is a henselian rational function field over K.*

Proof: Let us choose an element x of F^h such that $[F^h : K(x)^h]$ is minimal. We will show that $[F^h : K(x)^h] = 1$ which yields the assertion of our lemma.

Assume the contrary. Then the finite extension $F^h|K(x)^h$ which is purely wild since it is immediate, contains a minimal purely wild extension $E|K(x)^h$. According to (127) and Corollary 7.23, it is generated by a root of a minimal polynomial $X^p - (1 + f(x))$ with $f(x) \in \mathcal{M}_{K(x)}$. By an application of Lemma 7.24, we may in addition assume that $\mathbf{h}_K(x : f(x)) = 1$. By virtue of Lemma 11.58, the latter yields $K(x)^h = K(f(x))^h$. Since $f(x) = \vartheta^p - 1 \in K(\vartheta)$, we have $x \in K(\vartheta)^h$ and

$$E = K(x)^h(\vartheta) = K(x, \vartheta)^h = K(\vartheta)^h$$

with

$$[F^h : K(\vartheta)^h] = [F^h : E] < [F^h : K(x)^h] ,$$

which is a contradiction to the minimum assumption on x. This completes the proof of our lemma. □

We will now generalize this result to tame ground fields of rank 1 and transcendence degree 1. In view of Lemma 7.5, we have to consider only function fields which do not lie in the completion of the ground field, so we may use the pull down principle of the foregoing section.

Lemma 7.26 *Every henselian function field F of transcendence degree 1 over a tame field (K, v) of rank 1 and mixed characteristic is henselian rational.*

Proof: If F lies in the completion of K, our assertion follows from Lemma 7.5; so let us assume now that F is not contained in the completion of K. We know that $F.\tilde{K}|\tilde{K}$ is henselian rational by virtue of Lemma 7.25. But K being a tame field, the extension $\tilde{K}|K$ is tame. Thus our assertion follows from Theorem 7.21. □

Now we are able to prove the remaining mixed characteristic case of Theorem 7.1, i.e.

Theorem 7.27 *Let K be a tame field of mixed characteristic and F an immediate function field over K. If its transcendence degree over K is 1, then F^h is a henselian rational function field over K. In the general case of transcendence degree ≥ 1, given any immediate extension N of F which is a tame field (such an extension does always exist), there is a finite immediate extension F_1 of F within N such that F_1^h is a henselian rational function field over K.*

Proof: Let us first consider the case $\mathrm{trdeg}(F|K) = 1$. Let P be the place that is associated to the valuation on F^h. Since F is a valued field of mixed characteristic, there exist places P_1, P_2, P_3 where P_1 and P_3 may be trivial and P_2 has rank 1, such that $P = P_1P_2P_3$ and $\mathrm{char}\,(FP_1) = 0$, $\mathrm{char}\,(FP_1P_2) = p > 0$. Note that since $(F, P)^h|(K, P)$ is immediate, we have $v_{P_1}(F^h) = v_{P_1}(K)$, $v_{P_2}(F^hP_1) = v_{P_2}(KP_1)$ and that $(F^hP_1P_2, P_3)|(KP_1P_2, P_3)$ is an immediate extension.

We distinguish three cases:

case 1: $\mathrm{trdeg}\,(FP_1|KP_1) = 0$.

(K,P) being a tame field, the same holds for (KP_1, P_2P_3) by virtue of Lemma 6.16. On the other hand, $(F,P)|(K,P)$ and thus also $(FP_1, P_2P_3)|(KP_1, P_2P_3)$ are immediate extensions. $FP_1|KP_1$ being algebraic, we must consequently have $FP_1 = KP_1$. So (F, P_1) is an immediate function field over (K, P_1). Since char $(FP_1) = 0$, by our discussion of Theorem 7.2 (cf. page 126), the henselization $(F, P_1)^{h(P_1)}$ which by Lemma 2.16 may be chosen inside of $(K, P)^{h(P)}$, is a henselian rational function field over (K, P_1). Consequently, $F^{h(P)} = (F^{h(P_1)})^{h(P)}$ is a henselian rational function field over (K,P).

<u>case 2</u>: trdeg $(FP_1|KP_1) = 1$ and trdeg $(FP_1P_2|KP_1P_2) = 0$.
A similar argument as in case 1 shows that $FP_1P_2 = KP_1P_2$; this yields that (FP_1, P_2) is an immediate function field of transcendence degree 1 over (KP_1, P_2). By the previous lemma, the henselization $(FP_1, P_2)^{h(P_2)}$ which by Lemma 2.16 may be chosen inside of $(F^{h(P)}P_1, P_2)$, is a henselian rational function field over (KP_1, P_2). Let us choose $x \in F^{h(P)}$ such that

$$(FP_1)^{h(P_2)} = KP_1(xP_1)^{h(P_2)} = (K(x)P_1)^{h(P_2)} \;.$$

Since $(FP_1P_2, P_3) = (KP_1P_2, P_3)$ is already henselian, we may infer from Lemma 2.16 that

$$F^{h(P)}P_1 = (FP_1)^{h(P_2P_3)} = (FP_1)^{h(P_2)}$$
$$K(x)^{h(P)}P_1 = (K(x)P_1)^{h(P_2P_3)} = (K(x)P_1)^{h(P_2)}$$

which shows that

$$F^{h(P)}P_1 = K(x)^{h(P)}P_1 \;.$$

Consequently, since we have $v_{P_1}(F^h) = v_{P_1}(K)$, the extension

$$(F^{h(P)}, P_1)|(K(x)^{h(P)}, P_1)$$

is also immediate. Moreover, we know that it is algebraic and that $(K(x)^{h(P)}, P_1)$ is henselian like $(K(x), P)^{h(P)}$, by virtue of Lemma 2.16. But since xP_1 is transcendental over KP_1, the field $(K(x)^{h(P)}, P_1)$ is defectless by Theorem 3.1. It follows that $F^{h(P)}|K(x)^{h(P)}$ must be trivial, or in other words that $F^{h(P)}$ is a henselian rational function field over K.

<u>case 3</u>: trdeg $(FP_1P_2|KP_1P_2) = 1$.
Since $(F,P)|(K,P)$ is immediate, this yields that (FP_1P_2, P_3) is an immediate function field of transcendence degree 1 over (KP_1P_2, P_3). By the already proved part of Theorem 7.2 (the equal characteristic case), the henselization $(FP_1P_2, P_3)^{h(P_3)}$ which by Lemma 2.16 may be chosen inside of $(K^{h(P)}P_1P_2, P_3)$ is a henselian rational function field over (KP_1P_2, P_3). To show that $F^{h(P)}$ is henselian rational, one proceeds as in case 2, replacing P_1 by P_1P_2 and P_2 by P_3.

The second assertion of our theorem follows from the first by induction on the transcendence degree of $F|K$. We show that there is always an immediate extension of F which is tame. Let N be a maximal immediate algebraic extension of F. Since K is tame by hypothesis and $F|K$ is immediate, the value group $v(F) = v(K)$ is p–divisible and the residue field $\overline{F} = \overline{K}$ is perfect. By virtue of Lemma 6.2 it follows that N is a tame field. The remainder of the proof of the second assertion may be transferred wordly from the corresponding part of the proof of the equal characteristic case of Theorem 7.2. □

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8 On the model theory of valued fields.

In this section, we turn to model theoretic investigations about valued fields. The results presented here should not be considered a complete description of the model theory of tame and separably tame fields. In particular, details on the latter and questions about quantifier elimination have to be postponed to a subsequent paper.

We will use the ordinary language $\mathcal{L}_0 = \{0, 1, +, \cdot\}$ for the residue fields and the language $\mathcal{L} = \{0, 1, +, \cdot, V\}$ for valued fields (K, v), where V is either a unitary predicate $\mathcal{O}(x)$ denoting the assertion that $x \in \mathcal{O}_K$, or a binary predicate $v(x, y)$ denoting that $v(x) \leq v(y)$. Which of these predicates is chosen, does actually not influence our results. Also, adding the inverse function to $\mathcal{L}_0$ and $\mathcal{L}$ is optional and does not influence our results.

We use the following notation. By "$v(K) \prec_\exists v(L)$" we will always mean that $v(K)$ is existentially closed in $v(L)$ with respect to the language of ordered groups. Analogously, "$\overline{K} \prec_\exists \overline{L}$" means that $\overline{K}$ is existentially closed in $\overline{L}$ with respect to the language of fields. In contrast to this, "$(K, v) \prec_\exists (L, v)$" means that the valued field (K, v) is existentially closed in the valued field extension (L, v) with respect to the language of valued fields. The same shall hold for elementary embeddings, denoted by "$\prec$", and first order equivalence, denoted by "$\equiv$".

We want to prove model theoretic results by embedding lemmas; as a model theoretic justification we cite Korollar 2.19 of [PRE]:

Lemma 8.1 *Let $\mathcal{A}$ be a common $\mathcal{L}$–substructure with universe A of the $\mathcal{L}$–structures $\mathcal{A}^*$ and $\mathcal{B}$.*
(1) If $\mathcal{A}^*$ is κ–saturated with $\kappa >$ card$(\mathcal{B})$ and
(a) if $\mathcal{A} \prec_\exists \mathcal{B}$, or
(b) if every finitely generated $\mathcal{L}(A)$–substructure of $\mathcal{B}$ is embeddable into $\mathcal{A}^*$,
then $\mathcal{B}$ is embeddable as $\mathcal{L}(A)$–structure into $\mathcal{A}^*$.
(2) If $\mathcal{B}$ is embeddable as $\mathcal{L}(A)$–structure into $\mathcal{A}^*$ and if $\mathcal{A} \prec \mathcal{A}^*$, then $\mathcal{A} \prec_\exists \mathcal{B}$.

Instead of saying "embeddable as $\mathcal{L}(A)$–structure" we will use the "more algebraic" expression "embeddable over $\mathcal{A}$".

In our setting, the role of the models $\mathcal{A} \subset \mathcal{B}$ will be played by an extension $(L, v)|(K, v)$ of valued fields, and we will look for an embedding of (L, v) over (K, v) into a κ–saturated elementary extension $\mathcal{A}^*$ of $\mathcal{A}$ with $\kappa >$ card$(\mathcal{B})$. Our general approach to this problem is to break up the extension $(L, v)|(K, v)$ into an extension without transcendence defect and an immediate extension; the most general form of this approach will be discussed in the section on Ax–Kochen–Ershov–classes. In the next section, we will consider extensions of the first sort.

8.1 The Ax–Kochen–Ershov–principle for extensions without transcendence defect of algebraically complete fields.

We need the following preparation which gives a precise description of the constructed embedding relative to given embeddings of the value group and the residue field:

Lemma 8.2 (Embedding Lemma II)

Let (K,v) be a defectless field (the valuation is allowed to be trivial), $(L,v)|(K,v)$ an extension without transcendence defect and $(K^\star,v^\star)$ a $|L|^+$– saturated henselian extension of (K,v). Assume that $v(L)/v(K)$ is torsion free and that $\overline{L}|\overline{K}$ is separable. If

$$\rho\colon v(L) \longrightarrow v^\star(K^\star)$$

is an embedding over $v(K)$ and

$$\sigma\colon \overline{L} \longrightarrow K^\star/v^\star$$

is an embedding over $\overline{K}$, then there exists an embedding

$$\iota\colon (L,v) \longrightarrow (K^\star,v^\star)$$

over (K,v) that respects ρ and σ (in the sense of Lemma 3.5).

Proof: By our Embedding Lemma I (Lemma 3.5) we know that every finitely generated subextension of $L|K$ can be embedded over (K,v) into $(K^\star,v^\star)$ respecting both embeddings ρ and σ. Using the saturation property of $(K^\star,v^\star)$ we have to deduce from this the assertion of our Embedding Lemma II. To do so we will work in an enlarged language $\mathcal{L}'$ consisting of the ordinary language $\mathcal{L}$ of valued fields enriched by predicates

$$\begin{array}{lcl} \mathcal{P}_\delta(X) & , & \delta \in \rho(v(L)) \\ \mathcal{Q}_d(X) & , & d \in \sigma(\overline{L}) \end{array}$$

which are interpreted in $(K^\star,v^\star)$ such that

$$\begin{array}{lcl} \mathcal{P}_\delta(a) & \Longleftrightarrow & v^\star(a) = \delta \\ \mathcal{Q}_d(a) & \Longleftrightarrow & a/v^\star = d \end{array}$$

for all $a \in K^\star$ and in (L,v) such that

$$\begin{array}{lcl} \mathcal{P}_\delta(b) & \Longleftrightarrow & \rho(v(b)) = \delta \\ \mathcal{Q}_d(b) & \Longleftrightarrow & \sigma(\overline{b}) = d \end{array}$$

for all $b \in L$. Note that these interpretations coincide on K.

We show that $(K^\star,v^\star)$ remains $|L|^+$–saturated in the enriched language $\mathcal{L}'$. To this end, we choose a subset $\Delta \subset K^\star$ of all values δ in $\rho(v(L))$ and a subset $D \subset K^\star$ of representatives for all residues d in $\sigma(\overline{L})$. We compute

$$\begin{array}{lcl} |\Delta| & = & |\rho(v(L))| = |v(L)| \leq |L| < |L|^+ \ , \\ |D| & = & |\sigma(\overline{L})| = |\overline{L}| \leq |L| < |L|^+ \ , \end{array}$$

hence $|\Delta \cup D| < |L|^+$. Consequently, if we add the elements from $\Delta \cup D$ as constants to the language $\mathcal{L}$ we get a language $\mathcal{L}^\star$ which satisfies

$$||\mathcal{L}^\star|| < |L|^+ \ .$$

The new constants are interpreted in $K^\star$ by the corresponding elements from $\Delta \cup D \subset K^\star$. From [CHK], Proposition 5.1.1 (iii), p. 215, we deduce that $(K^\star,v^\star)$ remains $|L|^+$–saturated in this new language $\mathcal{L}^\star$. Now the predicates $\mathcal{P}_\delta$ and $\mathcal{Q}_d$ become definable in the language $\mathcal{L}^\star$, and this shows that $(K^\star, v^\star)$ is also $|L|^+$–saturated in the language $\mathcal{L}'$, as asserted.

An embedding ι respects the predicates $\mathcal{P}_\delta$ and $\mathcal{Q}_d$ if and only if it satisfies

$$\rho(v(b)) = \delta \iff \mathcal{P}_\delta(b) \iff \mathcal{P}_\delta(\iota b) \iff v^\star(\iota b) = \delta \;,$$
$$\sigma(\overline{b}) = d \iff \mathcal{Q}_d(b) \iff \mathcal{Q}_d(\iota b) \iff \iota b/v^\star = d \;,$$

which expresses the property of ι to respect the embeddings ρ and σ.
By Lemma 3.5 we know that for every finitely generated subextension of $L|K$ (and even for its henselization) there exists such an embedding ι over (K, v) into $(K^\star, v^\star)$ which respects the predicates $\mathcal{P}_\delta$ and $\mathcal{Q}_d$. The saturation property of $(K^\star, v^\star)$ now yields an embedding of (L, v) into $(K^\star, v^\star)$ which respects the predicates and thus the embeddings ρ and σ. This completes the proof of our lemma. □

Before applying Embedding Lemma II, we want to state that the conditions

"$v(L)/v(K)$ is torsion free" and "$\overline{L}|\overline{K}$ is separable"

do in particular hold if $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$:

Lemma 8.3 *Let $\Gamma \subset \Delta$ be an extension of ordered abelian groups and $k_2|k_1$ an extension of fields. If $\Gamma \prec_\exists \Delta$, then Γ is pure in Δ, i.e. Δ/Γ is torsion free. If $k_1 \prec_\exists k_2$, then the extension $k_2|k_1$ is regular.*

Proof: Cf. Proposition (2.3) of [VDD], p. 23, for the regularity. The easy proof of the assertion for ordered abelian groups is left to the reader. □

Now we are able to prove the following Ax–Kochen–Ershov–principle for extensions without transcendence defect:

Theorem 8.4 *If (L, v) is an extension without transcendence defect of the algebraically complete field (K, v), then the "side conditions"*

$$v(K) \prec_\exists v(L) \quad \textit{and} \quad \overline{K} \prec_\exists \overline{L} \tag{137}$$

imply

$$(K, v) \prec_\exists (L, v) \; .$$

Proof: We choose (K^*, v^*) to be an $|L|^+$–saturated elementary extension of (K, v). Then K^*/v^* is a $|\overline{L}|^+$–saturated elementary extension of $\overline{K}$ and $v^*(K^*)$ is a $|v(L)|^+$–saturated elementary extension of $v(K)$. Since "algebraically complete" is elementarily definable by a scheme of first order axioms (cf. [DEL1], p. 21), (K^*, v^*) is algebraically complete like (K, v) and thus henselian. By our hypothesis (137) and by Lemma 8.1, there exist embeddings

$$\rho\colon v(L) \longrightarrow v^*(K^*)$$

over $v(K)$ and

$$\sigma\colon \overline{L} \longrightarrow K^*/v^*$$

over $\overline{K}$. By the preceding Embedding Lemma II, there exists an embedding

$$\iota\colon (L,v) \longrightarrow (K^*,v^*)$$

over (K,v) that respects ρ and σ. By Lemma 8.1, this shows that (K,v) is existentially closed in (F,v). □

Note that it is not in general true that under the conditions of the above theorem, the strengthened side conditions

$$v(K) \prec v(L)\ , \quad \overline{K} \prec \overline{L}$$

would imply $(K,v) \prec (L,v)$. A counterexample is given in section 10 (cf. page 175).

8.2 Model theory of tame fields.

In this section we will treat the model theory of tame fields. We apply the Structure Theorem 7.1 of the foregoing section and the auxiliary results of section 6. We have already shown in Lemma 6.2 that in positive characteristic, the class of tame fields coincides with the class of algebraically maximal perfect fields.

The following auxiliary embedding lemma is put into a rather general form. Note that by Corollary 11.53, the condition "appr(x,K) is transcendental" does automatically hold if K is algebraically maximal.

Lemma 8.5 (Embedding Lemma III)
Let $(K(x),v)|(K,v)$ be a nontrivial immediate extension of valued fields. If appr(x,K) *is transcendental, then $(K(x),v)^h$ can be embedded over (K,v) into every $|K|^+$–saturated henselian extension $(K,v)^\star$ of (K,v).*

Proof: Since $(K,v)^\star$ is $|K|^+$–saturated, Lemma 11.13 shows that appr(x,K) is realized by an element x' in $(K,v)^\star$. By Theorem 11.51, the homomorphism induced by $x \mapsto x'$ is an embedding of $(K(x),v)$ over (K,v) into $(K,v)^\star$ since appr(x,K) is transcendental. By the universal property of the henselization, this embedding can be prolongated to an embedding of $(K(x),v)^h$ over (K,v) into $(K,v)^\star$ which is henselian by hypothesis. □

Note that the lemma becomes false if the condition on the approximation type of x over K is omitted, even if we require in addition that K is henselian. It is known that there exist henselian fields K which admit nontrivial algebraic immediate extensions (cf. the examples of F. K. Schmidt and of Ostrowski in [RIB1], Exemple 1, page 244, and Exemple 2, page 246). Hence there exist also nontrivial finite simple immediate extensions $(K(x),v)|(K,v)$. On the other hand, $K^\star$ may be a regular extension of K (e.g., this is always the case if $(K,v)^\star$ is an elementary extension of (K,v)), in which case $K(x)$ is certainly not embeddable into $K^\star$ over K.

The model theoretic application of Embedding Lemma III is:

Corollary 8.6 *Let (K,v) be a henselian field and $(K(x),v)|(K,v)$ an immediate extension such that* appr (x,K) *is transcendental. Then $(K,v) \prec_\exists (K(x),v)^h$. In particular, an algebraically maximal field (K,v) is existentially closed in every immediate henselian rational function field $(K(x),v)^h$.*

Proof: Apply Embedding Lemma III with $(K,v)^\star$ a $|K|^+$–saturated elementary extension of (K,v); since "henselian" is an elementary property, $(K,v)^\star$ will also be henselian. □

Using this corollary and infering our Structure Theorem 7.1, we deduce:

Corollary 8.7 *A tame field K is existentially closed in every immediate extension L of transcendence degree 1.*

Proof: It suffices to prove the assertion for every finitely generated subextension $F_0|K$ of $L|K$. For this, it suffices to show it for the henselization $F = F_0^h$. F is an immediate henselian function field over K. By Theorem 7.1, $F = K(x)^h$ for a suitable element $x \in F$. Now Corollary 8.6 shows that (K,v) is existentially closed in (F,v). □

As a consequence of Embedding Lemma II, Theorem 7.1 and Embedding Lemma III, we will now derive our main embedding lemma and our main theorem on tame fields. Note that instead of the following embedding lemma, we could also use the previous corollary together with Corollary 6.10 to deduce Theorem 8.9 below. Nevertheless, it seems worthwile to state the embedding lemma for tame fields in the most general form.

Lemma 8.8 (Embedding Lemma IV)
Let (K,v) be a valued field such that its henselization is a tame field (the valuation is allowed to be trivial), (L,v) an extension of (K,v) and $(K^\star,v^\star)$ a tame $|L|^+$– saturated extension field of (K,v). Assume that $v(L)/v(K)$ is torsion free and $\overline{L}|\overline{K}$ is separable.

If $\rho\colon v(L) \longrightarrow v^\star(K^\star)$ is an embedding over $v(K)$ and $\sigma\colon \overline{L} \longrightarrow K^\star/v^\star$ is an embedding over $\overline{K}$, then there exists an embedding $\iota\colon (L,v) \longrightarrow (K^\star,v^\star)$ over (K,v) that respects ρ and σ (in the sense of Lemma 3.5). Note that by virtue of Lemma 8.3, our conditions on $v(L)/v(K)$ and $\overline{L}|\overline{K}$ are always satisfied if $\overline{K} \prec_\exists \overline{L}$ and $v(K) \prec_\exists v(L)$.

Proof: We may assume from the start that L is henselian. Indeed, if we are able to show the existence of an embedding ι for L^h which has the required properties then the restriction of ι to L is the desired embedding of L. Here, we note once and for all that for the existence of the desired embedding, it always suffices to show its existence for a suitably enlarged field. Furthermore, we may assume that K is henselian. This is seen as follows:
Since we assume L to be henselian, L includes the henselization K^h of K. By the universal property of the henselization (cf. property H3 of the definition given in [RIB1] on page 175), K^h admits an embedding over K into the field $K^\star$ which is henselian by assumption. Since $K^h|K$ is immediate, this embedding trivially respects ρ and σ. Through this embedding we identify K^h with its image in $K^\star$. Hence we may from now on assume that K is henselian. Consequently, K is algebraically maximal and by Lemma 6.2, it is a tame field.

In view of what we have said above and by Lemma 6.5 we may now assume that L is a tame field too. By an application of Lemma 6.8, we obtain an intermediate field L_0 of $L|K$ which is a tame field such that $L|L_0$ is an immediate extension and $L_0|K$ is an extension without transcendence defect. In this situation, Embedding Lemma II (Lemma 8.2) yields an embedding of L_0 over K into $K^\star$ which respects ρ and σ. Through this embedding, we identify L_0 with its image in $K^\star$ and consequently, we may replace K by L_0 and assume from now on that $L|K$ is immediate.

Note that in this situation, we may forget about ρ and σ; since $L|K$ is immediate, every valuation preserving embedding of any intermediate field over K into $K^\star$ will respect ρ and σ. Let L_1 be a maximal tame intermediate field admitting a valuation preserving embedding over K into $K^\star$; it exists by Zorn's Lemma. Assume that there exists an element $y \in L \setminus L_1$. Let L_2 be the relative algebraic closure of $L_1(y)$ in the tame field L; by Lemma 6.6, L_2 is a tame field too; note for the application of Lemma 6.6 that $\overline{L} = \overline{L_1(y)}$ since $L|K$ and thus also $L|L_1(y)$ are immediate extensions. By showing that the embedding of L_1 can be extended to an embedding of L_2, we want to deduce a contradiction to the maximality property of L_1 which will show $L = L_1$, thereby completing the proof of our lemma.

Through the valuation preserving embedding of L_1 over K into $K^\star$, we identify L_1 with its image in $K^\star$ and we may w.l.o.g. assume $L_1 = K$. Since $K^\star$ is $|L|^+$–saturated by assumption, it suffices to show the existence of a valuation preserving embedding for every finitely generated subextension $F|K$ of $L_2|K$. Again, by an enlargement of F we may assume that F is henselian. Then F is an immediate henselian function field of transcendence degree 1 over the tame field K. Hence by Theorem 7.1, it is a henselian rational function field, i.e. there exists an element $x \in F$ such that $F = K(x)^h$. Since by Lemma 6.2, the tame field K is in particular algebraically maximal, Embedding Lemma III shows the existence of a valuation preserving embedding of F over K into $K^\star$. By this, we have proved the existence of the desired embedding of L_2 and deduced the contradiction which completes the proof of our lemma. □

Before applying Embedding Lemma IV, we want to introduce some model theoretic notions. Given a valued field extension $L|K$, we will say that $L|K$ satisfies the *Ax–Kochen–Ershov–principle* if

$$v(K) \prec_\exists v(L) \wedge \overline{K} \prec_\exists \overline{L} \implies (K,v) \prec_\exists (L,v)\ . \tag{138}$$

Let us use the abbreviation *AKE* for "Ax–Kochen–Ershov". A valued field K will be called *AKE–field* if every valued field extension $L|K$ satisfies the AKE–principle. An elementary class of valued fields will be called *AKE–class* if every field contained in this class is an AKE–field.

The following is our main model theoretic result on tame fields:

Theorem 8.9 *Every tame field is an AKE–field.*

Proof: Any trivially valued field is an AKE–field. Now let (K,v) be a nontrivially valued tame field and $(L,v)|(K,v)$ an extension with $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$. If we take $(K,v)^*$ to be an elementary $|L|^+$–saturated extension of (K,v), then $(K,v)^*$ is henselian like (K,v). $v^*(K^*)$ is $|L|^+$–saturated and thus also $|v(L)|^+$–saturated, and K^*/v^* is $|L|^+$–saturated and thus also $|\overline{L}|^+$–saturated. Hence the "side conditions" $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$ together with Lemma 8.1 imply the existence of embeddings $\rho: v(L) \longrightarrow v^*(K^*)$ over $v(K)$ and $\sigma: \overline{L} \longrightarrow K^*/v^*$ over $\overline{K}$. By Embedding Lemma IV we can now embed (L,v) over (K,v) into $(K,v)^*$. By virtue of Lemma 8.1, we get $(K,v) \prec_\exists (L,v)$. □

In view of algebraic applications, the foregoing theorem is somewhat the kernel of our model theoretic results. Nevertheless, we want to give a precise description of the main results that can be derived from Embedding Lemma IV. To do this in a rather general way, we need the following notations. Given two models M_1, M_2 of a theory with language $\mathcal{L}$, a substructure S of M_1 and an embedding ι of S in M_2. Then we write

$$M_1 \equiv M_2 \ \text{ over } \ \iota\colon S \longrightarrow \iota S$$

if $(M_1, a)_{a\in S} \equiv (M_2, \iota a)_{a\in S}$ which means that M_1 and M_2 are elementarily equivalent in the language which is obtained from $\mathcal{L}$ by adding constants for every element in S, these constants being interpreted in M_1 by the corresponding element $a \in S$ and in M_2 by the element ιa. If the above situation holds for a substructure S which is a substructure of both M_1 and M_2, ι being the identity, then we will simply write

$$M_1 \equiv_S M_2 \ .$$

Theorem 8.10 *Let (K, v) be a tame field and let (K_1, v_1) and (K_2, v_2) be two tame extension fields of (K, v). Assume that $v_1(K_1)/v(K)$ is torsion free and $(K_1/v_1)|(K/v)$ is separable. Then the conditions*

$$v_1(K_1) \equiv_{v(K)} v_2(K_2) \quad \textit{and} \quad K_1/v_1 \equiv_{K/v} K_2/v_2 \tag{139}$$

imply

$$(K_1, v_1) \equiv_{(K,v)} (K_2, v_2) \ .$$

Proof: Let us first assume that (K_1, v_1) is trivially valued. Then in particular, (K, v) is trivially valued and by $v_1(K_1) \equiv_{v(K)} v_2(K_2)$ it follows that also (K_2, v_2) is trivially valued. In this case, the assertion follows readily from the assumption $K_1/v_1 \equiv_{K/v} K_2/v_2$. A symmetric argument works if (K_2, v_2) is trivially valued.

We assume from now on that (K_1, v_1) and (K_2, v_2) are not trivially valued; in particular, they will not be finite models. Let $\kappa > \max\{\aleph_0, |K|\}$ be a cardinal and let $(K_i, v_i)^* = (K_i^*, v_i^*)$ be κ^+–saturated elementary extensions of (K_i, v_i) for $i = 1, 2$. Since $(K_i, v_i)^* \equiv_{(K,v)} (K_i, v_i)$, it suffices to show $(K_1, v_1)^* \equiv_{(K,v)} (K_2, v_2)^*$. Note that also $v_i^*(K_i^*) \equiv_{v(K)} v_i(K_i)$ and $K_i^*/v_i^* \equiv_{K/v} K_i/v_i$ $(i = 1, 2)$, which shows that the hypothesis $v_1(K_1) \equiv_{v(K)} v_2(K_2)$ and $K_1/v_1 \equiv_{K/v} K_2/v_2$ implies $v_1^*(K_1^*) \equiv_{v(K)} v_2^*(K_2^*)$ and $K_1^*/v_1^* \equiv_{K/v} K_2^*/v_2^*$. Consequently, we may assume from the start that both (K_1, v_1) and (K_2, v_2) are κ^+–saturated. If we are able to show that there exist submodels $(L_i, v_i) \prec (K_i, v_i)$ $(i = 1, 2)$, both containing (K, v), such that $(L_1, v_1) \cong (L_2, v_2)$ over (K, v), then we are done since by $(K_i, v_i) \equiv_{(K,v)} (L_i, v_i)$ this would imply $(K_1, v_1) \equiv_{(K,v)} (K_2, v_2)$.

We construct the fields L_i and the isomorphism by induction using a "back and forth" procedure. Let ι_0 be the identity $\iota_0\colon (K, v) \longrightarrow (K, v)$ and put $L_i^0 := K$ $(i = 1, 2)$. Assume that we have constructed subfields $(L_i^n, v_i) \subset (K_i, v_i)$ $(i = 1, 2)$ of cardinality κ and an isomorphism $\iota_n\colon (L_1^n, v_1) \longrightarrow (L_2^n, v_2)$ over (K, v) for $n \in I\!N$, such that

$$\begin{array}{rcl} v_1(K_1) \equiv v_2(K_2) & \text{over} & \rho_n\colon v_1(L_1^n) \longrightarrow v_2(L_2^n)\ , \\ K_1/v_1 \equiv K_2/v_2 & \text{over} & \sigma_n\colon L_1^n/v_1 \longrightarrow L_2^n/v_2 \end{array}$$

where ρ_n and σ_n are the isomorphisms of the value groups and the residue fields which are induced by ι_n. Assume further that the already constructed fields (L_1^μ, v_1), $1 \le \mu \in 2I\!N+1$, resp. (L_2^ν, v_2), $2 \le \nu \in 2I\!N$, form elementary chains of elementary submodels of (K_1, v_1) resp. (K_2, v_2). Then we proceed as follows:

Assume that $n \in 2I\!N$. We choose an arbitrary element $a_n \in K_1 \backslash L_1^n$. Using a strong version of the downward Löwenheim–Skolem Theorem (cf. Theorem 3.1.6, page 109 in [CHK]), we find that there exists an elementary submodel $(L_1^{n+1}, v_1) \prec (K_1, v_1)$ of cardinality κ and containing $L_1^n(a_n)$. If $n \geq 2$, then $(L_1^{n-1}, v_1) \prec (K_1, v_1)$, and we may view (K_1, v_1) as a model of

$$\mathrm{Th}(K_1, v_1) \cup \text{elementary Diagram}(L_1^{n-1}, v_1)$$

with respect to the language $\mathcal{L}_n$ which is obtained from $\mathcal{L}$ by adding a constant symbol for every element of L_1^{n-1}. In this way, the model (L_1^{n+1}, v_1) that we obtain by application of the downward Löwenheim–Skolem Theorem, will satisfy in addition:

$$(L_1^{n-1}, v_1) \prec (L_1^{n+1}, v_1) .$$

Note that the condition on the cardinality of the language as required in the downward Löwenheim–Skolem Theorem, is fulfilled since the language of valued fields is countable which yields that

$$||\mathcal{L}_n|| = |L_1^{n-1}| = \kappa .$$

Now we have to prolongate ι_n to an embedding of (L_1^{n+1}, v_1) into (K_2, v_2); the isomorphic image of L_1^{n+1} will be defined to be L_2^{n+1}. We have

$$v_1(L_1^{n+1}) \equiv_{v_1(L_1^{n+1})} v_1(K_1) \quad \text{and} \quad L_1^{n+1}/v_1 \equiv_{L_1^{n+1}/v_1} K_1/v_1 ,$$

and in view of our induction hypothesis, this yields that

$$v_1(L_1^{n+1}) \equiv v_2(K_2) \quad \text{over} \quad \rho_n\colon v_1(L_1^n) \longrightarrow v_2(L_2^n) , \tag{140}$$

$$L_1^{n+1}/v_1 \equiv K_2/v_2 \quad \text{over} \quad \sigma_n\colon L_1^n/v_1 \longrightarrow L_2^n/v_2 . \tag{141}$$

We note that since (K_2, v_2) is κ^+–saturated with $\kappa = |L_2^n|$, it remains κ^+–saturated when we add the elements of L_2^n as constants to the language $\mathcal{L}$ of valued fields; the same holds analogously for $v_2(K_2)$ and K_2/v_2 : we may add the elements of $v_2(L_2^n)$ resp. L_2^n/v_2 to the language of ordered groups resp. fields without loosing the κ^+–saturatedness. Consequently, all the three models are κ^+–universal (cf. [CHK], Theorem 5.1.14, page 221). In view of (140) and (141) this shows the existence of embeddings $\rho_{n+1}\colon v_1(L_1^{n+1}) \longrightarrow v_2(K_2)$ prolongating ρ_n and $\sigma_{n+1}\colon L_1^{n+1}/v_1 \longrightarrow K_2/v_2$ prolongating σ_n. Note that the embeddings ρ_{n+1} and σ_{n+1} can be chosen such that

$$v_1(K_1) \equiv v_2(K_2) \quad \text{over} \quad \rho_{n+1}\colon v_1(L_1^{n+1}) \longrightarrow v_2(K_2^{n+1}) ,$$

$$K_1/v_1 \equiv K_2/v_2 \quad \text{over} \quad \sigma_{n+1}\colon L_1^{n+1}/v_1 \longrightarrow K_2^{n+1}/v_2 ;$$

this is actually a consequence of Lemma 5.1.10 of [CHK], page 219.

At this point, let us identify L_1^n with L_2^n through the isomorphism ι_n. In view of Lemma 8.3, we may now apply Embedding Lemma IV which yields the existence of an embedding $\iota_{n+1}\colon (L_1^{n+1}, v_1) \longrightarrow (K_2, v_2)$ prolongating $\iota_n = \mathrm{id}_{L_1^n}$ and inducing ρ_{n+1} and σ_{n+1}. This completes our induction step in the case $n \in 2I\!N$. For $n \in 2I\!N + 1$ we just reverse the direction of the embeddings, i.e. we interchange the indices 1 and 2.

Having constructed the subfields L_i^n of K_i $(i = 1, 2)$ for all natural numbers n, where each (L_1^{2m-1}, v_1) is an elementary submodel of (K_1, v_1) and (L_1^{2m+1}, v_1), and every (L_2^{2m}, v_2) is an elementary submodel of (K_2, v_2) and (L_2^{2m+2}, v_2), for all $m \in I\!N$, $m \geq 1$, we set

$$L_1 = \bigcup_{m \geq 1} L_1^{2m-1} \quad \text{and} \quad L_2 = \bigcup_{m \geq 1} L_2^{2m} .$$

Since the union of an elementary chain of elementary submodels is again an elementary submodel, we have

$$(L_i, v_i) \prec (K_i, v_i) \quad (i = 1, 2)\;.$$

Furthermore, the union of the partial isomorphisms ι_n yields an isomorphism

$$\iota\colon (L_1, v_1) \longrightarrow (L_2, v_2)$$

over (K, v). This completes the proof of our theorem. □

From this theorem we derive the following two corollaries:

Corollary 8.11 *If $(L, v)|(K, v)$ is an extension of tame fields with $v(K) \prec v(L)$ and $\overline{K} \prec \overline{L}$, then $(K, v) \prec (L, v)$.*

Proof: This is an immediate consequence of Theorem 8.10 where we set $(K_1, v_1) = (L, v)$ and $(K_2, v_2) = (K, v)$. Note that in view of Lemma 8.3, the condition "$v_1(K_1)/v(K)$ torsion free" and the condition "$(K_1/v_1)|(K/v)$ separable" of Theorem 8.10 are fulfilled by our hypothesis. □

From now on, we will only consider fields of positive characteristic.

Corollary 8.12 *If (K_1, v_1) and (K_2, v_2) are tame fields of positive characteristic with $v_1(K_1) \equiv v_2(K_2)$ and $K_1/v_1 \equiv K_2/v_2$, then*

$$(K_1, v_1) \equiv (K_2, v_2)\;.$$

Proof: By hypothesis, $\mathrm{char}(K_i) = \mathrm{char}(K_i/v_i)$, $i = 1, 2$. Furthermore, $K_1/v_1 \equiv K_2/v_2$ yields $\mathrm{char}(K_1/v_1) = \mathrm{char}(K_2/v_2)$. Hence $\mathrm{char}(K_1) = \mathrm{char}(K_2)$. To apply Theorem 8.10, we set K equal to $\mathbb{F}_p$ which is a subfield of both K_1 and K_2. Note that $\mathbb{F}_p$ together with the trivial valuation is a tame field. The restrictions of v_1 and of v_2 to $\mathbb{F}_p$ are both trivial. This shows that they coincide on $\mathbb{F}_p$, i.e. $v_1(\mathbb{F}_p) = \{0\}$ and $\mathbb{F}_p/v_1 = \mathbb{F}_p$. Consequently, $v_1(K_1)/v_1(\mathbb{F}_p)$ is torsion free and $(K_1/v_1)|(\mathbb{F}_p/v_1)$ is separable and moreover, our hypothesis $v_1(K_1) \equiv v_2(K_2)$ and $K_1/v_1 \equiv K_2/v_2$ implies $v_1(K_1) \equiv_{v_1(\mathbb{F}_p)} v_2(K_2)$ and $K_1/v_1 \equiv_{\mathbb{F}_p/v_1} K_2/v_2$. Now the assertion of our corollary follows from Theorem 8.10. □

In view of Lemma 6.3, as an immediate consequence of the preceding corollary we get the following criterion for decidability:

Corollary 8.13 *Let (K, v) be a tame field of positive characteristic and let $\mathbf{T}$ be its theory. If the theories $v(\mathbf{T})$ of its value group $v(K)$ and $\overline{\mathbf{T}}$ of its residue field $\overline{K}$ both admit recursive first order axiomatizations then so does $\mathbf{T}$ and is thus decidable.*

The following corollary is a consequence of Corollary 8.11 and Corollary 8.12:

Corollary 8.14 *Let $\mathbf{T}$ be an elementary theory of perfect fields of positive characteristic, given by the axiom*
1) "K is a perfect valued field of characteristic $p > 0$"
and optional axioms
2) on the value group of K,

3) on the residue field of K.
Let $\mathcal{K}$ be the elementary class defined by $\mathbf{T}$ and assume that $v(\mathcal{K})$ and $\overline{\mathcal{K}}$ are model complete elementary classes. Then the theory $\mathbf{T}^*$ of algebraically maximal valued fields satisfying $\mathbf{T}$ is the model companion of $\mathbf{T}$.

Proof: It follows from Corollary 8.11 that $\mathbf{T}^*$ is model complete. For every model K of $\mathbf{T}$, any maximal immediate algebraic extension is a model of $\mathbf{T}^*$ (by Lemma 6.2); note that it is an extension of K having the same value group and residue field. □

If we work in a three–sorted language and use the notion of "K^*–model completion" in the sense of Ziegler [ZIE] (cf. his introduction) then we can state what we have proved as follows: the theory $\mathbf{T}^*$ of all algebraically maximal perfect fields of positive characteristic is the model companion of the theory $\mathbf{T}$ of all perfect valued fields of positive characteristic. Note that it is not a model completion since there exist perfect valued fields of positive characteristic which admit two nonisomorphic maximal immediate algebraic extension, both being models of the model companion.

Elementary classes of tame fields of positive characteristic admit prime models relative to prime models of the elementary classes of their value groups and their residue fields:

Theorem 8.15 *Elementary classes $\mathcal{K}$ of tame fields of positive characteristic admit prime models in the following sense:*
If there exists a cardinal κ, a model $\Gamma \in v(\mathcal{K})$ and a model $k \in \overline{\mathcal{K}}$, both of cardinality at most κ, such that Γ is elementarily embeddable into every κ^+–saturated model in $v(\mathcal{K})$ and k is elementarily embeddable into every κ^+–saturated model in $\overline{\mathcal{K}}$, then there exists a model (K,v) of $\mathcal{K}$ of cardinality at most κ, having value group Γ and residue field k, such that (K,v) is elementarily embeddable into every κ^+–saturated model of $\mathcal{K}$. Moreover, we may assume that (K,v) admits a valuation transcendence basis over its prime field.

Proof: By Lemma 6.4, there exists a tame field (K,v) of cardinality at most κ having value group Γ and residue field k and admitting a valuation transcendence basis over its prime field. If $(K^\star,v^\star)$ is a κ^+–saturated model of $\mathcal{K}$ then $v^\star(K^\star)$ and $K^\star/v^\star$ are κ^+–saturated models of $v(\mathcal{K})$ and $\overline{\mathcal{K}}$ respectively. Hence by hypothesis, there exists an elementary embedding of Γ into $v^\star(K^\star)$ (over the group $\{0\}$), and an elementary embedding of k into $K^\star/v^\star$ (over the prime field k). The valuation v is trivial on the prime field $\mathbb{F}_p$ of K, so $v(\mathbb{F}_p) = \{0\}$ and $\overline{\mathbb{F}_p} = \mathbb{F}_p$ which shows that $v(K)/v(\mathbb{F}_p)$ is torsion free and $\overline{K}|\overline{\mathbb{F}_p}$ is separable. Now Embedding Lemma IV shows the existence of an embedding of (K,v) into $(K^\star,v^\star)$ (over the trivially valued field $\mathbb{F}_p$). By virtue of Theorem 8.11, this embedding is elementary. This shows that (K,v) is elementarily embeddable into every κ^+–saturated model of $\mathcal{K}$ and this in turn shows that (K,v) is a model of $\mathcal{K}$. □

The prime models that we have constructed in the foregoing proof have the special property that they admit a valuation transcendence basis over their prime field. The following Corollary confirms the representative role that is played by models which have this property.

