THE VALUATION THEORY OF DEEPLY RAMIFIED FIELDS AND ITS CONNECTION WITH DEFECT EXTENSIONS

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Abstract. We study in detail the valuation theory of deeply ramified fields and introduce and investigate several other related classes of valued fields. Further, a classification of defect extensions of prime degree of valued fields that was earlier given only for the characteristic equal case is generalized to the case of mixed characteristic by a unified definition that works simultaneously for both cases. It is shown that deeply ramified fields and the other valued fields we introduce only admit one of the two types of defect extensions, namely the ones that appear to be more harmless in open problems such as local uniformization and the model theory of valued fields in positive characteristic. We use our knowledge about such defect extensions to give a new, valuation theoretic proof of the fact that algebraic extensions of deeply ramified fields are again deeply ramified. We also prove finite descend, and under certain conditions, even infinite descend. The classes of valued fields under consideration can be seen as generalizations of the class of tame valued fields. Our paper supports the hope that it will be possible to generalize to deeply ramified fields several important results that have been proven for tame fields and were at the core of partial solutions of the two open problems mentioned above.

1. Introduction

The main topics of this paper are the defect of valued field extensions, which lies at the heart of longstanding open problems in algebraic geometry and model theoretic algebra, and the valuation theory of deeply ramified fields. By studying the latter in depth, we will exibit the connection with the former. On the one hand, this enables us to better understand deeply ramified fields, and on the other hand, it shows us a possible direction in our attempt to tame the defect.

Our interest in the defect owes its existence to the following well known deep open problems in positive characteristic:

1) resolution of singularities in arbitrary dimension,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, and of its perfect hull.

Both problems are connected with the structure theory of valued function fields of positive characteristic p. The main obstruction here is the phenomenon of the **defect**, which we will define now.

Date: 4. 2. 2022.

2010 *Mathematics Subject Classification.* 12J10, 12J25.

Key words and phrases. deeply ramified fields, semitame fields, tame fields, defect, higher ramification groups.

The first author was partially supported by Opus grant 2017/25/B/ST1/01815 from the National Science Centre of Poland.

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the valuation ideals of L and K. The value group of the valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av.

We will say that a valued field extension $(L|K,v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski,

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{1}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p}=\operatorname{char}Kv$ if it is positive and $\tilde{p}=1$ otherwise. The factor $d(L|K,v)=\tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. We call $(L|K,v)$ a **defect extension** if $d(L|K,v)>1$, and a **defectless extension** if $d(L|K,v)=1$. Nontrivial defect only appears when $\operatorname{char}Kv=p>0$, in which case $\tilde{p}=p$.

Throughout this paper, when we talk of a **defect extension** $(L|K,v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. Then it follows from (1) that $[L:K]=p=\operatorname{char}Kv$ and that $(vL:vK)=1=[Lv:Kv]$; the latter means that $(L|K,v)$ is an **immediate extension**, i.e., the canonical embeddings $vK\hookrightarrow vL$ and $Kv\hookrightarrow Lv$ are onto.

Via ramification theory, the study of defect extensions can be reduced to the study of purely inseparable extensions and of Galois extensions of degree $p=\operatorname{char}Kv$. To this end, we fix an extension of v from K to its algebraic closure $\tilde{K}$. We denote the separable-algbraic closure of K by K^{sep}. The **absolute ramification field of** (K,v) (with respect to the chosen extension of v), denoted by (K^r,v), is the ramification field of the normal extension $(K^{\mathrm{sep}}|K,v)$. If $(K(a)|K,v)$ is a defect extension, then $(K^r(a)|K^r,v)$ is a defect extension with the same defect (see Proposition 2.14). On the other hand, $K^{\mathrm{sep}}|K^r$ is a p-extension, so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p.

Galois defect extensions of degree p of valued fields of characteristic $p>0$ (valued fields of **equal characteristic**) have been classified by the first author in [16]. There the extension is said to have **dependent** defect if it is related to a purely inseparable defect extension of degree p in a way that we will explain in Section 3.3, and to have **independent** defect otherwise. Note that the condition for the defect to be dependent implies that the purely inseparable defect extension does not lie in the completion of (K,v), hence if (K,v) lies dense in its perfect defect extensions of prime degree with dependent defect.

The classification of defect extensions is important because work by M. Temkin (see e.g. [27]) and by the first author indicates that dependent defect appears to be more harmful to the above cited problems than independent defect. In the paper [5], S. D. Cutkosky and O. Piltant give an example of an extension of valued function fields consisting of a tower of two Galois defect extensions of prime degree where strong monomialization fails. As the valuation on these extensions is defined by use of generating sequences, it is hard to determine whether they have dependent or independent defect. However, work of Cutkosky, L. Ghezzi and S. ElHitti shows that both of them have dependent defect (see e.g. [6]); this again lends credibility to the hypothesis that dependent defect is the more harmful one.

