

LARGE IMPERFECT FIELDS ARE EXISTENTIALLY CLOSED IN FUNCTION FIELDS AFTER FINITE CONSTANT EXTENSION

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1. INTRODUCTION

For notions and notation used in this introduction, see Section 2. In [8], the following question is considered: if $F|K$ is a function field that admits a rational place, under which additional assumptions does it follow that K is existentially closed in F in the language of rings? The following statement is part of [8, Theorem 17]:

Theorem 1. *If K is a perfect large field, then K is existentially closed in every extension field L which admits a K -rational place.*

If local uniformization in positive characteristic holds, then by [1, Proposition 2.3] the condition “perfect” in Theorem 1 can be dropped and we obtain a principle that has been introduced in [1] as a basic condition for the main result in that article, weaker than local uniformization:

(R4) If K is a large field, then K is existentially closed in every extension field L which admits a K -rational place.

In this note our goal is to prove the following result. Our proof in Section 3 is a slight modification of a proof suggested to us by Arno Fehm. Likewise, also Corollary 10 and Open Problem 2 are due to him.

Theorem 2. *Take a large field K and a function field $F|K$ which admits a rational place. Then there is a finite purely inseparable extension $K'|K$ such that K'' is existentially closed in $F.K''$ for every purely inseparable algebraic extension $K''|K'$.*

Furthermore, there is a finite extension $K'|K$ such that K'' is existentially closed in $F.K''$ for every algebraic extension $K''|K'$.

Hence we can say: *if $L|K$ is a function field, then (R4) holds after a finite purely inseparable constant extension.*

If $L|K$ is not finitely generated, then for arbitrary elementary sentences valid in L , ever larger purely inseparable extensions of K may be needed. However, Theorem 2 implies Theorem 1.

The following is part of [4, Lemma 9]:

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Lemma 3. *Take a large field K and a function field $F|K$. Then $F|K$ admits a discrete rational place if and only if K is existentially closed in F .*

Therefore, Theorem 2 is equivalent to:

Theorem 4. *Take a large field K and a function field $F|K$ which admits a rational place. Then there is a finite purely inseparable extension $K'|K$ such that for every purely inseparable extension K'' of K' , the function field $F.K''|K''$ admits a discrete rational place.*

Note that if $F.K'|K'$ admits a discrete rational place P , then FP is a finite purely inseparable extension of K . Hence one is led to:

Open problem 1: Assume that $F|K$ admits a discrete place P whose residue field is a finite purely inseparable extension of K . Under which additional assumptions does it follow that there is a finite purely inseparable extension $K'|K$ such that the unique extension of P to $F.K'$ has residue field K' ?

It is known that K' may not exist for a function field F over a field K of characteristic $p > 0$ and infinite degree of imperfection. Indeed, take a sequence $(a_i)_{i \in \mathbb{N}}$ in K of p -independent elements, set $s = \sum_n a_i t^{ip} \in K[[t]]$, and take $F = K(t, s^{1/p})$. Then the residue field of F under the t -adic valuation is $K_1 = K(a_0^{1/p})$, but extending the constant field to K_1 will yield residue field $K_2 = K_1(a_1^{1/p})$, and so on. This is a construction given by Masayoshi Nagata in [12, Appendix, Example (E3.1), pp. 206-207], cf. [9, Example 3.23 and Theorem 3.24].

We do not know whether there is such an example with a field K of finite degree of imperfection.

If we want to avoid the purely inseparable extension appearing in Theorem 2, we have to invoke local uniformization, which presently is not known to hold in positive characteristic for dimensions higher than three.

Theorem 5. *For an algebraic function field F over a large field K the following conditions are equivalent:*

- (1) *There exists a rational place P admitting local uniformization.*
- (2) *K is existentially closed in F .*

The implication (1) \Rightarrow (2) has already been stated in [8, Theorem 19]. However, its proof uses [8, Theorem 13], but the proof of that theorem is flawed. By [4, Lemma 9], the fact that the large field K is existentially closed in F implies that $F|K$ admits a discrete rational place Q . However, in order to prove Theorem 13 of [8], one would have to show that Q can be chosen as close to P as we want in the patch topology.

2. PRELIMINARIES

For basic facts from valuation theory, see [3].

Let $L|K$ be a field extension. We will say that K is **existentially closed** in L if every existential sentence in the ring (or field) language with parameters from K holds in K if it holds in L .