Corollary 8.16 *For every tame field (L,v) of arbitrary characteristic, there exists a subfield $(K,v) \prec (L,v)$ such that (K,v) admits a valuation transcendence basis over its prime field and $(L,v)|(K,v)$ is immediate.*

Proof: According to Lemma 6.8, for every tame field (L,v) there exists a subfield (K,v) of (L,v) admitting a valuation transcendence basis over its prime field, such that $(L,v)|(K,v)$ is immediate. In view of Corollary 8.11, the latter fact shows $(K,v) \prec (L,v)$. □

As a final example, we want to treat here the theory of tame fields of positive characteristic with divisible value groups and fixed finite residue field:

Corollary 8.17 *The theory* $\mathbf{T}_q$ *of tame fields of positive characteristic with divisible value group and fixed residue field* $\mathbb{F}_q$ $(q = p^r)$ *is model complete, complete and decidable. Moreover, it possesses a model having transcendence degree 1 over* $\mathbb{F}_q$ *which admits an elementary embedding into every* $\aleph_1$*–saturated model.*

Proof: Since the theory of divisible ordered abelian groups is model complete, complete and decidable, and since the same holds trivially for the theory of the finite field $\mathbb{F}_q$ which has only $\mathbb{F}_q$ as a model, model completeness and completeness follow readily from Corollary 8.11, Corollary 8.12 and Corollary 8.13. The prime model is constructed as follows:
Let $\mathbb{F}_q(t)$ be valued such that $v(t) = 1$. $\sqrt{\mathbb{F}_q(t)}$ admits a unique prolongation v of this valuation, and its residue field must be a purely inseparable algebraic extension of $\mathbb{F}_q$, hence it is equal to $\mathbb{F}_q$. Now we take (K_0, v) to be a maximal algebraic extension of $(\sqrt{\mathbb{F}_q(t)}, v)$ still having residue field $\mathbb{F}_q$. In particular, (K_0, v) will have divisible value group (which consequently is equal to $\mathbb{Q}$) and will be algebraically maximal. Then by Lemma 6.2, (K_0, v) is a tame field; moreover it admits $\{t\}$ as a valuation transcendence basis. Note that $|K_0| = \aleph_0$. Now it can be proved as in the proof of Theorem 8.15 that (K_0, v) admits an elementary embedding into every $\aleph_1$–saturated model of $\mathbf{T}_q$. □

8.3 Model theoretic results for separable–algebraically complete and separably tame fields.

We will first deal with the completion of a valued field.

Theorem 8.18 *Let* (K,v) *be a henselian field. Assume that* $(L,v)|(K,v)$ *is a separable subextension of* $(K^c,v)|(K,v)$*. Then* (K,v) *is existentially closed in* (L,v)*. In particular, every algebraically complete field and more generally, every inseparably defectless field is existentially closed in its completion.*

Proof: It suffices to show that (K,v) is existentially closed in every subfield (F,v) of (L,v) which is finitely generated over K. Equivalently, it suffices to show that (K,v) is existentially closed in $(F,v)^h$ (note that $F^h \subset K^c$ since the completion of a henselian field is again henselian; cf. Lemma 5.12). By Theorem 7.5,

$$F^h = K(x_1, \ldots, x_n)^h \;,$$

where $x_1, \ldots, x_n$ is any separating transcendence basis of $F|K$. Let $F_0 = K$ and $F_i = K(x_1, \ldots, x_i)$, $1 \leq i \leq n$, where the henselization is taken within F^h. Now it suffices to show $(F_{i-1},v)^h \prec_\exists (F_i,v)^h$ for $1 \le i \le n$ (observe that $F_0^h = K$). As x_i is an element of the completion K^c of $(F_{i-1},v)^h$ and is transcendental over F_{i-1}^h, its approximation type over F_{i-1}^h must be transcendental too (being an element of the completion, x_i is uniquely determined by its approximation type and consequently, there cannot exist an algebraic element with the same approximation type). Hence by Corollary 8.6, $(F_{i-1},v)^h \prec_\exists (F_{i-1}^h(x_i),v)^h$ for $1 \le i \le n$, which in view of $(F_{i-1}^h(x_i),v)^h = (F_{i-1}(x_i),v)^h = (F_i,v)^h$ proves our assertion.

The second assertion of our theorem follows from the first and the fact that if (K,v) is algebraically complete or at least inseparably defectless, then $K^c|K$ is a separable extension. The latter is a consequence of Lemma 2.14. $\square$

Now we turn to extensions without transcendence defect and prove an analogue of Theorem 8.4. We do not know whether the cofinality condition may be dropped.

Theorem 8.19 *Let (K,v) be a separable–algebraically complete field and $(L,v)|(K,v)$ a separable extension without transcendence defect. Let $v(K)$ be cofinal in $v(L)$. If $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, then (K,v) is existentially closed in (L,v).*

Proof: The compositum $L.K^c$, taken in the completion L^c, is an immediate extension of L (note that K^c is contained in L^c since $v(K)$ is supposed to be cofinal in $v(L)$). Thus $v(K^c) = v(K) \prec_\exists v(L) = v(L.K^c)$ and $\overline{K^c} = \overline{K} \prec_\exists \overline{L} = \overline{L.K^c}$. By Corollary 4.20 K^c is an algebraically complete field. By Theorem 8.4, it follows from our side conditions that

$$(K^c,v) \prec_\exists (L.K^c,v)\ .$$

Let us now take a $|L.K^c|^+$–saturated elementary extension

$$(K^c,v)^*|(K,v)^*$$

of the valued field extension $(K^c,v)|(K,v)$ (where the subfield K is given by a predicate added to the language of valued fields and where K^* is the range of this predicate within $(K^c)^*$). We note that $(K^c,v)^*$ is a subfield of the completion of $(K,v)^*$ since the property of K^c to be contained in the completion of K is elementary in the language of valued fields with the added predicate for the subfield:

$$\forall x \forall y \exists z:\ z \in K \wedge (\, y \neq 0 \longrightarrow v(x-z) > v(y)\,)$$

expresses this property.

Since $(K^c,v) \prec_\exists (L.K^c,v)$, Lemma 8.1 shows that $(L.K^c,v)$ may be embedded over (K^c,v) into $(K^c,v)^*$ and may thus be considered to be a subfield of the completion of $(K,v)^*$; by this embedding, the latter also holds for the smaller field $(L.K^*,v)$, which is a separable extension of K^*. Theorem 8.18 now shows

$$(K,v)^* \prec_\exists (L.K^*,v)\ .$$

Since $(K,v) \prec (K,v)^*$, we obtain $(K,v) \prec_\exists (L.K^*,v)$ and consequently also $(K,v) \prec_\exists (L,v)$, as asserted. $\square$

Theorem 8.20 *Let (K,v) be a separably tame field and $(L,v)|(K,v)$ a separable extension. If $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, then $(K,v) \prec_\exists (L,v)$.*

Proof: The perfect hull $\sqrt{K}$ of K admits a unique prolongation v of the valuation of K, and with this valuation it is a subextension of the completion of K, according to Lemma 6.13. By Lemma 6.14, $(\sqrt{K}, v)$ is a tame field. Both $\sqrt{K}$ and $L.\sqrt{K}$ are valued subfields of the perfect hull $(\sqrt{L}, v)$ of (L, v) whose value group is the p–divisible hull of $v(L)$ and whose residue field is the perfect hull of $\overline{L}$. Now $v(\sqrt{K}) = v(K)$ is p–divisible and $\overline{\sqrt{K}} = \overline{K}$ is perfect, hence by our side conditions, we obtain $v(\sqrt{K}) \prec_\exists v(L.\sqrt{K})$ and $\overline{\sqrt{K}} \prec_\exists \overline{L.\sqrt{K}}$ (cf. the proof of Lemma 6.5). According to the Ax–Kochen–Ershov–principle for tame fields (Theorem 8.9), this yields

$$(\sqrt{K}, v) \prec_\exists (L.\sqrt{K}, v) .$$

Let us now take a $|L.\sqrt{K}|^+$–saturated elementary extension

$$(\sqrt{K}, v)^* | (K, v)^*$$

of the valued field extension $(\sqrt{K}, v)|(K, v)$. From now on, the proof is an analogue of the proof of Theorem 8.19. □

8.4 A criterion for Ax–Kochen–Ershov–classes.

Recall the definition of AKE–classes given on page 151. Let $\mathcal{K}$ be an elementary class of valued fields. We will call it a *weak AKE–class* if the implication (138) does hold whenever (K, v) **and** (L, v) are members of $\mathcal{K}$.

Theorem 8.21 *$\mathcal{K}$ is a weak AKE–class if*
a) every field in $\mathcal{K}$ is algebraically complete,
b) given $(L, v) \in \mathcal{K}$ and K' relatively algebraically closed in L such that $\overline{L}|\overline{K'}$ is algebraic and $v(L)/v(K')$ is a torsion group, then $(K', v) \in \mathcal{K}$ with $\overline{L} = \overline{K'}$ and $v(L) = v(K')$,
c) if $(K, v) \in \mathcal{K}$, then every immediate henselian function field of transcendence degree 1 *over (K, v) is henselian rational.*

A weak AKE–class $\mathcal{K}$ is an AKE–class, if the following condition is satisfied:
d) given $(K, v) \in \mathcal{K}$ and some extension (L, v) of (K, v) with $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$, then there exists an extension $(L', v) \in \mathcal{K}$ of (L, v) with $v(K) \prec_\exists v(L')$ and $\overline{K} \prec_\exists \overline{L'}$.

Proof: Let $\mathcal{K}$ satisfy a), b), c). Given (K, v), $(L, v) \in \mathcal{K}$ with $(K, v) \subset (L, v)$ and $v(K) \prec_\exists v(L) \wedge \overline{K} \prec_\exists \overline{L}$, we want to show that $(K, v) \prec_\exists (L, v)$.

Take $\mathcal{T}$ to be a maximal set of algebraically valuation–independent elements over K in L. With this choice, $v(L)/v(K(\mathcal{T}))$ is a torsion group and $\overline{L}|\overline{K(\mathcal{T})}$ is algebraic. Let K' be the relative algebraic closure of $K(\mathcal{T})$ within L. Then by condition b), we have $(K', v) \in \mathcal{K}$ with $\overline{L} = \overline{K'}$ and $v(L) = v(K')$ which shows that the extension $L|K'$ is immediate. On the other hand, $\mathcal{T}$ is a valuation transcendence basis of $K'|K$ by construction which shows that according to Lemma 2.21, this extension has no transcendence defect. Since (K, v) is algebraically complete by condition a), Theorem 8.4 now shows $(K, v) \prec_\exists (K', v)$, hence it remains to prove $(K', v) \prec_\exists (L, v)$; in view of $(K', v) \in \mathcal{K}$ we may change notation and assume that we have to prove $(K, v) \prec_\exists (L, v)$ under the additional assumption that $(L, v)|(K, v)$ be immediate.

By virtue of Lemma 8.1, we only have to show that $(K,v) \prec_\exists (L_0,v)$ for every subextension L_0 of $L|K$ which is finitely generated over K; but $(K,v) \prec_\exists (L_0,v)$ is true if $(K,v) \prec_\exists (K',v)$ where K' is the relative algebraic closure of L_0 in L. Again by b), $(K',v) \in \mathcal{K}$. Since L_0 was finitely generated over K, the extension $K'|K$ has a finite transcendence basis $\{t_1,\ldots,t_n\}$. Let us put $K_0 = K$ and K_i to be the relative algebraic closure of $K(t_1,\ldots,t_i)$ in L for $1 \leq i \leq n$; then $K_n = K'$ and by condition b), every (K_i,v) is a member of $\mathcal{K}$. Moreover, $\mathrm{trdeg}(K_{i+1}|K_i) = 1$ for $0 \leq i < n$. Hence it now remains to show that a field $(K,v) \in \mathcal{K}$ is existentially closed in every immediate extension of transcendence degree 1. So we may again change notation and assume that we have to prove $(K,v) \prec_\exists (L,v)$ under the additional assumption that $(L,v)|(K,v)$ be immediate and of transcendence degree 1.

Again by virtue of Lemma 8.1, we only have to show that $(K,v) \prec_\exists (L_0,v)$ for every subextension L_0 of $L|K$ which is finitely generated over K; but $(K,v) \prec_\exists (L_0,v)$ is true if $(K,v) \prec_\exists (L_0,v)^h$. The latter is an immediate henselian function field of transcendence degree 1 over (K,v), so by condition c), it is henselian rational. Since (K,v) is algebraically complete by condition a), Corollary 8.6 now yields $(K,v) \prec_\exists (L_0,v)^h$. This completes the proof of the first part of our theorem.

Now suppose that $\mathcal{K}$ is a weak AKE–class satisfying condition d). Let be given an extension $(L,v)|(K,v)$ with $(K,v) \in \mathcal{K}$ such that $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$. By condition d), there is an extension $(L',v) \in \mathcal{K}$ of (L,v) with $v(K) \prec_\exists v(L')$ and $\overline{K} \prec_\exists \overline{L'}$. By what we have already shown, $(K,v) \prec_\exists (L',v)$ and consequently also $(K,v) \prec_\exists (L,v)$. □

Note that condition a) is necessary for an AKE–class; this is seen as follows: every member (K,v) of an AKE–class must be existentially closed in every maximal immediate extension since the side conditions are trivially fulfilled; on the other hand, a field which is existentially closed in some maximal immediate extension must be algebraically complete (cf. Lemma 10.2 below). We do not know whether condition a) is necessary for weak AKE–classes (but it does not seem likely): a model complete class of nonhenselian fields would be a counterexample, but we do not know how to construct such classes.

Condition b) may be reasonably weakened:

Lemma 8.22 *The above criterion remains true if condition b) is restricted to immediate extensions, i.e. to the case where $\overline{K'} = \overline{L}$ and $v(K') = v(L)$ (we will denote this version by b_i)), and one of the following conditions is added:*
b') given an extension $(L,v)|(K,v)$ with (K,v), $(L,v) \in \mathcal{K}$ such that $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$ then there exists an elementary extension (L',v) of (L,v) and an intermediate field $(K',v) \in \mathcal{K}$ such that $(L',v)|(K',v)$ is an immediate extension and $(K',v)|(K,v)$ is an extension without transcendence defect, or
b") given an extension $(L,v)|(K,v)$ with (K,v), $(L,v) \in \mathcal{K}$ such that $v(K) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$ then there exists an elementary extension

$$(K,v)^* \subset (L,v)^* \;=\; (\,(K,v) \subset (L,v)\,)^*$$

of the pair of models $(K,v) \subset (L,v)$ which admits an intermediate field $(K',v) \in \mathcal{K}$ such that $(L,v)^*|(K',v)$ is an immediate extension and $(K',v)|(K,v)^*$ is an extension without transcendence defect.

Proof: The restricted version of b) still serves to split up immediate extensions. In the proof of the foregoing lemma, the original version was also used to find the intermediate field (K', v) splitting up the extension into an immediate part and a part without transcendence defect. To find such an intermediate field, we may enlarge (L, v) as long as the side conditions are preserved; but since elementary properties of value groups and residue fields can be encoded in the valued fields themselves, $(L, v) \prec (L', v)$ implies $v(L) \prec v(L')$ and $\overline{L} \prec \overline{L'}$ so that the original side conditions will imply that also $v(K) \prec_\exists v(L')$ and $\overline{K} \prec_\exists \overline{L'}$. Hence b) may be replaced by its restricted version together with b').

We will show the same for b") in the place of b'). Note that $(K, v)^*$ and $(L, v)^*$ are elementary extensions of (K, v) and (L, v); hence they are also members of the elementary class $\mathcal{K}$. Consequently, conditions a), b") and c) will yield that the extension $(L, v)^*|(K, v)^*$ satisfies the AKE–principle (138). Now in the language of pairs of models, the property $\mathcal{M} \prec_\exists \mathcal{N}$ is elementary; moreover, $v(k) \prec_\exists v(L)$ and $\overline{K} \prec_\exists \overline{L}$ are also elementary properties of the pair $(K, v) \subset (L, v)$. This shows: if the extension $(L, v)|(K, v)$ satisfies the side conditions, then so does the extension $(L, v)^*|(K, v)^*$ which yields $(K, v)^* \prec_\exists (L, v)^*$; this in turn gives $(K, v) \prec_\exists (L, v)$. □

Now we will apply the foregoing criteria to well known elementary classes of valued fields.

Corollary 8.23 *The following elementary classes are AKE–classes:*
1) henselian fields of residue characteristic 0
2) henselian formally $\wp$–adic fields
3) henselian finitely ramified fields
4) algebraically closed valued fields
5) algebraically maximal Kaplansky fields
6) algebraically maximal perfect fields
7) tame fields.

Note that the class of tame fields contains all fields mentioned under 1), 4), 5) and 6) (cf. the Introduction). The tame fields constitute one branch of valued fields which admit a good model theory; they are too big to cause (too hard) problems. On the other hand, the class of henselian finitely ramified fields which includes all henselian formally $\wp$–adic fields, constitutes another such branch; here, the fields are too small to cause problems. We will now prove the above corollary by considering the conditions of our criteria for both classes.

For tame fields, condition a) follows from Lemma 6.2, and condition b) is a consequence of the more general Lemma 6.6. Condition c) is a special case of our main theorem on henselian function fields over tame fields (Theorem 7.1), and condition d) follows from Lemma 6.5.

Now let us consider finitely ramified fields (K, v). If P is the place associated to v, then it admits a decomposition $P = Q\overline{Q}$ (where Q may be trivial) such that $\mathrm{char}(KQ) = 0$ and $\overline{Q}$ is a place on KQ with value group $\mathbb{Z}$ and residue characteristic $p > 0$. By the Lemma of Ostrowski, (K, Q) is a defectless field. Let us show that also $(KQ, \overline{Q})$ is a defectless field; then it follows by Lemma 2.17 that every finitely ramified field is a defectless field and hence every henselian finitely ramified field satisfies condition a). First we note that by Lemma 7.4, a henselian field of characteristic 0 with value group isomorphic to $\mathbb{Z}$ must be algebraically maximal. Let $(L,\overline{Q})$ be a finite extension of $(KQ,\overline{Q})^h$. By Lemma 3.15, there is a finite tame extension $N|KQ^h$ such that the extension $L.N|N$ is a tower of extensions of prime characteristic; but all the intermediate fields of this tower, as well as N itself, are henselian fields of characteristic 0 with value group isomorphic to $\mathbb{Z}$, hence no extension in the tower can be immediate and since they are of prime characteristic, they are all defectless. Consequently, in view of the multiplicativity of the defect, the extension $L.N|N$ is defectless. But by Lemma 2.11, it has the same defect as the extension $(L,\overline{Q})|(KQ,\overline{Q})^h$ which proves this extension to be defectless and thereby $(KQ,\overline{Q})$ to be a defectless field.

Postponing condition b) and skipping condition c) which was already shown in the introduction to be true, we want to show now that condition d) is satisfied. Let $(L,v)|(K,v)$ be an extension of a finitely ramified field (K,v) with $v(K)\prec_\exists v(L)$. The latter shows that the smallest positive element in $v(K)$ remains the smallest positive element in $v(L)$, and this shows that also (L,v) is finitely ramified. The fact that $(L,v)^h$ is then a henselian finitely ramified field with $v(L^h)=v(L)$ and $\overline{L^h}=\overline{L}$ now proves condition d) to be true. In view of the fact that a relatively algebraically closed subfield of a henselian field is again henselian, these considerations also show that the restricted version b$_i$) holds for henselian finitely ramified fields. Note that b) is true for henselian formally $\wp$–adic fields (the proof is left to the reader). However, b) does not hold for the class of henselian finitely ramified fields; since the construction of an example is quite tedious, we will not give it here.

As a conclusion of this section, we will now prove that b') holds for the class of henselian finitely ramified fields, thereby completing the proof of our corollary. Nevertheless, the result is much more general:

Lemma 8.24 *Condition b') holds for the class of all henselian fields and also for the class of henselian finitely ramified fields.*

Proof: We put $K_0=K$ and $L_0=L$ and define valued fields (K_i,v) and (L_i,v) by induction on i as follows. Let (K_{i+1},v) be a henselian extension without transcendence defect of (K_i,v) such that $v(K_{i+1})=v(L_i)$ and $\overline{K_{i+1}}=\overline{L_i}$; the construction of such extension fields is straightforward. Furthermore, let (L_{i+1},v) be a $|L_i|^+$–saturated elementary extension of (L_i,v). Excluding a trivial case, we may assume that L is infinite; again, it is straightforward to show that with this assumption, $|K_{i+1}|\leq|L_i|$ for every $i\geq 0$. Being elementary extensions of (L,v), all (L_i,v) are henselian fields. We may now deduce from Lemma 8.2 that there is an embedding of (K_1,v) into (L_1,v) over (K,v) which respects the trivial embeddings $v(L)\subset v(L_1)$ and $\overline{L}\subset\overline{L_1}$, and by induction on i we get in the same way embeddings (K_{1+1},v) into (L_{i+1},v) prolongating the embedding of (K_i,v) into (L_i,v) and respecting the trivial embeddings $v(L_i)\subset v(L_{i+1})$ and $\overline{L_i}\subset\overline{L_{i+1}}$. The conditions on the value groups and residue fields of Lemma 8.2 are fulfilled by virtue of Lemma 8.3 and the fact that

$$v(K_i)\prec_\exists v(L_i)=v(K_{i+1})\ \text{ and }\ \overline{K_i}\prec_\exists\overline{L_i}=\overline{K_{i+1}}\ ;$$

for $i=1$, this follows from our hypothesis that $v(K)\prec_\exists v(L)$ and $\overline{K}\prec_\exists\overline{L}$, and for $i>0$, it follows from $(L_{i-1},v)\prec(L_i,v)$.

Now we define (L',v) and (K',v) to be the union over all (L_i,v) resp. (K_i,v); since the union of an elementary chain is an elementary extension of every member of the chain, we have $(L,v)\prec(L',v)$. Moreover, the union over all embeddings gives an embedding of (K',v) into (L',v). We identify (K',v) with its image under that embedding. Then we have that by our construction, $v(K')=v(L')$ and $\overline{K'}=\overline{L'}$, hence $(L',v)|(K',v)$ is an immediate extension, and again by our construction, $(K', v)|(K, v)$ is an extension without transcendence defect. Finally, as a union over henselian fields, (K', v) is also henselian.

For the class of henselian finitely ramified fields, we do the same construction. Then (K', v) is a finitely ramified field because of $v(K) \prec_\exists v(K')$. □

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9 An application: Places of algebraic function fields over perfect fields.

In this section we want to give an application of Theorem 8.9. The fact that tame fields are AKE–fields will serve to deduce a generalization of two results that are proved in [KP] by means of the AKE–principle for henselian fields of residue characteristic 0.
The following theorem generalizes the Main Theorem of [KP]:

Theorem 9.1 *Let $F|k$ be a function field in n variables with perfect ground field k. Let Q be a place of $F|k$ and*

$$x_1, ..., x_m, x_{m+1}, ..., x_{m+s} \in F \; .$$

Then there exists a place P of $F|k$ with a finitely generated residue field over k such that

$$\begin{array}{rcll} x_iQ & = & x_iP & \text{for} \quad 1 \le i \le m \; , \\ v_Q(x_i) & = & v_P(x_i) & \text{for} \quad m+1 \le i \le m+s \; . \end{array}$$

Moreover, if r_1 and d_1 are natural numbers satisfying

$$\dim(Q) \le d_1 \le n-1 \; , \qquad \mathrm{rr}(Q) \le r_1 \le n - d_1 \; ,$$

then P may be chosen to satisfy in addition:

(1) $\dim(P) = d_1$ and FP is a subfield of a purely transcendental extension of the perfect hull of FQ, finitely generated over k,

(2) $\mathrm{rr}(P) = r_1$ and $v_P(F)$ is a finitely generated subgroup of a discrete lexicographic extension of the p–divisible hull of $v_Q(F)$, where $p = \mathrm{char}(k) > 0$ or $p = 1$ if $\mathrm{char}(k) = 0$.

Proof: The case $\mathrm{char}(k) = 0$ is proved in [KP]. Hence we will assume $\mathrm{char}(k) = p > 0$ throughout this proof. We use the notation of [KP].
Let $d = \dim(Q)$. We choose $u_1, ..., u_d \in F$ such that $\overline{u_1}, ..., \overline{u_d}$ form a transcendence basis of $FQ|k$. Let $r = \mathrm{rr}(Q)$. We choose $z_1, ..., z_r \in F$ such that the values $v_Q(z_1), ..., v_Q(z_r)$ form a maximal set of rationally independent elements in $v_Q(F)$. According Lemma 2.19, the elements

$$u_1, ..., u_d, z_1, ..., z_r$$

are algebraically independent over k.
Now let $(\sqrt{F}, Q)$ be the perfect hull of (F, Q) and let (L, Q) be a maximal immediate algebraic extension of (F, Q). By virtue of Lemma 6.2, (L, Q) is a tame field. $v_Q(L)$ is the p–divisible hull of $v_Q(F)$, and LQ is the perfect hull of FQ.
Let K' be the relative algebraic closure of $k(u_1, ..., u_d, z_1, ..., z_r)$ in L, and let Q' be the restriction of Q to K'. According to Lemma 6.2, (K', Q') is a tame field with

$$K'Q' = LQ \quad \text{and} \quad v_{Q'}(K') = v_Q(L) \; .$$

Hence (K', Q') is existentially closed in (L, Q) by Theorem 8.9.
We write $K'.F = K'(t_1, ..., t_{n'}, y)$, where $n' = n - (d + r)$ and $t_1, ..., t_{n'}$ are algebraically independent over K' (since K' is perfect, $K'.F$ is separably generated over K', hence it suffices to take one element y which is algebraic over $K'(t_1, ..., t_{n'})$). Let $f \in K'[T_1, ..., T_{n'}, Y]$

be irreducible and normed in Y such that $f(\underline{t}, y) = 0$. We choose $x'_1, ..., x'_{m+s} \in K'$ such that

$$\begin{array}{rcll} x'_i Q & = & x_i Q & \text{for} \quad 1 \le i \le m \;, \\ v_Q(x'_i) & = & v_Q(x_i) & \text{for} \quad m+1 \le i \le m+s \;. \end{array}$$

We write the elements x_i as follows:

$$x_i = \frac{g_i(\underline{t}, y)}{h_i(\underline{t})} \quad \text{for} \quad 1 \le i \le m+s \;,$$

where g_i and h_i are polynomials over K'. Since (K', Q') is existentially closed in (L, Q), there exist elements

$$t'_1, ..., t'_{n'}, y' \in K'$$

such that

(1) $f(\underline{t'}, y') = 0$ and $\dfrac{\partial f}{\partial Y}(\underline{t'}, y') \neq 0$,

(2) $h_i(\underline{t'}) \neq 0$ for $1 \le i \le m+s$,

(3) $\dfrac{g_i(\underline{t'}, y')}{h_i(\underline{t'})} Q = x'_i Q$ for $1 \le i \le m$,

(4) $v_Q\left(\dfrac{g_i(\underline{t'}, y')}{h_i(\underline{t'})}\right) = v_Q(x'_i)$ for $m+1 \le i \le m+s$,

since these assertions are true in L for $\underline{t}, y$ in the place of $\underline{t'}, y'$.

Now let K_1 be the subfield of K' which is generated over k by the following elements:

— $u_1, ..., u_d, z_1, ..., z_r$,
— $x'_1, ..., x'_{m+s}, t'_1, ..., t'_{n'}, y'$,
— the coefficients of f, g_i and h_i for $1 \le i \le m+s$.

K_1 is a finite extension of $k(u_1, ..., u_d, z_1, ..., z_r)$. Hence according to Lemma 2.20, $v_Q(K_1)$ is a finitely generated subgroup of $v_Q(L)$ of rational rank r, and $K_1 Q$ is a subfield of LQ and finitely generated of transcendence degree d over k. Let P_1 be the restriction of Q to K_1.

As in [KP] one constructs now a suitable extension (K, P) of (K_1, P_1): the extensions (K_2, P_2), (K_3, P_3) and (K_4, P_4) are constructed like in [KP]; furthermore, it suffices also here to take (K, P) equal to the henselization of (K_4, P_4) since in the sequel only the Implicit Function Theorem is required which holds in every henselian field according to [PZ]. The remainder of the proof is like in [KP]. □

We have seen that the proof for the case of positive characteristic differs from the proof for characteristic 0 only in so far as we have to extend (F, Q) by passing to a maximal immediate algebraic extension of the perfect hull of F; we are doing nothing else in the case of characteristic 0, but in the case of positive characteristic this procedure may enlarge value group and residue field. This leads to the revised formulation of the Main Theorem which nevertheless comprises the Main Lemma of [KP] as a special case and seems to be the "natural" generalization of the theorem in view of the fact that nonperfect algebraically complete fields are not in general AKE–fields, as we will show in the next section. Moreover it cannot be expected that Lemma 6.6 which we have used at a crucial point in our proof, remains true for suitable classes of nonperfect AKE–fields. Furthermore it is also a nontrivial problem whether for such fields algebraically complete algebraic extension exist which are not "too large", i.e. which do not extend value group and residue field too much. Probably it is not possible to dispense with the enlargement of value group and residue field that we took into the bargain here.

Our following supplement to Theorem 3 of [KP] shows again the consequences of this enlargement. Instead of giving the obvious reformulation of this Theorem here which would assume $(l,q)=1$ *and* $(l,p)=1$ and which would thus comprise the original Theorem as the special case where $p=1$, it is more convenient to formulate the following supplement since in the case of positive characteristic $p>0$ it suffices to choose always $q=p$.

Theorem 9.2 *Let Q be a place of $F|k$ and let $x\in F$ be nonzero. Then there exists a place P of $F|k$ with $v_P(F)=\mathbb{Z}$ and FP being a subfield of the perfect hull of FQ, finitely generated over k, such that for every integer l the following holds: if l is prime to p and does not divide $v_Q(x)$ in $v_Q(F)$, then l does also not divide $v_P(x)$ in $v_P(F)$.*

Proof: Let $m=\dim(Q)$. We choose $u_1,...,u_m$ like in the foregoing proof and put

$$x_1=u_1,\ \ .\ .\ .\ ,\ x_m=u_m,\ x_{m+1}=x\ .$$

We proceed now like in the modification on p. 188 in [KP]. If $v_Q(x)\neq 0$ we put $z_1=x$, otherwise we choose an arbitrary $z_1\in F$ with $v_Q(z_1)\neq 0$. As in the foregoing proof we choose the tame field (L,Q) to be a maximal immediate algebraic extension of the perfect hull $(\sqrt{F},Q)$ of (F,Q). Let K' be the relative algebraic closure of $k(u_1,...,u_d,z_1)$ in L; by virtue of Lemma 6.6, K' is a tame field with $K'Q=LQ$ and $v_Q(K')$ pure in $v_Q(L)$. Note that $v_Q(K')$ is a subgroup of $\mathbb{Q}v(z_1)$ and thus archimedean. On the other hand, $v_Q(K')$ is dense since it is p–divisible. It follows that $v_Q(K')$ is dense regular. By Corollary 3 of [WEI5], this and the fact that $v_Q(K')$ is pure in $v_Q(L)$ implies that $v_Q(K')$ is existentially closed in $v_Q(L)$. From Theorem 8.9 we infer that (K',Q) is existentially closed in (L,Q).

We proceed like in the foregoing proof. Firstly it follows that $v_Q(K_1)$ is a finitely generated subgroup of $v_Q(K')$, hence isomorphic to $\mathbb{Z}$. In the sequel we put $r_1=1$, so we do not enlarge the value group. K_1Q is a subfield of $LQ=\sqrt{F}Q=\sqrt{FQ}$, finitely generated of transcendence degree $d=\mathrm{trdeg}(FQ|k)$ over k. We also do not enlarge the residue field, i.e. we put $d_1=\dim(Q)$. Hence we obtain a place P of F such that $FP\subseteq K_1Q$ and $v_P(F)\subseteq v_Q(K_1)$ have the same properties as K_1Q and $v_Q(K_1)$. Now if l divides $v_P(x)$ in $v_P(F)$, it also divides it in $v_Q(K_1)$ and henceforth in $v_Q(L)$. Thus if $(p,l)=1$, it also divides $v_Q(x)$ in $v_Q(F)$. □

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10 Beyond perfect fields.

In this section we will show the existence of algebraically complete fields of positive characteristic which are not AKE–fields and even admit an immediate function field of transcendence degree 1 in which they are not existentially closed. This shows that Corollary 8.7 does not hold any more if the condition "tame" is replaced by "algebraically complete". In view of Corollary 8.6, this yields that Theorem 7.1 does not remain true if we omit the condition that the ground field K is perfect, not even in the case of a function field of rank 1. However, this does not prove that a good structure theory for immediate henselian function fields like the one given in Theorem 7.1, is really necessary for a field to satisfy the AKE–principle.

Before proving negative results, we want to state at least one positive result which covers also the case of nonperfect defectless fields of positive characteristic. It is a consequence of what we have shown in section 7.

Lemma 10.1 *Let K be a defectless field of positive characteristic and F an immediate function field of transcendence degree 1 over K. Assume that its rank is finite. Then F^h is a finite immediate separable extension of a suitable henselian rational function field over K and a finite immediate simple purely inseparable extension of another suitable henselian rational function field over K.*

Proof: By Corollary 2.14, F admits a separating transcendence basis $\{z\}$ over the defectless field K. Thus $F|K(z)$ and also $F^h|K(z)^h$ are finite immediate separable extensions. This proves the first part of our assertion.

From Lemma 7.15 we infer that $(F.K^{1/p^\nu})^h$ is a henselian rational function field $K^{1/p^\nu}(x)^h$ for a suitable $\nu \in I\!N$. This implies that $(F^{p^\nu}.K)^h$ is equal to the henselian rational function field $K(x^{p^\nu})^h$. Now for the separating transcendent element z of $F|K$ we know that

$$F = F^{p^\nu}.K(z)\ .$$

This implies

$$F^h = (F^{p^\nu}.K(z))^h = K(x^{p^\nu})^h(z)$$

showing that F is a simple purely inseparable extension of the henselian rational function field $K(x^{p^\nu})^h$ of degree p^ν. Since $F^h|K$ is immediate, this extension must be immediate too. □

At this point, we want to use the occasion to put our considerations in a more general setting. We define the following six basic properties of a valued field (K,v), three of them being of algebraic nature and elementarily definable (cf. [DEL1]), the other three being of model theoretic nature:

- **(Alg 1)**: (K,v) is henselian
- **(Alg 2)**: (K,v) is algebraically maximal
- **(Alg 3)**: (K,v) is algebraically complete
- **(Mod 1)**: there exists at least one maximal immediate extension (M,v) of (K,v) such that $(K,v) \prec_\exists (M,v)$

- **(Mod 2)**: (K,v) is existentially closed in every immediate extension
- **(Mod 3)**: (K,v) is an AKE–field

There is the following general connection between these properties:

Lemma 10.2 *For any valued field K, the following holds:*

$$\textbf{(Mod 3)} \Longrightarrow \textbf{(Mod 2)} \Longrightarrow \textbf{(Mod 1)} \Longrightarrow \\ \Longrightarrow \textbf{(Alg 3)} \Longrightarrow \textbf{(Alg 2)} \Longrightarrow \textbf{(Alg 1)}\ .$$

Proof: First we consider the well known and trivial implications:
(Mod 3) $\Longrightarrow$ (Mod 2): If $L|K$ is an immediate extension then $v(L)=v(K)$ and $\overline{L}=\overline{K}$, so the conditions of the AKE–principle are trivially fulfilled.
(Mod 2) $\Longrightarrow$ (Mod 1): Take any maximal immediate extension of K.
(Alg 3) $\Longrightarrow$ (Alg 2): A finite immediate extension is defectless only if it is trivial.
(Alg 2) $\Longrightarrow$ (Alg 1): The henselization of a field is an immediate algebraic extension.

The only new and nontrivial part of the lemma is the implication **(Mod 1) $\Longrightarrow$ (Alg 3)**. But note that **(Mod 2) $\Longrightarrow$ (Alg 3)** was already shown by Delon in [DEL1]. Let us now assume that (M,v) is a maximal immediate extension of (K,v) and (K,v) is existentially closed in (M,v). A maximal field is algebraically complete, and "algebraically complete" is an elementary property which may be expressed by a scheme of $\forall\exists$–formulas; these are inherited by (K,v) which shows that it is also algebraically complete.

Another proof which does not rely on the axiomatization of "algebraically complete" is the following. Let $(L,v)|(K,v)$ be an arbitrary finite extension. We take

$$(L^*,v^*)|(K^*,v^*)$$

to be a κ–saturated elementary extension of $(L,v)|(K,v)$, i.e. we equip (L,v) with an additional predicate for the subfield K, take (L^*,v^*) to be a κ–saturated elementary extension of it w.r.t. the enlarged language, and we denote by (K^*,v^*) the subfield of (L^*,v^*) indicated by the new predicate. Then (L^*,v^*) and (K^*,v^*) are κ–saturated elementary extensions of (L,v) resp. (K,v). We choose κ such that $\kappa\geq|M|^+$. Since by assumption (K,v) is existentially closed in (M,v), according to Lemma 8.1 we may embed (M,v) over (K,v) into (K^*,v^*), and we identify it with its image in (K^*,v^*). Since $(L^*,v^*)|(K^*,v^*)$ is an elementary extension of $(L,v)|(K,v)$ and $n=[L:K]<\infty$ we have $[L^*:K^*]=n$ and thus also $[L.M:M]=n$.
The extension $(M,v)|(K,v)$ is immediate, and we will prove the same for $(L.M,v^*)|(L,v)$. Since $L.M|M$ is algebraic and $v(M)=v(K)$ we know that $v^*(L.M)/v(K)$ is a torsion group, and hence also $v^*(L.M)/v(L)$. For the same reason $\overline{M}=\overline{K}$ yields that $\overline{L.M}|\overline{K}$ is algebraic, and hence also $\overline{L.M}|\overline{L}$. On the other hand, since (L^*,v^*) is an elementary extension of (L,v) we know by Lemma 8.3 that $v(L)$ is pure in $v^*(L^*)$ and that $\overline{L}$ is relatively algebraically closed in $\overline{L^*}$. Combining these facts we get

$$v^*(L.M)=v(L) \ \text{ and } \ \overline{L.M}=\overline{L}$$

showing that $(L.M,v^*)|(L,v)$ is immediate, as contended.

Now (M,v) is maximally valued and thus a defectless field. Consequently,

$$\begin{aligned}[L:K] &= n = [L.M:M] = [\overline{L.M}:\overline{M}]\cdot(v^*(L.M):v(M)) \\ &= [\overline{L}:\overline{K}]\cdot(v(L):v(K))\end{aligned}$$

which shows that $(L,v)|(K,v)$ is defectless and that the prolongation of the valuation v from K to L is unique. Since (L,v) was an arbitrary finite extension of (K,v), this shows that (K,v) is a henselian defectless field or in other words, (K,v) is an algebraically complete field. □

On the other hand, the theorems of Ax – Kochen, Ershov and Ziegler show that

a) For every valued field K with residue characteristic 0 and for every finitely ramified field K the implication

(Alg 1) $\Longrightarrow$ **(Mod 3)**

holds, hence all stated properties are equivalent for such fields.

b) For every Kaplansky–field K, the implication

(Alg 2) $\Longrightarrow$ **(Mod 3)**

holds, hence all stated properties with the exception of **(Alg 1)** are equivalent for Kaplansky–fields.

Moreover, our investigations have shown

c) For every perfect field K of positive characteristic and more generally, for every valued field whose value group is divisible by its residue characteristic and whose residue field is perfect, the implication

(Alg 2) $\Longrightarrow$ **(Mod 3)**

holds, hence all stated properties with the exception of **(Alg 1)** are equivalent for such fields.

But we will show in this section that the implication

(Alg 3) $\Longrightarrow$ **(Mod1)**

does not hold in general. To this end we will consider fields K such that $K|K^p$ has the valuation basis $1,t,...,t^{p-1}$, and we will deal with the following property:

$$\begin{aligned}&\forall x\exists y\exists x_0,...,x_{p-1}:\ (x_0=0\vee v(x_0)=0)\wedge \\ &\quad\wedge\, x=y^p-y+x_0+tx_1^p+...+t^{p-1}x_{p-1}^p\end{aligned} \tag{142}$$

(which is elementary and contains but one element $t\in K$). In other words, a valued field K satisfies (142) if and only if

$$K=\wp(K)+(\mathcal{O}_K^\times\cup\{0\})+tK^p+...+t^{p-1}K^p\ . \tag{143}$$

In the sequel, we will always work with henselian fields (K,v). For such fields, $\mathcal{M}_K\subset\wp(K)$ by Lemma 3.22, and (143) is equivalent to

$$K=\wp(K)+\mathcal{O}_K+tK^p+...+t^{p-1}K^p\ . \tag{144}$$

If the residue field $\overline{K}$ is closed under Artin–Schreier–extensions, then Lemma 3.22 yields that (144) in turn is equivalent to

$$K = \wp(K) + tK^p + ... + t^{p-1}K^p \tag{145}$$

by virtue of Hensel's Lemma. Note that if $K = K^p$ is perfect then $t \in K$, thus $K = tK^p$ and (142), (143), (144) and (145) are trivially fulfilled.

There are important fields having property (142):

Lemma 10.3 *a) Every maximally valued field M of characteristic $p > 0$ has property (142) if $\{1, t, ..., t^{p-1}\}$ is a valuation basis of $M|M^p$. Hence in particular, the power series field $\mathbb{F}_p((t))$ has the property (142).*
b) If $\mathbb{F}_p(t)$ is valued such that $v(t) = 1$ then $(\mathbb{F}_p(t), v)^h$ has the property (142). More generally, every henselian field K of characteristic $p > 0$ with value group $\mathbb{Z}$ has property (142) if $\{1, t, ..., t^{p-1}\}$ is a valuation basis of $K|K^p$.