An analogous classification of Galois defect extensions of degree p of valued fields of characteristic 0 with residue fields of characteristic $p > 0$ (valued fields of **mixed characteristic**) has so far not been given. But such a classification is important for instance for the study of infinite algebraic extensions of the field $\mathbb{Q}_p$ of p-adic numbers, which in contrast to $\mathbb{Q}_p$ itself may well admit defect extensions. Indeed, $\mathbb{Q}_p^{ab}$, the maximal abelian extension of $\mathbb{Q}_p$, is such a field. Other examples will be given in Section 7. Moreover, we wish to study the valuation theory of deeply ramified fields (such as $\mathbb{Q}_p^{ab}$), which will be introduced below, in full generality without restriction to the equal characteristic case. For these fields in particular it is important to work out the similarities between the equal and the mixed characteristic cases.

The obvious problem for the definition of "dependent defect" in the mixed characteristic case is that a field of characteristic 0 has no nontrivial inseparable extensions. However, there is a characterization of independent defect equivalent to the one given in [16] that readily works also in the mixed characteristic case, and we use it to give a unified definition, as follows. Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p. For every σ in its Galois group $\mathrm{Gal}\,(L|K)$, with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\} \tag{2}$$

This set is a final segment of vK and independent of the choice of σ (see Theorems 3.4 and 3.5); we denote it by $\Sigma_{\mathcal{E}}$. We say that $\mathcal{E}$ has **independent defect** if

$$\Sigma_{\mathcal{E}} \;=\; \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\} \text{ for some proper convex subgroup } H_{\mathcal{E}} \text{ of } vK; \tag{3}$$

otherwise we will say that $\mathcal{E}$ has **dependent defect**. If (K, v) has rank 1 (i.e., its value group is order isomorphic to a subgroup of $\mathbb{R}$), then condition (3) just means that $\Sigma_{\mathcal{E}}$ consists of all positive elements in vK.

That our definition of "independent defect" in mixed characteristic is the right one is supported by the following observation. Take a valued field of positive characteristic. If it lies dense in its perfect hull, then by what we have said before, all Galois defect extensions must have independent defect. If the field is complete and of rank 1, then it is a perfectoid field. What about perfectoid fields of mixed characteristic? They share with their tilts, which are perfectoid fields of positive characteristic, isomorphic absolute Galois groups. Hence we expect that also perfectoid fields in mixed characteristic admit only independent defect extensions. This indeed holds with our definition. Similarly, the Fontaine-Wintenberger Theorem states that the fields $\mathbb{Q}_p(p^{1/p^n} \mid n \in \mathbb{N})$ and $\mathbb{F}_p((t))(t^{1/p^n} \mid n \in \mathbb{N})$ have isomorphic absolute Galois groups. Both are deeply ramified (and even semitame) fields (definitions are given below), and as such are independent defect fields, as we will show in Theorem 1.10.

For our purposes, the properties of completeness and rank 1 are irrelevant, and we prefer to work with a more flexible (and first order axiomatizable) notion. In fact, all perfectoid fields are deeply ramified, in the sense of [9]. Take a valued field (K, v) with valuation ring $\mathcal{O}_K$. Choose any extension of v to K^{sep} and denote the valuation ring of K^{sep} with respect to this extension by $\mathcal{O}_{K^{\mathrm{sep}}}$. Then (K, v) is a

deeply ramified field if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} \;=\; 0\,, \tag{4}$$

where $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra. This definition does not depend on the chosen extension of the valuation from K to K^{sep}.

According to [9, Theorem 6.6.12 (vi)], a nontrivially valued field (K,v) is deeply ramified if and only if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) if $\mathrm{char}\, Kv = p > 0$, then the homomorphism

$$\mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \ni x \mapsto x^p \in \mathcal{O}_{\hat{K}}/p\mathcal{O}_{\hat{K}} \tag{5}$$

is surjective, where $\mathcal{O}_{\hat{K}}$ denotes the valuation ring of the completion of (K,v).

Axiom (DRvr) means that modulo $p\mathcal{O}_{\hat{K}}$ every element in $\mathcal{O}_{\hat{K}}$ is a p-th power.

By altering axiom (DRvg) we will now introduce new classes of valued fields, one of them containing the class of deeply ramified fields, and one contained in it in the case of positive residue characteristic. We will call (K,v) a **generalized deeply ramified field**, or in short a **gdr field**, if it satisfies axiom (DRvr) together with:

(DRvp) if $\mathrm{char}\, Kv = p > 0$, then vp is not the smallest positive element in the value group vK.

Note that (DRvg) implies (DRvp).

If $\mathrm{char}\, Kv = p > 0$, then (DRvg) certainly holds whenever vK is divisible by p. We will call (K,v) a **semitame field** if it satisfies axiom (DRvr) together with:

(DRst) if $\mathrm{char}\, Kv = p > 0$, then the value group vK is p-divisible.