Take a field K . A **K -variety** is an integral separated K -scheme V of finite type; if V has dimension one, then it is called a **K -curve**. As usual the function field

of a K -variety V is denoted by $K(V)$ and for any extension field L of K the set of L -rational points of V is denoted by $V(L)$. The subset V_{sm} of smooth points of V , called the **smooth locus** of V , is Zariski-open and thus a K -variety if it is nonempty (see [10, Corollary 8.2]); the latter holds if and only if $K(V)|K$ is separable. If $V = V_{\text{sm}}$, then the K -variety V is said to be **smooth**.

A K -variety V is **geometrically integral** if the scheme $V \times_K \tilde{K}$ is a \tilde{K} -variety, where \tilde{K} denotes the algebraic closure of K . According to [6, Corollaire 4.6.3], this is the case if and only if the extension $K(V)|K$ is **regular**, that is, $K(V)$ and \tilde{K} are linearly disjoint over K . This holds if and only if K is relatively algebraically closed in $K(V)$ and $K(V)|K$ is separable, i.e., linearly disjoint from the perfect hull of K (cf. [11, Chapter VIII, §4]).

A **model** of a given function field $F|K$ is a K -variety V with the property $K(V) \cong_K F$. A place P of a function field $F|K$ is called **rational** if it is the identity on K and $FP = K$. The existence of a rational place P on $F|K$ implies the existence of a model V possessing a K -rational point. This is seen as follows. Take a transcendence base T in the valuation ring \mathcal{O}_P of the place P and denote by A the integral closure of $K[T]$ in \mathcal{O}_P . Then A is finitely generated over K and $V := \text{Spec}(A)$ is a model with the K -rational point $x := \mathcal{M}_P \cap A$, where \mathcal{M}_P denotes the maximal ideal of \mathcal{O}_P . However, in general such a point x need not be regular.

If there is a model V for which the center of P on V is regular, we say that P admits **local uniformization**.

Following Florian Pop (see [13, 14]), a field K is called a large field if it satisfies one of the following equivalent conditions:

- (LF) *For every smooth K -curve C the set $C(K)$ is infinite or empty.*
- (LF') *In every smooth K -variety V the set $V(K)$ is Zariski-dense or empty.*
- (LF'') *For every function field $F|K$ in one variable the set of rational places is infinite or empty.*

Various characterizations of large fields using existential closedness are given by [8, Theorem 15].

Theorem 6. *The following conditions are equivalent:*

- 1) *K is a large field,*
- 2) *K is existentially closed in every function field F in one variable over K which admits a K -rational place,*
- 3) *K is existentially closed in the henselization $K(t)^h$ of the rational function field $K(t)$ with respect to the t -adic valuation,*
- 4) *K is existentially closed in the field $K((t))$ of formal Laurent series,*
- 5) *K is existentially closed in every extension field which admits a discrete K -rational place.*

3. PROOFS

One of the main ingredients in the proofs of Theorem 2 and Theorem 5 is a geometric characterization of existential closedness of a field within a finitely generated extension, due to Pop [14, Fact 2.3]:

Lemma 7. *Let K be a field and V a K -variety. Then $K \prec_{\exists} K(V)$ if and only if $V(K)$ is Zariski-dense in V .*

The following result is well known, but the only reference we know is [2, Corollary 3.1.3], where it is derived from a more involved context. For the convenience of the reader, we will give a direct proof.

Lemma 8. *Let $F|K$ be an algebraic function field with the property $K \prec_{\exists} F$, then $F|K$ is a regular extension.*

Proof. Suppose that K is not relatively algebraically closed in L . Then there exists an irreducible polynomial $f(X) = X^n + c_{n-1}X^{n-1} + \dots + c_0 \in K[X]$ which has a root in L . The existential sentence

$$\exists x : x^n + c_{n-1}x^{n-1} + \dots + c_0 = 0$$

with parameters in K expresses the existence of a root of f . By our choice of f , it holds in L but not in K . Consequently, K is not existentially closed in L .

Now suppose that $L|K$ is not separable, i.e., not linearly disjoint from $K^{1/p^\infty}|K$. Then there are K -linearly independent elements $x_1, \dots, x_n \in L$ which are not K^{1/p^∞} -linearly independent. Choose m minimal such that there are $y_1, \dots, y_n \in K^{1/p^m}$ with $\sum_i x_i y_i = 0$. Then $m \geq 1$, and by the minimality of m , $x_1^{p^{m-1}}, \dots, x_n^{p^{m-1}}$ are K -linearly independent. But $x_1^{p^{m-1}}, \dots, x_n^{p^{m-1}}$ are not $K^{1/p}$ -linearly independent since $\sum_i x_i^{p^{m-1}} y_i^{p^{m-1}} = 0$ with $y_i^{p^{m-1}} \in K^{1/p}$. This proves that $L|K$ is not linearly disjoint from $K^{1/p}|K$.