Proof: a) Let $x \in M$. We have to show that there exist $y, x_1, ..., x_{p-1} \in M$ such that

$$x - (y^p - y) - tx_1^p - ... - t^{p-1}x_{p-1}^p \in \mathcal{O}_M$$

or equivalently that

$$\delta = \mathrm{dist}_M(x, \wp(M) + tM^p + ... + t^{p-1}M^p) \geq 0\ . \tag{146}$$

The lemma that will follow this proof will tell us that this distance is assumed, i.e. there are elements $y, x_1, ..., x_{p-1} \in M$ such that

$$v(x') = \delta$$

for

$$x' = x - (y^p - y) - tx_1^p - ... - t^{p-1}x_{p-1}^p\ .$$

For any such x', we carry through the following procedure if $v(x') < 0$. We write

$$x' = (x'_0)^p + t(x'_1)^p + ... + t^{p-1}(x'_{p-1})^p$$

and we note that

$$v(x') = \min_i v(t^i x_i^p) \leq pv(x'_0) \tag{147}$$

since $\{1, t, ..., t^{p-1}\}$ is a valuation basis. From $v(x') < 0$, it follows

$$v(x') < v(x'_0) \tag{148}$$

and thus

$$v\left(x' - ((x'_0)^p - x'_0) - t(x'_1)^p - ... - t^{p-1}(x'_{p-1})^p\right) = v(x'_0) > v(x')$$

which yields

$$v(\, x - ((x'_0 + y)^p - (x'_0 + y)) - t(x'_1 + x_1)^p - \ldots - t^{p-1}(x'_{p-1} + x_{p-1})^p\,) > v(x')\ .$$

But in our present case where $v(x') = \delta$, the result of this procedure would be a contradiction to the definition of δ. Thus $\delta = v(x') \geq 0$ which proves part a) of our lemma.

b) Since the value group of K is $\mathbb{Z}$, a finite repetition of the procedure described in the proof of a) will provide for any element $x \in K$ suitable elements $y, x_1, ...x_{p-1} \in K$ such that

$$x - (y^p - y) - tx_1^p - ... - t^{p-1}x_{p-1}^p \in \mathcal{O}_K \ .$$

□

Lemma 10.4 *Let M be any maximally valued field of characteristic $p > 0$. We consider an additive function*

$$\mathcal{F}(X_1, ..., X_n) = t_1\mathcal{F}_1(X_1) + ... + t_n\mathcal{F}_n(X_n)$$

where $t_1, ..., t_n$ are different elements chosen from a fixed valuation basis of an arbitrary finite subextension of $M|M^p$, and the functions $\mathcal{F}_i(X_i)$ are either equal to X_i^p or to $X_i^p - X_i$. Then for every $x \in M$, the value

$$\delta := \sup_{\underline{x} \in M^n} v(x - \mathcal{F}(\underline{x}))$$

is assumed by a suitable $\underline{x} = (x_1, ..., x_n) \in M^n$.

Proof: First we prove:
Assume that for given elements $t, y \in M$ and $z \in \mathcal{O}_M$,

$$v(y - t(z^p - z)) > v(y) \tag{149}$$

Then there exists $\tilde{z} \in M$ such that

$$y - t(\tilde{z}^p - \tilde{z}) = 0 \ .$$

Indeed, inequality (149) implies that

$$v(y/t - (z^p - z)) > v(y/t) \tag{150}$$

and thus

$$v(y/t) = v(z^p - z) \geq 0 \ . \tag{151}$$

As a maximally valued field, M is henselian; hence by Lemma 3.22, (150) and (151) yield the existence of an element $\tilde{z} \in M$ satisfying $\tilde{z}^p - \tilde{z} - y/t = 0$, which proves our assertion.

Now let the situation be as described in the hypothesis of our lemma, and let us assume that there is no $\underline{x} \in M^n$ such that $x = \mathcal{F}(\underline{x})$ since otherwise $\delta = \infty$ is assumed by $\underline{x}$ and our lemma is proved. Then for every $\underline{x} \in M^n$, every i such that $\mathcal{F}_i(X_i) = X_i^p - X_i$ and every $z \in \mathcal{O}_M$ we have

$$v\left(x - \mathcal{F}(\underline{x})\right) = v\left(x - \left(\mathcal{F}(\underline{x}) + t_i\mathcal{F}_i(z)\right)\right) \ . \tag{152}$$

This is seen as follows:
If "<" would hold in (152) we could put $y = x - \mathcal{F}(\underline{x})$ and $t = t_i$ and would get by what we have shown in the beginning, that there is an element $\tilde{z}$ such that $y - t_i\mathcal{F}_i(\tilde{z}) = 0$ and hence $x = \mathcal{F}(\underline{\tilde{x}})$ for $\underline{\tilde{x}} = (x_1, ..., x_{i-1}, x_i + \tilde{z}, x_{i+1}, ..., x_n)$ contrary to our assumption.

If ">" would hold in (152) we could put $y = x - (\mathcal{F}(\underline{x}) + t_i\mathcal{F}_i(z))$ and $t = -t_i$, and could derive a contradiction in the same way.

Now let λ be an ordinal number and $\{\underline{x}^{(\rho)} = (x_1^{(\rho)}, ..., x_n^{(\rho)}) \in M^n\}_{\rho<\lambda}$ a sequence such that

$$\{\delta_\rho = v(x - \mathcal{F}(\underline{x}^{(\rho)}))\}_{\rho<\lambda}$$

is a monotonically increasing sequence of values δ_ρ with supremum δ. (If λ is not a limit ordinal, then there is nothing more to show.) For every i such that $\mathcal{F}_i(X_i) = X_i^p - X_i$, we may assume that $x_i^{(\rho)} = x_i^{(\sigma)}$, whenever $\rho < \sigma < \lambda$ and $v(x_i^{(\rho)} - x_i^{(\sigma)}) \geq 0$. Indeed, otherwise $x_i^{(\sigma)}$ may be replaced by $x_i^{(\rho)}$ because from (152) where we put $z = -x_i^{(\sigma)} + x_i^{(\rho)}$, we deduce that

$$v\left(x - (\mathcal{F}(\underline{x}^{(\sigma)}) - t_i\mathcal{F}_i(x_i^{(\sigma)}) + t_i\mathcal{F}_i(x_i^{(\rho)}))\right) = v\left(x - \mathcal{F}(\underline{x}^{(\sigma)})\right) .$$

Consequently, if $x_i^{(\rho)} \neq x_i^{(\sigma)}$ then we may assume $v(x_i^{(\rho)} - x_i^{(\sigma)}) < 0$ and thus

$$v\left(\mathcal{F}_i(x_i^{(\rho)}) - \mathcal{F}_i(x_i^{(\sigma)})\right) = v\left((x_i^{(\rho)} - x_i^{(\sigma)})^p\right) < v\left(x_i^{(\rho)} - x_i^{(\sigma)}\right) < 0 . \tag{153}$$

Now we compute for $\rho < \sigma < \lambda$:

$$\begin{aligned}\delta_\rho &= \min(\delta_\sigma, \delta_\rho) = v\left((x - \mathcal{F}(\underline{x}^{(\sigma)})) - (x - \mathcal{F}(\underline{x}^{(\rho)}))\right) = v\left(\mathcal{F}(\underline{x}^{(\rho)}) - \mathcal{F}(\underline{x}^{(\sigma)})\right)\\ &= \min_i \left(v(t_i(\mathcal{F}_i(x_i^{(\rho)}) - \mathcal{F}_i(x_i^{(\sigma)}))\right) = \min_i \left(v(t_i(x_i^{(\rho)} - x_i^{(\sigma)})^p)\right) ,\end{aligned}$$

using that the t_i are different elements from a fixed valuation basis of $M|M^p$; the last equation holds by virtue of (153). Hence for every i,

$$\delta_\rho \leq v(t_i(x_i^{(\sigma)} - x_i^{(\rho)})^p) .$$

This shows that for every i and $\rho < \sigma < \lambda$:

$$v(x_i^{(\sigma)} - x_i^{(\rho)}) \geq (\delta_\rho - v(t_i))/p =: \gamma_{\rho,i} . \tag{154}$$

For the model theoretic notions that we will use in the sequel, see subsection 11.1. For every i, let $\mathbf{S}_i$ be the set of all sentences

$$\text{“}v(X - x_i^{(\rho)}) \geq \gamma_{\rho,i}\text{”} , \quad (\rho < \lambda) . \tag{155}$$

in the language $\mathcal{L}(M, X)$. This set is finitely satisfiable in M: given a finite subset $\mathbf{S}_i' \subset \mathbf{S}_i$, let

$$\mu := \max\{\rho \mid \text{“}v(X - x_i^{(\rho)}) \geq \gamma_{\rho,i}\text{”} \in \mathbf{S}_i'\} .$$

Now the element $x_i^{(\mu)}$ realizes $\mathbf{S}_i'$ in M since $v(x_i^{(\mu)} - x_i^{(\mu)}) = v(0) = \infty$ and by (154), $v(x_i^{(\mu)} - x_i^{(\rho)}) \geq \gamma_{\rho,i}$ for all $\rho < \mu$. From Lemma 11.20 we now infer the existence of immediate approximation types $\mathbf{A}_i$ such that $\mathbf{S}_i \subset \mathbf{S}_{\geq}(\mathbf{A}_i)$ for every i. Since M is maximally valued, there exist elements x_i such that x_i realizes $\mathbf{A}_i$, for every i, according to Corollary 11.54. Putting

$$\underline{x} = (x_1, ..., x_n)$$

we get $v(x_i - x_i^{(\rho)}) \geq \gamma_{\rho,i}$ for all $\rho < \lambda$ and thus in view of (154):

$$v(\mathcal{F}(\underline{x}^{(\rho)}) - \mathcal{F}(\underline{x})) = \min_i(v(t_i(x_i^{(\rho)} - x_i)^p)) \geq \delta_\rho$$

for all $\rho < \lambda$ and thereby

$$\begin{aligned} v(x - \mathcal{F}(\underline{x})) &= v(x - \mathcal{F}(\underline{x}^{(\rho)}) + \mathcal{F}(\underline{x}^{(\rho)}) - \mathcal{F}(\underline{x})) \\ &\geq \min(v(x - \mathcal{F}(\underline{x}^{(\rho)})), v(\mathcal{F}(\underline{x}^{(\rho)}) - \mathcal{F}(\underline{x}))) = \delta_\rho \end{aligned}$$

for all $\rho < \lambda$, hence

$$v(x - \mathcal{F}(\underline{x})) \geq \sup_\rho(\delta_\rho) = \delta$$

and thus

$$v(x - \mathcal{F}(\underline{x})) = \delta$$

by the definition of δ. This completes the proof of our lemma. □

Now we will construct a rather simple algebraically complete field which does not have the property (142).

Lemma 10.5 *There exists an algebraically complete extension (K, v) of $(\mathbb{F}_p(t), v_t)^h$ of transcendence degree 1 without transcendence defect such that $\mathbb{Z} = v_t(\mathbb{F}_p(t)) \prec_\exists v(K) = \mathbb{Q} \times \mathbb{Z}$ and $\overline{\mathbb{F}_p(t)} = \overline{K}$ and:*
$K|K^p$ has valuation basis $\{1, t, ..., t^{p-1}\}$ but (K, v) does not satisfy property (142).

Proof: We put $k = \mathbb{F}_p(t)^h$. Let x be transcendental over k and let v be the extension of v_t to $k(x)$ given by $v(x^{-1}) \gg v_t(k)$; hence $v(k(x)) = \mathbb{Z}v(x^{-1}) \times \mathbb{Z}v(t)$, lexicographically ordered. Note that $v(x) \ll v_t(k)$. Let (K_1, v) be any maximal tame extension of $(k(x), v)^h$ still admitting a coarsening of v with residue field $(k, v_t)^h$. Then $v(K_1)/v_t(k)$ is divisible by every prime but p. Thus we may choose elements $c_i \in K_1$ such that for all $i \geq 1$:

$$v(x)/p^i < v(tc_i^p) \ll v_t(k) \quad \text{and} \quad v(c_{i-1}) < v(c_i)\ . \tag{156}$$

We will now construct a purely inseparable algebraic extension (K_2, v) of (K_1, v) having p–basis $\{t\}$. We define recursively

$$\xi_1 = x^{1/p} \quad \text{and} \quad \xi_{i+1} = (\xi_i - tc_i^p)^{1/p} \tag{157}$$

and note that by virtue of (156),

$$\forall i \in \mathbb{N} : v(\xi_i) = v(x)/p^i < v(tc_i^p) \ll v_t(k)\ . \tag{158}$$

We put

$$(K_2, v) = (K_1(\xi_i \mid i \in \mathbb{N}), v)$$

where the prolongation of v from K_1 to K_2 is unique since the extension is purely inseparable. To prove that the p–basis of K_2 is $\{t\}$, let $a \in K_2$. Then $a \in K_1(\xi_1, ..., \xi_j) = K_1(\xi_j)$ for a suitable $j \in \mathbb{N}$. Now one deduces by induction that $\{t, \xi_j\}$ is a p–basis for $K(\xi_j)$ and that

$$\xi_j = (\xi_{j+1})^p + tc_j^p \in (K_1(\xi_{j+1}))^p + t(K_1(\xi_{j+1}))^p$$

which shows

$$a \in K_2^p + tK_2^p + ... + t^{p-1}K_2^p\ .$$

Hence $\{t\}$ is a p–basis of K_2 and K_2 has p–degree 1. On the other hand, every extension $K_1(\xi_i)|K_1$ is purely ramified of degree p^i, the value group of $K_1(\xi_i)$ being

$$v(K_1(\xi_i)) = v(K_1) + \mathbb{Z}(v(x)/p^i)$$

which shows that the valuation v on K_2 still admits a coarsening with residue field (k, v_t) and that $v(K_2)/v_t(k)$ is p–divisible and hence divisible (since $v(K_1)$ was already divisible by every prime but p). Consequently, $v(K_2)$ is a $\mathbb{Z}$–group and $v_t(k) \prec_\exists v(K_2)$. On the other hand, the residue field of (K_2, v) is equal to $k/v_t = \mathbb{F}_p$. The equality

$$[K_2 : K_2^p] = p = 1 \cdot p = [\overline{K_2} : \overline{K_2}^p] \cdot (v(K_2) : pv(K_2)) \tag{159}$$

implies by Proposition 1.43 of [DEL1], p. 25, that (K_2, v) is inseparably defectless.

Now we choose (K, v) to be a maximal immediate algebraic extension of (K_2, v). Then by what we have just shown, $K|K_2$ is separable. Consequently, residue field and value group being unchanged, equation (159) holds also for K in the place of K_2 showing that K is inseparably defectless. Since K is a maximal immediate algebraic extension, it is also algebraically maximal and Theorem 4.17 now shows that K is an algebraically complete field. Moreover, $v_t(\mathbb{F}_p(t)) \prec_\exists v(K_2) = v(K)$ and $\overline{\mathbb{F}_p(t)} = \overline{K_2} = \overline{K}$.
It remains to show that K does not satisfy property (142) for a suitable choice of the elements c_i. Let us choose $c_i \in K_1$ such that the partial sums

$$s_\mu = \sum_{i=1}^{\mu} c_i \tag{160}$$

determine an immediate transcendental approximation type $\mathbf{A}$ over K_1 (in the sense of Lemma 11.20). For instance, we may choose c_i to be an element having value $v(x)/q_i$ where q_i denotes the first prime integer such that (156) holds. Assume now that

$$x = y^p - y + x_0 + tx_1^p + ... + t^{p-1}x_{p-1}^p$$

with $y, x_0, x_1, ..., x_{p-1} \in K$. We will deduce from this a contradiction already under the assumption

$$\exists c \in k : v(c) < v(x_0) \tag{161}$$

or equivalently

$$w(x_0) \geq 0 \tag{162}$$

for the coarsening w of v which has rank 1 on K. This condition on x_0 is weaker than the condition used in (142).

Since $x_1 \in K$ is algebraic over K_1, we know that there exists a $\mu_0 \in \mathbb{N}$ such that for all $\mu \geq \mu_0$:

$$v(x_1 - s_\mu) \leq v(c_{\mu_0}) \;, \tag{163}$$

because otherwise the approximation type of x over K_1 would be equal to $\mathbf{A}$ which is impossible since $\mathbf{A}$ is transcendental. On the other hand, we may choose μ as large as to guarantee not only (163), but also

$$v(c_{\mu_0}) < \frac{-v(x)}{p^{\mu+1}} = v(\xi_{\mu+1}) \;. \tag{164}$$

Putting

$$\tilde{y} = y - \sum_{i=1}^{\mu+1} \xi_i \quad \text{and} \quad \tilde{x}_1 = x_1 - s_\mu \ ,$$

we have, according to (163) and (164):

$$v(\tilde{x}_1) \le v(c_{\mu_0}) < v(\xi_{\mu+1}) \tag{165}$$

and we compute

$$\begin{aligned}
\tilde{y}^p - \tilde{y} &= y^p - y + (-\sum_{i=1}^{\mu+1} \xi_i)^p + \sum_{i=1}^{\mu+1} \xi_i \\
&= y^p - y - \xi_1^p - \sum_{i=1}^{\mu} (\xi_{i+1}^p - \xi_i) + \xi_{\mu+1} \\
&= y^p - y - x + \sum_{i=1}^{\mu} t c_i^p + \xi_{\mu+1} \\
&= \xi_{\mu+1} - x_0 - (t\tilde{x}_1^p + t^2 x_2^p + \ldots + t^{p-1} x_{p-1}^p) \ . \qquad (166)
\end{aligned}$$

We note that by (158) and (161),

$$v(\xi_{\mu+1}) \ll v(t) \quad \text{and} \quad v(\xi_{\mu+1}) \ll v(x_0)$$

and thus

$$0 > v(\xi_{\mu+1} - x_0) = v(\xi_{\mu+1}) > v(t) + pv(\xi_{\mu+1}) > v(t) + pv(\tilde{x}_1) = v(t\tilde{x}_1^p) \ ;$$

here the last inequality follows from (165). Since the elements $t, \ldots, t^{p-1}$ are value–independent over K^p, it follows that the value of (166) is $\le v(t\tilde{x}_1^p)$ and thus negative; this yields $v(\tilde{y}) < 0$. Consequently,

$$\begin{aligned}
pv(\tilde{y}) &= v(\tilde{y}^p - \tilde{y}) = v(t\tilde{x}_1^p + t^2 x_2^p + \ldots + t^{p-1} x_{p-1}^p) \\
&= \min(v(t) + pv(\tilde{x}_1), 2v(t) + pv(x_2), \ldots, (p-1)v(t) + pv(x_{p-1})) \notin pv(K) \ .
\end{aligned}$$

We have deduced a contradiction since by construction, $\tilde{y}$ is an element of K. This proves that K cannot satisfy property (142) if the elements c_i are chosen as above. □

The foregoing example instantly produces a second interesting example. Taking w to be the coarsening of v which is of rank 1, we have actually deduced in the foregoing proof that the assumption that (K, w) satisfies the property (142) leads to a contradiction. Indeed, we have only used the assumption (162) on the element x_0 instead of an assumption which uses the valuation v. Since moreover the field (K, w) is algebraically complete like (K, v) by virtue of Lemma 2.15 and Lemma 2.17, we have proved:

Lemma 10.6 *There exists an algebraically complete field (K, w) of transcendence degree 1 over its embedded residue field, having value group $v(K) = \mathbb{Q}$, and an element $t \in K$ such that the residue field K/w has p–basis $\{t/w\}$ and $K|K^p$ has valuation basis $\{1, t, \ldots, t^{p-1}\}$, but (K, w) does not satisfy property (142).*

Note that these examples also show that a field which is relatively algebraically closed in an algebraically complete field that satisfies (142) does itself not necessarily satisfy (142), even if the extension is immediate. Indeed, every maximal immediate extension of our examples (K, v) or (K, w) is a maximally valued field and thus satisfies (142) according to Lemma 10.3, and K is relatively algebraically closed in every such extension since (K, v) and (K, w) are algebraically complete. This contrasts the behaviour of tame fields as described in section 6.

The model theoretic consequences of these examples are:

Corollary 10.7 *The algebraically complete fields (K, v) and (K, w) as constructed above do not satisfy* **(Mod 1)**. *Hence*

$$\textbf{(Alg 3)} \quad \not\Longrightarrow \textbf{(Mod 1)} \,.$$

In particular, there exist algebraically complete fields that are not AKE–fields. Moreover, there are algebraically complete fields (K, v) such that the theory

$$\{(K, v) \text{ is algebraically complete}\} \cup \{\mathrm{char}(K) = p\} \cup \mathrm{Th}(v(K)) \cup \mathrm{Th}(\overline{K}) \qquad (167)$$

(with respect to the language $\mathcal{L}$ of valued fields) is not complete. In particular, the theory

$$\{(K, v) \text{ is algebraically complete}\} \cup \{\mathrm{char}(K) = p\} \cup \{v(K) \equiv \mathbb{Z}\} \cup \{\overline{K} = \mathbb{F}_p\}$$

which is satisfied by the valued power series field $\mathbb{F}_p((t))$, is not complete.

Furthermore, the example (K, v) together with $(k, v) = (\mathbb{F}_p(t), v_t)^h$ shows that if $(K, v)|(k, v)$ is an extension of algebraically complete fields then the fact that $v(k) \subset v(K)$ and $\overline{K}|\overline{k}$ are elementary extensions does in general not imply that $(K, v)|(k, v)$ is an elementary extension, even if it has no transcendence defect. This contrasts our result for "$\prec_\exists$" (cf. Theorem 8.4).

Proof: Let (K, v) be one of the examples given above. If M_v is an arbitrary maximal immediate extension of (K, v), then by Lemma 10.3 it satisfies the elementary property (142), whereas (K, v) does not. This shows that for this field (K, v), the theory (167) with respect to the language $\mathcal{L}(\{t\})$ is not complete since K and M_v have the same characteristic, value group and residue field and M_v is algebraically complete. In order to eliminate the constant t, we note that by Lemma 10.3, assertion (142) holds in M_v for *every* valuation basis $\{1, t, \ldots, t^{p-1}$ of $M_v|M_v^p$. But for any valued field (L, v), the condition that $\{1, t, \ldots, t^{p-1}$ is a valuation basis of $L|L^p$, can be expressed over (L, v) by the conjunction of the following two first order sentences:

$$\begin{aligned} &\forall z \exists z_0, \ldots, z_{p-1} : \ z = z_0^p + \ldots + z_{p-1}^p t^{p-1} \\ &\forall z_0, \ldots, z_{p-1} : \ v\left(z_0^p + \ldots + z_{p-1}^p t^{p-1}\right) = v(z_0^p) \vee \ldots \\ &\qquad \ldots \vee v\left(z_0^p + \ldots + z_{p-1}^p t^{p-1}\right) = v(z_{p-1}^p t^{p-1}) \,. \end{aligned}$$

Consequently,

$$\forall t : \ \{1, t, \ldots, t^{p-1}\} \text{ valuation basis for } (L, v) \implies (L, v) \text{ satisfies } (142)$$

is a first order sentence in $\mathcal{L}$ (without constants) which is satisfied by $(L, v) = (M_v, v)$, but not by $(L, v) = (K, v)$.

Furthermore, (K,v) cannot be existentially closed in M_v. This is seen as follows: For x as above there are no elements $y, x_0, x_1, ..., x_{p-1}$ in K satisfying the assertion of (142). But according to Lemma 10.3 there must exist such elements in M_v. Hence the existential formula (with constants x, t)

$$\exists y \exists x_0, ..., x_{p-1} : \ (x_0 = 0 \vee v(x_0) = 0) \wedge x = y^p - y + x_0 + t x_1^p + ... + t^{p-1} x_{p-1}^p \qquad (168)$$

holds in M_v but not in (K,v). Since M_v was chosen to be an arbitrary maximal immediate extension of (K,v) we have shown that (K,v) does not satisfy **(Mod 1)** which implies by virtue of Lemma 10.2 that it is not an AKE–field.
For the last assertion of our corollary, we only have to note that $(k,v) = (\mathbb{F}_p(t), v_t)^h$ satisfies property (142) by Lemma 10.3, part b), whereas (K,v) does not. □

With the examples that we have constructed, we can even show a sharper result:

Lemma 10.8 *If (K,v) is one of the examples as constructed above, then there exists an immediate function field (F,v) of transcendence degree 1 generated by two elements over K such that (K,v) is not existentially closed in (F,v) and thus also not existentially closed in the henselian function field $(F,v)^h$. This shows that Corollary 8.7 is not true in general if the condition "K is tame" is replaced by "K is algebraically complete".*

Proof: Let us return to the old notation (K,v) and (K,w) for our examples and let M_v and M_w be maximal immediate extensions of (K,v) and (K,w) respectively. We use the notation of the proof of Lemma 10.5 and choose the elements c_i such that the partial sums (160) define an immediate transcendental approximation type $\mathbf{A}_v$ over (K,v) as well as an immediate transcendental approximation type $\mathbf{A}_w$ over (K,w) (the choice suggested in the proof of 10.5 satisfies both conditions). Now let z be an element of M_v or M_w having approximation type $\mathbf{A}_v$ or $\mathbf{A}_w$ resp. (Indeed, it can be shown that for a suitable choice of M_v and M_w we may view z as a common element of both.)
We will show that the existential formula (168) for x holds already in an immediate function field

$$(K(y,z),v)|(K,v) \ \text{ resp. } \ (K(y,z),w)|(K,w)$$

where $K(y,z)|K(z)$ is an immediate Artin–Schreier–extension. This implies the assertion of our lemma.
Let $\tilde{\mathbf{A}}_v$ and $\tilde{\mathbf{A}}_w$ be the immediate approximation types determined by the partial sums

$$\sum_{i=1}^{k} \xi_i \qquad (169)$$

over $(K(z),v)$ resp. $(K(z),w)$ (again in the sense of Lemma 11.20). Now we compute for all $k \in \mathbb{N}$, using (157) and (158):

$$\begin{aligned} v\left((\sum_{i=1}^{k} \xi_i)^p - \sum_{i=1}^{k} \xi_i - (x - t z^p) \right) &= v\left(\xi_1^p + \sum_{i=1}^{k-1} (\xi_{i+1}^p - \xi_i) - \xi_k - (x - t z^p) \right) \\ &= v\left(x - t(\sum_{i=1}^{k-1} c_i)^p - \xi_k - (x - t z^p) \right) \end{aligned}$$

$$= v\left(t(z - \sum_{i=1}^{k-1} c_i)^p - \xi_k\right)$$
$$= \min\left(v(t(z - \sum_{i=1}^{k-1} c_i)^p), v(\xi_k)\right)$$
$$= \min\left(v(tc_k^p), v(\xi_k)\right) = v(\xi_k) = v(x)/p^k$$

which holds for w in the place of v as well. It shows that $\tilde{\mathbf{A}}_v$ and $\tilde{\mathbf{A}}_w$ are approximation types which do not fix the value of the polynomial

$$Y^p - Y - (x - tz^p) . \tag{170}$$

We will now show that (170) is an associated minimal polynomial for both approximation types. In view of the fact that the degree of an immediate approximation type $\mathbf{A}$ over a henselian field K is always a power of p (cf. Corollary 11.82) and that the degree is 1 if and only if $\mathbf{A}$ is realized by an element in K (cf. Lemma 11.39), we just have to show that there exists no element in $(K(z), v)$ having approximation type $\tilde{\mathbf{A}}_v$ over (K, v) and no element in $(K(z), w)$ having approximation type $\tilde{\mathbf{A}}_w$ over (K, w). To show this, it suffices to prove that $\mathbf{S}_{\geq}(\tilde{\mathbf{A}}_w)$ is not realized in (K, w), since this set of sentences is a logical consequence of the set $\mathbf{S}_{\geq}(\tilde{\mathbf{A}}_v)$; indeed, $v(X - c) \geq v(d)$ implies $w(X - c) \geq w(d)$ because w is a coarsening of v. To deduce a contradiction, let us assume that there exists an element $y \in K(z)$ realizing the approximation type $\mathbf{S}_{\geq}(\tilde{\mathbf{A}}_w)$ in (K, w). Consequently, we would have

$$\forall k \in I\!N : w(y^p - y - (x - tz^p)) > w(x)/p^k ,$$

whence

$$w(y^p - y - (x - tz^p)) \geq 0 .$$

Choosing $x_0 \in K(z)$ such that

$$w(y^p - y - (x - tz^p) + x_0) > 0$$

with $w(x_0) = 0$ or $x_0 = 0$, we find by Hensel's Lemma that there exists y^* in $(K(z), w)^h$ such that

$$(y^*)^p - y^* = y^p - y - (x - tz^p) + x_0 ,$$

whence

$$x = (y - y^*)^p - (y - y^*) + x_0 + tz^p$$

showing that the existential formula (168) would hold in $(K(z), w)^h$. But from Corollary 8.6 we know that (K, w) is existentially closed in the immediate henselian rational function field $(K(z), w)^h$. Thus (168) would also hold in (K, w) contrary to what we have already proved about (K, w). This contradiction shows that (170) is an associated minimal polynomial for both approximation types $\tilde{\mathbf{A}}_v$ and $\tilde{\mathbf{A}}_w$, as asserted. Hence by virtue of Theorem 11.52, there are immediate extensions of the valuations v and w from $(K(z), v)$ resp. $(K(z), w)$ to the Artin–Schreier–extension $K(y, z)$ where y is taken to be any root of the polynomial (170). Now (168) is satisfied in both function fields $(K(y, z), v)$ and $(K(y, z), w)$. This completes the proof of our lemma. □

After this preparation we are able to show that also the Structure Theorem 7.1 for immediate function fields of transcendence degree 1 over perfect algebraically complete fields fails if the condition "perfect" is omitted. Note that this result is not immediate from Lemma 10.7 and we really need the more precise result given in the foregoing lemma.

Corollary 10.9 *If (K, v) is one of the examples as constructed above, then there exists an immediate function field (F, v) of transcendence degree 1 generated by two elements over (K, v) such that $(F, v)^h$ is not a henselian rational function field. This shows that Theorem 7.1 is not true in general if the condition "K is perfect" is omitted.*

Proof: Let $F = K(y, z)$ be as in the foregoing lemma. Then $(F, v)^h$ cannot be a henselian rational function field since otherwise $(K, v) \prec_\exists (F, v)$ according to Corollary 8.6. □

The function field F that we have constructed shows the following symmetry between a generating Artin–Schreier–extension and a generating purely inseparable extension of degree p: on the one hand, we have the immediate Artin–Schreier–extension

$$K(y, z)|K(z)$$

given by

$$y^p - y = x - tz^p \; , \tag{171}$$

which shows that $F|K$ is regular. On the other hand we have the immediate purely inseparable extension

$$K(y, z)|K(y)$$

given by

$$z^p = \frac{1}{t}(-y^p + y + x) \; .$$

From equation (171) it is immediately clear that the function field $K(y, z)$ becomes rational after constant field extension with $t^{1/p}$, namely

$$F(t^{1/p}) = K(t^{1/p})(y + t^{1/p}z) \; .$$

This shows that the ground field K, not being existentially closed in the function field F, may become existentially closed in the function field after a finite purely inseparable constant extension though this extension is linearly disjoint from the function field over K. The fact that a henselian function field may become rational after a constant field extension, corresponds in the above example to the fact that the degree of irrationality of a function field may be reduced by a constant field extension, but it is not immediately clear from such a correspondence that there exists a valuation under which the function field is an immediate extension of the (algebraically complete) ground field.

In our above example there exists also a separable constant extension $K'|K$ of degree p such that $(F.K')^h$ is henselian rational. To show this, we take a constant $d \in K$ and an element $s \in \widetilde{K(t)}$ satisfying

$$t = s^p - ds \; ,$$

and we put $K' = K(s)$. If we choose d with a sufficiently high value, then we will have $v(s^p) = v(t)$ and $v(dsz^p) > 0$. The former guarantees that $K'|K$ is defectless and hence linearly disjoint from $F|K$. From the latter we deduce by Hensel's Lemma that there is an element $y^\star \in K'(z)^h$ such that $(y^\star)^p - y^\star = -dsz^p$. If we put $\tilde{y} = y + sz + y^\star \in K(y,z)^h$, we get

$$\tilde{y}^p - \tilde{y} = x - tz^p + s^p z^p - sz - dsz^p = x - sz + (s^p - ds - t)z^p = x - sz$$

which shows

$$z \in K'(\tilde{y}) \; .$$

This in turn yields $y^\star \in K'(\tilde{y})^h$ and consequently

$$y = \tilde{y} - sz - y^\star \in K'(\tilde{y})^h \; .$$

Altogether we have proved that

$$K'(y,z)^h = K'(\tilde{y})^h$$

is henselian rational.

It can be shown that extension (171) could not be immediate if (K,v) resp. (K,w) would satisfy property (142). This generates some hope that the structure theorem 7.1 could be reestablished for henselian function fields over nonperfect fields which as a compensation satisfy axioms of the type that is indicated by (142) and which are fulfilled in every perfect field. Certainly it is to be expected that in general an infinite scheme of axioms is necessary since an equation like (171) is only a special case and additive polynomials in y of higher degree than p may play an important role. But the consideration of this role has to be postponed to a subsequent investigation.

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11 Appendix: Approximation types and distances.

To begin with, we want to state some basic properties of approximation types.

Lemma 11.1 *Let* $\mathbf{A}$ *be an approximation type over the valued field* (K,v) *and* $\Upsilon \subseteq \Lambda(\mathbf{A})$. *Let* Λ_Υ *denote the minimal initial segment of* $\Lambda(\mathbf{A})$ *which contains* Υ. *Assume* $c_\Upsilon \in K$ *is an element of* $\mathbf{A}_\alpha$ *for all* $\alpha \in \Upsilon$. *Then* $c_\Upsilon \in \mathbf{A}_\alpha$ *for all* $\alpha \in \Lambda_\Upsilon$, *and* $c_\Upsilon \in \mathbf{A}_\alpha^\circ$ *for all* $\alpha < \sup \Upsilon$. *Furthermore, for every* $c \in K$ *and every* $\beta \in v(K)$ *we have:*

$$\begin{aligned} \beta \in \Lambda_\Upsilon &\implies (c \in \mathbf{A}_\beta \Longleftrightarrow v(c_\Upsilon - c) \geq \beta)\;, \\ \beta < \sup \Upsilon &\implies (c \in \mathbf{A}_\beta^\circ \Longleftrightarrow v(c_\Upsilon - c) > \beta)\;, \\ \beta < \sup \Upsilon &\implies (c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ \Longleftrightarrow v(c_\Upsilon - c) = \beta)\;. \end{aligned}$$

If $c_\Upsilon \in \mathbf{A}_\alpha^\circ$ *for all* $\alpha \in \Upsilon$, *then the second and third implication will hold for all* $\beta \in \Lambda_\Upsilon$.

Proof: The assertion that $c_\Upsilon \in \mathbf{A}_\alpha$ for all $\alpha \in \Lambda_\Upsilon$ follows from (at 4), and the assertion that $c_\Upsilon \in \mathbf{A}_\alpha^\circ$ for all $\alpha < \sup \Upsilon$ follows from (at 1) and the definition of the supremum (if $\alpha < \sup \Upsilon$, then there exists $\beta \in \Upsilon$ such that $\beta > \alpha$ and thus $c_\Upsilon \in \mathbf{A}_\beta \subset \mathbf{A}_\alpha^\circ$ by (at 1)). Let $c \in K$ and $\beta \in \Lambda_\Upsilon$. Then $c_\Upsilon \in \mathbf{A}_\beta$; hence $v(c_\Upsilon - c) \geq \beta \Longrightarrow c \in \mathbf{A}_\beta$ by (at 3) and $c \in \mathbf{A}_\beta \Longrightarrow v(c_\Upsilon - c) \geq \beta$ by (at 2). Now assume in addition that the condition "$c_\Upsilon \in \mathbf{A}_\alpha^\circ$ for all $\alpha \in \Upsilon$" holds or that $\beta < \sup \Upsilon$. Then in either case, $c_\Upsilon \in \mathbf{A}_\beta^\circ$; hence $v(c_\Upsilon - c) > \beta \Longrightarrow c \in \mathbf{A}_\beta^\circ$ by (at 3°), and $c \in \mathbf{A}_\beta^\circ \Longrightarrow v(c_\Upsilon - c) > \beta$ by (at 2°); similarly, $v(c_\Upsilon - c) = \beta \Longrightarrow c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$ by (at 3) and (at 2°), and $c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ \Longrightarrow v(c_\Upsilon - c) = \beta$ by (at 2) and (at 3°). □

A trivial but helpful observation is the following:

Lemma 11.2 *Let* $\mathbf{A}$ *be an approximation type over the valued field* (K,v) *and* $\alpha \in v(K)$. *If* $\mathbf{A}_\alpha \neq \emptyset$, *then* $\mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ \neq \emptyset$. *In particular,*

$$\begin{aligned} \Lambda(\mathrm{appr}(x,K)) &= \{v(x-c) \in v(K) \mid c \in K\} \qquad (172)\\ \mathrm{dist}(x,K) &= \sup\{v(x-c) \in v(K) \mid c \in K\}\;. \qquad (173) \end{aligned}$$

Proof: Assume $\mathbf{A}_\alpha \neq \emptyset$. If $\mathbf{A}_\alpha^\circ = \emptyset$, then the assertion is clearly true. If $\mathbf{A}_\alpha^\circ \neq \emptyset$, let $c \in \mathbf{A}_\alpha^\circ$ and choose any element $c' \in K$ with $v(c') = \alpha$. By (at 0), $c \in \mathbf{A}_\alpha$, and by (at 3), $c + c' \in \mathbf{A}_\alpha$ since $v(c-(c+c')) = \alpha$. The latter shows by (at 2°) that $c + c' \notin \mathbf{A}_\alpha^\circ$, hence

$$c + c' \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ\;.$$

This yields the first assertion. The second assertion is seen as follows. By definition,

$$\Lambda(\mathrm{appr}(x,K)) = \{\alpha \in v(K) \mid \exists c \in K:\; v(x-c) \geq \alpha\}\;.$$

But by what we have shown,

$$\exists c \in K:\; v(x-c) \geq \alpha \in v(K) \Longrightarrow \exists c \in K:\; v(x-c) = \alpha\;.$$

□

The following lemma is just the transposition of (at 5_v) and (at 5_r) to the case $\mathbf{A} = \mathrm{appr}(x,K)$.

Lemma 11.3 *Let* $\mathbf{A} = \mathrm{appr}(x, K)$*.* $\mathbf{A}$ *is value–immediate iff*

$$\forall c \in K : \ v(x - c) \in v(K) \cup \{\infty\} \, .$$

$\mathbf{A}$ *is residue–immediate iff*

$$\forall c \in K : \ [v(x - c) \in v(K) \Longrightarrow \exists c' \in K : \ v(x - c') > v(x - c)] \, .$$

Consequently, if $v(K(x)) = v(K)$ *and if* $\{v(x - c) \mid c \in K\}$ *has no maximal element, then* $\mathrm{appr}(x, K)$ *is immediate.*

Lemma 11.4 *Let* $\mathbf{A}$ *be an approximation type over the valued field* (K, v) *with* $\mathbf{A}_\infty = \emptyset$*. Then:*

a) There is at most one element $c_0 \in K$ *which is contained in every* $\mathbf{A}_\alpha$*,* $\alpha \in v(K)$*. If such element* c_0 *exists, then* $\mathbf{A}$ *is residue–immediate, but not value–immediate, and* $\Lambda(\mathbf{A}) = v(K)$*.*

b) If $\Lambda(\mathbf{A})$ *has no greatest element (i.e. if* $\mathrm{dist}(\mathbf{A})$ *is not assumed), then* $\mathbf{A}$ *is residue–immediate.*

c) Suppose that $\Lambda(\mathbf{A})$ *admits a greatest element* γ*. Then there exists* $c_0 \in K$ *such that* $c_0 \in \mathbf{A}_\alpha$ *for all* $\alpha \in \Lambda(\mathbf{A})$*; more precisely, every element of* $\mathbf{A}_\gamma$ *(and thus also every element of* $\mathbf{A}_\gamma^\circ$*, if this set is nonempty) has this property.* $\mathbf{A}$ *is residue–immediate if and only if* $\mathbf{A}_\gamma^\circ \neq \emptyset$*, and* $\mathbf{A}$ *is value–immediate if and only if* $\mathbf{A}_\gamma^\circ = \emptyset$*.*

d) If $\Lambda(\mathbf{A})$ *does not admit a greatest element, and if there exists an element* $c_0 \in K$ *such that* $c_0 \in \mathbf{A}_\alpha$ *for all* $\alpha \in \Lambda(\mathbf{A})$*, then* $\mathbf{A}$ *is residue–immediate, but not value–immediate.*

e) $\mathbf{A}$ *is immediate if and only if there exists no* $c_0 \in K$ *such that* $c_0 \in \mathbf{A}_\alpha$ *for all* $\alpha \in \Lambda(\mathbf{A})$*.*

Proof:

a): Assume that there exists an element $c_0 \in K$ which is contained in every $\mathbf{A}_\alpha$, $\alpha \in v(K)$. Then by definition, $\Lambda(\mathbf{A}) = v(K)$. An application of Lemma 11.1 shows that for every $c \in K$ with $c \neq c_0$, we have $c \notin \mathbf{A}_\beta^\circ$, where $\beta = v(c_0 - c) \in v(K) = \Lambda(\mathbf{A})$. In view of (at 1) this shows that c cannot be included in $\mathbf{A}_\alpha$ for $\alpha > \beta$, hence c_0 is the only element which is included in all $\mathbf{A}_\alpha$, $\alpha \in v(K)$. Furthermore, for every $\alpha \in v(K)$, the set $\mathbf{A}_\alpha^\circ$ is nonempty since by Lemma 11.1, it contains c_0; this shows that $\mathbf{A}$ is residue–immediate. But since $c_0 \notin \mathbf{A}_\infty$ by hypothesis, there is no $\alpha \in v(K) \cup \{\infty\}$ such that $c_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$; this shows that $\mathbf{A}$ is not value–immediate.

b): Let $\alpha \in \Lambda(\mathbf{A})$. By our hypothesis that $\Lambda(\mathbf{A})$ has no greatest element, there exists $\beta \in \Lambda(\mathbf{A})$, $\beta > \alpha$, such that $\mathbf{A}_\beta \neq \emptyset$. Now $\mathbf{A}_\beta \subset \mathbf{A}_\alpha^\circ$ by (at 1), hence $\mathbf{A}_\alpha^\circ \neq \emptyset$. Since $\alpha \in \Lambda(\mathbf{A})$ was arbitrary, assertion b) is proved.

c): Suppose that $\Lambda(\mathbf{A})$ admits a greatest element γ. By definition of $\Lambda(\mathbf{A})$, there exists an element $c_0 \in \mathbf{A}_\gamma$. Hence by Lemma 11.1, $c_0 \in \mathbf{A}_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, and $c_0 \in \mathbf{A}_\alpha^\circ$ for all $\alpha < \gamma$; hence if $\mathbf{A}_\gamma^\circ \neq \emptyset$, then $\mathbf{A}_\alpha^\circ \neq \emptyset$ for all $\alpha \in \Lambda(\mathbf{A})$ which implies that $\mathbf{A}$ is residue–immediate. On the other hand, $\mathbf{A}_\gamma^\circ = \emptyset$ implies that $\mathbf{A}$ is not residue–immediate since $\mathbf{A}_\gamma \neq \emptyset$ by our choice of γ. Now if $\mathbf{A}_\gamma^\circ \neq \emptyset$ and $c_0 \in \mathbf{A}_\gamma^\circ$, then $c_0 \in \mathbf{A}_\alpha^\circ$ for all $\alpha \in \Lambda(\mathbf{A})$ by (at 4°); in this case, there is no $\alpha \in \Lambda(\mathbf{A})$ such that $c_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$, and $\mathbf{A}$ is thus not value–immediate. Finally, assume that $\mathbf{A}_\gamma^\circ = \emptyset$. From Lemma 11.1 we know that $c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$ whenever $c \in K$ with $\beta = v(c_0 - c) < \gamma$. If on the other hand, $v(c_0 - c) \geq \gamma$, then $c \in \mathbf{A}_\gamma$ by (at 3), but $c \notin \emptyset = \mathbf{A}_\gamma^\circ$. This shows $\mathbf{A}$ to be value–immediate.

d): Under the assumptions as stated in d), $\mathbf{A}$ is residue–immediate according to part b) of our lemma. By hypothesis, for every $\alpha \in \Lambda(\mathbf{A})$ there exists a value $\beta > \alpha$ such that the element c_0 is included in $\mathbf{A}_\beta$ and thus also in $\mathbf{A}_\alpha^\circ$ by virtue of (at 1). This shows that there exists no $\alpha \in v(K)$ such that $c_0 \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$, i.e. that $\mathbf{A}$ is not value–immediate.

e): If $\mathbf{A}$ admits an element $c_0 \in K$ such that $c_0 \in \mathbf{A}_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$, then part c) and d) of our lemma show that $\mathbf{A}$ cannot be immediate. Conversely, assume that $\mathbf{A}$ does not admit such element. Then by part c), $\Lambda(\mathbf{A})$ does not contain a greatest element, and by part b), $\mathbf{A}$ is thus residue–immediate. It remains to show that $\mathbf{A}$ is also value–immediate; to this end, let c be an arbitrary element of K. By hypothesis, there exists $\alpha \in \Lambda(\mathbf{A})$ such that $c \notin \mathbf{A}_\alpha$. Let $c_\alpha \in \mathbf{A}_\alpha$ (such element exists by definition of $\Lambda(\mathbf{A})$). Now $v(c_\alpha - c) < \alpha$ by (at 3) since $c \notin \mathbf{A}_\alpha$. Hence by Lemma 11.1, $c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$ for $\beta = v(c_\alpha - c)$. Since $c \in K$ was arbitrary, we have proved that $\mathbf{A}$ is value–immediate. □

Lemma 11.5 *Let* $\mathbf{A}$ *be an approximation type over the valued field* (K, v) *with* $\mathbf{A}_\infty \neq \emptyset$. *Then* $\mathbf{A}_\infty$ *consists of exactly one element* $c_0 \in K$, *and for all* $\alpha \in v(K)$, *we have* $c_0 \in \mathbf{A}_\alpha$ *and* $c_0 \in \mathbf{A}_\alpha^\circ$. *For every* $c \in K$, $c \neq c_0$, *we have*

$$c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ \;\; \text{for} \;\; \beta = v(c_0 - c) \; .$$

Consequently, $\mathbf{A}$ *is trivial:*

$$\mathbf{A} = \mathrm{appr}(c_0, K) \; ,$$

and $\mathbf{A}$ *is immediate with* $\Lambda(\mathbf{A}) = v(K)$.