We note:

Proposition 1.1. *The properties (DRvg), (DRvp) and (DRst) are first order axiomatizable in the language of valued fields, and so are the classes of semitame, deeply ramified and gdr fields of fixed characteristic.*

We will give the proof of this proposition and of almost all results that we will describe now in Section 6.

Let us mention at this point that it has been conjectured that the elementary theory of the perfect hull of $\mathbb{F}_p((t))$ is decidable, but no proof has been given so far. As a perfect valued field of positive characteristic, it is semitame, and understanding its valuation theory and in particular its defects may lay the basis for a future proof. Mastering the defect has already shown to be an efficient tool to prove results on local uniformization and the model theory of valued fields, as demonstrated in [12, 13, 14, 19].

The notion of "semitame field" is reminiscent of that of "tame field". Let us recall the definition of "tame". For the purpose of this paper we will slightly generalize the notion of "tame extension" as defined in [19] (there, tame extensions were only defined over henselian fields). A unibranched extension $(L|K,v)$ will be called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(**TE1**) The ramification index $(vE : vK)$ is not divisible by $\mathrm{char}\, Kv$.
(**TE2**) The residue field extension $Ev|Kv$ is separable.
(**TE3**) The extension $(E|K,v)$ is **defectless**, i.e., $[E:K] = (vE:vK)[Ev:Kv]$.

A henselian field (K,v) is called a **tame field** if its algebraic closure with the unique extension of the valuation is a tame extension, and a **separably tame field** if its separable-algebraic closure is a tame extension. The absolute ramification field (K^r,v) is the unique maximal tame extension of the henselian field (K,v) by [7, Theorem (22.7)] (see also [24, Proposition 4.1]). Hence a henselian field is tame if and only if its absolute ramification field is already algebraically closed; in particular, every tame field is perfect.

In contrast to tame and separably tame fields, we do not require semitame fields to be henselian; in this way they become closer to deeply ramified fields. The other fundamental difference to tame fields is that semitame fields may admit defect extensions, but as we will see in Theorem 1.10 below, only those with independent defect. This justifies the hope that many of the results that have been proved for tame fields and applied to the problems we have cited in the beginning (see [19, 20]) can be generalized (at least) to the case of (henselian) semitame fields.

All valued fields of residue characteristic 0 are semitame and gdr fields, and they are deeply ramified fields if and only if (DRvg) holds. Likewise, all henselian valued fields of residue characteristic 0 are tame fields. In the present paper, we are not interested in the case of residue characteristic 0, so we will always assume that $\operatorname{char} Kv = p > 0$. We will now summarize the basic facts about the connections between the properties we have introduced. The proofs will be provided in Section 6.

Theorem 1.2. *1) If (K,v) is a nontrivially valued field with $\operatorname{char} Kv = p > 0$, then the following logical relations between its properties hold:*

tame field $\Rightarrow$ separably tame field $\Rightarrow$ semitame field $\Rightarrow$
deeply ramified field $\Rightarrow$ gdr field.

2) For a valued field (K,v) of rank 1 with $\operatorname{char} Kv = p > 0$, the three properties "semitame field", "deeply ramified field" and "gdr field" are equivalent.

3) For a nontrivially valued field (K,v) of characteristic $p > 0$, the following properties are equivalent:
a) (K,v) is a semitame field,
b) (K,v) is a deeply ramified field,
c) (K,v) is a gdr field,
d) (K,v) satisfies (DRvr),
e) the completion of (K,v) is perfect,
f) (K,v) is dense in its perfect hull,
g) (K^p,v) is dense in (K,v).

4) Every perfect valued field of positive characteristic is a semitame field.

We note that for valued fields of mixed characteristic, axiom (DVvr) can be substituted by a version where $\hat{K}$ is replaced by K (see Lemma 6.1), and even p can be replaced by elements of certain lower or higher values (see Propositions 6.4 and 6.11).

In [22] the equivalence of assertions a) and f) of part 3) of this theorem is used to show that every valued field of positive characteristic that has only finitely many Artin-Schreier extensions is a semitame field. This proves that a nontrivially valued field of positive characteristic that is definable in an NTP_2 theory is a semitame field, as it is shown in [4] that such a field has only finitely many Artin-Schreier extensions.

Take a valued field (K,v) of characteristic 0 with residue characteristic $p>0$. Decompose $v=v_0\circ v_p\circ\overline{v}$, where v_0 is the finest coarsening of v that has residue characteristic 0, v_p is a rank 1 valuation on Kv_0, and $\overline{v}$ is the valuation induced by v on the residue field of v_p (which is of characteristic $p>0$). The valuations v_0 and $\overline{v}$ may be trivial. With this notation, we have:

Proposition 1.3. *Under the above assumptions, the valued field (K,v) is a gdr field if and only if (Kv_0,v_p) is.*

Note that by part 2) of Theorem 1.2, (Kv_0,v_p) is already a semitame field once it is a gdr field.