The Frobenius induces an isomorphism from L onto L^p , from $K^{1/p}$ onto K , and from K onto $K^{1/p}$. Hence it follows that $L^p|K^p$ is not linearly disjoint from $K|K^p$. Choose K^p -linearly independent elements $a_1, \dots, a_n \in K$ which are not L^p -linearly independent. Then the existential sentence

$$\exists x_1 \exists x_2 \dots \exists x_n : x_1^p a_1 + \dots + x_n^p a_n = 0$$

with parameters in K holds in L but not in K , showing again that K is not existentially closed in L . \square

[5, Lemma 2.6.9] provides another criterion for regularity relevant in the present context:

Lemma 9. *Let $F|K$ be an algebraic function field which admits a rational place, then $F|K$ is regular.*

Proof of Theorem 2. By Lemma 9, $F|K$ is a regular extension. Hence there exists a geometrically integral, smooth model V of $F|K$. One can assume that V is affine: $V = \text{Spec}(A)$.

The place P has a unique extension P' to $F' := F.K^{1/p^\infty}$, and $F'P' = K^{1/p^\infty}$. By [14, Prop. 2.7] K^{1/p^∞} is a large field. Thus, by Theorem 1, $K^{1/p^\infty} \prec_{\exists} F'$. The

variety $W := V \times_K K^{1/p^\infty} = \text{Spec}(A \otimes_K K^{1/p^\infty}) = \text{Spec}(A \cdot K^{1/p^\infty})$ is a model of $F|K^{1/p^\infty}$. Lemma 7 then yields that $W(K^{1/p^\infty})$ is Zariski-dense in W .

Fix any $q \in W(K^{1/p^\infty})$ and consider the point $p := \phi(q)$, where $\phi : W \rightarrow V$ is the natural projection. Then $K' := A/p$ is a finite purely inseparable extension of K . Now let $K''|K'$ be any purely inseparable extension, $W'' := V \times_K K'' = \text{Spec}(A \cdot K'')$ and $\phi'' : W'' \rightarrow V$, $\psi : W \rightarrow W''$ the natural projections. Then $\phi''(\psi(q)) = p$ and therefore the point $q'' := \psi(q)$ satisfies $(A \cdot K'')/q'' = A/p \cdot K'' = K''$, that is, $q'' \in W''(K'')$. Since W'' is a smooth K'' -variety and K'' is large, $W''(K'')$ is Zariski-dense by **(LF')**. Lemma 7 now yields $K'' \prec_{\exists} F.K''$.

Replacing the perfect hull of K by its algebraic closure in the arguments above, gives a proof of the remaining assertion of the theorem. \square

Corollary 10. *For $F|K$ as in Theorem 2, there exists $n \in \mathbb{N}$ such that $K \prec_{\exists} F^{p^n} K$, where $p = \text{char}(K)$.*

Proof. If K' is as in the theorem, then $K' \subseteq K'' := K^{p^{-n}}$ with $p^n = [K' : K]$. It follows that $K'' \prec_{\exists} F.K''$, and therefore $K = (K'')^{p^n} \prec_{\exists} (F.K'')^{p^n} = F^{p^n}.K$. \square

Open problem 2: Are Theorem 2 and Corollary 10 also true if $F|K$ is not a function field, i.e., not finitely generated?

Proof of Theorem 5. Assume that the rational place P of the function field $F|K$ admits local uniformization. Then there exists a model V of $F|K$ such that P is centered in a regular, K -rational point $x \in V$. By Lemma 9, the existence of a rational place shows that the extension $F|K$ is separable. Therefore the smooth locus of V is Zariski-open, so we can assume V to be smooth. A regular, K -rational point is smooth, hence $V(K) \neq \emptyset$. Since K is large, $V(K)$ is dense in V , thus $K \prec_{\exists} F$ by Lemma 7.

Now assume that $K \prec_{\exists} F$. Lemma 8 yields that $F|K$ is separable. Therefore the smooth locus V_{sm} of any model V of $F|K$ is nonempty and Zariski-open. By Lemma 7, $V(K)$ is dense in V , which implies $V_{\text{sm}} \cap V(K) \neq \emptyset$. Take $x \in V_{\text{sm}} \cap V(K)$. By [7, Cor. A2] (see also [5, Lemma 16.1.2]) there exists a K -rational place P centered in x and since x is regular, P by definition admits local uniformization. \square

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