Proof: Let $c_0 \in \mathbf{A}_\infty$, hence $c_0 \in \mathbf{A}_\alpha$ and $c_0 \in \mathbf{A}_\alpha^\circ$ for all $\alpha \in v(K)$ in view of (at 4) and (at 1). The former implies $\Lambda(\mathbf{A}) = v(K)$, the latter shows that $\mathbf{A}$ is residue–immediate. From Lemma 11.1 we infer that $c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$ whenever $c_0 \neq c \in K$ with $\beta = v(c_0 - c)$, and by (at 1) it follows that $c \notin \mathbf{A}_\infty$, hence $\mathbf{A}_\infty$ consists only of the element c_0, and $c_0 \notin \emptyset = \mathbf{A}_\infty^\circ$. Hence for every $c \in K$, there exists a value $\beta \in v(K) \cup \{\infty\}$ with $c \in \mathbf{A}_\beta \setminus \mathbf{A}_\beta^\circ$, proving that $\mathbf{A}$ is value–immediate and thus immediate. On the other hand, Lemma 11.1 shows $c \in \mathbf{A}_\alpha \Longleftrightarrow c \in \mathrm{appr}(c_0, K)_\alpha$ for all $c \in K$ and all $\alpha \in v(K) \cup \{\infty\}$, and $c \in \mathbf{A}_\alpha^\circ \Longleftrightarrow c \in \mathrm{appr}(c_0, K)_\alpha^\circ$ for all $c \in K$ and all $\alpha \in v(K)$; this gives $\mathbf{A} = \mathrm{appr}(c_0, K)$. □

Corollary 11.6 *Let* $\mathbf{A}$ *be an arbitrary approximation type over the valued field* (K, v). *Then*

a) If $\mathbf{A}$ *is immediate, then* $\Lambda(\mathbf{A})$ *has no greatest element and consequently,* $\mathrm{dist}(\mathbf{A})$ *is not finitely assumed. If in addition* $\mathbf{A}$ *is nontrivial, then* $\mathrm{dist}(\mathbf{A})$ *is not finitely assumed.*

b) $\mathbf{A}$ *is value–immediate or residue–immediate.*

c) If $\mathrm{dist}(\mathbf{A})$ *is not finitely assumed by an element of* K, *then* $\mathbf{A}$ *is residue–immediate.*

d) If $\mathbf{A}$ *is not immediate, then there exists an element* $c_0 \in K$ *such that* $c_0 \in \mathbf{A}_\alpha$ *whenever* $\mathbf{A}_\alpha \neq \emptyset$ *and* $c_0 \in \mathbf{A}_\alpha^\circ$ *whenever* $\mathbf{A}_\alpha^\circ \neq \emptyset$.

Proof: a): This is a consequence of part c) of Lemma 11.4 together with Lemma 11.5.
b): This is a consequence of part b) and c) of Lemma 11.4 together with Lemma 11.5.

c): This is a consequence of part b) of Lemma 11.4 together with Lemma 11.5.

d): If $\mathbf{A}$ is not immediate, then in view of Lemma 11.5, we know that $\mathbf{A}_\infty = \emptyset$. Assume that $\Lambda(\mathbf{A})$ admits a greatest element γ. Then choose $c_0 \in \mathbf{A}_\gamma^\circ$ if this set is nonempty, otherwise in $\mathbf{A}_\gamma$. (at 0), (at 1) and (at 4) show that c_0 has the required properties. Now assume that $\Lambda(\mathbf{A})$ has no greatest element γ. Then choose c_0 by virtue of part e) of Lemma 11.4; (at 1) and (at 4) show that c_0 has the required properties. □

Furthermore, we state some facts about the restriction of an approximation type; the proof is straightforward.

Lemma 11.7 *Let $\mathbf{A}_L$ be an approximation type over the valued field L which contains the field K. Then the restriction $\mathbf{A}_K$ of $\mathbf{A}_L$ to K is an approximation type over K with $\Lambda(\mathbf{A}_K) = \Lambda(\mathbf{A}_L) \cap v(K)$ and consequently,*

$$\mathrm{dist}(\mathbf{A}_L) \geq \mathrm{dist}(\mathbf{A}_K)\ .$$

The following lemma treats approximation types of elements in defectless extensions:

Lemma 11.8 *Let $L|K$ be a nontrivial finite extension admitting a valuation basis, and let $x \in L \setminus K$. Then* $\mathrm{appr}(x, K)$ *is not immediate, and there exists an element $c \in K$ such that*

$$v(x - c) \geq \mathrm{dist}(x, K)\ .$$

Here, equality holds if $\mathrm{appr}(x, K)$ *is value–immediate.*

Proof: By Lemma 2.7 we may assume that $L|K$ admits a valuation basis which contains 1. We write

$$x = c_0 + c_1 y_1 + \ldots + c_n y_n$$

with $c_0, \ldots, c_n \in K$ and $y_1, \ldots, y_n$ elements of the valuation basis, different from 1. For all $c \in K$, we have

$$\begin{aligned} v(x-c) &= v(c_0 - c + c_1 y_1 + \ldots + c_n y_n) = \min\{v(c_0 - c), v(c_1 x_1), \ldots, v(c_n x_n)\} \\ &\leq \min\{v(c_1 x_1), \ldots, v(c_n x_n)\}\ , \end{aligned}$$

where the latter value is assumed for $c = c_0$. Hence

$$\begin{aligned} v(x - c_0) &= \max\{v(x-c) \mid c \in K\} \\ &\geq \sup\{v(x-c) \in v(K) \mid c \in K\} = \mathrm{dist}(x, K)\ . \end{aligned}$$

If $\mathrm{appr}(x, K)$ is value–immediate, then $v(x - c_0) \in v(K)$ and $\mathrm{dist}(x, K)$ is assumed by $c_0 \in K$. Since $x \notin K$ by hypothesis, the distance must be finitely assumed, and part a) of Corollary 11.6 now shows that $\mathrm{appr}(x, K)$ is not immediate. □

11.1 Comparison of our notion of "approximation types" with other possible concepts, and the realization of approximation types.

We want to compare our notion of approximation types with the classical concept of pseudo Cauchy sequences as defined by Ostrowski in [OS] and used by Kaplansky in his important paper "Maximal fields with valuations" [KAP1], by Schilling in [SCH] and by Ribenboim in [RIB1]. Though this comparison is mainly of historical interest, the concept of pseudo Cauchy sequences may still be adequate for the construction of valuation theoretical examples.

A second basic notion seems to be of higher theoretical interest in connection to approximation types. As indicated already in the name "approximation type", model theoretic *types* are closely related to approximation types. In the following, we will first discuss this relation.

As we have already explained in section 8, we are working in a fixed language $\mathcal{L}$ of valued fields. Given a valued field (K, v), we extend $\mathcal{L}$ to a language $\mathcal{L}(K)$ by adding to $\mathcal{L}$ a constant symbol for every element in K. The *elementary diagram* of (K, v) is the collection of all sentences in $\mathcal{L}(K)$ that hold in (K, v). Furthermore, we add to $\mathcal{L}(K)$ a symbol X for one variable and obtain $\mathcal{L}(K, X)$. Let $\mathbf{T}$ be a collection of sentences in $\mathcal{L}(K, X)$. If this collection is consistent with the elementary diagram of (K, v) and if it is maximal with respect to this property, then $\mathbf{T}$ is called a *type* (or *1–type*) over (K, v). An arbitrary collection $\mathbf{S}$ of sentences in $\mathcal{L}(K, X)$ is *realized (over* (K, v)*)* in a valued field (L, v), if (L, v) contains (K, v) and there exists an element $x \in L$ such that all sentences of $\mathbf{S}$ hold in (L, v) when X is replaced by x. In this case, we say that x *realizes* $\mathbf{S}$. $\mathbf{S}$ is called *finitely satisfiable in* (K, v), if every finite subset $\mathbf{S}_0 \subset \mathbf{S}$ is realized in (K, v). We will use the following basic properties of types:

Lemma 11.9 *A collection* $\mathbf{S}$ *of sentences in* $\mathcal{L}(K, X)$ *is consistent with the elementary diagram of* (K, v) *if and only if it is finitely satisfiable in* (K, v).

Proof: Let $\mathbf{S}'$ be the union of $\mathbf{S}$ and the elementary diagram of (K, v). Then $\mathbf{S}$ is consistent with the elementary diagram of (K, v) if and only if $\mathbf{S}'$ is consistent. By the Completeness Theorem 1.3.21 of [CHK], p. 66, $\mathbf{S}'$ is consistent if and only if it has a model. It remains to show that $\mathbf{S}'$ has a model if and only if it is finitely satisfiable over (K, v).

Let us first assume that $\mathbf{S}'$ has a model $\mathcal{M}$. By [CHK], Proposition 3.1.3, p. 108, any model of the elementary diagram of (K, v) is an elementary extension of (K, v) with respect to the language $\mathcal{L}(K)$, i.e. $(K, v) \prec \mathcal{M}$. If $\mathbf{S}'_0 \subset \mathbf{S}'$ is a finite subset, we may form the conjunction $\phi(X)$ over all sentences in $\mathbf{S}'_0$. Now the $\mathcal{L}(K)$–sentence

$$\exists x : \ \phi(x)$$

holds in $\mathcal{M}$, so it must hold in (K, v) too. This shows that $\mathbf{S}'_0$ is realized by an element in (K, v), and we have proved that $\mathbf{S}'$ is finitely satisfiable over (K, v).

For the converse, let us assume now that $\mathbf{S}'$ is finitely satisfiable over (K, v), i.e. (K, v) is a model for every finite subset of $\mathbf{S}'$. Then by the Compactness Theorem (cf. [CHK], Theorem 1.3.22, p. 33), $\mathbf{S}'$ has a model. □

Lemma 11.10 *If* $\mathbf{T}$ *is a type over* (K, v) *and* (K^*, v^*) *is a* $|K|^+$*–saturated elementary extension of* (K, v)*, then* $\mathbf{T}$ *is realized in* (K^*, v^*)*. Note that such extension* (K^*, v^*) *always exists.*

Proof: Since (K^*, v^*) is an elementary extension of (K, v), its theory with respect to the language $\mathcal{L}(K)$ is equal to the elementary diagram of (K, v) (which is nothing else but the theory of (K, v) with respect to the language $\mathcal{L}(K)$); this is true in view of [CHK], Theorem 1.3.22, p. 33, since the theory of a model is complete (since a sentence either holds or does not hold in the model). Hence the given type $\mathbf{T}$ is consistent with the $\mathcal{L}(K)$–theory of (K^*, v^*). Now the assertion follows immediately from the definition of "α–saturated" models as given in [CHK], chapter 5, p. 214.

The existence of a $|K|^+$–saturated elementary extension of (K, v) follows from Lemma 5.1.4 of [CHK], p. 216, together with the fact that every α–saturated model is also β–saturated for every $\beta \leq \alpha$ (which follows immediately from the definition). $\square$

Now we will detect the connection between approximation types and types by discussing through which sets $\mathbf{S}$ of sentences in $\mathcal{L}(K, X)$ an approximation type $\mathbf{A}$ can be (uniquely) determined, in a sense that we have to make precise now. We say that $\mathbf{S}$ *determines* $\mathbf{A}$, if the following holds: if $(K(x), v)|(K, v)$ is an extension of valued fields such that every sentence of $\mathbf{S}$ holds for x in the place of X, then $\mathbf{A} = \operatorname{appr}(x, K)$. Let us begin with the trivial observation that every approximation type $\mathbf{A}$ over K is uniquely determined by the set

$$\mathbf{S}(\mathbf{A}) = \mathbf{S}_>(\mathbf{A}) \cup \mathbf{S}_=(\mathbf{A}) \cup \mathbf{S}_<(\mathbf{A})$$

of sentences in $\mathcal{L}(K, X)$, where

$$\begin{aligned} \mathbf{S}_>(\mathbf{A}) &:= \{\text{“}v(X - c) > v(d)\text{”} \mid c, d \in K \wedge c \in \mathbf{A}^\circ_{v(d)}\} \\ \mathbf{S}_=(\mathbf{A}) &:= \{\text{“}v(X - c) = v(d)\text{”} \mid c, d \in K \wedge c \in \mathbf{A}_{v(d)} \setminus \mathbf{A}^\circ_{v(d)}\} \\ \mathbf{S}_<(\mathbf{A}) &:= \{\text{“}v(X - c) < v(d)\text{”} \mid c, d \in K \wedge c \notin \mathbf{A}_{v(d)}\} \,. \end{aligned}$$

This holds even in the following strong sense: if $\mathbf{A}$ and $\mathbf{A}'$ are approximation types, then

$$\mathbf{S}(\mathbf{A}) \subset \mathbf{S}(\mathbf{A}') \Longrightarrow \mathbf{A} = \mathbf{A}' \,.$$

We will also consider the following set

$$\mathbf{S}_\geq(\mathbf{A}) = \{\text{“}v(X - c) \geq v(d)\text{”} \mid c, d \in K \wedge c \in \mathbf{A}_{v(d)}\}$$

of sentences in $\mathcal{L}(K, X)$. The significance of this set will be shown in Lemma 11.12 below. Also $\mathbf{S}_=(\mathbf{A})$ plays a central role:

Lemma 11.11 *Every value–immediate approximation type* $\mathbf{A}$ *is determined already by the set* $\mathbf{S}_=(\mathbf{A})$.

Proof: We assume that $\mathbf{A}$ is value–immediate, hence for all elements $c \in K$, there exists $d \in K$ such that $c \in \mathbf{A}_{v(d)} \setminus \mathbf{A}^\circ_{v(d)}$ and consequently, "$v(X - c) = v(d)$" $\in \mathbf{S}_=(\mathbf{A})$. We have to show that the sentences of the sets $\mathbf{S}_>(\mathbf{A})$ and $\mathbf{S}_<(\mathbf{A})$ are logical consequences of the sentences in $\mathbf{S}_=(\mathbf{A})$ and the elementary diagram $\mathbf{D}$ of (K, v). But this is immediately seen to be true since in view of "$v(X-c) = v(d)$" $\in \mathbf{S}_=(\mathbf{A})$ we have: "$v(X-c) > v(d')$" $\in \mathbf{S}_>(\mathbf{A})$ whenever "$v(d') < v(d)$" $\in \mathbf{D}$, and "$v(X-c) < v(d')$" $\in \mathbf{S}_<(\mathbf{A})$ whenever "$v(d') > v(d)$" $\in \mathbf{D}$. $\square$

Lemma 11.12 *Every immediate approximation type* $\mathbf{A}$ *is determined by the set* $\mathbf{S}_=(\mathbf{A})$ *and also by the set* $\mathbf{S}_\geq(\mathbf{A})$*. Both sets are consistent with the elementary diagram* $\mathbf{D}$ *of* (K,v)*; in fact, this is already true for residue–immediate approximation types.*

Proof: It was already shown in the previous lemma that $\mathbf{A}$ is determined by the set $\mathbf{S}_=(\mathbf{A})$. We have to show that it is also determined by $\mathbf{S}_\geq(\mathbf{A})$. For this, it suffices to show that the sentences of $\mathbf{S}_=(\mathbf{A})$ are logical consequences of the sentences of $\mathbf{S}_\geq(\mathbf{A})$ and the elementary diagram $\mathbf{D}$ of (K,v). To this end, we assume that "$v(X-c)=v(d)$" $\in \mathbf{S}_=(\mathbf{A})$, i.e. $c\in \mathbf{A}_{v(d)}\setminus \mathbf{A}^\circ_{v(d)}$ and thus "$v(X-c)\geq v(d)$" $\in \mathbf{S}_\geq(\mathbf{A})$.

Let us first assume that $d=0$. Then the sentence "$d=0$" is contained in $\mathbf{D}$, and together with "$v(X-c)\geq v(d)$" it implies "$v(X-c)=v(d)$".

Let us now assume that $d\neq 0$. By hypothesis, $\mathbf{A}$ is residue–immediate and hence there exists an element $c'\in K$ such that $c'\in \mathbf{A}^\circ_{v(d)}$. Since $\mathbf{A}$ is value–immediate, there exists $d'\in K$ such that $c'\in \mathbf{A}_{v(d')}\setminus \mathbf{A}^\circ_{v(d')}$. Consequently, "$v(X-c')\geq v(d')$" $\in \mathbf{S}_\geq(\mathbf{A})$, and $v(d')>v(d)$ by virtue of (at 4°). $c'\in \mathbf{A}^\circ_{v(d)}$ yields $v(c'-c)\leq v(d)$ by (at 3°) because of $c\notin \mathbf{A}^\circ_{v(d)}$. On the other hand, $c,c'\in \mathbf{A}_{v(d)}$ and (at 2) yield $v(c'-c)\geq v(d)$. So we have shown that $v(c'-c)=v(d)$. Both sentences, "$v(d')>v(d)$" and "$v(c'-c)=v(d)$" are sentences of $\mathbf{D}$. Together with "$v(X-c')\geq v(d')$", they imply the sentence "$v(X-c)=v(d)$". This poves that the sentences of $\mathbf{S}_=(\mathbf{A})$ are logical consequences of the sentences of $\mathbf{S}_\geq(\mathbf{A})$, as desired.

It remains to show that both sets $\mathbf{S}_=(\mathbf{A})$ and $\mathbf{S}_\geq(\mathbf{A})$ are consistent with $\mathbf{D}$ (under the only condition that $\mathbf{A}$ is a residue–immediate approximation type). According to Lemma 11.9, it suffices to show that they are finitely satisfiable in (K,v). Let $\mathbf{S}_0$ be a finite subset of $\mathbf{S}_=(\mathbf{A})$ or of $\mathbf{S}_\geq(\mathbf{A})$, and let

$$\alpha_0=\max\{v(d)\mid d\in K \text{ and "}v(X-c)=v(d)\text{"}\in \mathbf{S}_0 \text{ or "}v(X-c)\geq v(d)\text{"}\in\mathbf{S}_0\}\;.$$

Since $\mathbf{A}$ is residue–immediate by hypothesis, there exists an element $c_0\in K$ with $c_0\in \mathbf{A}^\circ_{\alpha_0}$. By (at 0) and (at 4), $c_0\in \mathbf{A}_\alpha$ for all $\alpha\leq\alpha_0$. For given $c,d\in K$ with "$v(X-c)=v(d)$" or "$v(X-c)\geq v(d)$" in $\mathbf{S}_0$, we will now compute the value of c_0-c.

If "$v(X-c)\geq v(d)$"$\in \mathbf{S}_0$, then $c\in \mathbf{A}_{v(d)}$ and $v(d)\leq\alpha_0$; the latter implies $c_0\in \mathbf{A}_{v(d)}$. Hence by (at 2), $v(c_0-c)\geq v(d)$. Consequently, for the case where $\mathbf{S}_0\subset \mathbf{S}_\geq(\mathbf{A})$ we have proved that $\mathbf{S}_0$ is realized by c_0 in (K,v).

If "$v(X-c)=v(d)$"$\in \mathbf{S}_0$, then $c\in \mathbf{A}_{v(d)}\setminus \mathbf{A}^\circ_{v(d)}$ and $v(d)\leq\alpha_0$. As before, it is shown that $v(c_0-c)\geq v(d)$. On the other hand, from $c\notin \mathbf{A}^\circ_{v(d)}$ and (at 3°) we obtain $v(c_0-c)\leq v(d)$. This shows $v(c_0-c)=v(d)$. Consequently, for the case where $\mathbf{S}_0\subset \mathbf{S}_=(\mathbf{A})$ we have proved that $\mathbf{S}_0$ is realized by c_0 in (K,v). This completes the proof of the lemma. □

This lemma shows that for an immediate approximation type $\mathbf{A}$, we may actually forget about the second part of the map $\mathbf{A}$ which sends $\alpha\in v(K)\cup\{\infty\}$ to $\mathbf{A}^\circ_\alpha$. As soon as we know that the approximation type is immediate, all the remaining information is carried already by the first part of the map. We will use this principle when we are working with immediate approximation types.

Since by the preceding lemma, for every immediate approximation type $\mathbf{A}$ the set $\mathbf{S}_\geq(\mathbf{A})$ is consistent with the elementary diagram $\mathbf{D}$ of (K,v), there exists by Zorn's Lemma a maximal set $\mathbf{T}$ of sentences in $\mathcal{L}(K,X)$ which contains $\mathbf{S}_\geq(\mathbf{A})$ and is consistent with $\mathbf{D}$

(since $\mathbf{T}$ is maximal, it thus contains $\mathbf{D}$). By definition, $\mathbf{T}$ is a type over (K, v). If we take (K^*, v^*) to be a $|K|^+$–saturated extension of (K, v), then by Lemma 11.10 there exists an $x \in K^*$ that realizes $\mathbf{T}$ over (K, v). $(K(x), v)$ — where v denotes the restriction of v^* — is a valued subfield of (K^*, v^*) containing (K, v). Every existential sentence with constants from K which holds in $(K(x), v)$ will trivially hold in (K^*, v^*) too; so if we take the latter to be an elementary extension of (K, v), the sentence will also hold in (K, v). We conclude that (K, v) is existentially closed in $(K(x), v)$. Then in particular, $x \in K$ or x is transcendental over K; this is shown in Lemma 8.3. We summarize what we have proved:

Lemma 11.13 *For every immediate approximation type* $\mathbf{A}$ *over* K*, there exists a simple valued field extension* $(K(x), v)|(K, v)$ *such that* $\mathbf{A} = \mathrm{appr}(x, K)$ *and* $(K, v) \prec_\exists (K(x), v)$*. Note that the latter property implies that* x *is transcendental over* K *if* $x \notin K$*.* x *can be found in every* $|K|^+$*–saturated extension of* (K, v)*.*

Under additional assumptions, it is possible to show the same for approximation types which are not immediate. This will be discussed in the sequel.

Lemma 11.14 *Let* $\mathbf{A}$ *be an approximation type over a valued field* (K, v) *whose residue field* $\overline{K}$ *is infinite. Then the set* $\mathbf{S}_=(\mathbf{A})$ *is consistent with the elementary diagram of* (K, v)*. Consequently, if* $\mathbf{A}$ *is value–immediate, then it is realized in every* $|K|^+$*–saturated elementary extension of* (K, v) *and thus, there exists a simple valued field extension* $(K(x), v)|(K, v)$ *such that* $\mathbf{A} = \mathrm{appr}(x, K)$ *and* $(K, v) \prec_\exists (K(x), v)$*. Again,* $x \in K$ *or* x *is transcendental over* K*.*

Proof: Let $\mathbf{S}_0$ be a finite subset of $\mathbf{S}_=(\mathbf{A})$ and let

$$\alpha_0 = \max\{v(d) \mid d \in K \wedge \text{“}v(X - c) = v(d)\text{”} \in \mathbf{S}_=(\mathbf{A})\} .$$

Furthermore, let

$$\mathcal{C} = \{c \in K \mid \text{“}v(X - c) = v(d)\text{”} \in \mathbf{S}_0 \wedge v(d) = \alpha_0\} ,$$

and let $c_0 \in \mathcal{C}$. By (at 2), we know that $v(c - c_0) \geq \alpha_0$ for every $c \in \mathcal{C}$. Since $\mathcal{C}$ is finite and by hypothesis, $\overline{K}$ is infinite, there exists an element $c' \in K$ with value α_0 such that $v(c' - (c - c_0)) = v(c') = \alpha_0$ for all $c \in \mathcal{C}$. We claim that the element $c_0 + c' \in K$ realizes $\mathbf{S}_0$. Indeed, if $c \in \mathcal{C}$, then $v((c_0 + c') - c) = v(c' - (c - c_0)) = \alpha_0 = v(d)$. If “$v(X - c) = v(d)$” $\in \mathbf{S}_0$ with $c \notin \mathcal{C}$, then $v(d) < \alpha_0$ and as in the proof of Lemma 11.12 it can be shown that $v(c_0 - c) = v(d)$; consequently, $v((c_0 + c') - c) = v(c' + (c_0 - c)) = v(d)$. This proves that $c_0 + c'$ realizes $\mathbf{S}_0$ in K, proving that $\mathbf{S}_=(\mathbf{A})$ is finitely satisfiable in (K, v) and hence consistent with the elementary diagram of (K, v), according to Lemma 11.9. We conclude that $\mathbf{S}_=(\mathbf{A})$ is realized in every $|K|^+$–saturated elementary extension of (K, v). The same is true for $\mathbf{A}$ if $\mathbf{A}$ is value–immediate, since then according to Lemma 11.11, it is determined by $\mathbf{S}_=(\mathbf{A})$. □

In view of the fact that every approximation type is residue–immediate if it is not value–immediate (cf. part b) of Corollary 11.6), it only remains to deal with the case of a residue–immediate approximation type.

Lemma 11.15 *Every residue–immediate approximation type* $\mathbf{A}$ *is determined by*

$$\mathbf{S}_>(\mathbf{A}) \cup \mathbf{S}_<(\mathbf{A}) .$$

Proof: Assume that $\mathbf{A}$ is a residue–immediate approximation type over K. We will show that the sentences of the set $\mathbf{S}_=(\mathbf{A})$ are logical consequences of the sentences in $\mathbf{S}_>(\mathbf{A})$ and the elementary diagram $\mathbf{D}$ of (K, v), which yields the assertion of our lemma. To this end, assume that "$v(X - c) = v(d)$" $\in \mathbf{S}_=(\mathbf{A})$, i.e. $c \in \mathbf{A}_{v(d)} \setminus \mathbf{A}^\circ_{v(d)}$. $\mathbf{A}$ being residue–immediate by assumption, there exists an element $c' \in \mathbf{A}^\circ_{v(d)}$ and we have "$v(X - c') > v(d)$" $\in \mathbf{S}_>(\mathbf{A})$. By (at 3°) we know that $v(c' - c) \leq v(d)$, and by (at 2) we conclude that $v(c' - c) = v(d)$; hence "$v(c' - c) = v(d)$" $\in \mathbf{D}$. Now "$v(X - c) = v(d)$" is a logical consequence of "$v(X - c') > v(d)$" and "$v(c' - c) = v(d)$". □

Lemma 11.16 *Let* $\mathbf{A}$ *be a residue–immediate approximation type over a valued field* (K, v) *whose value group* $v(K)$ *is dense. Then the set* $\mathbf{S}(\mathbf{A})$ *is consistent with the elementary diagram of* (K, v)*. Hence* $\mathbf{A}$ *is realized in every* $|K|^+$*–saturated elementary extension of* (K, v) *and thus, there exists a simple valued field extension* $(K(x), v)|(K, v)$ *such that* $\mathbf{A} = \mathrm{appr}(x, K)$ *and* $(K, v) \prec_\exists (K(x), v)$*. Again,* $x \in K$ *or* x *is transcendental over* K*.*

Proof: In view of the foregoing lemma, we only have to show that $\mathbf{S}_>(\mathbf{A}) \cup \mathbf{S}_<(\mathbf{A})$ is consistent with the elementary diagram of (K, v). By virtue of Lemma 11.9 it suffices to show that every finite set $\mathbf{S}_0 \subset \mathbf{S}_>(\mathbf{A}) \cup \mathbf{S}_<(\mathbf{A})$ is realized in (K, v). Let

$$\alpha_0 := \min\{v(d) \mid \text{“}v(X - c) > v(d)\text{”} \in \mathbf{S}_>(\mathbf{A})$$

($\alpha_0 = -\infty$ if the set is empty), and choose an element $c' \in \mathbf{A}^\circ_{\alpha_0}$ which exists in view of our definition of α_0. Furthermore, let

$$\beta_0 := \min\{v(c - c'), v(d) \mid \text{“}v(X - c) < v(d)\text{”} \in \mathbf{S}_<(\mathbf{A}) \wedge v(c - c') > \alpha_0 \wedge v(d) > \alpha_0\}$$

($\beta_0 = \infty$ if the set is empty). We have $\beta_0 > \alpha_0$, and since $v(K)$ is dense by assumption, there exists an element $\tilde{c} \in K$ such that $\beta_0 > v(\tilde{c}) > \alpha_0$. We put $c_0 := \tilde{c} + c'$ and we will prove that c_0 realizes $\mathbf{S}_0$. From Lemma 11.1 it follows that c_0 realizes all sentences "$v(X - c) > v(d)$" $\in \mathbf{S}_0$. Now let "$v(X - c) < v(d)$" $\in \mathbf{S}_0$. Let us assume first that $v(c - c') > \alpha_0$; for such c we have $c \in \mathbf{A}_{\alpha_0}$ by (at 3) and thus $v(d) > \alpha_0$, by the definition of $\tilde{c}$, and we compute

$$v(c_0 - c) = \min(v(\tilde{c}), v(c - c')) = v(\tilde{c}) < \beta_0 \leq v(d) .$$

Now let us assume that $v(c - c') \leq \alpha_0 < v(\tilde{c})$; for such c we have $c \in \mathbf{A}_{v(c-c')}$ by Lemma 11.1, which shows that necessarily $v(c - c') < v(d)$, hence $v(c - c') < v(\tilde{c})$, and we compute

$$v(c_0 - c) = \min(v(\tilde{c}), v(c - c')) = v(c - c') < v(d) .$$

We have shown that c_0 realizes all sentences in $\mathbf{S}_0$, as contended. The further assertions of the lemma follow as in the corresponding lemma on value–immediate approximation types that we have proved above. □

To show that in any case a given approximation type is realized in some simple valued field extension, we have to consider whether an approximation type over K can always be prolongated to a given overfield L of K which has an infinite residue field and a dense value group (e.g. $L = \tilde{K}$). Indeed:

Lemma 11.17 *Let* $\mathbf{A}$ *be an approximation type over the valued field* K *and let* L *be a field extension of* K*. Then there exists an approximation type* $\mathbf{A}_L$ *over* L *whose restriction to* K *coincides with* $\mathbf{A}$*.* $\mathbf{A}_L$ *may be chosen to be immediate if* $\mathbf{A}$ *is immediate.*

Proof: We distinguish the following two cases:
case 1: There exists $c_0 \in L$ such that $\mathbf{A} = \mathrm{appr}(c_0, K)$. Then we take $\mathbf{A}_L$ to be $\mathrm{appr}(c_0, L)$ which by Lemma 11.5 is an immediate approximation type and whose restriction to K is just $\mathrm{appr}(c_0, K) = \mathbf{A}$.
case 2: There exists no $c_0 \in L$ such that $\mathbf{A} = \mathrm{appr}(c_0, K)$; in particular, $\mathbf{A}_\infty = \emptyset$ by Lemma 11.5. Then we define $\mathbf{A}_L$ as follows: for every $\alpha \in v(L)$, we put

$$\begin{aligned}(\mathbf{A}_L)_\alpha &= \{c \in L \mid \exists \beta \in v(K),\ \beta \geq \alpha,\ \exists c' \in \mathbf{A}_\beta : v(c' - c) \geq \alpha\}\\ (\mathbf{A}_L)^\circ_\alpha &= \{c \in L \mid \exists \beta \in v(K),\ \beta \geq \alpha,\ \exists c' \in \mathbf{A}^\circ_\beta : v(c' - c) > \alpha\}\end{aligned}$$

The proof that $\mathbf{A}_L$ is an approximation type over L is straightforward and thus left to the reader. Note that $\Lambda(\mathbf{A}_L)$ is just the minimal initial segment of $v(L)$ containing $\Lambda(\mathbf{A})$.

Now assume in addition that $\mathbf{A}$ is immediate; we want to show that $\mathbf{A}_L$ is immediate too. By our definition of $\mathbf{A}_L$, we also have $(\mathbf{A}_L)_\infty = \emptyset$. In view of Lemma 11.4 it thus suffices to prove that there is no element $c_0 \in L$ such that $c_0 \in (\mathbf{A}_L)_\alpha$ for all $\alpha \in \Lambda(\mathbf{A}_L)$. In order to derive a contradiction, let us assume the existence of such element c_0. Now we will apply Lemma 11.1 where we put $\Upsilon = \Lambda(\mathbf{A}_L)$ and $c_\Upsilon = c_0$. Using the fact that $\mathbf{A}$ is immediate by hypothesis and consequently, $\Lambda(\mathbf{A})$ and thus also $\Lambda(\mathbf{A}_L)$ have no greatest element, we now deduce from Lemma 11.1 that for all $c \in L$ and all $\beta \in v(L)$,

$$\begin{aligned}c \in \mathbf{A}_\beta &\iff v(c_0 - c) \geq \beta\\ c \in \mathbf{A}^\circ_\beta &\iff v(c_0 - c) > \beta\end{aligned}$$

which in particular shows that $\mathbf{A} = \mathrm{appr}(c_0, K)$ in contradiction to our assumption of case 2. This is the desired contradiction, and our lemma is proved. □

Corollary 11.18 *For every approximation type* $\mathbf{A}$ *over a valued field* K *there exists a simple valued field extension* $(K(x), v)|(K, v)$ *such that* $\mathbf{A} = \mathrm{appr}(x, K)$.

Proof: By the foregoing lemma, there exists a prolongation $\mathbf{A}'$ of $\mathbf{A}$ which is an approximation type over the algebraically complete valued field $\tilde{K}$. This field has the infinite residue field $\overline{\tilde{K}}$ and the divisible and thus dense value group $v(\tilde{K})$. Hence one of the above lemmata guarantees that $\mathbf{A}' = \mathrm{appr}(x, \tilde{K})$ for some simple valued field extension $(\tilde{K}(x), v)|(\tilde{K}, v)$. Since the restriction of $\mathrm{appr}(x, \tilde{K})$ to K is just $\mathrm{appr}(x, K)$, it follows that $(K(x), v)|(K, v)$ is a simple valued field extension with $\mathbf{A} = \mathrm{appr}(x, K)$. □

At this point, let us introduce an observation which is a corollary to part e) of Lemma 11.4 and will be useful for the next lemma:

Corollary 11.19 *For every approximation type* $\mathbf{A}$ *over a valued field* K *there exists an immediate approximation type* $\mathbf{A}'$ *over* K *such that* $\mathbf{S}_\geq(\mathbf{A}) \subseteq \mathbf{S}_\geq(\mathbf{A}')$.

Proof: If $\mathbf{A}$ is already immediate, there is nothing to show. If this is not the case, then we know by part e) of Lemma 11.4 and Lemma 11.5 that there is an element $c_0 \in K$ such that $c_0 \in \mathbf{A}_\alpha$ for all $\alpha \in \Lambda(\mathbf{A})$. We put $\mathbf{A}' = \mathrm{appr}(c_0, K)$. Assume that "$v(X-c) \geq v(d)$" $\in \mathbf{S}_\geq(\mathbf{A})$, i.e. $c \in \mathbf{A}_{v(d)}$ and $v(d) \in \Lambda(\mathbf{A})$. Hence $c_0 \in \mathbf{A}_{v(d)}$ and by (at 2), $v(c-c_0) \geq v(d)$ or equivalently, $c \in (\mathbf{A}')_{v(d)}$. This shows $\mathbf{S}_\geq(\mathbf{A}) \subseteq \mathbf{S}_\geq(\mathbf{A}')$, and our corollary is proved. □

For the construction of immediate approximation types, we need the following

Lemma 11.20 *Let K be a valued field and $\mathbf{S}$ a set of sentences in $\mathcal{L}(K, X)$ of the form "$v(X-c) \geq v(d)$" $(c, d \in K)$. Assume that $\mathbf{S}$ is consistent with the elementary diagram of K or equivalently, that $\mathbf{S}$ is finitely satisfiable in K. Then there exists an immediate approximation type $\mathbf{A}$ over K such that $\mathbf{S} \subset \mathbf{S}_\geq(\mathbf{A})$.*

Consequently, a given sequence of partial sums

$$s_k = \sum_{i=1}^{k} c_i$$

where c_i, $i \in I\!N$, are elements of K satisfying $\forall i: \ v(c_{i+1}) > v(c_i)$, induces an immediate approximation type through

$$\mathbf{S} = \{\, \text{“}v(X - s_k) \geq v(c_{k+1})\text{”} \mid k \in I\!N \} \,.$$

Proof: Since $\mathbf{S}$ is consistent with the elementary diagram of (K, v), it is realized in every $|K|^+$–saturated valued extension field of (K, v). Let x be an element realizing $\mathbf{S}$. Then $\mathbf{S} \subseteq \mathbf{S}_\geq(\mathrm{appr}(x, K))$. If $\mathrm{appr}(x, K)$ is not already immediate, then according to the preceding corollary we may replace it by an immediate approximation type $\mathbf{A}'$ such that

$$\mathbf{S}_\geq(\mathrm{appr}(x, K)) \subseteq \mathbf{S}_\geq(\mathbf{A}')$$

and hence also $\mathbf{S} \subseteq \mathbf{S}_\geq(\mathbf{A}')$.

In the case of the partial sums s_k, we only have to show that $\mathbf{S}$ is finitely satisfiable in K. But this is trivial since every finite subset of $\mathbf{S}$ is realized by some s_k for large enough k. □

Furthermore, we want to clear up the connection of pseudo Cauchy sequences and approximation types.

Lemma 11.21 *For every pseudo Cauchy sequence in (K, v), there exists an immediate approximation type $\mathbf{A}$ such that in every valued field $(K(x), v)$ where x realizes $\mathbf{A}$, this element x is a limit of the given pseudo Cauchy sequence.*

Conversely, let $\mathbf{A} = \mathrm{appr}(x, K)$ be an immediate approximation type over K and $(a_\rho)_{\rho<\lambda}$ an arbitrary sequence of elements $a_\rho \in \mathbf{A}_{\alpha_\rho} \setminus \mathbf{A}^\circ_{\alpha_\rho}$, where λ is a limit ordinal and α_ρ, $\rho < \lambda$, is a monotonically increasing sequence of values in $\Lambda(\mathbf{A})$ (such sequences of elements a_ρ always exist). Then $(a_\rho)_{\rho<\lambda}$ is a pseudo Cauchy sequence with limit x (in $(K(x), v)$).

Proof: Let $(a_\rho)_{\rho<\lambda}$ be a pseudo Cauchy sequence. We define $\gamma_\rho = v(a_{\rho+1} - a_\rho)$ and take

$$\mathbf{S} = \{\text{“}v(X - a_\rho) \geq \gamma_\rho\text{”} \mid \rho < \lambda\} \,.$$

Every finite subset $\mathbf{S}_0$ of $\mathbf{S}$ is realized by some a_σ for high enough $\sigma < \lambda$ (σ should just be greater than all indeces occuring in $\mathbf{S}_0$, cf. [KAP1], Lemma 2, p. 304). Hence $\mathbf{S}$ is finitely satisfiable in (K, v), and we obtain by Lemma 11.20 an immediate approximation type $\mathbf{A}$ over K such that $\mathbf{S} \subset \mathbf{S}_{\geq}(\mathbf{A})$. If the element x in some valued field extension of K realizes this approximation type $\mathbf{A}$, then it also realizes $\mathbf{S}$ and is thus a limit of the pseudo Cauchy sequence $(a_\rho)_{\rho<\lambda}$ (cf. [KAP1], Definition, p. 304).

For the converse, let $\mathbf{A} = \mathrm{appr}(x, K)$ be an immediate approximation type over K. Sequences $(a_\rho)_{\rho<\lambda}$ with $a_\rho \in \mathbf{A}_{\alpha_\rho} \setminus \mathbf{A}^\circ_{\alpha_\rho}$ and $(\alpha_\rho)_{\rho<\lambda}$ monotonically increasing, can be found as follows: Suppose that the elements a_ρ, $\rho < \sigma$, are already constructed. If the values α_ρ are cofinal in $\Lambda(\mathbf{A})$, then σ must be a limit ordinal since $\Lambda(\mathbf{A})$ has no greatest element according to Corollary 11.6; in this case, we put $\lambda = \sigma$, and the sequence $(a_\rho)_{\rho<\lambda}$ has the required properties. If the values α_ρ are not cofinal in $\Lambda(\mathbf{A})$, then we may choose a value $\alpha^*_\sigma \in \Lambda(\mathbf{A})$ which is greater than all α_ρ, $\rho < \sigma$ (the latter condition is void for the starting value $\sigma = 0$). Furthermore, we choose an element $a_\sigma \in \mathbf{A}_{\alpha^*_\sigma}$. Since $\mathbf{A}$ is immediate, there exists a value $\alpha_\sigma \in \Lambda(\mathbf{A})$ such that $a_\sigma \in \mathbf{A}_{\alpha_\sigma} \setminus \mathbf{A}^\circ_{\alpha_\sigma}$, and in view of (at 1) we conclude that $\alpha_\sigma \geq \alpha^*_\sigma$ which shows that α_σ is greater than all α_ρ, $\rho < \sigma$, which is just the required property of the new element a_σ.