From Theorem 1.2 and Proposition 1.3 it can be deduced that the three properties "semitame", "deeply ramified" and "gdr" behave well for composite valuations.

Proposition 1.4. *Take an arbitrary valued field (K,v) and assume that $v=w\circ\overline{w}$ with w and $\overline{w}$ nontrivial. Then (K,v) is a gdr field if and only if (K,w) and $(Kw,\overline{w})$ are. If $\operatorname{char} Kw>0$, then for (K,v) to be a gdr field it suffices that (K,w) is a gdr field. The same holds for "semitame" and "deeply ramified" in place of "gdr".*

If $\operatorname{char} Kw=0$, then for (K,v) to be a gdr field it suffices that $(Kw,\overline{w})$ is a gdr field.

For deeply ramified fields, the first assertion of the next theorem has been proved before (see [9, Corollary 6.6.16 (i)]), based on their definition given in (4).

Theorem 1.5. *Every algebraic extension of a deeply ramified field is again deeply ramified. The same holds for semitame fields and for gdr fields.*

We will give the easy proof for the equal characteristic case in Proposition 6.7. The proof for the mixed characteristic case can be reduced to the study of Galois defect extensions of prime degree via the following theorem:

Theorem 1.6. *Take a valued field (K,v), fix any extension of v to $\tilde{K}$, and let (K^r,v) be the respective absolute ramification field of (K,v). Then (K^r,v) is a gdr field if and only if (K,v) is, and (K^r,v) is a semitame field if and only if (K,v) is. If (K,v) is a gdr field, then (K^r,v) is a deeply ramified field.*

Note that the last assertion holds since if (K^r,v) is a gdr field, then it is already a deeply ramified field because vK^r is divisible by every prime distinct from the residue characteristic. However, it is not true in general that this implies that (K,v) is deeply ramified, since (DRvg) always holds in (K^r,v) (as long as v is nontrivial), while it may not hold in (K,v).

Corollary 1.7. *1) Take an algebraic (not necessarily finite) extension $(L|K,v)$ of valued fields. If $K^r=L^r$ with respect to some extension of v from L to $\tilde{L}$, then (L,v) is a gdr field if and only if (K,v) is, and the same holds for "semitame" in place of "gdr".*

2) Take a valued field (K,v), fix any extension of v to $\tilde{K}$, and let (K^h,v) be the henselization of (K,v) in $(\tilde{K},v)$. Then (K^h,v) is a deeply ramified field if and only if (K,v) is, and the same holds for "gdr" and "semitame" in place of "deeply ramified".

Note that the assumption of part 1) holds in particular if $(L|K,v)$ is a tame extension. We see that we have infinite descend of the properties "gdr" and "semitame" through extensions in the absolute ramification field and in particular through tame extensions. If the lower field already satisfies (DRvg), then the descend also works for "deeply ramified". For all of the properties, we have finite descend in general:

Theorem 1.8. *Take a finite extension $(L|K,v)$. If (L,v) is a deeply ramified field, then so is (K,v). The same holds for "gdr" and "semitame" in place of "deeply ramified".*

The next theorem addresses the connection of the properties we have defined with the classification of the defect. Take a valued field (K,v) of residue characteristic $p>0$. We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if $\operatorname{char} K=0$, and set $(vK)_{vp}=vK$ otherwise. If (K,v) is of mixed characterisitc, then we set $K' := K(\zeta_p)$, where ζ_p is a primitive p-th roots of unity. Then we call (K,v) an **independent defect field** if for some extension of v to $\tilde{K}$, all Galois defect extensions of (K',v) of degree p have independent defect. (This definition does not depend on the chosen extension of v as all extensions are conjugate.) We will show in Theorem 1.10 that all gdr fields, and hence all deeply ramified and semitame fields, are independent defect fields.

Remark 1.9. If (K,v) is a gdr field of mixed characteristic, then it does not necessarily contain a primitive p-th root of unity. In this case, a condition on Galois defect extensions may not contain enough information. We also need information on extensions by p-th roots which will then not be Galois. This is why we pass to the field K' in our definition.

From Proposition 1.3 we see that in the case of fields (K,v) of mixed characteristic, (Kv_0,v_p) is essential for the gdr property. In analogy to the case of formally p-adic fields we call (Kv_0,v_p) the **core valued field**, v_p the **core valuation**, and Kv_0v_p the **core residue field**; so we set $\operatorname{crf}(K,v) := Kv_0v_p$. If (K,v) is of equal characteristic, we set $\operatorname{crf}(K,v) := Kv$.

Theorem 1.10. *1) Take a valued field (K,v) with $\operatorname{char} Kv=p>0$. Then (K,v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, $\operatorname{crf}(K,v)$ is perfect, and (K,v) is an independent defect field.*

2) A nontrivially valued field (K,v) is semitame if and only if every unibranched Galois extension of (K',v) of prime degree is either tame or an extension with independent defect.