Now we assume that $(a_\rho)_{\rho<\lambda}$ is a sequence with the described properties. We have to show that it is a pseudo Cauchy sequence, i.e. that $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$ whenever $\rho < \sigma < \tau < \lambda$. Indeed, by (at 1) we find that $a_\tau \in \mathbf{A}^\circ_{\alpha_\sigma}$ and $a_\sigma \in \mathbf{A}^\circ_{\alpha_\rho}$, and by (at 3○) we find that $v(a_\tau - a_\sigma) = \alpha_\sigma$ and $v(a_\sigma - a_\rho) = \alpha_\rho$. In view of $\alpha_\sigma > \alpha_\rho$, we conclude that $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$, as desired. □

Note that by the procedure described in the above proof, it is possible to construct pseudo Cauchy sequences $(a_\rho)_{\rho<\lambda}$ which have the property that the values $v(a_{\rho+1} - a_\rho)$, $\rho < \lambda$, are cofinal in $\Lambda(\mathbf{A})$. It can be shown that such sequences determine the approximation type $\mathbf{A}$ uniquely, i.e. there is no other approximation type over K from which the sequence can be derived in the described way, and there is no other approximation type which is connected to the sequence in the way as described in the first assertion of the lemma. Similarly, it can be shown that a pseudo Cauchy sequence which admits no limit in K, will yield a unique approximation type $\mathbf{A}$ in the sense of the lemma.

We should note that immediate approximation types can be easily distinguished by sequences of elements which need not be pseudo Cauchy sequences (but contain subsequences which are pseudo Cauchy sequences):

Lemma 11.22 *Let $\mathbf{A}$ be an immediate approximation type and $(\alpha_\rho)_{\rho<\lambda}$ some cofinal sequence in $\Lambda(\mathbf{A})$. Then $\mathbf{A}$ is uniquely determined by every sequence $(c_\rho)_{\rho<\lambda}$ of elements $c_\rho \in \mathbf{A}_{\alpha_\rho}$.*

Proof: According to Lemma 11.12, $\mathbf{A}$ is uniquely determined already by the sets $\mathbf{A}_\alpha$, $\alpha \in v(K)$. $\mathbf{A}$ being immediate, for every $c \in K$ there is a value $\alpha \in \Lambda(\mathbf{A})$ such that $c \in \mathbf{A}_\alpha \setminus \mathbf{A}^\circ_\alpha$. Choosing ρ such that $\alpha_\rho > \alpha$, we get $c_\rho \in \mathbf{A}_\alpha$ by (at 4), and $v(c_\rho - c) = \alpha$ by (at 2) and (at 3°). Hence every approximation type $\mathbf{A}'$ with $c_\rho \in \mathbf{A}'_{\alpha_\rho}$ will satisfy $c \in \mathbf{A}_\alpha \setminus \mathbf{A}^\circ_\alpha$ by (at 3) and (at 2°). Since $c \in K$ was arbitrary, this shows that $\mathbf{A}$ is uniquely determined by the sequence $(c_\rho)_{\rho<\lambda}$. □

Finally, we want to describe another concept which is closely connected to approximation types and which will also play a certain role in the next subsection. Any valued field extension (L,v) over (K,v) may also be viewed as a valued vector space over (K,v). Similarly, every type over (K,v) may be restricted to a type in the language of valued (K–)vector spaces by omitting all sentences of the type which are not sentences in the new language. We do not want to formalize this language here, but in any case one finds that given an $\mathcal{L}(K,X)$–sentence which uses multiplication and inversion only by elements from K, this sentence will contain the variable X only linearly: every rational function in X appearing in the sentence will just be a linear polynomial. Consequently, an element x which realizes the restricted type, will realize it already in the valued K–vector space $K+Kx$ (which is equal to K if $x\in K$ and equal to $K\oplus Kx$ if $x\notin K$). Now the elementary diagram of this vector space in the language of K–vector spaces (without valuation) is just a logical consequence of the elementary diagram of K (in the language of fields) together with a sentence expressing that $K+Kx$ is one–dimensional (if $x\in K$) resp. that $K+Kx$ is two–dimensional (if $x\notin K$). But if we view $K+Kx$ as a valued vector space, the nontrivial question arises: which sentences determine the valuation on $K+Kx$? To answer this question, we first observe that for any approximation type $\mathbf{A}$, the restriction of $\mathbf{S}(\mathbf{A})$ to the K–vector space language coincides with $\mathbf{S}(\mathbf{A})$: every assertion of $\mathbf{S}(\mathbf{A})$ is just an assertion about the valued K–vector space $K+Kx$. And in spite of the fact that it does not contain every assertion about the valuation on $K+Kx$ which we may think of, it indeed determines the valuation on $K+Kx$. To prove this fact, we first need the following lemma:

Lemma 11.23 *Let* $\mathbf{A}=\mathrm{appr}(x,K)$ *be an approximation type over* K*. If it is a value–immediate approximation type, then for every* $c\in K$ *there is an element* $d\in K$ *such that* $v(x-c)=v(d)$*, and "*$v(X-c)=v(d)$*"*$\in\mathbf{S}_=(\mathbf{A})\subset\mathbf{S}(\mathbf{A})$*. If* $\mathrm{appr}(x,K)$ *is not value–immediate, and if* $c_0\in K$ *is chosen as in part d) of Corollary 11.6, then* $v(x-c_0)\notin v(K)$*, and the following holds: for every* $c\in K$ *with* $v(c-c_0)\in\Lambda(\mathbf{A})$*, we have* $v(x-c)=v(c-c_0)$*, and "*$v(X-c)=v(c-c_0)$*"*$\in\mathbf{S}_=(\mathbf{A})\subset\mathbf{S}(\mathbf{A})$*; if on the other hand,* $v(c-c_0)\notin\Lambda(\mathbf{A})$*, we have* $v(x-c)=v(x-c_0)$*, and "*$v(X-c)=v(X-c_0)$*" is a logical consequence of* $\mathbf{S}(\mathbf{A})$*.*

Proof: If $\mathrm{appr}(x,K)$ is value–immediate, then the assertion follows by definition.

Now assume that $\mathbf{A}=\mathrm{appr}(x,K)$ is not value–immediate; hence it is residue–immediate by virtue of part b) of Corollary 11.6; in particular, this shows

$$\forall\alpha\in v(K):\ c_0\in\mathbf{A}_\alpha\Longrightarrow c_0\in\mathbf{A}_\alpha^\circ\ ,$$

and consequently, $v(x-c_0)\notin v(K)$. Let $c_0\in K$ be as in part d) of Corollary 11.6. Applying Lemma 11.1 with $\Upsilon=\Lambda(\mathbf{A})$ and $c_\Upsilon=c_0$, we find that $v(x-c)=v(c-c_0)$ whenever $v(c-c_0)\in\Lambda(\mathbf{A})$. In this case, $c\in\mathbf{A}_{v(c-c_0)}\setminus\mathbf{A}^\circ_{v(c-c_0)}$, which implies "$v(X-c)=v(c-c_0)$"$\in\mathbf{S}_=(\mathbf{A})$, as contended.

It remains to deal with the case of $v(c-c_0)\notin\Lambda(\mathbf{A})$. Then $v(x-c)=v(c-c_0)\in v(K)$ is impossible since it would yield $v(c-c_0)\in\Lambda(\mathbf{A})$ contrary to our assumption. On the other hand, $v(x-c_0)=v(c-c_0)$ is impossible since $v(x-c_0)\notin v(K)$. This shows $v(x-c)=v(x-c_0)$ as well as

$$v(x-c)<v(c-c_0)\ \text{ and }\ v(x-c_0)<v(c-c_0)\ ,$$

so that "$v(X-c)<v(c-c_0)$"$\in\mathbf{S}_<(\mathbf{A})$ and "$v(X-c_0)<v(c-c_0)$"$\in\mathbf{S}_<(\mathbf{A})$. These sentences imply the sentence "$v(X-c)=v(X-c_0)$". □

Lemma 11.24 *Let $K(x)$ be a valued field extension of K. Then the valuation on the K–vector space $K+Kx$ is uniquely determined by* $\mathrm{appr}(x,K)$*. In other words, the elementary diagram of the valued K–vector space $K+Kx$ is a logical consequence of the elementary diagram $\mathbf{D}$ of (K,v) together with $\mathbf{S}(\mathrm{appr}(x,K))$ and a sentence fixing the K–dimension of $K+Kx$.*

Proof: The only terms with constants in $K\cup\{X\}$ which can be built up in the vector space language, are just the linear polynomials in X with coefficients in K. Hence it suffices to show that "$v(a_1X+b_1)\geq v(a_2X+b_2)$" with $a_1,a_2,b_1,b_2\in K$ is a consequence of $\mathbf{S}(\mathrm{appr}(x,K))$ together with the elementary diagram $\mathbf{D}$ of (K,v), if $v(a_1x+b_1)\geq v(a_2x+b_2)$ holds in $K+Kx$, and that the same is true for "$v(a_1X+b_1)>v(a_2X+b_2)$". After division by a_1 and a_2 we have to deal with the sentence "$v(X+c_1)+v(a_1)\geq v(X+c_2)+v(a_2)$" resp. "$v(X+c_1)+v(a_1)>v(X+c_2)+v(a_2)$" where $c_1=b_1/a_1$ and $c_2=b_2/a_2$. Let us first assume that $\mathrm{appr}(x,K)$ is value–immediate. Then by the foregoing lemma, there are elements $d_1,d_2\in K$ such that the sentences "$v(X-c_i)=v(d_i)$", $i=1,2$, belong to $\mathbf{S}_=(\mathbf{A})$. If $v(a_1x+b_1)\geq v(a_2x+b_2)$ holds in $K+Kx$, then $v(d_1)+v(a_1)\geq v(d_2)+v(a_2)$ holds in K, and the corresponding sentence is contained in $\mathbf{D}$ (and similarly for "$>$" in the place of "$\geq$"). Now these sentences imply "$v(a_1X+b_1)\geq v(a_2X+b_2)$" resp. "$v(a_1X+b_1)>v(a_2X+b_2)$".

Now assume that $\mathrm{appr}(x,K)$ is not value–immediate. Let $c_0\in K$ be as in the foregoing lemma. If both $v(c_1-c_0)$ and $v(c_2-c_0)$ are elements of $\Lambda(\mathbf{A})$, then the proof works like above. If both $v(c_1-c_0)$ and $v(c_2-c_0)$ are not elements of $\Lambda(\mathbf{A})$, then $v(x-c_1)=v(x-c_0)$ and $v(x-c_2)=v(x-c_0)$, and the corresponding sentences in $\mathcal{L}(K,X)$ are logical consequences of $\mathbf{S}(\mathbf{A})$, as shown in the foregoing lemma. It follows that $v(x-c_1)=v(x-c_2)$, and if $v(a_1x+b_1)\geq v(a_2x+b_2)$ holds in $K+Kx$, then $v(a_1)\geq v(a_2)$ holds in K, and the corresponding sentence is contained in $\mathbf{D}$ (and similarly for "$>$" in the place of "$\geq$"). Consequently, the sentences "$v(a_1X+b_1)\geq v(a_2X+b_2)$" resp. "$v(a_1X+b_1)>v(a_2X+b_2)$" are shown to be consequences of $\mathbf{S}(\mathbf{A})\cup\mathbf{D}$ in the present case.

Finally, we have to discuss the remaining case where only one of the values $v(c_1-c_0)$, $v(c_2-c_0)$ belong to $\Lambda(\mathbf{A})$. We distinguish the following two cases:

<u>case 1:</u> Only $v(c_1-c_0)$ belongs to $\Lambda(\mathbf{A})$. According to the foregoing lemma, $v(x-c_1)=v(c_1-c_0)$ and $v(x-c_2)=v(x-c_0)$, and the corresponding $\mathcal{L}(K,X)$–sentences are consequences of $\mathbf{S}(\mathbf{A})$. Since $v(a_1),v(a_2)\in v(K)$ and $v(x-c_1)=v(c_1-c_0)\in v(K)$, it follows that $v(x-c_1)+v(a_1)\neq v(x-c_2)+v(a_2)$ because the foregoing lemma shows that $v(x-c_2)\notin v(K)$. So we only have to deal with the case $v(x-c_1)+v(a_1)>v(x-c_2)+v(a_2)$, i.e.

$$v(x-c_0)=v(x-c_2)<v(c_1-c_0)+v(a_1)-v(a_2)\;.$$

We choose $d\in K$ such that $v(d)=v(c_1-c_0)+v(a_1)-v(a_2)$ so that the corresponding sentence is contained in $\mathbf{D}$. Then "$v(X-c_0)<v(d)$"$\in\mathbf{S}_<(\mathbf{A})\subset\mathbf{S}(\mathbf{A})$. Both sentences together with those that we have noted above, imply "$v(a_1X-b_1)>v(a_2X-b_2)$" and thus also "$v(a_1X-b_1)\geq v(a_2X-b_2)$".

<u>case 2:</u> Only $v(c_2-c_0)$ belongs to $\Lambda(\mathbf{A})$. This is the same as the first case, except that the role of c_1 and c_2 is exchanged and thus "$<$" is replaced by "$>$". □

If $\mathrm{appr}(x,K)$ is immediate, the valuation may be determined even on much larger subvectorspaces of $K(x)$ which actually depend on the degree of $\mathrm{appr}(x,K)$. For this, see Corollary 11.38 below.

11.2 The approximation type version of the theory of pseudo Cauchy sequences.

We will now develop the theory of immediate approximation types over a valued field (K,v). In view of Lemma 11.13, we may always assume that $\mathbf{A} = \mathrm{appr}(x,K)$, i.e. that $\mathbf{A}$ is realized by x in an extension $(K(x),v)$ of (K,v).

The proof of the following lemma for arbitrary approximation types is straightforward:

Lemma 11.25 *For every $c \in K$,*

$$\begin{aligned}
\mathrm{appr}(x+c,K)_\alpha &= c + \mathrm{appr}(x,K)_\alpha \,, \\
\mathrm{appr}(x+c,K)^\circ_\alpha &= c + \mathrm{appr}(x,K)^\circ_\alpha \,, \\
\mathrm{dist}(x+c,K) &= \mathrm{dist}(x,K) \,, \\
\mathrm{appr}(cx,K)_{\alpha+v(c)} &= c \cdot \mathrm{appr}(x,K)_\alpha \,, \\
\mathrm{appr}(cx,K)^\circ_{\alpha+v(c)} &= c \cdot \mathrm{appr}(x,K)^\circ_\alpha \,, \\
v(c) + \mathrm{dist}(x,K) &= \mathrm{dist}(cx,K) \,.
\end{aligned}$$

Furthermore, we note:

Lemma 11.26 (cf. Lemma 3 of [KAP1])
Let x, y be elements of a valued field extension of the valued field (K,v). If $v(x-y) \geq \mathrm{dist}(x,K)$, then

$$\mathrm{dist}(x,K) \leq \mathrm{dist}(y,K) \,.$$

If in addition $\mathrm{appr}(x,K)$ is immediate, or if $v(x-y) > \mathrm{dist}(x,K)$, then

$$\mathrm{dist}(x,K) = \mathrm{dist}(y,K) \quad \textit{and} \quad \mathrm{appr}(x,K) = \mathrm{appr}(y,K) \,.$$

On the other hand,

$$\mathrm{appr}(x,K) = \mathrm{appr}(y,K) \Longrightarrow v(x-y) \geq \mathrm{dist}(x,K) \,.$$

If $\mathrm{dist}(x,K)$ is finitely assumed and $\mathrm{appr}(x,K)$ or $\mathrm{appr}(y,K)$ is residue–immediate, then

$$\mathrm{appr}(x,K) = \mathrm{appr}(y,K) \Longleftrightarrow v(x-y) > \mathrm{dist}(x,K) \,.$$

Furthermore, if $(K,v) \subset (L,v) \subset (L(x),v)$, then

$$\mathrm{dist}(x,L) \geq \mathrm{dist}(x,K) \,,$$

and if ">" holds, then there exists an element $y \in L$ with

$$\mathrm{appr}(x,K) = \mathrm{appr}(y,K) \quad \textit{and} \quad \mathrm{dist}(x,K) = \mathrm{dist}(y,K) \,.$$

Proof: Assume that $v(x-y) \geq \mathrm{dist}(x,K)$. Then $c \in \mathrm{appr}(x,K)_\alpha$ implies $v(x-c) \geq \alpha$ and thus $v(y-c) \geq \alpha$, because $v(x-y) \geq \mathrm{dist}(x,K) \geq \alpha$. This shows $\mathrm{appr}(x,K) \subseteq \mathrm{appr}(y,K)$ which in particular yields

$$\Lambda(\mathrm{appr}(x,K)) \subseteq \Lambda(\mathrm{appr}(y,K))$$

and thus $\mathrm{dist}(x,K) \leq \mathrm{dist}(y,K)$.

Assume that $v(x-y) > \mathrm{dist}(x,K)$. Then $v(y-c) > \mathrm{dist}(x,K)$ would imply $v(x-c) \geq \alpha > \mathrm{dist}(x,K)$ for some $\alpha \in v(K)$, a contradiction; this proves $\mathrm{dist}(x,K) \geq \mathrm{dist}(y,K)$ and thus the equality.

Assume that $v(x-y) = \mathrm{dist}(x,K)$ and that $\mathrm{appr}(x,K)$ is immediate. Then $v(y-c) > \mathrm{dist}(x,K)$ would imply $\mathrm{dist}(x,K) = v(x-c) \in v(K) \cup \{\infty\}$, which in view of part a) of Lemma 11.6 yields that $v(y-c) > \mathrm{dist}(x,K) = \infty$, a contradiction; again, this shows the desired equality.

Now assume that $v(x-y) \geq \mathrm{dist}(x,K) = \mathrm{dist}(y,K)$. Then $c \in \mathrm{appr}(x,K)_\alpha$ implies $v(x-c) \geq \alpha$ and thus $v(y-c) \geq \alpha$, because $v(x-y) \geq \mathrm{dist}(x,K) \geq \alpha$. Since the situation is symmetric, this proves

$$\mathrm{appr}(x,K)_\alpha = \mathrm{appr}(y,K)_\alpha$$

for all $\alpha \in v(K) \cup \{\infty\}$. From Lemma 11.1 we know that $\mathrm{appr}(x,K)_\alpha$ determines all sets $\mathrm{appr}(x,K)_\beta$ and $\mathrm{appr}(x,K)^\circ_\beta$ for $\alpha > \beta \in v(K)$. The same holds for y in the place of x, hence the equalities that we just proved yield $\mathrm{appr}(x,K) = \mathrm{appr}(y,K)$ if $\mathrm{dist}(x,K)$ is not finitely assumed. If the latter is not the case, our hypothesis says $v(x-y) > \mathrm{dist}(x,K)$. Then $c \in \mathrm{appr}(x,K)^\circ_\gamma$ implies $v(x-c) > \gamma$ and thus $v(y-c) > \gamma$, because $v(x-y) > \mathrm{dist}(x,K) = \gamma$. Since the situation is symmetric, this proves $\mathrm{appr}(x,K)^\circ_\gamma = \mathrm{appr}(y,K)^\circ_\gamma$ and completes the proof of $\mathrm{appr}(x,K) = \mathrm{appr}(y,K)$.

Conversely, assume $\mathrm{appr}(x,K) = \mathrm{appr}(y,K)$. Then $c \in \mathrm{appr}(x,K)_\alpha$ implies $v(x-c) \geq \alpha$ and $v(y-c) \geq \alpha$ and thus $v(x-y) \geq \alpha$; since this is true for every $\alpha \in \Lambda(\mathrm{appr}(x,K))$, this shows $v(x-y) \geq \mathrm{dist}(x,K)$. If $\mathrm{dist}(x,K)$ is finitely assumed, hence $\gamma := \mathrm{dist}(x,K) \in v(K)$, and if $\mathrm{appr}(x,K)$ is residue–immediate, then

$$\mathrm{appr}(x,K)^\circ_\gamma \neq \emptyset\ ,$$

and $c \in \mathrm{appr}(x,K)^\circ_\gamma$ implies $v(x-c) > \gamma$ and $v(y-c) > \gamma$ and thus $v(x-y) > \gamma$.

The inequality of the distances in the last part of the lemma follows from Lemma 11.7. If “$>$” holds, then there exists a value $\gamma \in v(L)$ with $\gamma > \mathrm{dist}(x,K)$ and an element $y \in \mathrm{appr}(x,L)_\gamma$, i.e. $v(x-y) \geq \gamma > \mathrm{dist}(x,K)$. Now the assertion $\mathrm{appr}(x,K) = \mathrm{appr}(y,K)$ follows from the first part of our lemma. □

Theorem 11.27 (cf. Theorem 1 of [KAP1])
Let L be an immediate extension of K. Then for every element $x \in L$ it follows that $\mathbf{A} = \mathrm{appr}(x,K)$ is immediate and in particular, $v(x) < \mathrm{dist}(x,K)$.

Proof: Let us first show that $\mathbf{A}$ is residue–immediate. Assume $c \in \mathbf{A}_\alpha$ for $\alpha \in v(K)$; we have to show that $\mathbf{A}^\circ_\alpha \neq \emptyset$. If $c \in \mathbf{A}^\circ_\alpha$, we are done. Otherwise, we know that $v(x-c) = \alpha \in v(K)$, hence there exists $d \in K$ such that $v(d(x-c)) = 0$; now $\overline{d(x-c)} \in \overline{L} = \overline{K}$, hence there exists $d' \in K$ such that $\overline{d(x-c)} - \overline{d'} = 0$ which means $v(x-c-d'd^{-1}) > -v(d) = v(x-c)$. Since $c + d'd^{-1} \in K$, this shows that $\mathbf{A}^\circ_\alpha \neq \emptyset$. If we set $c = 0$, we also obtain that $v(x) < \mathrm{dist}(x,K)$.

Now we want to show that $\mathbf{A}$ is value–immediate. Let $c \in K$. Then $v(x-c) \in v(L) \cup \{\infty\} = v(K) \cup \{\infty\}$, and by definition, $c \in \mathbf{A}_{v(x-c)} \setminus \mathbf{A}^\circ_{v(x-c)}$. □

Given an immediate approximation type $\mathbf{A} = \mathrm{appr}(x, K)$ of an element x over K and a polynomial $f \in K[X]$, the computation of

$$\mathrm{appr}(f(x), K)$$

plays a key role in the theory of approximation types. The same is to be seen in the theory of pseudo Cauchy sequences, but it is more clandestine there. We want to concede some independent interest to this problem and thus we will attack it before using it as a tool in later proofs.

In the sequel, we will consider the following situation:

$$\left.\begin{array}{ll} \mathbf{A} = \mathrm{appr}(x, K) & \text{is an immediate approximation type} \\ & \text{of degree } \mathbf{d} \text{ over } K \text{ and} \\ & p = \mathrm{char}(\overline{K}) > 0 \text{ or } p = 1 \text{ if } \mathrm{char}(\overline{K}) = 0, \\ f \in K[X] & \text{is a polynomial of degree } n = \deg(f) \leq \mathbf{d}\,. \end{array}\right\} \tag{174}$$

From this assumption it follows that $\mathbf{A}$ fixes the value of every formal derivative f_i of f ($i > 0$), since every such derivative has a degree $< \mathbf{d}$. So we may define

$$\beta_i := v(f_i(c)) \;\text{ for }\; c \nearrow x\,. \tag{175}$$

For the convenience of the reader, we will state and prove some lemmata though we will take them over from Ostrowski and Kaplansky almost without change.

Lemma 11.28 *Let $\beta_1, ..., \beta_m$ be any elements of an ordered abelian group Γ, and let $\Upsilon \subset \Gamma$ an infinite subset without greatest element. Let $t_1, ..., t_m$ be distinct positive integers. Then there exists an element $\lambda \in \Upsilon$ and an integer k $(1 \leq k \leq m)$ such that*

$$\beta_i + t_i \cdot \gamma > \beta_k + t_k \cdot \gamma$$

for all $i \neq k$ and all $\gamma \in \Upsilon$, $\gamma > \lambda$.

For the proof, see [OS], p. 371, IV. In view of Corollary 11.2 and of part a) of Corollary 11.6, we may derive the following corollary:

Corollary 11.29 *Let $\mathbf{A} = \mathrm{appr}(x, K)$ be immediate. If $\beta_1, ..., \beta_m$ and $t_1, ..., t_m$ are as in the preceding lemma, then for $c \nearrow x$,*

$$\beta_i + t_i \cdot v(x - c) \neq \beta_j + t_j \cdot v(x - c)$$

whenever $i \neq j$. Hence there exists an integer k $(1 \leq k \leq m)$ such that

$$\beta_i + t_i \cdot v(x - c) > \beta_k + t_k \cdot v(x - c)$$

for all $i \neq k$ and $c \nearrow x$.

Lemma 11.30 *If p is prime and r is a positive integer prime to p, $r > 1$, then*

$$\binom{p^t r}{p^t}$$

is prime to p, for any integer $t \geq 0$.

Proof:

$$\binom{p^t r}{p^t} = \frac{p^t r(p^t r - 1)\cdot \ldots \cdot (p^t r - p^t + 1)}{p^t(p^t-1)\cdot \ldots \cdot 1}$$

In the numerator of this fraction, the first factor $p^t r$ is divisible by precisely p^t, while the remaining factors $p^t r - m$, $1 \le m \le p^t - 1$, are not divisible by p^t. Hence, for every such factor occuring in the numerator, the corresponding factor $p^t r - m - p^t(r-1)$ which occurs in the denominator will be divisible by p to precisely the same power. This gives the desired result. □

From now on we will assume (174) together with definition (175) of the values β_i.

Lemma 11.31 *If $i = p^t$, $j = p^t r$ with $r > 1$, $(r,p) = 1$, then*

$$\beta_i + i \cdot v(x-c) < \beta_j + j \cdot v(x-c)$$

for $c \nearrow x$.

Proof: We form a Taylor expansion for $f_i(x)$ (cf. [KAP1], equation (10), p. 310):

$$f_i(x) - f_i(c) = (i+1)(x-c)f_{i+1}(c) + \ldots$$
$$\ldots + \binom{j}{i}(x-c)^{j-i} f_j(c) + \ldots + \binom{n}{i}(x-c)^{n-i} f_n(c)\ .$$

For $c \nearrow x$, the values $v(f_{i+1}(c)), \ldots, v(f_n(c))$ will be be equal to $\beta_{i+1}, \ldots, \beta_n$ as defined in (175). By Lemma 11.29, among the terms on the right hand side there will be precisely one which has least value for all $c \nearrow x$. The value of this term must then equal the value of the left hand side, which in turn is not less than β_i if $c \nearrow x$ (since then, both values $v(f_i(x))$ and $v(f_i(c))$ are fixed, but the value $v(f_i(x) - f_i(c))$ is not). It follows that the term

$$\binom{j}{i}(x-c)^{j-i} f_j(c)$$

occuring on the right hand side, must also have value not less than β_i for $c \nearrow x$. But by Lemma 11.30, $\binom{j}{i}$ has value zero. Therefore,

$$\beta_i \le (j-i)\cdot v(x-c) + \beta_j$$

for $c \nearrow x$. Since $\Lambda(\mathbf{A})$ contains no greatest element according to Corollary 11.6, "$<$" will hold for $c \nearrow x$ in the above inequality, as contended. □

Corollary 11.32 *There is an integer $\mathbf{h} = \mathbf{h}(f)$ which is a power of p (including the case $\mathbf{h} = 1 = p^0$), such that for all $c \nearrow x$,*

$$\beta_i + i \cdot v(x-c) > \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c)$$

for $i \neq \mathbf{h}$, $1 \le i \le \deg(f)$. Furthermore: for $c \nearrow x$, all values $\beta_i + i \cdot v(x-c)$, $0 \le i \le \deg(f)$, are different.

Proof: This is an immediate consequence of Corollary 11.29 and Lemma 11.31. $\square$

We will call $\mathbf{h}(f)$ the *relative approximation degree of $f(x)$ in x* (over K); it will also be denoted by

$$\mathbf{h}_K(x : f(x)) .$$

The relative approximation degree is only defined if $\deg(f) \leq \deg(\mathbf{A})$ (which in particular is true if $\mathbf{A}$ is transcendental). Note that by Corollary 11.53 below, this is always the case for algebraically maximal ground field K and nontrivial $\mathbf{A}$. If defined, $\mathbf{h}_K(x : f(x))$ is always 1 or a power of $p = \mathrm{char}(\overline{K})$.

The following lemmata will give more detailed information on $\mathbf{h}(f)$.

Lemma 11.33 *Let $v(x - c) \geq 0$ for $c \nearrow x$. If k is an integer such that among all β_i, $i > 0$, the value of β_k is minimal, then $\mathbf{h}(f) \leq k$.*

Proof: By assumption, we have $\beta_j - \beta_k \geq 0$ for all $j > 0$. $\mathbf{h} = \mathbf{h}(f)$ is the unique index with

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) < \beta_i + i \cdot v(x - c)$$

for every $i > 0$, $i \neq \mathbf{h}$, and every $c \nearrow x$. Thus

$$0 \leq \beta_{\mathbf{h}} - \beta_k \leq (k - \mathbf{h}) \cdot v(x - c)$$

for $c \nearrow x$ which in view of $v(x-c) \geq 0$ for $c \nearrow x$ yields $k-\mathbf{h} \geq 0$ which is the assertion. $\square$

Lemma 11.34 *Let $p \geq 2$ and*

$$f(x) = \sum_{i=1}^{n} c_i x^i \ \in K[x]$$

with $v(x) = 0$. Assume that there exists $i_0 > 0$ such that $v(c_{i_0}) < v(c_i)$ for all $i > 0$, $i \neq i_0$, and write $i_0 = jp^m$ with $(p, j) = 1$. Then $\mathbf{h}(f) \leq p^m$, and for every c with $v(c) = 0$ we have $v(f_{\mathbf{h}}(c)) \geq v(c_{i_0})$.

Proof: For every $k \geq 0$ we have

$$f_k(X) = \sum_{i=k}^{n} \binom{i}{k} c_i X^{i-k} ,$$

hence for $v(c) = 0$ and $k \geq 1$,

$$v(f_k(c)) \geq \min_{k\leq i\leq n} v\left(\binom{i}{k} c_i c^{i-k}\right) \geq v(c_{i_0}) .$$

By Lemma 11.30, the binomial coefficient

$$\binom{jp^m}{p^m}$$

is not divisible by p which shows that

$$v(f_{p^m}(c)) = v(c_{i_0}) .$$

Observe that for all $c \nearrow x$ we have $v(c) = 0$ since $v(x) = 0$; this yields

$$\beta_{p^m} = v(c_{i_0}) \leq \beta_i$$

for all $i > 0$. The foregoing lemma now gives our assertion. $\square$

Corollary 11.35 *Let $p \geq 2$, $v(x) = 0$ and $e \geq 1$. Assume that all nonzero coefficients c_i of f, $i > 0$, have different values and that for all i with $p^e | i$, the coefficient c_i is equal to zero. Then $\mathbf{h}(f) < p^e$.*

In the following lemmata, we will give a more precise version of Kaplanskys Lemmata 8 and 9.

Lemma 11.36 *For the relative approximation degree $\mathbf{h} = \mathbf{h}(f)$ as defined above, the following holds:*

$$\forall c \in K,\ c \nearrow x:\ v(f(x) - f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) < \beta_i + i \cdot v(x - c) \tag{176}$$

whenever $i \neq \mathbf{h}$, $1 \leq i \leq \deg(f)$; hence

$$\forall c \in K,\ c \nearrow x:\ c \in \mathrm{appr}(x, K)_\gamma \Longleftrightarrow f(c) \in \mathrm{appr}(f(x), K)_{\beta_{\mathbf{h}} + \mathbf{h} \cdot \gamma} \tag{177}$$

In particular,

$$\mathrm{dist}(f(x), K) \geq \mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K)\ ,$$

and $\mathrm{dist}_K(f(x), f(K))$ is not finitely assumed by an element of K.

Proof: We consider the Taylor expansion

$$f(x) - f(c) = (x - c) f_1(c) + \ldots + (x - c)^n f_n(c) \tag{178}$$

with $c \in K$. According to Corollary 11.32, the value $v(f(x) - f(c))$ of the left hand side will be equal to the value $\mathbf{h} \cdot v(x - c) + v(f_{\mathbf{h}}(c))$ of the right hand side, for $c \nearrow x$. This yields equations (176) and (177) as well as

$$\mathrm{dist}_K(f(x), f(K)) \geq \mathbf{h} \cdot \mathrm{dist}(x, K) + \beta_{\mathbf{h}}\ ,$$

while $\mathrm{dist}(f(x), K) \geq \mathrm{dist}_K(f(x), f(K))$ follows from the definition of the distance. It remains to prove

$$\mathrm{dist}_K(f(x), f(K)) = \mathbf{h} \cdot \mathrm{dist}(x, K) + \beta_{\mathbf{h}}\ .$$

If $\mathrm{dist}(x, K) = \infty$, this equality follows immediately from the inequality that we have already proved. So let us assume from now on that $\mathrm{dist}(x, K) < \infty$. In order to deduce a contradiction, assume that there exists an element $c_0 \in K$ such that

$$v(f(x) - f(c_0)) \geq \mathbf{h} \cdot \mathrm{dist}(x, K) + \beta_{\mathbf{h}}\ ,$$

or equivalently,

$$v(f(x) - f(c_0)) > v(f(x) - f(c))$$

for all $c \nearrow x$, hence

$$\begin{aligned} v(f(c_0) - f(c)) &= \min\{v(f(x) - f(c)), v(f(x) - f(c_0))\} \\ &= v(f(x) - f(c)) \end{aligned}$$

for all $c \nearrow x$. Replacing x by c_0 in the Taylor expansion (178) and using the above computation for the value of $v(f(x)-f(c))$, we find

$$\begin{aligned} v((c-c_0)f_1(c_0)+\ldots+(c-c_0)^n f_n(c_0)) &= v(f(c_0)-f(c)) = v(f(x)-f(c)) \\ &= \mathbf{h}\cdot v(x-c)+v(f_{\mathbf{h}}(c)) \end{aligned}$$

for all $c \nearrow x$. By our assumption $\mathrm{dist}(x,K)<\infty$, we know from part a) of Corollary 11.6 that $v(x-c_0)<\mathrm{dist}(x,K)$, hence $v(c-c_0)$ will be equal to $v(x-c_0)$ and thus fixed for all $c \nearrow x$. On the other hand, the value $\mathbf{h}\cdot v(x-c)+v(f_{\mathbf{h}}(c))$ is not fixed for $c \nearrow x$, so we may conclude that the value

$$v(f_1(c_0)) + (c-c_0)f_2(c_0)+\ldots+(c-c_0)^{n-1}f_n(c_0))$$

is not fixed for $c \nearrow x$ which proves the existence of a polynomial of degree $n-1$ whose value is not fixed by $\mathbf{A}=\mathrm{appr}(x,K)$. But $n-1=\deg(f)-1$ which by hypothesis is smaller than the degree of $\mathbf{A}$, a contradiction. This proves the desired equality, and it shows that $\mathrm{dist}_K(f(x),f(K))$ is not finitely assumed. □

Lemma 11.37 *Let* $\mathbf{A}$*,* f *and* $\mathbf{h}$ *be as above. If* $\mathbf{A}$ *fixes the value of* f*, then* $v(f(x)) = v(f(c))$ *for* $c \nearrow x$*, and if* $\deg(f)<\mathbf{d}$*, then* $\mathrm{appr}(f(x),K)$ *is an immediate approximation type over* K *with*

$$\mathrm{dist}(f(x),K)=\mathrm{dist}_K(f(x),f(K))=\beta_{\mathbf{h}}+\mathbf{h}\cdot\mathrm{dist}(x,K)\;;$$

$\mathrm{appr}(f(x),K)$ *is thus determined by (177), in the sense of Lemma 11.22.*

Proof: Let $\mathbf{A}$ fix the value of f. In this case, the value of the right hand side of (178) can only eventually increase if $v(f(x))=v(f(c))$ for $c \nearrow x$. Now assume $\deg(f)<\mathbf{d}$. To show $\mathrm{dist}(f(x),K)=\mathrm{dist}_K(f(x),f(K))$, we have to show that for every $c'\in K$ there exists an element $c\in K$ such that $v(f(x)-f(c))\geq v(f(x)-c')$. Since $\deg(f(X)-c')=\deg(f(X))<\mathbf{d}$, $\mathbf{A}$ fixes the value of $f-c'$. From what we have just shown, putting $f-c'$ in the place of f, we infer that $v(f(x)-c')=v(f(c)-c')$ for $c \nearrow x$. Consequently, for such element $c\in K$ we get

$$v(f(x)-f(c))\geq \min\{v(f(x)-c'),v(f(c)-c')\}=v(f(x)-c')\;,$$

as desired. It follows that $\mathrm{dist}(f(x),K)=\beta_{\mathbf{h}}+\mathbf{h}\cdot\mathrm{dist}(x,K)$ is not finitely assumed by an element of K, since $\mathrm{dist}(x,K)$ is not finitely assumed by an element of K, as is shown in part a) of Corollary 11.6. According to part c) of Corollary 11.6, this shows that $\mathrm{appr}(f(x),K)$ is residue–immediate. To show that it is value–immediate, assume $c'\in K$. By what we have shown above, there is $c\in K$ such that $\alpha:=v(f(x)-c')=v(f(c)-c')\in v(K)$, hence $c'\in\mathrm{appr}(f(x),K)_\alpha\setminus\mathrm{appr}(f(x),K)^\circ_\alpha$. This proves that $\mathrm{appr}(f(x),K)$ is value–immediate. Altogether, we have shown that $\mathrm{appr}(f(x),K)$ is immediate. □

Corollary 11.38 *Let* $\mathbf{A}=\mathrm{appr}(x,K)$ *be an immediate approximation type of degree* $\mathbf{d}$ *over* K*. Then the valued vector space over* K *generated by the set* $\{x^i\mid 0\leq i<\mathbf{d}\}$ *has* $v(K)$ *as its value set and* $\overline{K}$ *as its residue set; in this sense, it is immediate over* K*; moreover, its valuation is uniquely determined by* $\mathbf{A}$*. In particular, if* $\mathbf{d}=\infty$ *or if* $\mathbf{d}=[K(x):K]<\infty$*, then* $K[x]|K$ *is immediate and the same is consequently true for* $K(x)|K$*.*

Proof: Firstly, we have to show that $v(f(x)) \in v(K)$ for every $f \in K[X]$ with $\deg(f) < \mathbf{d}$. But this is an immediate consequence of the preceding lemma which states that $v(f(x)) = v(f(c)) \in v(K)$ for $c \nearrow x$.
Secondly, we have to show $\overline{f(x)} \in \overline{K}$ for every $f \in K[X]$ with $\deg(f) < \mathbf{d}$ and $v(f(x)) = 0$. We choose $c \nearrow x$ such that

$$0 = v(f(x)) = v(f(c)) .$$

From the preceding lemma we know that $\mathrm{appr}(f(x), K)$ is an immediate approximation type over K; in particular, we may infer from part a) of Corollary 11.6 that $\mathrm{dist}(f(x), K)$ is not assumed by an element of K. Hence $0 \leq v(f(x) - f(c)) < \mathrm{dist}(f(x), K)$ and consequently there exists an element $c_f \in K$ such that $v(f(x) - c_f) > 0$ which yields $\overline{f(x)} = \overline{c_f} \in \overline{K}$. □

Lemma 11.39 *Let* $\mathbf{A}$, f *and* $\mathbf{h}$ *be as above. Assume that* $\mathbf{A}$ *does not fix the value of* f, *hence* $\deg(f) = \mathbf{d}$. *Then*

$$v(f(x)) \geq v(f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c) \quad \textit{for every } c \nearrow x$$

and consequently,

$$v(f(x)) \geq \mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K) .$$

In particular, this yields $v(f(x)) > v(f(c))$ *for all* $c \in K$, $c \neq x$.

If $\mathbf{d} = 1$ *(i.e.* $\mathbf{A}$ *is trivial) and* $f(X) = X - c_0$ *with* $c_0 \in K$, *then* $c_0 = x$ *and* $\mathbf{A}$ *is realized by* c_0 *in* K. *Conversely, if* $\mathbf{A}$ *is realized by the element* $c_0 \in K$, *then* $x = c_0$ *and* $\mathbf{A}$ *does not fix the value of* $X - c_0$ *and thus,* $\mathbf{d} = 1$.

Proof: Assume that $\mathbf{A}$ does not fix the value of f, i.e. $\deg(f) = \mathbf{d}$ since $\deg(f) \leq \mathbf{d}$ by hypothesis (174). We rewrite (178) as follows:

$$-f(c) = (x - c) f_1(c) + \ldots + (x - c)^n f_n(c) - f(x) .$$

In view of the fact that the value of the right hand side of (178) tends to $\beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K)$, this equation shows that $\mathrm{appr}(x, K)$ fixes the value of f whenever $v(f(x)) < \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K)$. In our present case, this proves $v(f(x)) \geq \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K)$. Since by hypothesis, $\mathbf{A}$ is immediate and according to part a) of Corollary 11.6, $\mathrm{dist}(x, K)$ is thus not finitely assumed by an element of K, we have $v(f(x)) > v(f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x - c)$ for every $c \nearrow x$, $c \neq x$. Together with Lemma 11.36, this proves the first part of our lemma.

Now assume that $\mathbf{d} = 1$ and $f(X) = X - c_0$ with $c_0 \in K$. Then $\mathbf{h} = 1$ and $\beta_{\mathbf{h}} = 0$ and by what we have just proved, we conclude that $v(x - c_0) = v(f(x)) \geq \mathrm{dist}(x, K)$, and by definition of the distance, "$=$" must hold. This shows that the distance is assumed by $c_0 \in K$. But the distance is not finitely assumed since $\mathrm{appr}(x, K)$ is immediate. We conclude that $v(x - c_0) = \infty$, i.e. $x = c_0$ and $\mathrm{appr}(x, K)$ is thus realized by c_0 in K.

Conversely, assume that the approximation type $\mathbf{A}$ is realized by the element $c_0 \in K$, hence $\mathbf{A} = \mathrm{appr}(c_0, K)$ and by Lemma 11.5, $x = c_0$. For every element $c \in \mathbf{A}_\alpha \setminus \mathbf{A}_\alpha^\circ$, we have $v(f(c)) = v(c - c_0) = \alpha$ which shows that $\mathbf{A}$ does not fix the value of $X - c_0$. This also shows that $\mathbf{d} = 1$. □

In the case $\deg(f) = \mathbf{d}$ we can only say that $\mathrm{appr}(f(x), f(K))$ is determined by (177); it may happen that

$$\mathrm{dist}(f(x), K) > \mathrm{dist}_K(f(x), f(K)) \;.$$

As an example, take x to be an Artin–Schreier–root of t^{-1} over $K = \sqrt{I\!\!F_p(t)}$, valued such that $t/v = 0$. For $f(X) = X^p - X - t^{-1}$, we have $\mathrm{dist}_K(f(x), f(K)) = \mathrm{dist}(x, K) = 0$, but $\mathrm{dist}(f(x), K) = \infty$ since $f(x) = 0$.