The classification of Galois defect extensions of prime degree in the equal characteristic case is also an important tool in the proof of Theorem 1.2 of [16], which we will state now. A valued field is called **algebraically maximal** (or **separable-algebraically maximal**) if it admits no nontrivial immediate algebraic (or separable-algebraic, respectively) extensions. Since henselizations are immediate separable-algebraic extensions, every separable-algebraically maximal field is henselian.

Theorem 1.11. *A valued field of positive characteristic is a henselian and defectless field if and only if it is separable-algebraically maximal and each finite purely inseparable extension is defectless.*

This theorem in turn is used in [15] for the construction of an example showing that a certain natural axiom system for the elementary theory of $\mathbb{F}_p((t))$ ("henselian defectless valued field of characteristic p with residue field $\mathbb{F}_p$ and value group a $\mathbb{Z}$-group") is not complete.

A full analogue of Theorem 1.11 in mixed characteristic is not presently known. But we are able to show in Section 4 that the Galois extensions of prime degree with independent defect in mixed characteristic have the same properties as the ones in equal characteristic that have been used in [16] for the proof of Theorem 1.11. As a consequence, we are able to prove:

Theorem 1.12. *Every algebraically maximal gdr field is a perfect, henselian and defectless field.*

The study of mixed characteristic independent defect fields that are not gdr fields is only at its infancy. We hope that the valuation theoretic proof of Theorem 1.5 will be a basis for further insight. At this point, we are able to prove:

Proposition 1.13. *1) If (K^r,v) is an independent defect field, then so is (K,v).*
2) A valued field (K,v) of equal positive characteristic is an independent defect field if and only if every immediate purely inseparable extension of (K,v) lies in its completion.

It is an important fact that the properties of valued fields of being henselian, tame, semitame, deeply ramified or gdr all are preserved under infinite algebraic extensions. In contrast to this, the properties of being a defectless or an independent defect field are not necessarily preserved, as will be shown in Corollary 7.3 by the construction of a suitable algebraic extension of $\mathbb{Q}_p$.

Conjectures: 1) If (K,v) is an independent defect field, then also (K^r,v) is an independent defect field.
2) A valued field (K,v) of mixed characteristic with residue characteristic p is an independent defect field if and only if for every $a \in \mathcal{O}_K$ for which the set $\{v(a-c^p) \mid c \in K\}$ has no maximal element there is some $c \in K$ such that $v(a-c^p) \geq vp$.

Continuing the work presented in [5], the idea is presently investigated to employ higher ramification groups for the study of the ramification theory of 2-dimensional valued function fields. When working over valued fields with arbitrary value groups, the classical ramification numbers have to be replaced by **ramification jumps** which can be understood as cuts (or equivalently, final segments) in the value group (cf. Section 2.4).

While dealing with defect extensions $\mathcal{E}$ of prime degree, in Theorem 3.5 we show that $\Sigma_{\mathcal{E}}$ is a ramification jump. This allows us to characterize independent defect via this ramification jump and its associated ramification ideal.

Moreover, for Galois defect extensions $(L|K,v)$ of prime degree we will compute in Section 5 the image of the valuation ideal $\mathcal{M}_L$ under the trace of the extension. This allows us to characterize independent defect in yet another way, see Theorem 5.2. In summary, we obtain the following equivalent conditions for independent defect. The equivalence of the first four conditions will be shown in Section 3.2. The notions of "ramification jump", "ramification ideal" and "idempotent distance" will be defined in Section 2.1.

Theorem 1.14. *Take a Galois defect extension $\mathcal{E} = (K(a)|K,v)$ of prime degree with Galois group G. Then the following assertions are equivalent:*

a) $\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ for some proper convex subgroup $H_{\mathcal{E}}$ of vK, i.e., $\mathcal{E}$ has independent defect,

b) the ramification jump $\Sigma_-(G)$ is equal to $\{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ for some proper convex subgroup $H_{\mathcal{E}}$ of vK,

c) the ramification ideal $I_-(G)$ is a nontrivial prime ideal of $\mathcal{O}_L$ (and the localization of $\mathcal{O}_L$ with respect to $I_-(G)$ is a valuation ring on L containing $\mathcal{O}_L$, i.e., the associated valuation is a coarsening of v).

d) the distance of $\mathcal{E}$ is idempotent,

e) the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ is a valuation ideal $\mathcal{M}_{H_{\mathcal{E}}}$ of K that is contained in $\mathcal{M}_K$.

If the rank of (K,v) is 1, then $H_{\mathcal{E}}$ can only be equal to $\{0\}$ and $I_-(G)$ can only be equal to $\mathcal{M}_L$.

2. Preliminaries

2.1. **Cuts, distances and defect.**

We recall basic notions and facts connected with cuts in ordered abelian groups and distances of elements of valued field extensions. For the details and proofs see Section 2.3 of [16] and Section 3 of [25].

Take a totally ordered set $(T,<)$. For a nonempty subset S of T and an element $t \in T$ we will write $S < t$ if $s < t$ for every $s \in S$. A set $S \subseteq T$ is called an **initial segment** of T if for each $s \in S$ every $t < s$ also lies in S. Similarly, $S \subseteq T$ is called a **final segment** of T if for each $s \in S$ every $t > s$ also lies in S. A pair (Λ^L, Λ^R) of subsets of T is called a **cut** in T if Λ^L is an initial segment of T and $\Lambda^R = T \setminus \Lambda^L$; it then follows that Λ^R is a final segment of T. To compare cuts in $(T,<)$ we will use the lower cut sets comparison. That is, for two cuts $\Lambda_1 = (\Lambda_1^L, \Lambda_1^R)$, $\Lambda_2 = (\Lambda_2^L, \Lambda_2^R)$ in T we will write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$, and $\Lambda_1 \leq \Lambda_2$ if $\Lambda_1^L \subseteq \Lambda_2^L$.

For any $s \in T$ define the following **principal cuts**:

$$\begin{aligned} s^- &:= (\{t \in T \mid t < s\}, \{t \in T \mid t \geq s\}), \\ s^+ &:= (\{t \in T \mid t \leq s\}, \{t \in T \mid t > s\}). \end{aligned}$$

We identify the element s with s^+. Therefore, for a cut $\Lambda = (\Lambda^L, \Lambda^R)$ in T and an element $s \in T$ the inequality $\Lambda < s$ means that for every element $t \in \Lambda^L$ we have $t < s$. Similarly, for any subset M of T we define M^+ to be a cut (Λ^L, Λ^R) in T such that Λ^L is the smallest initial segment containing M, that is,

$$M^+ = (\{t \in T \mid \exists\, m \in M \ t \leq m\}, \{t \in T \mid t > M\}).$$

Likewise, we denote by M^- the cut (Λ^L, Λ^R) in T such that Λ^L is the largest initial segment disjoint from M, i.e.,

$$M^- = (\{t \in T \mid t < M\}, \{t \in T \mid \exists\, m \in M \ t \geq m\}).$$

For every extension $(L|K,v)$ of valued fields and $z \in L$ define

$$v(z-K) := \{v(z-c) \mid c \in K\}.$$

The set $v(z-K)\cap vK$ is an initial segment of vK and thus the lower cut set of a cut in vK. However, it is more convenient to work with the cut

$$\mathrm{dist}\,(z,K)\;:=\;(v(z-K)\cap vK)^+\quad\text{in the divisible hull }\widetilde{vK}\text{ of }vK\,.$$

We call this cut the **distance of z from K**. The lower cut set of $\mathrm{dist}\,(z,K)$ is the smallest initial segment of $\widetilde{vK}$ containing $v(z-K)\cap vK$. If $(F|K,v)$ is an algebraic subextension of $(L|K,v)$ then $\widetilde{vF}=\widetilde{vK}$. Thus $\mathrm{dist}\,(z,K)$ and $\mathrm{dist}\,(z,F)$ are cuts in the same group and we can compare these cuts by set inclusion of the lower cut sets. Since $v(z-K)\subseteq v(z-F)$ we deduce that

$$\mathrm{dist}\,(z,K)\;\leq\;\mathrm{dist}\,(z,F)\,.$$

If $\mathrm{char}\,K=p>0$ and $z\in K$, then K^p is a subfield of K, and the expressions

$$v(z-K^p)\;\text{ and }\;\mathrm{dist}\,(z,K^p)$$

are covered by our above definitions. We generalize this to the case where $\mathrm{char}\,K=0$ with the same definitions but note that $v(z-K^p)\cap vK$ is not necessarily an initial segment of vK.

If S is any subset of an abelian group T, then for every $t\in T$ and $n\in\mathbb{Z}$ we set

$$t+nS\;:=\;\{t+ns\mid s\in S\}\,;$$

in particular, $-S=\{-s\mid s\in S\}$. If $\Lambda=(\Lambda^L,\Lambda^R)$ is a cut in a divisible ordered abelian group Γ and $n>0$, then $n\Lambda^L$ is again an initial segment of Γ; we denote by $n\Lambda$ the cut in Γ with the lower cut set $n\Lambda^L$. Further, we define $-\Lambda$ to be the cut $(-\Lambda^R,-\Lambda^L)$.

We say that the distance $\mathrm{dist}\,(z,K)$ is **idempotent** if

$$n\cdot\mathrm{dist}\,(z,K)\;=\;\mathrm{dist}\,(z,K)$$

for some natural number $n\geq 2$ (and hence for all $n\in\mathbb{N}$). The following characterization of cuts distances is a consequence of [16, Lemma 2.14]:

Lemma 2.1. *A cut in $\widetilde{vK}$ is idempotent if and only if it is equal to H^- or H^+ for some convex subgroup H of $\widetilde{vK}$.*

If y is another element of L then we define:

$$z\sim_K y\;:\Leftrightarrow\;v(z-y)\;>\;\mathrm{dist}\,(z,K)\,.$$

The next lemma shows, among other things, that the relation $\sim_K$ is symmetrical.