If on the other hand, (K, v) is existentially closed in $(K(x), v)$ (which is the situation of Lemma 11.13), then these difficulties do not appear:

Lemma 11.40 *Let* $\mathbf{A}$*,* f *and* $\mathbf{h}$ *be as above. Assume that* $\mathbf{A}$ *does not fix the value of* f *and that* $(K, v) \prec_\exists (K(x), v)$*. Then* $\mathrm{dist}(f(x), K)$ *is not finitely assumed by an element of* K*,*

$$\mathrm{dist}(f(x), K) = \mathrm{dist}_K(f(x), f(K)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot \mathrm{dist}(x, K)$$

and consequently, $\mathrm{appr}(f(x), K)$ *is determined by (177), in the sense of Lemma 11.22. However,* $\mathrm{appr}(f(x), K)$ *is not value–immediate, and* $v(f(x))$ *realizes the cut* $\mathrm{dist}(f(x), K)$ *in* $v(K(x))$*.*

Proof: Assume that there are elements $c, d \in K$ such that

$$\infty > v(d) \geq \mathrm{dist}_K(f(x), f(K)) \;\text{ and }\; v(f(x) - c) \geq v(d) \;.$$

Then the existential sentence

$$\exists x : \; v(f(x) - c) \geq v(d)$$

with constants from K holds in $(K(x), v)$, and by hypothesis, it must also hold in (K, v), hence there is $c' \in K$ with $v(f(c') - c) \geq v(d)$ which yields $v(f(x) - f(c')) \geq v(d) \geq \mathrm{dist}_K(f(x), f(K))$, and by definition of the distance, equality must hold here. But this is a contradiction since this distance is not finitely assumed, according to Lemma 11.36. This contradiction shows that $\mathrm{dist}(f(x), K)$ is not finitely assumed by an element of K and that $\mathrm{dist}(f(x), K) = \mathrm{dist}_K(f(x), f(K))$. On the other hand, we know from Lemma 11.39 that

$$v(f(x)) \geq \mathrm{dist}_K(f(x), f(K)) = \mathrm{dist}(f(x), K) \;.$$

Putting $c = 0$ in our above proof, we find that $v(f(x)) \notin v(K)$ (since the existence of d would yield a contradiction), so $\mathbf{A}$ cannot be value–immediate. Moreover, we find that there exists no element of $v(K)$ which lies between $v(x)$ and $\Lambda(\mathbf{A})$ (in $v(K(x))$), i.e. $v(x)$ realizes the cut $\mathrm{dist}(f(x), K)$ in $v(K(x))$. □

It can even be shown that $f(x)$ is algebraically valuation–independent over K. For this, cf. Corollary 11.50 below.

Up to this point, we only considered polynomials of degree $\leq \mathbf{d}$. To give a description of the behaviour of polynomials of higher degree, we assume that $\mathbf{A} = \mathrm{appr}(x, K)$ is an approximation type of degree $\mathbf{d} < \infty$ and that $f \in K[X]$ is of degree $\mathbf{d}$ such that $\mathbf{A}$ does not fix the value of f. Any polynomial $g \in K[X]$ of arbitrary degree may be written in a unique way as

$$g(X) = f(X)^k c_k(X) + ... + f(X) c_1(X) + c_0(X) \tag{179}$$

with polynomials $c_i \in K[X]$ of degree $< \mathbf{d}$ and k a nonnegative integer.

Lemma 11.41 *Let $g(x)$ be given as in (179). Then there exists an integer m, $0 \le m \le k$, such that for every $i \ne m$ and all $c \nearrow x$,*

$$m \cdot v(f(c)) + v(c_m(c)) = v(f(c)^m c_m(c)) < v(f(c)^i c_i(c)) = i \cdot v(f(c)) + v(c_i(c))$$

with constant values $v(c_m(c)) = v(c_m(x))$, $v(c_i(c)) = v(c_i(x))$. If **A** *fixes the value of g, then $m = 0$ and*

$$v(g(x)) = v(g(c)) = v(c_0(c)) = v(c_0(x))$$

for $c \nearrow x$. If **A** *does not fix the value of g, then $m \ge 1$ and*

$$v(g(x)) > v(g(c)) = m \cdot v(f(c)) + v(c_m(c))$$

for $c \nearrow x$, $c \ne x$.

Proof: From Lemma 11.39 we know that $v(f(c)) = \mathbf{h} \cdot v(x - c) + \beta_{\mathbf{h}}$ for every $c \nearrow x$, where the value $\beta_{\mathbf{h}} = v(f_{\mathbf{h}}(c))$ is fixed for $c \nearrow x$. Since the c_i are polynomials of degree $< \mathbf{d}$, their values will also be fixed for $c \nearrow x$. Hence we may write

$$v(f(c)^i c_i(c)) = i \cdot \mathbf{h} \cdot v(x - c) + \beta'_i \; .$$

From Corollary 11.29 we infer that for $c \nearrow x$, there is exactly one integer m, $1 \le m \le k$, such that $v(f(c)^m c_m(c))$ is the least value among all values $v(f(c)^i c_i(c))$, $1 \le i \le k$. This value is not fixed for $c \nearrow x$. Consequently, if **A** fixes the value of g, it must be greater than the (fixed) value of $c_0(c)$ for $c \nearrow x$, which yields $v(g(c)) = v(c_0(c)) = v(c_0(x))$. From Lemma 11.39 we know that $v(f(x)) > v(f(c))$ for all $c \nearrow x$, $c \ne x$, hence $v(f(x)^i c_i(x)) > v(f(c)^i c_i(c))$ for all $1 \le i \le k$ and $c \nearrow x$, $c \ne x$. This proves $v(g(x)) = v(g(c))$.

If **A** does not fix the value of g, then $v(f(c)^m c_m(c)) < v(c_0(c))$ and

$$v(g(c)) = v(f(c)^m c_0(c)) = m \cdot v(f(c)) + v(c_m(c)) = m \cdot \mathbf{h} \cdot v(x - c) + \beta'_m$$

for all $c \nearrow x$. The inequality $v(g(x)) > v(g(c))$ for $c \nearrow x$ is seen as follows. We have stated already that $v(f(x)^i c_i(x)) > v(f(c)^i c_i(c))$ for all $1 \le i \le k$ and $c \nearrow x$, $c \ne x$, which implies

$$\begin{aligned} v(g(x)) &\ge \min\{v(f(x)^k c_k(x)), ..., v(f(x)c_1(x)), v(c_0(x))\} \\ &> \min\{v(f(c)^k c_k(c)), ..., v(f(c)c_1(c)), v(c_0(c))\} = v(g(c)) \end{aligned}$$

in view of $v(f(c)^m c_m(c)) < v(c_0(c))$. This completes the proof of our lemma. □

This lemma also gives some information about $\mathrm{appr}(g(x), K)$:

Corollary 11.42 *Let the assumptions be as in the foregoing lemma. If* **A** *fixes the value of $g - c^*$ for every $c^* \in K$, then*

$$\mathrm{appr}(g(x), K) = \mathrm{appr}(c_0(x), K) \; .$$

If on the other hand, there exists an element $c^* \in K$ such that **A** *does not fix the value of the polynomial $g - c^*$, then there exists an integer m, $1 \le m \le k$, such that*

$$\begin{aligned} &\forall c \in K,\; c \nearrow x : \\ &\quad c \in \mathrm{appr}(x, K)_\gamma \Longleftrightarrow g(c) \in \mathrm{appr}(g(x), K)_{m \cdot v(f(c)) + v(c_m(c))} \end{aligned} \tag{180}$$

and

$$\forall c \in K,\ c \nearrow x:\ v(g(x)-c^*) > v(g(c)-c^*)\ ;$$

and consequently,

$$\mathrm{dist}(g(x),K) \geq \mathrm{dist}_K(g(x),g(K)) \geq m\cdot \mathrm{dist}_K(f(x),f(K)) + \beta(c_m)\ ,$$

where $\beta(c_m)$ denotes the fixed value of $c_m(c)$ for $c \nearrow x$.

Proof: If **A** fixes the value of $g-c^*$ for every $c^* \in K$, then in view of

$$g(X)-c^* = f(X)^k c_k(X) + ... + f(X)c_1(X) + (c_0(X)-c^*)\ , \tag{181}$$

it follows from the foregoing lemma that $v(g(x)-c^*) = v(c_0(x)-c^*)$ for every $c^* \in K$ which proves the equality of the approximation types of $g(x)$ and $c_0(x)$ in this case.

Now assume that **A** does not fix the value of $g-c^*$ for some $c^* \in K$. In this case, we infer from the foregoing lemma that

$$\forall c \in K,\ c \nearrow x:\ v(g(x)-c^*) > v(g(c)-c^*)\ ,$$

i.e. $v(g(x)-g(c)) = v(g(c)-c^*)$ for $c \nearrow x$. Moreover, the foregoing lemma shows, again in view of (181), that there exists some integer m, $1 \leq m \leq k$, such that $v(g(c)-c^*) = m\cdot v(f(c)) + v(c_m(c))$ for $c \nearrow x$. This shows (180). It follows that $\mathrm{dist}_K(g(x),g(K)) \geq m\cdot v(f(c))+v(c_m(c))$ for all $c \nearrow x$. From Lemma 11.39 we infer that $v(f(c)) = v(f(x)-f(c))$, hence $\mathrm{dist}_K(g(x),g(K)) \geq m\cdot \mathrm{dist}_K(f(x),f(K)) + \beta(c_m)$. □

Lemma 11.41 also enables us to describe the behaviour of $v(g(c))$ for $c \nearrow x$ in full generality; this has important consequences for the question whether a given approximation type is transcendental. From Lemma 11.41 together with Lemma 11.39, we may immediately derive:

Corollary 11.43 *Let g be any polynomial. If* appr (x,K) *does not fix the value of g, then there is some natural number k and a constant $\beta \in v(K)$ such that for $c \nearrow x$,*

$$v(g(c)) = k\cdot v(x-c) + \beta\ .$$

In particular, if there exists a set $M \subset K$ such that

$$\sup\{v(x-c) \mid c \in M\} = \mathrm{dist}\,(x,K)$$

and that $v(g(c))$ is constant for $c \in M$, then appr (x,K) *fixes the value of g.*

Lemma 11.44 *Assume f to be a polynomial of degree $< \mathbf{d} = \deg(\mathrm{appr}\,(x,K))$, and let $d' \geq 1$ be a natural number such that $d'\cdot\deg(f(x)) \leq \mathbf{d}$. Then*

$$\deg(\mathrm{appr}(f(x),K)) \geq d'\ .$$

In particular, if appr (x,K) *is transcendental, then so is* appr$(f(x),K)$.

Proof: Let g be a polynomial of degree $< d'$. Then $g(f(x))$ is a polynomial of degree $< \mathbf{d}$, hence $\mathrm{appr}\,(x,K)$ fixes the value of $g(f(x))$, i.e. there exists a set $M \subset K$ such that $\sup\{v(x-c) \mid c \in M\} = \mathrm{dist}\,(x,K)$ and that $v(g(f(c)))$ is constant for $c \in M$. But from Lemma 11.37 we know that

$$\sup\{v(f(x)-f(c)) \mid c \in M\} = \mathrm{dist}_K(f(x), f(K)) = \mathrm{dist}\,(f(x),K)\ .$$

By the criterion given in the foregoing corollary it now follows that $\mathrm{appr}\,(f(x),K)$ fixes the value of g. Since g was an arbitrary polynomial of degree $< d'$, this proves $\deg(\mathrm{appr}(f(x),K)) \geq d'$. □

Another consequence of Lemma 11.41 is the following normal form for polynomials whose value is fixed by a given approximation type:

Corollary 11.45 *Let* $\mathbf{A} = \mathrm{appr}(x,K)$ *be an immediate approximation type and* $g \in K[X]$*,* $\deg(g) = n$*. Then* g *may be represented in the form*

$$g(X) = \sum_{i=1}^{n} (X-c^*)^i\, c_i^*$$

with suitable elements $c^*, c_0^*, \ldots, c_n^* \in K$ *satisfying*

$$\forall i, 1 \leq i \leq n:\ i \cdot v(x-c^*) + v(c_i^*) \geq v(c_0^*) = v(g(c^*))\ . \tag{182}$$

If the value of g *is fixed by* $\mathbf{A}$*, then the elements may be chosen such that in addition,* $v(g(c^*)) = v(g(x))$ *and "*$\geq$*" may be replaced by "*$>$*".*

Proof: For polynomials f of degree $\deg(f) \leq \mathbf{d} = \deg(\mathbf{A})$, we consider (176) from Lemma 11.36. From Lemma 11.37 and Lemma 11.39 we know that $v(f(x)) \geq v(f(c))$ and hence $v(f(c)) \leq v(f(x)-f(c))$ for $c \nearrow x$. Moreover, if the value of f is fixed by $\mathbf{A}$, we know that $v(f(c)) < v(f(x)-f(c))$ for $c \nearrow x$ since the latter value is not fixed for $c \nearrow x$, $c \neq x$. If we take c^* to be any such $c \nearrow x$ and $c_i^* = f_i(c^*)$, these facts prove the assertions of the lemma in the case of polynomials of degree $\leq \mathbf{d}$.

Now let f be an associated minimal polynomial for $\mathbf{A}$ and let g be given in the form (179). We may apply to the polynomials f, c_i what we have just proved; we find that there exists an element c^* such that (182) holds for every monomial $f(x)^i c_i(x)$ in the place of $g(x)$ (with coefficients c_i^* which are derived from the corresponding coefficients for the polynomials f and c_i). If we choose $c^* \nearrow x$, then we may assume the assertions of Lemma 11.41 for $c = c^*$. In particular, the value of $g(c^*)$ is equal to the value of exactly one monomial and consequently, it is less or equal to the value of every monomial $(x-c)^i c_i^*$ which appears in any of the representations that we have got already for the monomials $f(x)^i c_i(x)$. Summing up these representations, we thus get a representation of $g(x)$ which satisfies (182). If in addition, $\mathbf{A}$ fixes the value of g, then we know that $v(g(c^*)) = v(c_0(c^*)) = v(g(x))$; furthermore, $v(c_0(c^*))$ is smaller than the value of all other monomials, and since $\mathbf{A}$ fixes the value of c_0, inequality (182) holds with "$>$" for c_0 in the place of g. This yields that (182) also holds with "$>$" for g, if $\mathbf{A}$ fixes the value of g. □

Based on the foregoing corollary, we can give a partial answer to the question: given $f(x) \in K[x]$, which elements $g(x) \in K[x]$ have the same approximation type as $f(x)$ over K?

Corollary 11.46 *Let* $\mathbf{A}$ *be an immediate approximation type of degree* $\mathbf{d}$ *and* $f,g\in K[X]$. *If*

$$g(x)=f(x)+\sum_{i=1}^{n}(x-c^*)^i\,c_i^*$$

for some $n\in I\!N$ *and elements* $c^*,c_0^*,\ldots,c_n^*\in K$ *with*

$$i\cdot v(x-c^*)+v(c_i^*)\geq \mathrm{dist}(f(x),K)$$

for $0\leq i\leq n$, *then*

$$\mathrm{appr}(f(x),K)=\mathrm{appr}(g(x),K)\;.$$

If $\mathbf{A}$ *fixes the value of* $f-g$ *(which in particular is always the case if* $\deg(f-g)<\mathbf{d}$*, then* g *is necessarily of the above form and it may in addition be assumed that*

$$i\cdot v(x-c^*)+v(c_i^*)>v(c_0^*)$$

for $1\leq i\leq n$.

Proof: The first part follows from Lemma 11.26, whereas the second part is a consequence of the preceding corollary. □

As an immediate consequence of Lemma 11.37, Lemma 11.39 and Lemma 11.41, we get the following observation:

Corollary 11.47 *Let* $\mathbf{A}=\mathrm{appr}(x,K)$ *be an immediate approximation type and* $g\in K[X]$ *be any polynomial.* $\mathbf{A}$ *does not fix the value of* g *if and only if* $v(g(x))>v(g(c))$ *for every* $c\nearrow x$, $c\neq x$.

From this corollary and Lemma 11.39 we may derive:

Corollary 11.48 *Let* L *be an immediate extension of* K. *If* $x\in L$ *is algebraic over* K *with minimal polynomial* $f\in K[X]$, *then* $\mathrm{appr}(x,K)$ *does not fix the value of* f *and is thus of degree* $\leq[K(x):K]$. *If this degree is 1, then* $x\in K$.

Proof: $\mathrm{appr}(x,K)$ is immediate by virtue of Lemma 11.27. By hypothesis, $f(x)=0$, but $f(c)\neq 0$ for all $c\in K$, $c\neq x$. Hence $v(f(x))>v(f(c))$ for all $c\in K$, and the first assertion follows by an application of Corollary 11.47. Now assume that $\mathrm{appr}(x,K)$ is of degree 1 over K, and let $f(X)=c_1X+c_2\in K[X]$ be a polynomial of degree 1 whose value is not fixed by $\mathrm{appr}(x,K)$. Then also the value of $X-c_0$ is not fixed where $c_0=-c_2/c_1$. By Lemma 11.39 we conclude that $x=c_0\in K$. □

Again, if (K,v) is existentially closed in $(K(x),v)$ then we can prove much more about the approximation type and distance of $g(x)$:

Corollary 11.49 *Let the hypothesis be as in Lemma 11.41 and assume in addition that* (K,v) *is existentially closed in* $(K(x),v)$. *Furthermore, choose the integer* m *according to Lemma 11.41. Then for* $i\neq m$,

$$m\cdot v(f(x))+v(c_m(x))=v(f(x)^m c_m(x))<v(f(x)^i c_i(x))=i\cdot v(f(x))+v(c_i(x))$$

and thus

$$v(g(x)) = m \cdot v(f(x)) + v(c_m(x)) \ .$$

Moreover, $\mathrm{dist}(g(x), K)$ *is not finitely assumed by an element of* K*,*

$$\begin{aligned} \mathrm{dist}(g(x), K) &= \mathrm{dist}_K(g(x), g(K)) = m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m) \\ &= m \cdot \mathrm{dist}(f(x), K) + \beta(c_m) \ , \end{aligned}$$

and consequently, $\mathrm{appr}(g(x), K)$ *is determined by (180), in the sense of Lemma 11.22. Note in particular that* $\mathrm{appr}(g(x), K)$ *is of the form*

$$\mathrm{dist}(g(x), K) = m \cdot \mathbf{h} \cdot \mathrm{dist}(x, K) + \beta \ ,$$

where $\beta \in v(K)$ *and* $\mathbf{h} = \mathbf{h}(f)$ *according to Lemma 11.40.*

Proof: Assume the contrary, i.e. $v(f(x)^i c_i(x)) \geq v(f(x)^m c_m(x))$ for some $i \neq m$. From Lemma 11.41 we know that

$$v(f(c)^i c_i(c)) < v(f(c)^m c_m(c)) \tag{183}$$

for all $c \nearrow x$, i.e. for every $c \in \mathbf{A}_\alpha$ for some suitable $\alpha < \mathrm{dist}(x, K)$. We fix some $c_\alpha \in \mathbf{A}_\alpha$ and some $d_\alpha \in K$ having value $v(d_\alpha) = \alpha$. Since $(K, v) \prec_\exists (K(x), v)$ by hypothesis, the existential formula

$$\exists x : \ v(f(x)^i c_i(x)) \geq v(f(x)^m c_m(x)) \wedge v(x - c_\alpha) \geq v(d_\alpha)$$

which holds in $K(x)$ (for x), must also hold in K, i.e.

$$v(f(c)^i c_i(c)) \geq v(f(c)^m c_m(c))$$

for some c which satisfies $v(c - c_\alpha) \geq v(d_\alpha) = \alpha$ and thus

$$v(x - c) \geq \min\{v(x - c_\alpha), v(c - c_\alpha)\} \geq \alpha$$

which shows $c \in \mathbf{A}_\alpha$ by (at 3). By this, we have deduced a contradiction to (183). This contradiction proves the first part of our corollary.

In view of

$$\mathrm{dist}_K(f(x), f(K)) = \mathrm{dist}(f(x), K)$$

which we infer from Lemma 11.40, it remains now to prove

$$\mathrm{dist}(g(x), K) = \mathrm{dist}_K(g(x), g(K)) = m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m)$$

and that $\mathrm{dist}(g(x), K)$ is not finitely assumed by an element of K. Using Lemma 11.41, we choose $\alpha \in K$, $\alpha < \mathrm{dist}(x, K)$ such that $v(c_m(c)) = v(c_m(x))$ and $v(g(x)) > v(g(c)) = m \cdot v(f(c)) + v(c_m(c))$ for all $c \in K$ with $c \in \mathbf{A}_\alpha$; furthermore, we choose some $c_\alpha \in \mathbf{A}_\alpha$ and some $d_\alpha \in K$ with $v(d_\alpha) = \alpha$. To deduce a contradiction, we assume that there exist elements $c', d' \in K$ such that $v(g(x) - c') \geq v(d') \geq m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m)$. Now the existential sentence

$$\exists x \ : v(g(x) - c') \geq v(d') \wedge v(x - c_\alpha) \geq v(d_\alpha)$$

with constants from K holds in $(K(x), v)$, and by hypothesis, it must also hold in (K, v), hence there is $c \in K$ with $v(g(c) - c') \geq v(d')$ and $v(c - c_\alpha) \geq v(d_\alpha) = \alpha$. This yields $c \in \mathbf{A}_\alpha$ by (at 3), and

$$v(g(x) - g(c)) \geq v(d') \geq m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m) .$$

On the other hand, by our choice of α, we may conclude that

$$\begin{aligned} v(g(x) - g(c)) = v(g(c)) &= m \cdot v(f(c)) + v(c_m(c)) = m \cdot v(f(x) - f(c)) + v(c_m(c)) \\ &< m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m) , \end{aligned}$$

a contradiction. This contradiction shows that $\mathrm{dist}(g(x), K)$ is not finitely assumed by an element of K and that

$$\mathrm{dist}(g(x), K) \leq m \cdot \mathrm{dist}_K(f(x), f(K)) + \beta(c_m)$$

which in view of the inequalities shown in Corollary 11.42, proves the remaining equalities.

□

An immediate consequence of this corollary is the following

Corollary 11.50 *Let* $\mathbf{A}$ *and* f *be as in (174) and assume in addition that* $(K, v) \prec_\exists (K(x), v)$. *If* $\mathbf{A}$ *does not fix the value of* f, *then* $f(x)$ *is algebraically valuation–independent over* K *and even over the valued ring* $K[x]_{<\mathbf{d}} = \{g(x) \in K[x] \mid \deg(g) < \mathbf{d}\}$.

Proof: Given any $g(x)$ as in (179), we have to show that

$$v(g(x)) = \min_i v(f(x)^i c_i(x)) .$$

But this is just the assertion of the foregoing corollary. □

Now we are able to show:

Theorem 11.51 (cf. Theorem 2 of [KAP1])
For every transcendental immediate approximation type $\mathbf{A}$ *over* (K, v) *there exists a simple immediate extension* $(K(x), v)$ *such that*

$$\mathrm{appr}(x, K) = \mathbf{A} .$$

For every such extension, x *must be transcendental over* K, *and the following holds for every* $g \in K[X]$:

$$c \nearrow \mathbf{A} \Longrightarrow v(g(c)) = v(g(x)) . \tag{184}$$

If $(K(y), w)$ *is another valued field extension of* (K, v) *such that* $\mathrm{appr}(y, K) = \mathbf{A}$, *then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *which sends* x *to* y, *is valuation preserving.*

Proof: According to Lemma 11.13, there exists a purely transcendental extension $(K(x), v)$ of (K, v) such that $\mathbf{A} = \mathrm{appr}(x, K)$. By Corollary 11.38, $K(x)|K$ is an immediate extension. Given another element y such that $\mathbf{A} = \mathrm{appr}(x, K) = \mathrm{appr}(y, K)$, we want to show that the isomorphism between $K(x)$ and $K(y)$ induced by $x \mapsto y$ is valuation preserving. For this, we only have to show that $v(g(x)) = v(g(y))$ for every $g \in K[X]$. By hypothesis, $\mathbf{A}$ fixes the value of every polynomial $g \in K[X]$. From Lemma 11.37 we may thus infer that $v(g(x)) = v(g(c)) = v(g(y))$ holds for every $c \nearrow x$; this proves the desired equality and thereby also (184). In particular, $g(x) = 0 \Longleftrightarrow g(y) = 0$ and since x is transcendental over K, the element y must also be transcendental over K. □

Theorem 11.52 (cf. Theorem 3 of [KAP1])
For every algebraic immediate approximation type $\mathbf{A}$ *over* (K, v) *of degree* $\mathbf{d}$ *with associated minimal polynomial* $f(X) \in K[X]$ *and* y *a root of* f*, there exists a prolongation* v *of the valuation of* K *such that* $(K(y), v)$ *is an immediate extension of* (K, v) *with* $\mathrm{appr}(y, K) = \mathbf{A}$*. This prolongation will satisfy*

$$\forall g \in K[X], \deg(g) < \mathbf{d}: \; c \nearrow \mathbf{A} \Longrightarrow v(g(c)) = v(g(y)) \; . \tag{185}$$

If $(K(z), v)$ *is another valued field extension of* (K, v) *such that* $\mathrm{appr}(z, K) = \mathbf{A}$*, then any field isomorphism between* $K(y)$ *and* $K(z)$ *over* K *which sends* y *to* z*, preserves the valuation. (Note that there exists such an isomorphism if and only if* z *is also a root of* f*.)*

Proof: By Lemma 11.13 there exists a simple extension $(K(x), v)|(K, v)$ such that $\mathrm{appr}(x, K) = \mathbf{A}$, and by Corollary 11.38 we know that the vector space $K \oplus Kx \oplus \ldots \oplus Kx^{\mathbf{d}-1}$ equiped with the valuation v is immediate over K. Sending x to y, we get an isomorphism from this vector space onto $K(y) = K \oplus Ky \oplus \ldots \oplus Ky^{\mathbf{d}-1}$, and through this isomorphism, a vector space valuation v on $K(y)$ may be defined such that the isomorphism is valuation preserving and hence $\mathrm{appr}(y, K) = \mathrm{appr}(x, K) = \mathbf{A}$; consequently, it also satisfies property (185). We have to show that $(K(y), v)$ is a valued field. Since it is already a valued vector space, we only need to show that $v(ab) = v(a) + v(b)$ for all elements $a, b \in K(y)$. Suppose $a = r(y)$ and $b = s(y)$ with $r, s \in K[X]$ satisfying $0 \leq \deg(r) < \mathbf{d}$ and $0 \leq \deg(s) < \mathbf{d}$. By our construction,

$$v(r(x)s(x)) = v(r(x)) + v(s(x)) = v(r(y)) + v(s(y)) \; .$$

Now let $g(X) := r(X)s(X)$ and write $g(X)$ in the form (179). Since $r(y)s(y) = c_0(y)$ and $v(c_0(y)) = v(c_0(x))$, it suffices now to show that

$$v(r(x)s(x)) = v(c_0(x)) \; .$$

But this is true by Lemma 11.41 since $\mathbf{A}$ fixes the values of r and s (because their degree is $< \mathbf{d}$) and thus also the value of $g = rs$.

The last assertion of our theorem is seen like the corresponding assertion of Theorem 11.51: if $g \in K[X]$ with $\deg(g) < \mathbf{d}$ then $v(g(y)) = v(g(c)) = v(g(z))$ for every $c \nearrow x$. Hence an isomorphism over K sending y to z will preserve the valuation. □

As immediate consequences we get:

Corollary 11.53 *If K is algebraically maximal and $\mathbf{A}$ is a nontrivial approximation type over K, then $\mathbf{A}$ is transcendental and hence it fixes the value of every polynomial in $K[X]$.*

Corollary 11.54 (cf. Theorem 4 of [KAP1])
A valued field K is maximal if and only if for every immediate approximation type $\mathbf{A}$ over K there exists an element $a \in K$ such that $\mathbf{A} = \mathrm{appr}(a, K)$.

11.3 The relative approximation degree.

In this section, we will assume

$$\left.\begin{array}{l}(K(x),v)|(K,v) \text{ a nontrivial immediate extension} \\ \mathrm{appr}(x,K) \text{ a transcendental approximation type}\end{array}\right\} \qquad (186)$$

with

$$\mathrm{dist}\,(x,K) < \infty \;, \qquad (187)$$

unless stated otherwise. Note that by Corollary 11.53 and by Corollary 11.48, the condition "$\mathrm{appr}(x,K)$ is transcendental" is always fulfilled if (K,v) is algebraically maximal and $(K(x),v)|(K,v)$ is a nontrivial immediate extension. By Corollary 11.48, condition (186) implies that x is transcendental over K.

We will consider a given element

$$y \in K(x)^h \setminus K$$

and ask for the degree

$$[K(x)^h : K(y)^h] \;.$$

To treat this question and in particular to define the relative approximation degree of x over y, we require that

$$y \in K[x]^c \setminus K^c \;, \qquad (188)$$

(where "$.^c$" denotes the completion). In particular, this implies

$$\mathrm{dist}\,(y,K) < \infty$$

(and together with (186), this in turn implies (187)).

Condition (188) holds if the rank of (K,v) is 1:

Lemma 11.55 *Assume (186), (187), and let the rank of (K,v) be 1. Then every $y \in K(x)^h \setminus K$ satisfies (188).*

Proof: From Lemma 7.8, we may infer $y \in K[x]^c$. It remains to prove $y \notin K^c$. To deduce a contradiction, let us assume $y \in K^c$. Then K is dense in $K(y)$ and also in $K(y)^h$ since $K(y)$ being of rank 1 like K, it is dense in its henselization. Let $g(X) \in K(y)^h[X]$ be the minimal polynomial of x over $K(y)^h$. We may choose polynomials $\tilde{g}(X) \in K[X]$ with coefficients arbitrarily close to the corresponding coefficients of g. By the continuity of roots (cf. Theorem 4.5 of [PZ]) and in view of hypothesis 187, we find a suitable $\tilde{g}$ with a suitable root $\tilde{x} \in \tilde{K}$ such that

$$v(x - \tilde{x}) \geq \mathrm{dist}\,(x,K) \;.$$

By Lemma 11.26, this implies

$$\text{appr}\,(x, K) = \text{appr}\,(\tilde{x}, K)\ .$$

Since $\tilde{x}$ is algebraic over K, it follows by Corollary 11.48 that appr $(\tilde{x}, K)$ and hence appr (x, K) is an algebraic approximation type over K, a contradiction to hypothesis (186). This shows $y \notin K^c$. □

Condition (188) yields the existence of some $f(x) \in K[x]$ such that $v(y - f(x)) \geq$ dist (y, K); this implies

$$\text{appr}\,(y, K) = \text{appr}\,(f(x), K)$$

and dist $(f(x), K) =$ dist (y, K) (cf. Lemma 11.26). Assuming (186), (187) and (188), we define

$$\mathbf{h}_K(x : y) := \mathbf{h}_K(x : f(x))$$

and call $\mathbf{h}_K(x : y)$ the *relative approximation degree of* y *in* x (over K). Note that for $y \in K[x]$, condition (186) suffices for the definition of $\mathbf{h}_K(x : y)$. We should remark that the definition of the relative approximation degree can be reasonably generalized if the results of section 11.5 are involved; however, this should be postponed to a subsequent paper.

Lemma 11.56 $\mathbf{h}_K(x : y)$ *is welldefined, i.e. it does not depend on the choice of* $f(x)$ *(as long as* $v(y - f(x)) \geq$ dist (y, K) *is satisfied).*

Proof: If $g(x)$ is another polynomial in $K[x]$ such that $v(y - g(x)) \geq$ dist (y, K), then by Lemma 11.26,

$$\text{appr}\,(g(x), K) = \text{appr}\,(y, K) = \text{appr}\,(f(x), K)\ ,$$

hence $\mathbf{h}(g) = \mathbf{h}(f)$ by the following lemma. □

Lemma 11.57 *Assume (186) and let* $f(x), g(x) \in K[x]$.
If appr $(f(x), K) =$ appr $(g(x), K)$, *then* $\mathbf{h}(f) = \mathbf{h}(g)$.

Proof: From appr $(f(x), K) =$ appr $(g(x), K)$ it follows that

$$v(f(x) - g(x)) \geq \text{dist}\,(f(x), K) = \text{dist}\,(g(x), K)\ .$$

Since appr(x, K) is transcendental, it fixes the value of $f - g$, hence by Lemma 11.37,

$$v(f(c) - g(c)) = v(f(x) - g(x)) \geq \text{dist}\,(f(x), K) \quad \text{for} \quad c \nearrow x\ .$$

This implies

$$g(c) \in \text{appr}\,(f(x), K)_\alpha \Longleftrightarrow f(c) \in \text{appr}\,(f(x), K)_\alpha$$

for $c \nearrow x$. Adopting the notation of Lemma 11.36 with $\mathbf{h} = \mathbf{h}(f)$, it follows from this lemma together with the above equivalence and the equality of the approximation types that

$$\begin{aligned} c \in \text{appr}\,(x, K)_\gamma \quad &\Longleftrightarrow \quad f(c) \in \text{appr}\,(f(x), K)_{\beta_{\mathbf{h}} + \mathbf{h}\cdot\gamma} \\ &\Longleftrightarrow \quad g(c) \in \text{appr}\,(f(x), K)_{\beta_{\mathbf{h}} + \mathbf{h}\cdot\gamma} = \text{appr}\,(g(x), K)_{\beta_{\mathbf{h}} + \mathbf{h}\cdot\gamma} \end{aligned}$$

for $c \nearrow x$. (Here $\beta_{\mathbf{h}}$ denotes the constant value of $f_{\mathbf{h}}(c)$ for $c \nearrow x$). From this together with the cited lemma we may conclude that $\mathbf{h} = \mathbf{h}(g)$ which gives the assertion. □

Lemma 11.58 *In the situation (186),*

$$[K(x)^h : K(f(x))^h] \leq \mathbf{h}_K(x : f(x)) \; .$$

Proof: We consider the Taylor expansion of $f(x)$ for an arbitrary $c \in K$:

$$f(x) = \sum_{i=0}^{\deg(f)} f_i(c)(x-c)^i \; .$$

From Lemma 11.32 we know that for $c \nearrow x$ and $1 \leq i \leq \deg(f)$, $i \neq \mathbf{h}$, we have

$$f_i(c) + i \cdot v(x-c) > f_{\mathbf{h}}(c) + \mathbf{h} \cdot v(x-c) \; .$$

We choose such $c \in K$ and an element $d \in K$ with $v(d) = -v(x-c)$ and put $z = d \cdot (x-c)$; hence $v(z) = 0$ and $K(x) = K(z)$. Furthermore, we let

$$f^*(Z) = \sum_{i=0}^{\deg(f)} f_i(c) d^{-i} Z \; ;$$

hence $f^*(z) = f(x)$. Let us consider the polynomial

$$F(Z) = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} (f^*(Z) - f^*(z)) \in \mathcal{O}_{K(f(x))}[Z] \; .$$

We find

$$\overline{F(Z)} = \overline{Z}^{\mathbf{h}} - \overline{z}^{\mathbf{h}} \; .$$

Using a “strong” version of Hensel’s Lemma (cf. [RIB1], Théorème 4, version 3, on p. 186) we deduce that there is a factorization

$$F(Z) = G(Z)H(Z)$$

over $K(f(x))^h$ with

$$\overline{G(Z)} = \overline{Z}^{\mathbf{h}} - \overline{z}^{\mathbf{h}}$$

and

$$\deg G(Z) = \deg \overline{G(Z)} = \mathbf{h} \; .$$

Every zero of $F(Z)$ that has residue $\overline{z}$ cannot be a zero of $H(Z)$ since $\overline{H(Z)} = 1$, hence it must appear as a zero of $G(Z)$; in particular, $G(z) = 0$. Since $G(Z) \in K(f(x))^h[Z]$ and $\deg G(Z) = \mathbf{h}$, this shows that

$$[K(x)^h : K(f(x))^h] = [K(z)^h : K(f(x))^h] \leq \mathbf{h} = \mathbf{h}_K(x : f(x)) \; ,$$

as asserted. □

Adding assumptions (187) and (188), we can prove the above lemma also for y in the place of $f(x)$. For the proof, we need the following

Lemma 11.59 *Assume (186), (187), (188), and let* $v(y-f(x)) \geq \operatorname{dist}(y,K)$. *Then there exists an element* $z \in \widetilde{K(y)}$ *such that*

$$[K(y,z)^h : K(y)^h] \leq \mathbf{h} = \mathbf{h}_K(x:y)$$

and

$$v(x-z) \geq \frac{1}{\mathbf{h}}\left(v(y-f(x)) - \beta_{\mathbf{h}}\right) \ ,$$

where $\beta_{\mathbf{h}}$ *denotes the constant value of* $f_{\mathbf{h}}(c)$ *for* $c \nearrow x$.

Proof: We put $r := y - f(x)$. We choose $c, d \in K$ and $F(X)$ as in the foregoing proof. Then

$$\begin{aligned} v(r) \geq \operatorname{dist}(y,K) > v(y-f(c)) &= v(f(x)-f(c)) \\ &= v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) = v(f_{\mathbf{h}}(c)d^{-\mathbf{h}}) \ . \end{aligned}$$

This shows that

$$F^\circ(Z) := F(Z) - \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} r = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)}(f^*(Z) - y) \in \mathcal{O}_{K(y)}[Z]$$

has the same residue as $F(Z)$. We find as for $F(Z)$, that $F^\circ(Z)$ admits a factorization

$$F^\circ(Z) = G^\circ(Z) H^\circ(Z)$$

over $K(y)^h$ with $\overline{G^\circ(Z)} = \overline{Z}^{\mathbf{h}} - \overline{z}^{\mathbf{h}}$, G° normed, $\deg G^\circ(Z) = \deg \overline{G^\circ(Z)} = \mathbf{h}$ and $\overline{H^\circ(Z)} = 1$. In particular,

$$v(F^\circ(t)) = v(G^\circ(t)) \ \text{ if } \ v(t) \geq 0 \ .$$

Consequently, from

$$-f_{\mathbf{h}}(c) d^{-\mathbf{h}} \cdot F^\circ(d(x-c)) = r$$

it follows that

$$v(d^{\mathbf{h}} r) - \beta_{\mathbf{h}} = v(F^\circ(d(x-c))) = v(G^\circ(d(x-c))) \ .$$

Hence there must exist a root z_{j_0} of

$$G^\circ(Z) = \prod_{1 \leq j \leq \mathbf{h}} Z - z_j \ , \quad z_j \in \widetilde{K(y)}$$

with

$$v(d(x-c) - z_{j_0}) \geq \frac{1}{\mathbf{h}}\left(v(d^{\mathbf{h}} r) - \beta_{\mathbf{h}}\right)$$

which is equivalent to

$$v(x - (d^{-1} z_{j_0} + c)) \geq \frac{1}{\mathbf{h}}\left(v(r) - \beta_{\mathbf{h}}\right) \ .$$

Now $z := d^{-1} z_{j_0} + c$ is the element of our assertion, since it satisfies $K(y,z) = K(y, z_{j_0})$ and thus $[K(y,z)^h : K(y)^h] \leq \mathbf{h}$. □

Lemma 11.60 *Assume (186), (187), (188). If* $K(x)^h | K(y)^h$ *is separable, then*

$$[K(x)^h : K(y)^h] \leq \mathbf{h}_K(x:y) \ .$$

Proof: Let

$$\alpha := \max\{v(\sigma x - x) \mid \sigma \in \mathrm{Gal}(K(y)^h) \wedge \sigma x \neq x\} .$$

Then $\alpha < \infty$. Using the foregoing lemma, we choose $f(x) \in K[x]$ and $z \in \widetilde{K(y)}$ such that

$$v(y - f(x)) \geq \mathrm{dist}\,(y, K) \ , \ \ v(x - z) \geq \frac{1}{\mathbf{h}}\,(v(y - f(x)) - \beta_{\mathbf{h}}) > \alpha \ ,$$

and $[K(y,z)^h : K(y)^h] \leq \mathbf{h} = \mathbf{h}_K(x : y)$. In view of our separability condition, we may deduce by Krasners Lemma that $x \in K(y)^h(z)$, hence $[K(x,y)^h : K(y)^h] \leq \mathbf{h}$. Since $y \in K(x)^h$ by assumption, $K(x,y)^h = K(x)^h$ and thus $[K(x)^h : K(y)^h] \leq \mathbf{h}$, as asserted.

□

Lemma 11.61 *Assume (186), (187), (188). Then (186), (187) hold also for y in the place of x. So if $z \in K(y)^h$ with $z \in K[y]^c \setminus K^c$, then $\mathbf{h}_K(y : z)$ is defined. In this situation, $\mathbf{h}_K(x : z) = \mathbf{h}_K(x : y) \cdot \mathbf{h}_K(y : z)$.*

Proof: It was already mentioned that (188) implies that (187) holds for y in the place of x. Moreover, as a subextension of the immediate extension $K(x)^h|K$, the extension $K(y)|K$ is also immediate. For the definition of $\mathbf{h}_K(x : y)$ we have already used the fact that there exists some polynomial $f(x)$ such that $\mathrm{appr}\,(y,K) = \mathrm{appr}\,(f(x),K)$; by Lemma 11.44, this approximation type is transcendental since $\mathrm{appr}\,(x,K)$ is. We have proved that (186) too holds for y in the place of x.

Let us now prove the multiplicativity. Since $\mathbf{h}_K(y : z) = \mathbf{h}_K(y : g(y))$ whenever $v(z - g(y)) \geq \mathrm{dist}\,(z,K)$, it suffices to show our assertion under the additional assumption $z = g(y) \in K[y]$. Furthermore, because of $y \in K[x]^c$ we may choose $f(x) \in K[x]$ so near to y that $v(g(y) - g(f(x))) \geq \mathrm{dist}\,(g(y),K)$, hence it suffices to show our assertion under the assumption $y = f(x) \in K[x]$, $z = g(f(x)) \in K[x]$. Since by assumption, K is algebraically maximal, $\mathrm{appr}\,(x,K)$ fixes the value of every polynomial over K, and thus we know from Lemma 11.37 that $f(c) \nearrow f(x)$ whenever $c \nearrow x$. Again since K is algebraically maximal, $\mathrm{appr}\,(f(x),K)$ fixes the value of every polynomial over K, and thus for $f(c) \nearrow f(x)$,

$$\begin{aligned} v(g(f(x)) - g(f(c))) &= v(g_{\mathbf{h}_1}(f(c))) + \mathbf{h}_1 \cdot v(f(x) - f(c)) \\ &= v(g_{\mathbf{h}_1}(f(c))) + \mathbf{h}_1 \cdot (v(f_{\mathbf{h}_2}(c)) + \mathbf{h}_2 \cdot v(x - c)) \\ &= \beta + \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot v(x - c) \end{aligned}$$

where $\mathbf{h}_1 = \mathbf{h}_K(f(x) : g(f(x)))$, $\mathbf{h}_2 = \mathbf{h}_K(x : f(x))$ and $\beta = v(g_{\mathbf{h}_1}(f(c))) + \mathbf{h}_1 \cdot v(f_{\mathbf{h}_2}(c))$. This shows

$$\mathbf{h}_K(x : g(f(x))) = \mathbf{h}_1 \cdot \mathbf{h}_2 = \mathbf{h}_2 \cdot \mathbf{h}_1 = \mathbf{h}_K(x : f(x)) \cdot \mathbf{h}_K(f(x) : g(f(x))) \ ,$$

as asserted.