Lemma 2.2. *Take a valued field extension $(L|K,v)$ and elements $z,y\in L$.*

1) If $z\sim_K y$, then $v(z-c)=v(y-c)$ for all $c\in K$ such that $v(z-c)\in vK$, $v(z-K)=v(y-K)$, $\mathrm{dist}\,(z,K)=\mathrm{dist}\,(y,K)$, and $y\sim_K z$.

2) If $(K(z)|K,v)$ is immediate, then $v(z-K)$ has no largest element and is a subset of vK.

Proof. 1): This is part (1) of Lemma 2.17 in [16].

2): The first assertion follows from [10, Theorem 1]. To prove the second assertion, take $c\in K$; we wish to show that $v(z-c)\in vK$. By assumption there is $d\in K$ such that $v(z-d)>v(z-c)$. Hence $v(z-c)=\min\{v(z-c),v(z-d)\}=v(c-d)\in vK$. $\square$

For any $\alpha \in vK$ and each cut Λ in vK we set $\alpha + \Lambda := (\alpha + \Lambda^L, \alpha + \Lambda^R)$. An immediate consequence of the above definitions is the following lemma:

Lemma 2.3. *Take an extension $(L|K,v)$ of valued fields. Then for every element $c \in K$ and $y, z \in L$,*
1) $\operatorname{dist}(z+c, K) = \operatorname{dist}(z, K)$,
2) $\operatorname{dist}(cz, K) = vc + \operatorname{dist}(z, K)$.

Here are some important properties of distances in valued field extensions. For the proof of the next lemma see [2, Lemma 7] and [16, Lemma 2.5].

Lemma 2.4. *Take any immediate extension $(F|K,v)$ and a finite defectless unibranched extension $(L|K,v)$. Then the extension of v from F to $F.L$ is unique, $(F.L|F,v)$ is defectless, $(F.L|L,v)$ is immediate, and for every $a \in F \setminus K$ we have that*

$$\operatorname{dist}(a, K) = \operatorname{dist}(a, L).$$

Moreover, $F|K$ and $L|K$ are linearly disjoint, i.e.,

$$[F.L : F] \;=\; [L : K]\,.$$

For the proof of the following results see [2, Lemmas 5 and 9].

Lemma 2.5. *Take a unibranched extension $(F|K,v)$ and an extension of v to the algebraic closure of F. Take K^h to be the henselization of K with respect to this fixed extension of v. Then for every $a \in F$ we have that $[K(a):K] = [K^h(a):K^h]$ as well as*

$$d(K(a)|K,v) \;=\; d(K^h(a)|K^h,v) \quad \text{and} \quad \operatorname{dist}(a,K) \;=\; \operatorname{dist}(a,K^h)\,.$$

A valued field (K,v) is said to be **separably defectless** if every finite separable extension of (K,v) is defectless, and **inseparably defectless** if every finite purely inseparable extension of (K,v) is defectless. The following is Lemma 4.15 of [16].

Lemma 2.6. *Every finite extension of an inseparably defectless field is again an inseparably defectless field.*

For the proof of the next proposition, see [16], Proposition 2.8.

Proposition 2.7. *Take a henselian field (K,v) and a tame extension N of K. Then for any finite extension $L|K$,*

$$d(L|K,v) \;=\; d(L.N|N,v)\,.$$

In particular, (K,v) is defectless (separably defectless, inseparably defectless) if and only if (K^r,v) is defectless (separably defectless, inseparably defectless).

For the following theorem, see [10, Theorem 1] and [16, Theorem 2.19].

Theorem 2.8. *If $(L|K,v)$ is an immediate extension of valued fields, then for every element $a \in L \setminus K$ the set $v(a-K)$ is an initial segment of vK without maximal element. In particular, $va < \operatorname{dist}(a,K)$.*

The following partial converse of this theorem also holds (see [16, Lemma 2.21]):

Lemma 2.9. *Assume that $(K(a)|K,v)$ is a unibranched extension of prime degree such that $v(a-K)$ has no maximal element. Then the extension $(K(a)|K,v)$ is immediate and hence a defect extension.*

The property that the set $v(a-K)$ has no maximal element does not in general imply that $(K(a)|K,v)$ is immediate. However, the next lemma (see e.g. [25, Lemma 2.1]) shows that if in addition (K,v) is henselian and a is algebraic over K, then $(K(a)|K,v)$ is a defect extension.

Lemma 2.10. *If $(L|K,v)$ is a finite defectless unibranched extension, then for every element $a \in K$ the set $v(a-K)$ admits a maximal element.*

We will need a version of Lemma 2.9 that also works for extensions that are not assumed to be unibranched.