□

Corollary 11.62 *Assume (186), (187), and let the rank of (K,v) be 1. Then*

$$K(x)^h = K(y)^h$$

implies

$$\mathbf{h}_K(x : y) = 1 \ .$$

Proof: By Lemma 11.55, also (188) holds; by virtue of Lemma 11.61, (186), (187) hold also for y in the place of x. Again by Lemma 11.55, it follows that $x \in K[y]^c \setminus K^c$. If $K(x)^h = K(y)^h$, we have by the foregoing lemma that $1 = \mathbf{h}_K(x:x) = \mathbf{h}_K(x:y)\cdot\mathbf{h}_K(y:x)$, which yields $\mathbf{h}_K(x:y) = 1$. □

Lemma 11.63 *Assume (186), (187), and let the rank of* (K,v) *be 1. If*

$$K(x)^h|K(y)^h$$

is not separable, then

$$\mathbf{h}_K(x:y) \geq p = \mathrm{char}(K)\ .$$

Proof: If $K(x)^h|K(y)^h$ is not separable, then there exists a subextension $L|K(y)^h$ such that $x \notin L$ and $(K(x)^h)^p \subseteq L$; the latter yields $K(x^p)^h \subseteq L$. Since $K(x)^h = L(x)$ is an extension of $K(x^p)^h$ of degree p, we may conclude $L = K(x^p)^h$, hence $y \in K(x^p)^h$. By Lemma 7.8, $y \in K[x^p]^c$. By Lemma 11.55, $y \in K[x]^c \setminus K^c$ and thus $y \in K[x^p]^c \setminus K^c$. Hence by Lemma 11.61,

$$\mathbf{h}_K(x:y) = \mathbf{h}_K(x:x^p)\cdot\mathbf{h}_K(x^p:y) \geq p$$

since $\mathbf{h}_K(x:x^p) = p$. □

Corollary 11.64 *Assume (186), (187), and let the rank of* (K,v) *be 1. Then*

$$K(x)^h = K(y)^h \iff \mathbf{h}_K(x:y) = 1\ .$$

Proof: By the foregoing lemma, $\mathbf{h}_K(x:y) = 1$ implies that $K(x)^h|K(y)^h$ is separable. Since the rank of (K,v) is assumed to be 1, y satifies (188) by virtue of Lemma 11.55. Now our assertion follows from Lemma 11.60 and Corollary 11.62. □

From now on, we will always assume (186), (187), (188). As usual, let $f(x) \in K[x]$ such that $v(y - f(x)) \geq \mathrm{dist}\,(y,K)$. An element $d \in K$ will be called *approximation coefficient of y in x* (over K), if

$$\forall c \nearrow x:\ v(f(x) - f(c)) < v(f(x) - f(c) - d\cdot(x-c)^{\mathbf{h}}) \tag{189}$$

with $\mathbf{h} = \mathbf{h}_K(x:y)$.

Lemma 11.65 *If d satisfies (189) for some $f(x)$ with $v(y - f(x)) \geq \mathrm{dist}\,(y,K)$, then it satisfies (189) for every such $f(x)$; in other words: approximation coefficients are independent of the choice of $f(x)$. If d satisfies (189), then it also satisfies*

$$\forall c \nearrow x:\ v(y - f(c)) < v(y - f(c) - d\cdot(x-c)^{\mathbf{h}})\ . \tag{190}$$

Proof: If $g(x)$ is another element of $K[x]$ with $v(y - g(x)) \geq \mathrm{dist}\,(y,K)$, then

$$v(f(x) - g(x)) \geq \mathrm{dist}\,(y,K) = \mathrm{dist}\,(f(x),K) = \mathrm{dist}\,(g(x),K)\ .$$

Since $\mathrm{appr}\,(x,K)$ is transcendental, we have for all $c \nearrow x$:

$$v(f(c) - g(c)) = v(f(x) - g(x)) \geq \mathrm{dist}\,(f(x),K)$$

by Lemma 11.53. Hence for all $c \nearrow x$:

$$v(f(x) - f(c)) = v(g(x) - g(c))$$

and

$$\begin{aligned} & v(g(x) - g(c) - d \cdot (x-c)^{\mathbf{h}}) \\ \geq\ & \min\{v(f(x) - f(c) - d \cdot (x-c)^{\mathbf{h}}), v(f(x) - g(x)), v(f(c) - g(c))\} \\ \geq\ & \min\{v(f(x) - f(c) - d \cdot (x-c)^{\mathbf{h}}), \operatorname{dist}(f(x), K)\} \\ >\ & v(f(x) - f(c)) = v(g(x) - g(c)) \end{aligned}$$

which shows that d fulfills equation (189) also with g in the place of f. Replacing $g(x)$ by y and $g(c)$ by $f(c)$ in the above deduction, one obtains a proof of the second assertion. □

Lemma 11.66 *d is an approximation coefficient of y in x if and only if*

$$v(f_{\mathbf{h}}(c)) < v(f_{\mathbf{h}}(c) - d) \quad \textit{for} \quad c \nearrow x\ .$$

Consequently, there exists an approximation coefficient of y in x.

Proof: By definition of $\mathbf{h} = \mathbf{h}_K(x : y) = \mathbf{h}_K(x : f(x))$ we have

$$v(f(x) - f(c) - f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) > v(f(x) - f(c))$$

for all $c \nearrow x$, hence (189) holds for $c \nearrow x$ if and only if

$$v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}} - d \cdot (x-c)^{\mathbf{h}}) > v(f(x) - f(c)) = v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}})$$

which is equivalent to

$$\forall c \nearrow x:\ v(f_{\mathbf{h}}(c)) < v(f_{\mathbf{h}}(c) - d)\ .$$

Since $K(x)|K$ is assumed to be an immediate extension, there exists some $d \in K$ such that $v(f_{\mathbf{h}}(x) - d) > v(f_{\mathbf{h}}(x))$. Since $\operatorname{appr}(x, K)$ is transcendental, for $c \nearrow x$ we have $v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d)$ and $v(f_{\mathbf{h}}(c)) = v(f_{\mathbf{h}}(x))$ and thus

$$v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d) > v(f_{\mathbf{h}}(x)) = v(f_{\mathbf{h}}(c))\ .$$

Hence d is an approximation coefficient for y in x by the first part of our proof. □

Lemma 11.67 *Let $y_i \in K[x]^c \setminus K^c$ with equal approximation degree $\mathbf{h} = \mathbf{h}_K(x : y_i)$, $1 \leq i \leq m$. Assume that $d_i \in K$ is an approximation coefficient of y_i in x and let k_i be elements in K such that*

$$v\left(\sum_{i=1}^{m} k_i d_i\right) = \min_{1 \leq i \leq m} (v(k_i d_i))\ . \tag{191}$$

Then the following will hold:

$$\mathbf{h}_K\left(x : \sum_{i=1}^{m} k_i y_i\right) = \mathbf{h}\ .$$

Proof: We choose $f^{(i)}(x) \in K[x]$ with $v(y_i - f^{(i)}(x)) \geq \mathrm{dist}\,(y_i, K)$. We put

$$g(x) := \sum_{i=1}^{m} k_i f^{(i)}(x) \in K[x]$$

and show that $\mathbf{h}_K(x : g(x)) = \mathbf{h}$. Indeed, for $c \nearrow x$ and $1 \leq j \neq \mathbf{h}$ we have

$$\begin{aligned}
v(g_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) &= v\left(\sum_{i=1}^{m} k_i f_{\mathbf{h}}^{(i)}(c)\right) + \mathbf{h}\cdot v(x-c) = v\left(\sum_{i=1}^{m} k_i d_i\right) + \mathbf{h}\cdot v(x-c)\\
&= \min_{1\leq i\leq m}\left(v(k_i d_i)\right) + \mathbf{h}\cdot v(x-c) = \min_{1\leq i\leq m}\left(v(k_i f_{\mathbf{h}}^{(i)}(c)(x-c)^{\mathbf{h}})\right)\\
&< \min_{1\leq i\leq m}\left(v(k_i f_j^{(i)}(c)(x-c)^{j})\right)\\
&\leq v\left(\sum_{i=1}^{m} k_i f_j^{(i)}(c)\right) + j\cdot v(x-c) = v(g_j(c)(x-c)^j)\;.
\end{aligned}$$

Now it also follows that

$$\begin{aligned}
\mathrm{dist}\,(g(x), K) &= \mathrm{dist}\,(g(x), g(K)) = \sup_{c\in K} v(g(x)-g(c))\\
&= \sup_{c\in K} v(g_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) = \sup_{c\in K}\min_{1\leq i\leq m}\left(v(k_i f_{\mathbf{h}}^{(i)}(c)(x-c)^{\mathbf{h}})\right)\\
&= \min_{1\leq i\leq m}\sup_{c\in K}(v(k_i f_{\mathbf{h}}^{(i)}(c)(x-c)^{\mathbf{h}})) = \min_{1\leq i\leq m}\mathrm{dist}\,(k_i f^{(i)}(x), K)\\
&= \min_{1\leq i\leq m}\left(v(k_i) + \mathrm{dist}\,(f^{(i)}(x), K)\right) = \min_{1\leq i\leq m}\left(v(k_i) + \mathrm{dist}\,(y_i, K)\right)\\
&\leq \min_{1\leq i\leq m}\left(v(k_i) + v(y_i - f^{(i)}(x))\right) \leq \min_{1\leq i\leq m} v(k_i y_i - k_i f^{(i)}(x))\\
&\leq v\left(\sum_{i=1}^{m}(k_i y_i - k_i f^{(i)}(x))\right) = v\left(\sum_{i=1}^{m} k_i y_i - g(x)\right)
\end{aligned}$$

(where the first equation is taken from Lemma 11.37, in view of our hypothesis that $\mathrm{appr}\,(x, K)$ be transcendental). This shows

$$\mathrm{dist}\left(\sum_{i=1}^{m} k_i y_i, K\right) = \mathrm{dist}\,(g(x), K)$$

and

$$v\left(\sum_{i=1}^{m} k_i y_i - g(x)\right) \geq \mathrm{dist}\left(\sum_{i=1}^{m} k_i y_i, K\right)\;;$$

hence

$$\mathbf{h}_K(x : \sum_{i=1}^{m} k_i y_i) = \mathbf{h}_K(x : g(x)) = \mathbf{h}\;,$$

as asserted. □

11.4 Classes of associated minimal polynomials.

Given an immediate algebraic approximation type $\mathbf{A} = \mathrm{appr}(x, K)$, we want to consider the *class of all associated minimal polynomials for* $\mathbf{A}$, which we will denote by $\mathrm{amp}(\mathbf{A})$. As usual, we let $\mathbf{d} = \deg(\mathbf{A})$ and fix one associated minimal polynomial $f(X) \in K[X]$. It turns out that $\mathrm{amp}(\mathbf{A})$ contains precisely all normed polynomials g of degree $\mathbf{d}$ for which $g(x)$ has the same approximation type $\mathrm{appr}(f(x), K)$ over K as $f(x)$, and they are of a special form:

Lemma 11.68 *Let* $\mathbf{A}$ *and* f *as described, and let* $g \in K[X]$. *Then* $g \in \mathrm{amp}(\mathbf{A})$ *if and only if* $\mathrm{appr}(f(x), K) = \mathrm{appr}(g(x), K)$, *and this is the case if and only if*

$$g(X) = f(X) + \sum_{i=1}^{\mathbf{d}-1} (X - c^*) c_i^*$$

for suitable elements $c^*, c_0^*, \ldots, c_{\mathbf{d}-1}^* \in K$ *which satisfy*

$$i \cdot v(x - c^*) + v(c_i^*) > v(c_0^*) \geq \mathrm{dist}(f(x), K) \quad \text{for } i = 1, \ldots, \mathbf{d} - 1\ .$$

In particular, if $g(x)$ *is chosen as in Corollary 11.46 with* $n < \mathbf{d}$, *then it is also an associated minimal polynomial for* $\mathbf{A}$.

Proof: In view of the fact that if $g \in \mathrm{amp}(\mathbf{A})$, then both f and g are normed polynomials of degree $\mathbf{d}$ and consequently, $f - g$ is of degree $< \mathbf{d}$, the entire lemma is just an application of Corollary 11.46. □

Corollary 11.69 *If the distance of* $\mathbf{A} = \mathrm{appr}(x, K)$ *is* ∞, *then* $\mathbf{A}$ *admits a unique associated minimal polynomial which is the minimal polynomial of* x *over* K, *and the* $K[X]$*–ideal generated by this polynomial is equal to the set of all polynomials* $g \in K[X]$ *whose value is not fixed by* $\mathbf{A}$.

Proof: If $\mathbf{A}$ has distance ∞, then Lemma 11.36 shows that

$$\mathrm{dist}(f(x), K) = \infty\ .$$

In view of the foregoing lemma we conclude that f is the only associated minimal polynomial for $\mathbf{A}$. From Lemma 11.39 we know that $v(f(x)) = \infty$, i.e. $f(x) = 0$. Since f is irreducible over K (cf. page 31), this shows that f is the minimal polynomial of x over K. Furthermore, any polynomial g may be represented in the form (179) (cf. page 202), and Lemma 11.41 shows that $\mathbf{A}$ fixes the value of $g \not\equiv 0$ if and only if $c_0(X) \not\equiv 0$ (since in this case, the values of all terms except $c_0(c)$ tend to ∞ for $c \nearrow x$). But $c_0(X) \equiv 0$ holds if and only if g lies in the $K[X]$–ideal generated by f. □

Now we want to determine easy normal forms for associated minimal polynomials. The idea is to generalize Kaplansky's Lemma 10 (cf. [KAP1], p. 311) to general rank, using the fact that henselian fields are separable–algebraically closed in their completion:

Lemma 11.70 *If* K *is a henselian field of arbitrary rank, then* $K^c \cap \tilde{K}$ *is purely inseparable over* K.

Proof: Assume that $K^c \cap \tilde{K}$ contains a nontrivial finite separable extension L of K. Take N to be the normal hull of L over K. Let v be prolongated to $N.K^c$ (in fact, this prolongation is uniquely determined since by Lemma 5.12, K^c is henselian too). Let $a \in L \setminus K$ and let $b \neq a$ be a conjugate of a over K. Then

$$\infty \neq v(a-b) \in v(N.K^c) \subset v(\widetilde{K^c}) = v(\tilde{K}) \;,$$

hence there is an element $\alpha \in v(K)$ with $\alpha \geq v(a-b)$, and since $a \in K^c$, there is an element $c \in K$ with $v(a-c) > \alpha \geq v(a-b)$. For $\sigma \in \mathrm{Gal}(L|K)$ with $\sigma a = b$, this yields

$$v(\sigma(a-c)) = v(b-c) = \min\{v(a-b), v(a-c)\} = v(a-b) < v(a-c) \;,$$

showing that v and $v \circ \sigma$ are two distinct prolongations of the valuation v from K to N, a contradiction to our hypothesis that K should be henselian. This contradiction proves that $K^c \cap \tilde{K}$ is purely inseparable over K. □

Furthermore, we need the following details and definitions about cuts in ordered abelian groups:

Lemma 11.71 *Let $\delta = (\Lambda, \Lambda')$ be a cut in the ordered abelian group Γ. Then the set*

$$\mathcal{I}(\delta,\Gamma) := \{\gamma \in \Gamma \mid \gamma + \Lambda = \Lambda\}$$

is a convex subgroup of Γ. $\Lambda/\mathcal{I}$ is again an initial segment of the ordered abelian group $\Gamma/\mathcal{I}$. Let $\delta/\mathcal{I}$ denote the corresponding cut $(\Lambda/\mathcal{I}, \Gamma/\mathcal{I} \setminus \Lambda/\mathcal{I})$ in the ordered abelian group $\Gamma/\mathcal{I}$. Then for every $\alpha \in \Gamma$,

$$\begin{aligned} \alpha \geq \delta \;&\Longleftrightarrow\; \alpha/\mathcal{I} \geq \delta/\mathcal{I} \\ \alpha > \delta \;&\Longleftrightarrow\; \alpha/\mathcal{I} > \delta/\mathcal{I} \end{aligned}$$

and $\delta/\mathcal{I} = (\Lambda/\mathcal{I}, \Lambda'/\mathcal{I})$. Moreover,

$$\mathcal{I}(\delta/\mathcal{I}, \Gamma/\mathcal{I}) = \{0\} \;.$$

Furthermore, if i is a nonzero integer and $\gamma \in \Gamma$, then

$$\mathcal{I}(i \cdot \delta + \gamma, \Gamma) = \mathcal{I}(\delta, \Gamma) \;.$$

Proof: We abbreviate $\mathcal{I} = \mathcal{I}(\delta,\Gamma)$. The straightforward proof that $\mathcal{I}$ is a group, is left to the reader. It remains to show that $\mathcal{I}$ is convex in Γ. Since Λ is an initial segment of Γ, we have $\Lambda + \alpha \subseteq \Lambda + \beta$ whenever $\alpha \leq \beta$. Hence if $\alpha, \beta \in \mathcal{I}$ and $\gamma \in \Gamma$ with $\alpha \leq \gamma \leq \beta$, then

$$\Lambda = \Lambda + \alpha \subseteq \Lambda + \gamma \subseteq \Lambda + \beta = \Lambda$$

which shows that $\Lambda + \gamma = \Lambda$, i.e. $\gamma \in \mathcal{I}$. This proves that $\mathcal{I}$ is convex in Γ. Thus $\Gamma/\mathcal{I}$ is again an ordered abelian group, and the natural epimorphism "$/\mathcal{I}$" preserves the relation "$\geq$". From this it follows that $\Lambda/\mathcal{I}$ is again an initial segment of $\Gamma/\mathcal{I}$ and that $\Lambda/\mathcal{I} \leq \Lambda'/\mathcal{I}$. Moreover, it shows

$$\begin{aligned} \alpha \geq \delta \;&\Longrightarrow\; \alpha/\mathcal{I} \geq \delta/\mathcal{I} \\ \alpha > \delta \;&\Longleftarrow\; \alpha/\mathcal{I} > \delta/\mathcal{I} \;. \end{aligned}$$

It remains to show the converses. If $\alpha/\mathcal{I} \geq \delta/\mathcal{I}$, then there must exist an element $\gamma \in \mathcal{I}$ such that $\alpha + \gamma \geq \delta$, hence $\alpha \geq \delta - \gamma = \delta$ by definition of $\mathcal{I}$. If $\alpha/\mathcal{I} \leq \delta/\mathcal{I}$, then there must exist an element $\gamma \in \mathcal{I}$ such that $\alpha+\gamma \leq \delta$, hence $\alpha \leq \delta - \gamma = \delta$. Now it also follows that $\Gamma/\mathcal{I} \setminus \Lambda/\mathcal{I} = \Lambda'/\mathcal{I}$, hence $\delta/\mathcal{I} = (\Lambda/\mathcal{I}, \Lambda'/\mathcal{I})$.

The fact that the invariance subgroup of $\delta/\mathcal{I}$ is trivial, is seen as follows: if $\gamma/\mathcal{I}+\delta/\mathcal{I} = \delta/\mathcal{I}$ for an element $\gamma \in \Gamma$, this means that

$$\gamma + \mathcal{I} + \Lambda + \mathcal{I} = \Lambda + \mathcal{I} \;,$$

and in view of $\Lambda + \mathcal{I} = \Lambda$, this yields $\gamma + \Lambda = \Lambda$, i.e. $\gamma \in \mathcal{I}$ and thus $\gamma/\mathcal{I} = 0$.

Finally, let i be a nonzero integer and $\gamma \in \Gamma$. The already proved property of an invariance subgroup to be a convex subgroup shows that $\alpha \in \mathcal{I}(i \cdot \delta) \Longleftrightarrow i \cdot \alpha \in \mathcal{I}(i \cdot \delta)$. Using this, we compute

$$\begin{aligned} \alpha + \Lambda = \Lambda \;&\Longleftrightarrow\; i \cdot \alpha + i \cdot \Lambda = i \cdot \Lambda \\ &\Longleftrightarrow\; \alpha + i \cdot \Lambda = i \cdot \Lambda \\ &\Longleftrightarrow\; \alpha + i \cdot \Lambda + \gamma = i \cdot \Lambda + \gamma \;. \end{aligned}$$

For positive i, this proves $\mathcal{I}(i \cdot \delta + \gamma, \Gamma) = \mathcal{I}(\delta, \Gamma)$. For negative i, we thus have

$$\mathcal{I}(i \cdot \delta + \gamma, \Gamma) = \mathcal{I}((-i) \cdot (-\delta) + \gamma, \Gamma) = \mathcal{I}(-\delta, \Gamma) \;,$$

and it remains to show that

$$\mathcal{I}(\delta, \Gamma) = \mathcal{I}(-\delta, \Gamma) \;.$$

Since (Λ, Λ') is a partition of Γ, we have $\alpha + \Lambda = \Lambda$ if and only if $\alpha + \Lambda' = \Lambda'$. The latter is true if and only if $\alpha + -\Lambda' = -\Lambda'$. Hence α is an element of $\mathcal{I}(\delta, \Gamma)$ if and only if it is an element of $\mathcal{I}(-\delta, \Gamma)$; this proves the desired equality. □

$\mathcal{I}(\delta, \Gamma)$ will be called the *invariance subgroup of the cut* δ in Γ; we will also denote it by $\mathcal{I}(\delta)$ if there is no danger of confusion. If $\Lambda/\mathcal{I}$ admits a maximal element, then the cut δ will be called a *weakly distinguished cut*. If in this case, δ is the distance of an approximation type $\mathbf{A}$, then $\mathbf{A}$ will be called a *weakly distinguished approximation type*. If $\Lambda/\mathcal{I}$ admits $0/\mathcal{I}$ as maximal element (which implies that $0/\mathcal{I}$ realizes the cut $\delta/\mathcal{I}(\delta)$), then the cut δ will be called a *distinguished cut*, and $\mathbf{A}$ will be called a *distinguished approximation type*. This name is chosen since distinguished approximation types are corresponding to distinguished pseudo Cauchy sequences in the sense of Ribenboim [RIB1], p. 105.

Lemma 11.72 *Let $\delta = (\Lambda, \Lambda')$ be a cut in the ordered abelian group Γ, i a positive integer and $\gamma \in \Gamma$. Then δ is weakly distinguished if and only if $i \cdot \delta + \gamma$ is, and both cuts have the same invariance subgroup $\mathcal{I}$; moreover, if $\gamma_\delta/\mathcal{I}$ is a maximal element of $\Lambda/\mathcal{I}$ for some $\gamma_\delta \in \Gamma$ (which implies that $\delta/\mathcal{I}$ is realized by $\gamma_\delta/\mathcal{I}$), then $(i \cdot \gamma_\delta + \gamma)/\mathcal{I}$ is a maximal element of $i \cdot \Lambda + \gamma$ (which implies that $i \cdot \delta + \gamma$ is realized by $(i \cdot \gamma_\delta + \gamma)/\mathcal{I}$).*

If δ is weakly distinguished, and $\gamma_\delta \in \Gamma$ such that $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$, then the distinguished cut $\delta - \gamma_\delta = (\Lambda - \gamma_\delta, \Lambda' - \gamma_\delta)$ is represented by the convex subgroup $\mathcal{I} = \mathcal{I}(\delta)$ of Γ, i.e. $\mathcal{I}$ is cofinal in $\Lambda - \gamma_\delta$. Conversely, if there exists an element $\gamma_\delta \in \Gamma$ and a convex subgroup $\mathcal{I}$ of Γ such that $\mathcal{I}$ is cofinal in $\Lambda - \gamma_\delta$, then δ is weakly distinguished with $\mathcal{I} = \mathcal{I}(\delta)$ and $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$.

Proof: From Lemma 11.71 we infer $\mathcal{I}(i\cdot\delta+\gamma)=\mathcal{I}(\delta)$, whence

$$(i\cdot\Lambda+\gamma)/\mathcal{I}(i\cdot\delta+\gamma)=i\cdot\Lambda/\mathcal{I}(\delta)+\gamma/\mathcal{I}(\delta)$$

which for positive i shows that $\Lambda/\mathcal{I}(\delta)$ admits $\gamma_\delta/\mathcal{I}(\delta)$ as maximal element if and only if $(i\cdot\Lambda+\gamma)/\mathcal{I}(i\cdot\delta+\gamma)$ admits $(i\cdot\gamma_\delta+\gamma)/\mathcal{I}(\delta)$ as maximal element. This proves the first part of our assertion.

Now assume δ to be a weakly distinguished cut and $\gamma_\delta\in\Gamma$ such that $\gamma_\delta/\mathcal{I}$ is the maximal element of $\Lambda/\mathcal{I}$. Consequently, $(\Lambda-\gamma_\delta)/\mathcal{I}$ admits $0/\mathcal{I}$ as maximal element which shows that $\delta-\gamma_\delta$ is distinguished. In particular, $(\Lambda-\gamma_\delta)/\mathcal{I}$ contains no positive elements, i.e. there are no elements $\beta\in\Lambda-\gamma_\delta$ with $\beta>\mathcal{I}$. Now let $\beta\in\mathcal{I}$. Note that w.l.o.g. γ_δ may be chosen to be an element of Λ. Then $0=\gamma_\delta-\gamma_\delta\in\Lambda-\gamma_\delta$, and since $\beta+\Lambda=\Lambda$, β being an element of the invariance subgroup $\mathcal{I}$ of δ in Γ, we find

$$\beta=\beta+0\;\in\;\beta+(\Lambda-\gamma_\delta)=(\beta+\Lambda)-\gamma_\delta=\Lambda-\gamma_\delta\;.$$

Since $\beta\in\mathcal{I}$ was arbitrary, we deduce that $\mathcal{I}\subset\Lambda-\gamma_\delta$. Together with what we have shown before, this proves that $\mathcal{I}$ is a final segment of $\Lambda-\gamma_\delta$ and hence cofinal in $\Lambda-\gamma_\delta$.

Now assume that there exists an element $\gamma_\delta\in\Gamma$ and a convex subgroup $\mathcal{I}$ of Γ such that $\mathcal{I}$ is cofinal in $\Lambda-\gamma_\delta$. In view of the first assertion of our lemma that we have already proved, and in view of the equality $\mathcal{I}(\delta-\gamma_\delta)=\mathcal{I}(\delta)$, we may assume w.l.o.g. that $\gamma_\delta=0$. Thus we have to show that $\mathcal{I}$ is the invariance subgroup of the cut which is represented by $\mathcal{I}$. But this is immediately seen to be true since the biggest convex subgroup $\mathcal{I}'$ of Γ such that $\mathcal{I}+\mathcal{I}'$ is cofinal in $\mathcal{I}$, is just $\mathcal{I}$. □

The following lemma deals with a characteristic interval U_δ that we use in section 4.

Lemma 11.73 *Given a negative cut $\delta=(\Lambda,\Lambda')$ in Γ (i.e. $0\in\Lambda'$), the set*

$$U_\delta=\{\alpha\in\Gamma\mid\Lambda<\alpha<-\Lambda\}$$

is an interval in Γ containing 0 and closed under $\alpha\mapsto-\alpha$. We find

$$U_\delta=\Lambda'\cap-\Lambda'\;.$$

If U_δ is a group, hence a convex subgroup of Γ, then

$$U_\delta=\mathcal{I}(\delta)=\mathcal{I}(-\delta)\;,$$

and $-\delta$ is a distinguished cut. If in addition, Γ is divisible, then δ is not weakly distinguished.

Conversely, if $-\delta$ is distinguished, then δ is a negative cut, and

$$\mathcal{I}(\delta)=U_\delta=\{\alpha\in\Gamma\mid\delta<\alpha<-\delta\}\;.$$

Proof: Since $0\in\Lambda'$ by hypothesis, we have $\Lambda<0$ and $0=-0<-\Lambda$, hence $0\in U_\delta$. If $\Lambda<\alpha<-\Lambda$, then $\Lambda=-(-\Lambda)<-\alpha<-\Lambda$ which proves that U_δ is closed under $\alpha\mapsto-\alpha$. Since $\alpha>\Lambda\Leftrightarrow\alpha\in\Lambda'$, the latter shows

$$\alpha\in U_\delta\Longleftrightarrow\alpha\in\Lambda'\wedge-\alpha\in\Lambda'\;,$$

whence $U_\delta = \Lambda' \cap -\Lambda'$. Consequently, if U_δ is nonempty, then it is cofinal in $-\Lambda'$. If in addition U_δ is a group, then by Lemma 11.72, U_δ is the invariance subgroup of $-\delta$ in Γ, and $-\delta$ is weakly distinguished with $-\delta/\mathcal{I}(-\delta) = 0$, hence distinguished. By Lemma 11.71, $\mathcal{I}(\delta) = \mathcal{I}(-\delta)$. Furthermore, let Γ be divisible; then so is $\Gamma/\mathcal{I}(\delta)$. From Lemma 11.71 we know that $(\Lambda/\mathcal{I}(\delta), \Lambda'/\mathcal{I}(\delta))$ is a cut in $\Gamma/\mathcal{I}(\delta)$, and from Lemma 11.72 we infer that $0/\mathcal{I}(\delta)$ is the maximal element of $-\Lambda'/\mathcal{I}(\delta)$, hence the minimal element of $\Lambda'/\mathcal{I}(\delta)$. Since $\Gamma/\mathcal{I}(\delta)$ is divisible, this shows that $\Lambda/\mathcal{I}(\delta)$ has no maximal element and thereby proves that δ is not weakly distinguished.

For the converse, assume that $-\delta = (-\Lambda', \Lambda)$ is distinguished. Then by Lemma 11.72, $\mathcal{I}(\delta)$ is cofinal in $-\Lambda'$. In particular, this shows that $-\Lambda'$ contains nonnegative elements, hence $-\delta$ is a positive and δ is a negative cut. Furthermore, we may deduce that $-\Lambda'$ admits no maximal element and Λ' admits no least element, which yields

$$U_\delta = \{\alpha \in \Gamma \mid \delta < \alpha < -\delta\} .$$

Finally, since both U_δ and $\mathcal{I}(\delta)$ are cofinal in $-\Lambda'$ and are closed under $\alpha \mapsto -\alpha$, we conclude that

$$U_\delta = \mathcal{I}(\delta) .$$

□

For the investigation of associated minimal polynomials, we need a further lemma.

Lemma 11.74 *Given the cut $\delta = (\Lambda, \Lambda')$ in the ordered abelian group Γ, let $0 < \eta \notin \mathcal{I}(\delta)$. Then there exists an element $\gamma \in \Lambda$ such that $\gamma + \eta > \Lambda$ (i.e. $\gamma + \eta \geq \delta$).*

Proof: Since $\eta \notin \mathcal{I}$, we have $\Lambda \neq \Lambda + \eta$. On the other hand, $\Lambda \subset \Lambda + \eta$ since Λ is an initial segment and $\eta > 0$ by hypothesis. Consequently, there exists an element in $\Lambda + \eta \setminus \Lambda$ which we may write as $\gamma + \eta$ for some element $\gamma \in \Lambda$. Since Λ is an initial segment, it follows that $\gamma + \eta > \Lambda$, as asserted. □

Lemma 11.75 *Given the cut $\delta = (\Lambda, \Lambda')$ in the ordered abelian group Γ, suppose that δ is not weakly distinguished. Let i, j be natural numbers, $j > i > 0$, and $\alpha_i, \alpha_j \in \Gamma$. If there exists $\beta \in \Lambda$ such that*

$$\alpha_j + j \cdot \beta > \alpha_i + i \cdot \beta ,$$

then there exists an element $\beta_0 \in \Lambda$ such that

$$\forall \beta \geq \beta_0 : \ \alpha_j + j \cdot \beta > \alpha_i + i \cdot \Lambda .$$

Proof: First we show that there exist $\alpha \in \Lambda$ and $\eta \in \Gamma$, $\eta > \mathcal{I}$, such that

$$\alpha_j - \alpha_i + (j - i) \cdot \alpha \geq i \cdot \eta > \mathcal{I} .$$

Indeed, by assumption there exists $\beta \in \Lambda$ such that

$$\alpha_j - \alpha_i + (j - i) \cdot \beta > 0 .$$

Since δ is assumed to be not weakly distinguished, $\Lambda/\mathcal{I}$ has no greatest element, hence there exists an element $\alpha \in \Lambda$ such that $\alpha/\mathcal{I} > \beta/\mathcal{I}$, i.e. $\alpha - \beta > \mathcal{I}$, whence

$$\alpha_j - \alpha_i + (j-i)\cdot\alpha > (j-i)\cdot(\alpha-\beta) > \mathcal{I} \; .$$

If $\Lambda/\mathcal{I}$ has no smallest positive element, then there exists an element $\eta > \mathcal{I}$ such that $(j-i)\cdot(\alpha-\beta) > i\cdot\eta > i\cdot\mathcal{I} = \mathcal{I}$. If on the other hand, $\Lambda/\mathcal{I}$ possesses a smallest positive element, say $\eta/\mathcal{I}$, then by our hypothesis that $\Lambda/\mathcal{I}$ has no greatest element, we may choose $\alpha \in \Lambda$ such that $\alpha - \beta > i\cdot\eta$, whence again, $(j-i)\cdot(\alpha-\beta) > i\cdot\eta > i\cdot\mathcal{I} = \mathcal{I}$. This proves the existence of the required element η.

Since $\eta \notin \mathcal{I}$, by the preceding lemma there exists $\gamma \in \Lambda$ such that $\gamma + \eta > \Lambda$. Putting $\beta_0 = \max\{\gamma, \alpha\}$ we get $\beta_0 + \eta \geq \gamma + \eta > \Lambda$ and consequently $i\cdot\beta_0 + i\cdot\eta > i\cdot\Lambda$. Hence for all $\beta \geq \beta_0$, we have $\beta \geq \alpha$ and

$$\begin{aligned} \alpha_j + j\cdot\beta &= \alpha_j + i\cdot\beta + (j-i)\cdot\beta \\ &\geq \alpha_i + \alpha_j - \alpha_i + i\cdot\beta_0 + (j-i)\cdot\alpha \\ &\geq \alpha_i + i\cdot\beta_0 + i\cdot\eta > \alpha_i + i\cdot\Lambda \; , \end{aligned}$$

as asserted. □

We will apply invariance subgroups as a tool in the following general situation. Given an immediate algebraic approximation type $\mathbf{A} = \mathrm{appr}(x,K)$ of degree $\mathbf{d} < \infty$, let $\delta = \mathrm{dist}(x,K) = (\Lambda,\Lambda')$ (where $\Lambda = \Lambda(\mathbf{A})$), and $\mathcal{I} = \mathcal{I}(\delta, v(K))$. Furthermore, let $w_\delta = w_\mathcal{I}$ be a coarsening of the valuation v (of $K(x)$) whose restriction to K is the coarsening of v on K which corresponds to the convex subgroup $\mathcal{I}$ of the value group $v(K)$. Given any convex subgroup $\mathcal{I}$ of $v(K)$, the coarsening $w_\mathcal{I}$ which corresponds to $\mathcal{I}$ is the unique one which satisfies the following conditions:

$$\forall c \in K:\; w_\mathcal{I}(c) \mapsto v(c) + \mathcal{I}$$

induces an isomorphism

$$w_\mathcal{I}(K) \cong v(K)/\mathcal{I} \; ,$$

and

$$\forall c \in \mathcal{O}^\times_{(K,w_\mathcal{I})}:\; v/w_\mathcal{I}(c/w_\mathcal{I}) \mapsto v(c)$$

induces an isomorphism

$$v/w_\mathcal{I}(K/w_\mathcal{I}) \cong \mathcal{I} \; .$$

The valuation $w_\mathcal{I}$ is uniquely determined on K, but may have two different prolongations to $K(x)$ both being coarsenings of v, if the rank of $v(K(x))$ is greater than the rank of $v(K)$. Note that w_δ is the trivial valuation on K iff $\delta = \infty$. Finally, by $\delta/\mathcal{I}$ we will denote the cut $(\Lambda/\mathcal{I}, \Lambda'/\mathcal{I})$ in $w_\delta(K)$, cf. Lemma 11.71. For this situation, we note the following

Lemma 11.76 *The approximation type* $\mathrm{appr}(x,K)$ *is weakly distinguished if and only if there exists an element $b \in K$ such that* $\mathrm{appr}(bx,K)$ *is distinguished. If $x \in K$, then* $\mathrm{appr}(x,K)$ *is distinguished.*

If $\mathbf{A} = \mathrm{appr}(x,K)$ is distinguished and $x \notin K$, then for every $c \nearrow x$, the w_δ–residue $(x-c)/w_\delta$ does not lie in the residue field K/w_δ but is an element of the completion

$(K/w_\delta)^{c(v/w_\delta)}$ *of* K/w_δ *with respect to the induced valuation* v/w_δ *(in particular, this holds for every* $c \in K$ *with* $w_\delta(x-c) \geq 0$*).*

Conversely, if there exists an element $c \in K$ *and a coarsening* w *of* v *such that* $(x-c)/w \in (K/w)^{c(v/w)} \setminus K/w$*, then* $\mathbf{A}$ *is distinguished with* $x \notin K$ *and* $w = w_\delta$ *(on* K*).*

Proof: The first assertion follows immediately from Lemma 11.72 if we choose $b \in K$ such that $v(b) = -\gamma_\delta$. The second assertion is trivial since $x \in K$ yields $\Lambda(\mathrm{appr}(x,K)) = v(K)$ and thus $\mathcal{I} = v(K)$ which shows $\Lambda(\mathrm{appr}(x,K))/\mathcal{I} = \{0\}$.

Let $\mathcal{I}$ be a convex subgroup of $v(K)$ and $w_\mathcal{I}$ a coarsening of v (on $K(x)$) which corresponds to $\mathcal{I}$. We will show: $\mathcal{I}$ is cofinal in $\Lambda(\mathrm{appr}(x,K))$ if and only if for some $c \in K$,

$$(x-c)/w_\mathcal{I} \in (K/w)_\mathcal{I}^{c(v/w_\mathcal{I})} \setminus K/w_\mathcal{I}\;.$$

By the second of the above mentioned isomorphisms,

$$\mathcal{I} \subset \Lambda(\mathrm{appr}(x,K))$$

is equivalent to the fact that the set

$$\{v/w_\mathcal{I}((x-c)/w_\mathcal{I}) \mid x-c \in \mathcal{O}^\times_{(K,w_\mathcal{I})}\}$$

is cofinal in $v/w_\mathcal{I}(K/w_\mathcal{I})$. In this case, there exists some $c \in K$ such that $x-c \in \mathcal{O}^\times_{(K,w_\mathcal{I})}$, and in view of $(x-c)/w_\mathcal{I} - c'/w_\mathcal{I} = (x-(c+c'))/w_\mathcal{I}$, the cofinality of the above set is equivalent to the property of

$$\Lambda(\mathrm{appr}((x-c)/w_\mathcal{I}, K/w_\mathcal{I}))$$

to be cofinal in $v/w_\mathcal{I}(K/w_\mathcal{I})$, i.e. the property of $(x-c)/w_\mathcal{I}$ to be an element of the completion of $K/w_\mathcal{I}$ with respect to the induced valuation $v/w_\mathcal{I}$.

Furthermore, the existence of an element $\beta \in \Lambda(\mathrm{appr}(x,K))$ with $\beta > \mathcal{I}$ is equivalent to the existence of an element $c_\beta \in K$ with $w_\mathcal{I}(x-c_\beta) > 0$, i.e. $(x-c-(c_\beta - c))/w_\mathcal{I} = (x-c_\beta)/w_\mathcal{I} = 0$, and this is equivalent to $(x-c)/w_\mathcal{I} \in K/w_\mathcal{I}$. Together with what we have proved already, this shows that $\mathcal{I}$ is cofinal in $\Lambda(\mathrm{appr}(x,K))$ and $x \notin K$ if and only if for some $c \in K$,

$$(x-c)/w_\mathcal{I} \in (K/w)_\mathcal{I}^{c(v/w_\mathcal{I})} \setminus K/w_\mathcal{I}\;.$$

Now let $\delta = (\Lambda, \Lambda')$ be the distance of $\mathbf{A}$. If δ is distinguished, then by virtue of Lemma 11.72, the convex subgroup $\mathcal{I} = \mathcal{I}(\delta)$ of $v(K)$ is cofinal in Λ. Then by what we have shown, there exists $c \in K$ such that

$$(x-c)/w_\delta \in (K/w)_\delta^{c(v/w_\delta)} \setminus K/w_\delta\;.$$

The same must be true for every c' in the place of c if $v(x-c') \geq v(x-c)$ or $w_\delta(x-c') \geq 0$. This is seen as follows: If $v(x-c') > v(x-c)$ then $w_\delta(x-c') \geq w_\delta(x-c) = 0$. Thus we have $w_\delta(x-c') \geq 0$ in both cases, and this implies $w_\delta(c-c') \geq 0$. Consequently, $(x-c')/w_\delta = (x-c)/w_\delta + (c-c')/w_\delta \in (K/w_\delta)^{c(v/w_\delta)}$. On the other hand, $(x-c')/w_\delta)$ cannot be an element of $K/w_\delta)$ since otherwise this would also hold for $(x-c)/w_\delta)$. We have herewith proved that the above property holds for every $c \nearrow x$ and every $c \in K$ with $w_\delta(x-c) \geq 0$.

For the converse, assume that there exists an element $c \in K$ and a coarsening w of v such that $(x-c)/w \in (K/w)^{c(v/w)} \setminus K/w$, and let $w = w_{\mathcal{I}}$ for a suitable convex subgroup $\mathcal{I}$ of $v(K)$. By what we have shown in the beginning, $\mathcal{I}$ is cofinal in Λ. Then Lemma 11.72 shows that the cut δ is distinguished with invariance subgroup $\mathcal{I}(\delta) = \mathcal{I}$, proving moreover that $w_\delta = w_{\mathcal{I}}$. This completes the proof of our lemma. □

Now we are able to prove normal form theorems which generalize Lemma 10 of [KAP1]. Beforehand, note that for our investigation of the associated minimal polynomials for an immediate approximation type $\mathbf{A}$, we may always assume that $\mathbf{A} = \mathrm{appr}(x, K)$ with $(K, v) \prec_\exists (K(x), v)$ according to Lemma 11.13. By Lemma 11.40, h being as in that lemma, this yields for every associated minimal polynomial of $\mathbf{A}$:

$$\mathrm{dist}(f(x), K) = h \cdot \mathrm{dist}(x, K) + v(f_{\mathbf{h}}(c)) \; . \tag{192}$$

Theorem 11.77 *Let $\mathbf{A} = \mathrm{appr}(x, K)$, $x \notin K$, be a weakly distinguished algebraic approximation type of degree $\mathbf{d}$ and distance δ with associated minimal polynomial $f \in K[X]$. Let $w = w_\delta$ the coarsening of v (on $K(x)$) which we have defined on page 223. Choose the integer h to be as in Lemma 11.36. Then there exists $b \in K$ such that for every $c \nearrow x$ we have: $\omega_c := b(x-c)/w$ is finite and an element of*

$$(K/w)^{c(v/w)} \setminus K/w \; ,$$

and it is a zero of the polynomial

$$\tilde{f}(X) = a_c f(c)/w + X^h + \sum_{i=h+1}^{\mathbf{d}} X^i (a_c b^{-i} f_i(c))/w$$

where $a_c = b^h f_{\mathbf{h}}(c)^{-1}$, and the residues $(a_c b^{-i} f_i(c))/w$ and $a_c f(c)/w$ are finite. $\tilde{f}$ is of degree $\mathbf{d}$ and irreducible over K/w, and it is the unique associated minimal polynomial for appr$(\omega_c, K/w)$. We may put $b = 1$ if $\mathbf{A}$ is distinguished.

Furthermore,

$$g(X) = f(c) + \sum_{i=h}^{\mathbf{d}} (X-c)^i f_i(c)$$

is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$.

Proof: $\mathbf{A}$ being weakly distinguished by hypothesis, we choose $\gamma_\delta \in \Gamma$ according to Lemma 11.72 such that $\mathcal{I}$ is cofinal in $\Lambda - \gamma_\delta$. Let $b \in K$ such that $v(b) = -\gamma_\delta$ (we may put $b = 1$ if $\gamma_\delta = 0$, i.e. if $\mathbf{A}$ is distinguished). Then $\mathrm{dist}(bx, K) = \delta - \gamma_\delta$ is distinguished, and for every $c \nearrow x$ we will have $v(b(x-c)) \in \mathcal{I}$, thus $w(b(x-c)) \geq 0$ and $\omega_c = b(x-c)/w \neq \infty$. Since appr$(bx, K)$ is distinguished, we infer from Lemma 11.76 that ω_c is an element of $(K/w)^{c(v/w)} \setminus K/w$. Furthermore, putting $a_c = b^h f_{\mathbf{h}}(c)^{-1}$ we get

$$v((b(x-c))^i a_c b^{-i} f_i(c)) > v((b(x-c))^h)$$

for all $i \neq h$, $1 \leq i \leq \mathbf{d}$, and all $c \nearrow x$ by the definition of h (cf. Lemma 11.36). This shows that all residues $\omega_c^i (a_c b^{-i} f_i(c))/w$ and thus all residues $(a_c b^{-i} f_i(c))/w$ are finite for $c \nearrow x$. By Lemma 11.39, we know that $v(f(x)) > v(f(c)) = v(f_{\mathbf{h}}(c)) + h \cdot v(x-c)$ for all $c \nearrow x$. Hence

$$v(a_c f(c)) = h \cdot v(b) + h \cdot v(x-c) = h \cdot v(b(x-c)) \; .$$

Firstly, this shows $a_c f(c)/w \neq \infty$. Secondly: since the values $v(b(x-c))$ are cofinal in the convex subgroup $\mathcal{I}$ for $c \nearrow x$, the same holds for the values $h \cdot v(b(x-c))$. Consequently, $v(a_c f(x)) > \mathcal{I}$, i.e. $a_c f(x)/w = 0$. Since on the other hand,

$$a_c f(x) = a_c f(c) + (b(x-c))^h + \sum_{\substack{1 \leq i \leq \mathbf{d} \\ i \neq h}} (b(x-c))^i a_c b^{-i} f_i(c) \ ,$$

by the finiteness that we have shown above we get

$$0 = a_c f(c) + \omega_c^h + \sum_{\substack{1 \leq i \leq \mathbf{d} \\ i \neq h}} \omega_c^i (a_c b^{-i} f_i(c))/w \ . \tag{193}$$

We want to deduce from this that $\tilde{f}(\omega_c) = 0$, where $\tilde{f}$ is defined as in the assertion of our theorem. If $h = 1$, then this is already the assertion. Suppose now that $h > 1$. By the definition of h we have

$$v(f_{\mathbf{h}}(c)) + h \cdot v(x-c) < v(f_i(c)) + i \cdot v(x-c)$$

for $i \neq h$ and all $c \nearrow x$. Consequently, if $i < h$, then for $c \nearrow x$,

$$\begin{aligned} v(a_c b^{-i} f_i(c)) &= (h-i) \cdot v(b) + v(f_i(c)) - v(f_{\mathbf{h}}(c)) \\ &> (h-i) \cdot v(b) + (h-i) \cdot v(x-c) \\ &= (h-i) \cdot v(b(x-c)) \ . \end{aligned}$$

The latter values are cofinal in $\mathcal{I}$. On the other hand, the values of $a_c = b^h f_{\mathbf{h}}(c)$ and of $f_i(c)$ are fixed for $c \nearrow x$ ($f_{\mathbf{h}}$ and f_i having degree $< \mathbf{d}$), hence

$$v(a_c b^{-i} f_i(c)) > \mathcal{I} \ \text{ for } i < h \text{ and } c \nearrow x \ , \tag{194}$$

i.e. $(a_c b^{-i} f_i(c))/w = 0$, which proves that the sum in (193) has to range only over $i > h$, as asserted in the theorem.

To show the irreducibility of the polynomial $\tilde{f}(X)$, assume that there is a factorization $\tilde{f} = \tilde{h}_1 \tilde{h}_2$ over K/w and let $h_1, h_2 \in K[X]$ be foreimages of $\tilde{h}_1, \tilde{h}_2$ with respect to the residue map "$/w$". Now we have

$$\begin{aligned} (a_c f(x) - h_1(b(x-c)) h_2(b(x-c)))/w &= a_c f(x)/w - h_1(b(x-c)) h_2(b(x-c))/w \\ &= 0 - \tilde{h}_1(\omega_c) \tilde{h}_2(\omega_c) = \tilde{f}(\omega_c) = 0 \end{aligned}$$

for all $c \nearrow x$, hence

$$\begin{aligned} & v(a_c f(x) - h_1(b(x-c)) h_2(b(x-c))) \\ & \quad > \ \mathcal{I} = h \cdot \mathcal{I} = h \cdot (\gamma_\delta + \mathcal{I}) + v(f_{\mathbf{h}}(c)) - h \cdot \gamma_\delta - v(f_{\mathbf{h}}(c)) \\ & \quad = \ h \cdot \operatorname{dist}(x, K) + v(f_{\mathbf{h}}(c)) + v(a_c) \ , \end{aligned}$$

whence by (192) and by Lemma 11.25,

$$\begin{aligned} v(a_c f(x) - h_1(b(x-c)) h_2(b(x-c))) &> \operatorname{dist}(f(x), K) + v(a_c) \\ &= \operatorname{dist}(a_c f(x), K) \end{aligned}$$

which shows $\mathrm{appr}(h_1(b(x-c))h_2(b(x-c)),K) = \mathrm{appr}(a_c f(x))$, according to Lemma 11.26. Since $\mathrm{appr}(x,K)$ does not fix the value of $f(X)$ by hypothesis, it also does not fix the value of $a_c f(X)$, and by the equality of the approximation types it follows that it does not fix the value of $h_1(b(X-c))h_2(b(X-c))$ either. But the degree of this polynomial is $\leq \mathbf{d}$, hence it must be equal to $\mathbf{d}$ since $\mathrm{appr}(x,K)$ is of degree $\mathbf{d}$, and the polynomial must be irreducible over K as remarked in section 2, page 31. This shows that one of the polynomials h_1, h_2 must be a constant. The same must hold for $\tilde{h}_1, \tilde{h}_2$ which proves that $\tilde{f}$ is irreducible over K/w.

Now let $g \in K[X]$ be as in the assertion of the theorem. According to Lemma 11.68, to prove that g is an associated minimal polynomial for $\mathbf{A}$, it suffices to prove that $\mathrm{appr}(f(x),K) = \mathrm{appr}(g(x),K)$, and according to Lemma 11.26, this is equivalent to

$$v(f(x) - g(x)) \geq \mathrm{dist}(f(x),K) \; .$$

Using again the Taylor expansion (178) for f, we find

$$f(X) - g(X) = \sum_{1\leq i<h} (X-c)^i f_i(c) \; .$$

From (194) and the fact that $v(b(x-c)) \in \mathcal{I}$ for $c \nearrow x$, it follows that also $v((b(x-c))^i a_c b^{-i} f_i(c)) > \mathcal{I}$ for $i < h$ and every $c \nearrow x$, whence

$$\begin{aligned}
v(f(x)-g(x)) &= v\left(a_c^{-1} \sum_{1\leq i<h} (b(x-c))^i a_c b^{-i} f_i(c)\right) \\
&= v\left(\sum_{1\leq i<h} (b(x-c))^i a_c b^{-i} f_i(c)\right) - v(a_c) \\
&> \mathcal{I} - v(a_c) = \mathcal{I} - h\cdot v(b) + v(f_{\mathbf{h}}(c)) \\
&= \mathcal{I} + h\cdot\gamma_\delta + v(f_{\mathbf{h}}(c)) = h\cdot(\gamma_\delta + \mathcal{I}) + v(f_{\mathbf{h}}(c)) \; ,
\end{aligned}$$

whence

$$v(f(x)-g(x)) > h\cdot \mathrm{dist}(x,K) + v(f_{\mathbf{h}}(c)) = \mathrm{dist}(f(x),K)$$

for every $c \nearrow x$; the last equation holds by (192). This completes the proof of our theorem. □

Corollary 11.78 *Let* $\mathbf{A} = \mathrm{appr}(x,K)$ *be a distinguished approximation type of degree* $\mathbf{d}$ *and distance* δ*. For every* $c \nearrow x$*, it induces a distinguished approximation type*

$$\mathbf{A}/w_\delta = \mathrm{appr}((x-c)/\delta, K/w_\delta)$$

which is of the same degree $\mathbf{d}$ *and has distance* ∞*.*

Proof: Let $w = w_\delta$. Theorem 11.77 shows that for $c \nearrow x$, the element $(x-c)/w$ lies in the completion of K/w, hence its approximation type $\mathrm{appr}((x-c)/w, K/w)$ over K/w has distance ∞. Moreover, we infer from the theorem that this approximation type has an associated minimal polynomial of degree $\mathbf{d}$ and thus is itself of degree $\mathbf{d}$. □

For henselian ground fields, the assertion of Theorem 11.77 may be supplemented as follows:

Theorem 11.79 *Let the situation be as in the foregoing theorem and assume in addition that K is henselian. Let*

$$p_w = \max\{1, \mathrm{char}(K/w)\} .$$

Then $h = \mathbf{d} = p_w^e$ for some integer $e \geq 0$. Thus

$$b^{\mathbf{d}} f(c)/w + \omega_c^{\mathbf{d}} = 0$$

for all $c \nearrow x$, and

$$g(X) = f(c) + (X - c)^{\mathbf{d}}$$

is an associated minimal polynomial for $\mathbf{A}$ for all $c \nearrow x$. In particular, if $\mathbf{d} > 1$, then K/w has positive characteristic.

Proof: We choose b as in the proof of Theorem 11.77. According to that theorem, $b(x - c)/w$ is an element of $(K/w)^{c(v/w)}$ for $c \nearrow x$, and it is algebraic over K. Since K is assumed to be henselian, Lemma 2.15 shows that $(K/w, v/w)$ is also henselian. By Lemma 11.70, ω_c must be purely inseparable over K. The irreducibility assertion of Theorem 11.77 for $\tilde{f}$ thus shows that $\tilde{f}(X)$ must be of the form $a_c f(c)/w + X^h$. But Theorem 11.77 also asserts that its degree is $\mathbf{d}$, hence $h = \mathbf{d}$ and $\mathbf{d} = p_w^e$ since $\tilde{f}$ is the minimal polynomial of ω_c which is purely inseparable over K/w. The last assertion now follows immediately from Theorem 11.77. □

Now we turn to the remaining case of $\mathbf{A}$ not being weakly distinguished.

Theorem 11.80 *Let $\mathbf{A} = \mathrm{appr}(x, K)$ be an algebraic approximation type of degree $\mathbf{d}$ and distance δ with associated minimal polynomial $f \in K[X]$. Assume that $\mathbf{A}$ is not weakly distinguished. Let h be as in Lemma 11.36 and $p = \max\{1, \mathrm{char}(\overline{K})\}$. Then $h = \mathbf{d} = p^e$ for some integer $e \geq 0$, and*

$$g(X) = f(c) + \sum_{i=0}^{e} (X - c)^{p^i} f_{p^i}(c)$$

is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$. (Note that g is additive if the characteristic of K is positive.)

Let $\mathcal{I} = \mathcal{I}(\delta)$, the coarsening $w = w_\delta$ of v and the cut $\delta/\mathcal{I}$ in $v(K)/\mathcal{I}$ be as explained on page 223. Then

$$g(X) = f(c) + \sum_{i=0}^{e} (X - c)^{p^i} \epsilon_i f_{p^i}(c)$$

is an associated minimal polynomial for $\mathbf{A}$ for all $c \nearrow x$, where $v(f_{p^i}(c)) = v(f_{p^i}(x))$ and

$$\epsilon_i = \begin{cases} 1 & \text{if } w(f_{p^i}(c)) = (p^e - p^i) \cdot (\delta/\mathcal{I}) \\ 0 & \text{else.} \end{cases}$$

In particular, if the cut $\delta/\mathcal{I}$ is not rational, then

$$g(X) = f(c) + (X - c)^{p^e}$$

is an associated minimal polynomial for $\mathbf{A}$ for all $c \nearrow x$.

Proof: As in the proof of Lemma 11.36, we consider the Taylor expansion (178) of f, keeping in mind that $\mathbf{A}$ fixes the value of all derivatives f_i, $i > 0$, since their degree is $< \mathbf{d}$. By Lemma 11.31 we know: if $i = p^t$, $j = p^t r$ with $r > 1$, $(r,p) = 1$, then

$$(j-i)\cdot v(x-c) + v(f_j(c)) - v(f_i(c)) > 0$$

for all $c \nearrow x$. Now it follows from Lemma 11.75 that there exists an element $\beta_0 \in \Lambda$ such that

$$v(f_j(c)) + j\cdot v(x-c) > v(f_i(c)) + i\cdot\Lambda$$

for all $c \nearrow x$ (such that $v(x-c) \geq \beta_0$). On the other hand, if $i \neq h$ then $v(f_i(c)) + i\cdot v(x-c) > v(f_{\mathbf{h}}(c)) + h\cdot v(x-c)$ which yields $v(f_i(c)) + i\cdot\Lambda \geq v(f_{\mathbf{h}}(c)) + h\cdot\Lambda$ and thus by (192),

$$v(f_j(c)) + j\cdot v(x-c) > v(f_{\mathbf{h}}(c)) + h\cdot\Lambda = \mathrm{dist}(f(x),K)\;. \tag{195}$$

For any j such that $h < j \leq \mathbf{d}$, we have by the choice of h that

$$(j-h)\cdot v(x-c) + v(f_j(c)) - v(f_{\mathbf{h}}(c)) > 0$$

for all $c \nearrow x$, and in the same way as above, (195) may also be deduced for such j. Altogether, if we put $h = p^e$ according to Lemma 11.36, it follows in view of the Taylor expansion (178):

$$v\left(f(x) - f(c) - \sum_{i=0}^{e}(x-c)^{p^i} f_{p^i}(c)\right) = v\left(\sum_{j\neq p^\nu \vee j>h}(x-c)^j f_j(c)\right) \geq \mathrm{dist}(f(x),K)\;,$$

hence by Lemma 11.26, $\mathrm{appr}(f(x),K) = \mathrm{appr}(g(x),K)$ for

$$g(X) = f(c) + \sum_{i=0}^{e}(X-c)^{p^i} f_{p^i}(c)\;,$$

which in view of Lemma 11.68 shows that g is an associated minimal polynomial for $\mathbf{A}$ whenever $c \nearrow x$. In particular, $\mathbf{A}$ does not fix the value of this polynomial, hence its degree cannot be less than $\mathbf{d}$. This proves $\mathbf{d} = h = p^e$. Note that this yields $f_{\mathbf{h}}(c) = 1$ and thus $v(f_{\mathbf{h}}(c)) = 0$ for all $c \in K$.

Now let $j = p^i < h$, $i \geq 0$. Then by the choice of h, we have

$$j\cdot v(x-c) + v(f_j(c)) > h\cdot v(x-c) + v(f_{\mathbf{h}}(c)) = h\cdot v(x-c)$$

and thus also

$$j\cdot w(x-c) + w(f_j(c)) \geq h\cdot w(x-c)$$

for all $c \nearrow x$. This shows

$$j\cdot(\delta/\mathcal{I}) + w(f_j(c)) \geq h\cdot(\delta/\mathcal{I})\;.$$

If "$>$" holds here, then we know that

$$j\cdot w(x-c) + w(f_j(c)) > h\cdot(\delta/\mathcal{I})$$

for all $c \nearrow x$ and thus also

$$j\cdot v(x-c) + v(f_j(c)) > h\cdot\delta = \mathrm{dist}(f(x),K)$$

for all $c \nearrow x$. As we have shown before in this proof, for such indices j we may omit the summand $(X-c)^j f_j(c)$ from the polynomial $g(X)$ without loosing the property $\mathrm{appr}(f(x),K) = \mathrm{appr}(g(x),K)$. But for $j \neq h$, the equation

$$j \cdot (\delta/\mathcal{I}) + w(f_j(c)) = h \cdot (\delta/\mathcal{I})$$

is only possible if

$$\delta/\mathcal{I} = \frac{w(f_j(c))}{h-j}$$

is a rational cut in $\Gamma/\mathcal{I}$. Consequently, if this is not the case, then all summands

$$(X-c)^j f_j(c)\ , \quad j \neq h$$

may be omitted from the polynomial $g(X)$ without loosing the property that $f(x)$ and $g(x)$ have the same approximation type over K. But if $\delta/\mathcal{I}$ is rational, then the above equation yields the criterion that we have used in the formulation of our theorem for those summands that have to appear in $g(X)$. □

Whenever we have obtained a polynomial g from the assertion of Theorem 11.77, Theorem 11.79 or Theorem 11.80, $g(x)$ having the same approximation type as $f(x)$ over K, then $\mathbf{A}$ does not fix the value of g since by hypothesis, it does not fix the value of f. Then from Theorem 11.52 we know that there is a simple immediate extension $K(y)|K$ of degree $\mathbf{d}$ such that $\mathrm{appr}(y,K) = \mathrm{appr}(x,K)$ and y is a root of the polynomial g.

Corollary 11.81 *Let the situation be as in Theorem 11.80 and assume in addition that there exists no nontrivial immediate algebraic extension of K of degree $< p^e$. If $e > 1$, then δ must be rational, and we will necessarily have $\epsilon_1 \neq 0$, i.e. $v(f_1(c)) = (p^e-1)\cdot\delta$ for $c \nearrow x$.*

Proof: If the assertion were not right, we could write

$$g(X) = f(c) + \sum_{i=0}^{e-1} Z^{p^i} \epsilon_{i+1} f_{p^{i+1}}(c)$$

where $Z = (X-a)^p$. Then the immediate extension $K(y)|K$ which we have introduced above and which is of degree p^e, would admit a nontrivial immediate subextension $K(z)|K$ where $z = (y-a)^p$, which is of degree p^{e-1} contradicting our hypothesis. □

Theorem 11.79 and Theorem 11.80 together show:

Corollary 11.82 *If K is a henselian field and $\mathbf{A}$ is an immediate approximation type over K, then the degree of $\mathbf{A}$ is a power of p.*

Corollary 11.83 *Let K be a henselian field, $(K(y),v)|(K,v)$ be an immediate extension and $(K(y'),v)|(K,v)$ an arbitrary extension of degree $p = \mathrm{char}(\overline{K})$. If*

$$\mathrm{appr}(y,K) = \mathrm{appr}(y',K)\ ,$$

then $(K(y'),v)|(K,v)$ is also an immediate extension.

Proof: The approximation type $\mathrm{appr}(y',K) = \mathrm{appr}(y,K)$ is immediate by Theorem 11.27, because $K(y)|K$ is immediate by assumption. Since $[K(y'):K] = p$, $\mathrm{appr}(y',K)$ is of degree $\leq p$ by Corollary 11.48. If it were of degree less than p, it could only be of degree 1 by Corollary 11.82. But by Lemma 11.39 this means that $y' \in K$, contrary to our assumption that $[K(y'):K] = p$. Thus $\mathrm{appr}(y',K)$ is an immediate approximation type of degree p over K. By Theorem 11.52, there exists an immediate extension of the valuation v from K to $K(y')$. Since K is henselian by assumption, this extension must coincide with the given valuation v on $K(y')$ which shows that $(K(y'),v)|(K,v)$ is immediate. □

In Corollary 11.78, we have stated a correlation between a given distinguished approximation type over K and certain approximation types on the residue field K/w. This also covers the case of a weakly distinguished approximation type because it is always connected to a distinguished approximation type through multiplication with a constant. In the case where the given approximation type is not weakly distinguished, there is a correlation with certain approximation types on the valued field (K, w_δ), as we will see below. Beforehand, we note the following easy observation:

Lemma 11.84 *Let $\mathbf{A}_v = \mathrm{appr}_v(x,K)$ be an approximation type over K of degree $\mathbf{d} \leq \infty$. Then for every $g \in K[X]$ of degree $< \mathbf{d}$,*

$$\mathcal{I}(\mathrm{dist}(g(x),K)) = \mathcal{I}(\mathrm{dist}(x,K)) \;.$$

If in addition $(K,v) \prec_\exists (K(x),v)$, then the assertion holds even for polynomials g of arbitrary degree.

Proof: If $g \in K[X]$ is of degree $< \mathbf{d}$, then by Lemma 11.37,

$$\mathrm{dist}(g(x),K) = i \cdot \mathrm{dist}(x,K) + \beta$$

for some integer $i > 0$ and $\beta \in v(K)$. If $(K,v) \prec_\exists (K(x),v)$, then according to Lemma 11.40, the same holds for polynomials g of arbitrary degree. Now our lemma follows from the last assertion of Lemma 11.71. □

Theorem 11.85 *Let $\mathbf{A}_v = \mathrm{appr}_v(x,K)$ be an approximation type with respect to the valuation v of K, of degree $\mathbf{d}$ and distance δ. Assume that $\mathbf{A}_v$ is not weakly distinguished, and let again $w = w_\delta$ denote a coarsening of v (on $K(x)$) corresponding to $\mathcal{I} = \mathcal{I}(\delta)$ (cf. page 223). Then the approximation type $\mathbf{A}_w = \mathrm{appr}_w(x,K)$ of x over K with respect to the valuation w is also an immediate approximation type of degree $\mathbf{d}$. More precisely, given a polynomial $g \in K[X]$, then $\mathbf{A}_v$ fixes the value of g if and only if $\mathbf{A}_w$ fixes the value of g. In particular, g is an associated minimal polynomial for $\mathbf{A}_v$ if and only if it is an associated minimal polynomial for $\mathbf{A}_w$. In particular, if $1 < \mathbf{d} < \infty$, then $\mathrm{char}(K/w) > 0$.*

Proof: By hypothesis, $\mathbf{A}_v$ is not weakly distinguished, i.e. $\Lambda(\mathbf{A}_v)/\mathcal{I}$ has no greatest element. But $\Lambda(\mathrm{appr}_w(x,K)) = \Lambda(\mathbf{A}_v)/\mathcal{I}$, so it follows that $\mathrm{dist}_w(x,K)$ is not assumed by an element of K. This proves that $\mathrm{appr}_w(x,K)$ is residue–immediate.

To prove that $\mathrm{appr}_w(x,K)$ is value–immediate, let $c \in K$; we have to show that there exists $\alpha \in w(K)$ with

$$c \in \mathrm{appr}_w(x,K)_\alpha \setminus \mathrm{appr}_w(x,K)^\circ_\alpha \;.$$

But this follows immediately from the same property of $\mathrm{appr}_v(x,K)$.

Now let $g\in K[X]$ be an arbitrary polynomial. By Corollary 11.47, $\mathbf{A}_v$ does not fix the v–value of g if and only if $v(g(x))\geq \mathrm{dist}_v(g(x),K)$, and $\mathbf{A}_w$ does not fix the w–value of g if and only if $w(g(x))\geq \mathrm{dist}_w(g(x),K)$. Thus, we have to show

$$v(g(x))\geq \mathrm{dist}_v(g(x),K)\Longleftrightarrow w(g(x))\geq \mathrm{dist}_w(g(x),K)\;.$$

In view of the fact that

$$w(g(x))=v(g(x))/\mathcal{I}\;\;\text{and}\;\;\mathrm{dist}_w(g(x),K)=\mathrm{dist}_v(g(x),K)/\mathcal{I}\;,$$

this equivalence holds if and only if the following equivalence holds:

$$v(g(x))\geq \mathrm{dist}_v(g(x),K)\Longleftrightarrow v(g(x))/\mathcal{I}\geq \mathrm{dist}_w(g(x),K)/\mathcal{I}\;.$$

But this equivalence follows from Lemma 11.71 and the equality

$$\mathcal{I}(\mathrm{dist}_v(g(x),K))=\mathcal{I}(\mathrm{dist}_v(x,K))$$

that we have deduced in the foregoing lemma; we may apply this lemma here since we may assume w.l.o.g. that $(K,v)\prec_\exists (K(x),v)$, according to Lemma 11.13. We have proved that for an arbitrary polynomial $g\in K[X]$, $\mathbf{A}_v$ fixes the v–value of g if and only if $\mathbf{A}_w$ fixes the w–value of g. From this result it follows at once that the degrees of $\mathbf{A}_v$ and $\mathbf{A}_w$ are equal and that a minimal associated polynomial of $\mathbf{A}_v$ is also such for $\mathbf{A}_w$, and vice versa.

The assertion on the characteristic of K/w follows by an application of the equation $\mathbf{d}=p^e$ of Theorem 11.80 to the approximation type $\mathbf{A}_w$ which is an approximation type over the valued field (K,w) whose residue field is just K/w. □

11.5 Distinguished approximation types in henselizations.

Our goal in this subsection is to show that every element in the henselization K^h has a weakly distinguished approximation type over K. To prove this, we will consider a very special type of immediate extensions $(K(x),v)|(K,v)$. We will call an element x *strictly distinguished* over K, if there exists a coarsening w of v such that the following three conditions hold:

1. $w(x)=0$,
2. x/w is an element of the completion of K/w ,
3. $\forall n\in I\!N:\;(1,x,...,x^n$ linearly independent over $K \Longrightarrow 1,x/w,...,(x/w)^n$ linearly independent over K/w).

The following lemma characterizes strictly distinguished elements via their approximation types:

Lemma 11.86 *Let* $\mathrm{appr}(x,K)$ *be of degree* $\mathbf{d}$ *with distance* δ. *Then the element* x *is strictly distinguished over* K *if and only if* $w_\delta(x)=0$ *and* $\mathrm{appr}(x,K)$ *is distinguished with* $\mathbf{d}=[K(x):K]$. *In this case,* $w=w_\delta$.

Proof: From Lemma 11.76 it follows that $w(x) = 0 \wedge x \in (K/w)^{c(v/w)}$ is equivalent to the property that $w_\delta(x) = 0$ and $\mathrm{appr}(x, K)$ is distinguished; in this case, the equality $w = w_\delta$ (on K) also follows from Lemma 11.76.

Now it suffices to show the equivalence of the additional conditions under the assumption that all other conditions hold. From Corollary 11.78 we infer that $\mathrm{appr}(x/w_\delta, K/w_\delta)$ is of the same degree $\mathbf{d}$ as $\mathrm{appr}(x, K)$, but of distance ∞. By virtue of Corollary 11.69, the associated minimal polynomial of $\mathrm{appr}(x/w_\delta, K/w_\delta)$ is the minimal polynomial of x/w_δ over K/w_δ if $\mathrm{appr}(x/w_\delta, K/w_\delta)$ is algebraic. If $\mathrm{appr}(x/w_\delta, K/w_\delta)$ is not algebraic, then $[(K/w_\delta)(x/w_\delta) : K/w_\delta] = \infty$ by virtue of Corollary 11.47. This shows $\mathbf{d} = [(K/w_\delta)(x/w_\delta) : K/w_\delta]$. Hence $[K(x) : K] = [(K/w_\delta)(x/w_\delta) : K/w_\delta]$ if and only if $\mathbf{d} = [K(x) : K]$. □

Strictly distinguished elements generate extensions with a nice property:

Lemma 11.87 *Let $(K(x), v)|(K, v)$ be an immediate extension and x be strictly distinguished over K. Then for every element $y \in K(x)$, the approximation type* $\mathrm{appr}(y, K)$ *is weakly distinguished.*

Proof: By Lemma 11.76, the case $y \in K$ is trivial. Now assume $y \notin K$, and let the coarsening w of v be as in the above definition of strictly distinguished elements. In the first step, we will prove the lemma under the assumption that y is a polynomial in x, say $y = f(x)$ with $f \in K[X]$ and $\deg(f) < [K(x) : K]$ if the latter is finite. (If x is algebraic over K, then this assumption is no loss of generality.) By Lemmata 11.25 and 11.72, for every $c \in K^\times$ we have that $\mathrm{appr}(y, K)$ is weakly distinguished if and only if $\mathrm{appr}(by, K)$ is weakly distinguished; after multiplication with a suitable element b we may thus assume that $f \in \mathcal{O}_{(K,w)}[X]$, but $f \notin \mathcal{M}_{(K,w)}[X]$. Consequently, $f/w \not\equiv 0$, and since $w(x) = 0$, we have $f(x)/w = f/w(x/w)$. By our assumption on the degree of f, the elements $1, x, ..., x^{\deg(f)}$ are linearly independent over K, and by condition 3) of the above definition of strictly distinguished elements, the same holds for the elements $1, x/w, \ldots, (x/w)^{\deg(f)}$ over K/w. Hence $f/w(x/w) \notin K/w$. But since x/w is an element of the completion of K/w, the element $f(x)/w = f/w(x/w)$ lies also in the completion of K/w. This shows in view of Lemma 11.76 that $\mathrm{appr}(f(x), K)$ is weakly distinguished.

In the second step, it remains to prove the lemma for the case where x is transcendental over K and $y = f(x)/g(x)$ with $f, g \in K[X]$. By a similar argument as above, we may assume that after multiplication of f and g with suitable elements from $K^\times$, $f, g \in \mathcal{O}_{(K,w)}[X] \setminus \mathcal{M}_{(K,w)}[X]$ and both $f(x)/w$ and $g(x)/w$ lie in the completion of $(K/w, v/w)$ and that $f(x)/w \notin K/w$ if $f(x) \notin K$, and $g(x)/w \notin K/w$ if $g(x) \notin K$. To avoid the case where $(f(x)/g(x))/w = (f(x)/w)/(g(x)/w) \in K/w$, we have to do the following consideration. If $m = \deg(g/w)$, then the m–th coefficient of g is not 0; hence there exists an element $d \in K$ such that the m–th coefficient of the polynomial $f - d \cdot g$ is 0. After multiplication of $f - d \cdot g$ with a suitable element from $K^\times$, we will have that $(f(x) - d \cdot g(x))/w$ lies in the completion of $(K/w, v/w)$ and that the degree of $(f - d \cdot g)/w$ cannot equal the degree of $g(x)/w$, which shows

$$\frac{(f(x) - d \cdot g(x))/w}{g(x)/w} \notin K/w \; .$$

But since $(f - d \cdot g)/g = (f/g) - d$ and

$$\mathrm{appr}(\frac{f(x)}{g(x)}, K) = \mathrm{appr}(\frac{f(x)}{g(x)} - d, K) \; ,$$

it follows by Lemma 11.76 that $\mathrm{appr}(f(x)/g(x), K)$ is weakly distinguished. This completes our proof. □

We need some more auxiliary results:

Lemma 11.88 *Let $L|K$ be an immediate extension of valued fields and assume that for every element $x \in L$, the approximation type* $\mathrm{appr}(x, K)$ *is weakly distinguished. If $L(y)|L$ is also an immediate extension of valued fields, and if* $\mathrm{appr}(y, L)$ *is weakly distinguished, then* $\mathrm{appr}(y, K)$ *is weakly distinguished too.*

Proof: By Lemma 11.26, $\mathrm{dist}(y, K) \leq \mathrm{dist}(y, L)$, the latter distance being weakly distinguished by hypothesis. If "$=$" holds, then there is nothing to prove. If "$<$" holds, then we infer from Lemma 11.26 that there exists an element $x \in L$ such that

$$\mathrm{appr}(y, K) = \mathrm{appr}(x, K) \ .$$

Since by hypothesis, $\mathrm{appr}(x, K)$ is weakly distinguished, this completes the proof. □

Lemma 11.89 *Let $(M, v)|(K, v)$ be an extension of valued fields generated by a set of elements $\{x_\nu \mid \nu < \tau\} \subset M$, where τ is an ordinal number, such that for every $\nu < \tau$, the element x_ν is strictly distinguished over $K_\nu := K(x_\mu | \mu < \nu)$ ($K_0 := K$). Then* $\mathrm{appr}(x, K)$ *is a weakly distinguished approximation type for every element $x \in M$.*

Proof: We prove the lemma by transfinite induction on $\rho < \tau$. The assertion holds trivially for the field K. Now assume $\rho \geq 1$ and that the assertion holds for every K_μ with $\mu < \rho$. If ρ is a limit ordinal, then $K_\rho = \bigcup_{\mu<\rho} K_\mu$ showing that the assertion holds for K_ρ too. Now let $\rho = \nu + 1$ be a successor ordinal. Then $K_\rho = K_\nu(x_\nu)$ where x_ν is strictly distinguished over K_ν. Let y be an arbitrary element of $K_\nu(x_\nu)$. By Lemma 11.87, the approximation type $\mathrm{appr}(y, K_\nu)$ is weakly distinguished. By our induction hypothesis, for every element $x \in K_\nu$, the approximation type $\mathrm{appr}(x, K)$ is weakly distinguished; in view of Lemma 11.88 this yields that also $\mathrm{appr}(y, K)$ is weakly distinguished. Hence the lemma holds for K_ρ, and the induction step is established. □

Lemma 11.90 *Let K be a valued field. The henselization K^h can be generated over K in the way as described in the hypothesis of the foregoing lemma.*

Proof: The henselization K^h can be generated over K by a transfinitely repeated adjunction of roots x of polynomials which satisfy the hypothesis of Hensel's Lemma. From the proof of the fact that stepwise complete valued fields are henselian it can be deduced that every such root x has a weakly distinguished approximation type over the field K' in which the polynomial is defined, and moreover, that there exists a coarsening w of the valuation v such that x/w is an element of the completion of K'/w with respect to v/w, but not an element of K'/w. Hence we only have to care for condition 3) of the definition of strictly distinguished elements. We modify our construction in the following way: we generate a field L over K by transfinitely repeated adjunction of roots x of irreducible polynomials f over already constructed fields K', which satisfy the following condition:

f/v admits x/v as a simple root, and for every proper coarsening w of v, either f/w remains irreducible or admits a root with v/w–residue x/v. Because of this last condition, such elements x will also satisfy condition 3). Indeed, if w is the coarsening of v such that x/w is an element of the completion of K'/w with respect to v/w, but not an element of K'/w, then f/w will be irreducible over K'/w; this is true since by our hypothesis on f, it would otherwise admit a root in K'/w with v/w–residue x/v which consequently must be equal to x/w contradicting $x/w \notin K'$. Moreover, $L \subset K^h$. It remains to show that L is henselian, since then we will have $L = K^h$ and the desired procedure to generate K^h over K. Assume that there exists an irreducible polynomial $f \in L[X]$ satisfying the conditions of Hensel's Lemma: f/v has a simple zero x/v. Among all coarsenings w of v, such that f/w admits an irreducible factor g_w of degree > 1 whose v/w–reduction $g_w/(v/w)$ admits x/v as a zero, we choose a coarsening w_0 for which g_{w_0} has least degree. Furthermore, we choose any $g^* \in L[X]$ with $g^*/w_0 = g_{w_0}$ and $\deg(g^*) = \deg(g_{w_0})$. Then g^* satisfies the above condition: g^*/v admits x/v as a simple zero, and for every coarsening w of v, the polynomial g^*/w is either irreducible or admits a zero whose v/w–residue is equal to x/v. By our construction of L, it must contain a root x of g^* with residue x/v and consequently, x/w_0 is a root of $g^*/w_0 = g_{w_0}$ in contradiction to our assumption that g_{w_0} is irreducible of degree > 1. This contradiction shows that L is henselian, as asserted, and our lemma is proved. □

As an immediate consequence, we now obtain the following corollary:

Corollary 11.91 *Let K be a valued field. Then for every element a of the henselization of K, the approximation type* $\mathrm{appr}(x, K)$ *is weakly distinguished.*

With the help of this corollary, we are able to prove:

Lemma 11.92 *Let $x, y \in \tilde{K}$, $y \notin K$. Assume*

$$\mathrm{appr}(y, K) = \mathrm{appr}(x, K)$$

and $x \in K^h$. Then

$$[K^h(y) : K^h] < [K(y) : K] .$$

In particular, $K(y)|K$ is not purely inseparable.

Proof: If $x \in K$, hence $\mathrm{dist}(x, K) = \infty$, then $\mathrm{appr}(y, K) = \mathrm{appr}(x, K)$ yields $x = y$ by Lemma 11.26, and the assertion follows trivially. Now let us assume $x \notin K$. Since $x \in K^h$, the foregoing corollary and Lemma 11.76 show that there exist elements $b, c \in K$ and a coarsening w of v such that $w(b(x - c)) = 0$ and

$$b(x - c)/w \in (K/w)^{c(v/w)} \setminus K/w ,$$

where "$c(v/w)$" denotes the completion with respect to the valuation v/w. By virtue of Lemma 11.25, $\mathrm{appr}(y, K) = \mathrm{appr}(x, K)$ implies

$$\mathrm{appr}(b(y - c), K) = \mathrm{appr}(b(x - c), K) .$$

In view of this equality and $K(b(y - c)) = K(y)$, we may assume from the start that $b = 1$ and $c = 0$. Now $\mathrm{appr}(y, K) = \mathrm{appr}(x, K)$ implies $w(y) = w(x) = 0$ and

$$\mathrm{appr}(y/w, K/w) = \mathrm{appr}(x/w, K/w)$$

(the straightforward proof of this equality is left to the reader). Since x/w is an element of the completion of K/w, this yields $y/w = x/w$.

Now we consider the fields K, K^h and $K^h(y)$ equipped with the valuation w. Since K^h is henselian for the valuation v, it is also henselian for the coarsening w by Lemma 2.15. Let $f(X) \in K[X]$ be the minimal polynomial of y over K. Our assertion is proved if we are able to show that f is reducible over K^h. At this point we may assume that all conjugates of y over K have the same value $v(y)$ since otherwise the inequality $[K^h(y) : K^h] < [K(y) : K]$ is immediately seen to be true. This assumption yields $f \in \mathcal{O}_{(K,w)}[X]$, and because of $w(y) = 0$, the reduced polynomial f/w is nontrivial. The minimal polynomial $g \in (K/w)[X]$ for $x/w = y/w$ over K/w has degree > 1 since $x/w \notin K/w$. Furthermore, it must divide f/w which satisfies $(f/w)(y/w) = f(y)/w = 0$. Since $x \in K^h$,

$$y/w = x/w \in K^h/w \;,$$

and g becomes reducible over K^h/w. From Lemma 2.16, we infer

$$K^h/w = (K/w)^{h(v/w)} \;.$$

In particular, this shows that x/w is a simple root of g since the henselization is a separable extension. Applying Hensel's Lemma to the henselian field (K^h, w), one concludes that f becomes reducible over K^h; indeed, f factors into two nontrivial polynomials where the roots of the first one all have w–residue y/w while there exists at least one root (in $\tilde{K}$) of the second polynomial which has as w–residue a root of g (in $\tilde{K}/w$) which is different from y/w. This proves our lemma. □

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13 List of notations.

(K,v), (K,P)	a valued field with valuation v resp. place P
$v(K)$	the value group of (K,v) (we always assume $\infty \notin v(K)$)
$\overline{K}$, KP, K/v	the residue field of (K,v)
$v(x)$	the value of $x \in K$
$\overline{x}$, xP, x/v	the residue of $x \in K$
P_v, v_P	the place associated to v resp. the valuation associated to P
$\mathcal{O}$, $\mathcal{O}_v$, $\mathcal{O}_K$	the valuation ring
$\mathcal{M}$, $\mathcal{M}_v$, $\mathcal{M}_K$	the valuation ideal
PQ, $v \circ w$	the composition of places resp. valuations
v/w	the valuation induced by v on the residue field K/w if w is a coarsening of v
K^h, $(K,v)^h$, $(K,P)^h$	the henselization
$K^{h(P)}$, $K^{h(v)}$	the henselization with respect to P resp. v
K^c, $(K,v)^c$, $(K,P)^c$	the completion
$K^{c(P)}$, $K^{c(v)}$	the completion with respect to P resp. v
$(K,v)^{hc}$, K^{hc}	the completion of the henselization, cf. p. 100
$(K,v)^r$, K^r	the (absolute) ramification field
$\tilde{K}$	the algebraic closure of K
K^{sep}	the separable–algebraic closure of K
$\sqrt{K}$	the perfect hull of K
$\frac{1}{p^\infty}\Gamma$	the p–divisible hull of the abelian group Γ
Γ	the divisible hull of the abelian group Γ
$\mathrm{rk}\Gamma$	the rank of the abelian group Γ
$\mathrm{rr}\Gamma$	the rational rank of the abelian group Γ
$\mathrm{d}(L\|K)$	the defect
$\mathrm{d}_c(L\|K)$	the completion defect
$\mathrm{d}_q(L\|K)$	the defect quotient (= quotient of d and d_c)
$\wp(x)$	$= x^p - x$
Tr	the trace

appr	the approximation type
dist, dist_R	the distance
$\mathbf{A}$	usually denotes approximation types
$\mathbf{d}$	the degree of an approximation type
$\mathbf{h}$, $\mathbf{h}_K$, $\mathbf{h}_K(x:y)$	the relative approximation degree
$\mathbf{S}(\mathbf{A})$, $\mathbf{S}_=(\mathbf{A})$,...	cf. p. 189
$\Lambda(\mathbf{A})$	the value set of the approximation type $\mathbf{A}$
amp($\mathbf{A}$)	the set of associated minimal polynomials for $\mathbf{A}$
sup	the supremum of a subset of an ordered abelian group, cf. p. 32
$\asymp$	cf. p. 32
$c \nearrow \mathbf{A}$, $\forall z \nearrow \mathbf{A}$	cf. p. 32
$c \nearrow x$, $\forall z \nearrow x$	cf. p. 32
$\mathcal{I}$, $\mathcal{I}(\delta)$, $\mathcal{I}(\delta,\Gamma)$	the invariance subgroup
U_δ	cf. p. 75
$\prec_\exists$	stands for "existentially closed in"
$\prec$	stands for "elementary extension"
$\equiv$	stands for "elementary equivalent"
Th($\mathcal{A}$)	the theory of the model $\mathcal{A}$
$\lvert K\rvert^+$	the successor cardinal to the cardinality of K

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14 Index.