Lemma 2.11. *Assume that $(K(a)|K,v)$ is an extension of degree at most $p = \operatorname{char} Kv$ and of rank 1 valued fields such that $v(a-K)$ has no maximal element but is bounded from above in vK. Then the extension $(K(a)|K,v)$ is a unibranched defect extension.*

Proof. Take a henselization (K^h,v) and consider the extension $(K^h(a)|K^h,v)$ which again is of degree at most p. Take any $b \in K^h$. Since $K^h(a)|K$ is algebraic, we know that $v(K^h(a))$ lies in the divisible hull $\widetilde{vK}$ of vK and thus there is some $\alpha \in vK$ such that $\alpha > v(a-b)$. Since (K,v) is of rank 1 by assumption, (K,v) lies dense in (K^h,v) (cf. [28, 32.11 and 32.18]). Therefore, there is some $c \in K$ such that $v(b-c) \geq \alpha > v(a-b)$, so that $v(a-b) = v(a-c) \in vK$. This shows that $v(a-K^h) = v(a-K)$.

Since the extension $(K^h(a)|K,v)$ is unibranched, we now obtain from Lemma 2.9 that it is a defect extension. Consequently, its degree is p and we find that $K(a)|K$ is linearly disjoint from $K^h|K$. Therefore, by [1, Lemma 2.1] also $(K(a)|K,v)$ is unibranched. Employing Lemma 2.9 again, we see that it is a defect extension. □

The next lemma follows from [10, Lemma 8] and [25, Lemma 5.2]. We use the Taylor expansion

$$f(X) = \sum_{i=0}^{n} \partial_i f(c)(X-c)^i \tag{6}$$

where $\partial_i f$ denotes the i-th **Hasse-Schmidt derivative** (also called **formal derivative**) of f.

Lemma 2.12. *Take a nontrivial extension $(K(a)|K,v)$ of degree p. Assume that $v(a-K)$ has no maximal element. Then for every nonconstant polynomial $f \in K[X]$ of degree $< p$ there is some $\gamma \in v(a-K)$ such that for all $c \in K$ with $v(a-c) \geq \gamma$ and all i with $1 \leq i \leq \deg f$, we have:*
the values $v\partial_i f(c)$ are fixed, equal to $v\partial_i f(a)$,
the values $v\partial_i f(c) + i \cdot v(x-c)$ are pairwise distinct,

$$v\partial_1 f(c) + v(x-c) \;<\; v\partial_i f(c) + i \cdot v(x-c) \tag{7}$$

whenever $i \neq 1$,

$$v(f(a) - f(c)) \;=\; v\partial_1 f(c) + v(a-c)\,, \tag{8}$$

and

$$\operatorname{dist}(f(a),K) \;=\; v\partial_1 f(c) + \operatorname{dist}(a,K)\,. \tag{9}$$

The following is Lemma 2.4 of [16].

Lemma 2.13. *Take a valued field (K,v), a finite extension $(L|K,v)$ and a coarsening w of v on L. If (K,v) is henselian, then so is (K,w). If $(L|K,v)$ is defectless, then so is $(L|K,w)$.*

2.2. **The absolute ramification field.**

Proposition 2.14. *Take an immediate unibranched extension $(K(a)|K,v)$. Extend v to the algebraic closure of K and let (K^h,v) be the henselization and (K^r,v) the absolute ramification field of (K,v) with respect to this extension. Then $(K^r(a)|K^r,v)$ is an immediate extension with*

$$[K^r(a):K^r] \;=\; [K^h(a):K^h] \;=\; [K(a):K]\,, \tag{10}$$

$$d(K^r(a)|K^r,v) \;=\; d(K^h(a)|K^h,v) \;=\; d(K(a)|K,v)\,, \tag{11}$$

$$\mathrm{dist}\,(a,K^r) \;=\; \mathrm{dist}\,(a,K^h) \;=\; \mathrm{dist}\,(a,K)\,. \tag{12}$$

If $(N|K,v)$ is any subextension of $(K^r|K,v)$, then $[N(a):K]=[N(a):K]$ and

$$d(N(a)|N,v) \;=\; d(K(a)|K,v) \;\; \textit{and} \;\; \mathrm{dist}\,(a,N) \;=\; \mathrm{dist}\,(a,K)\,. \tag{13}$$

Proof. Since $(K(a)|K,v)$ is a unibranched extension, we know from Lemma 2.5 that $[K^h(a):K^h]=[K(a):K]$ as well as $d(K^h(a)|K^h,v)=d(K(a)|K,v)$ and $\mathrm{dist}\,(a,K^h)=\mathrm{dist}\,(a,K)$. Since $(K(a)|K,v)$ is an immediate unibranched extension by assumption,

$$[K^h(a):K^h] \;=\; [K(a):K] \;=\; d(K(a)|K,v) \;=\; d(K^h(a)|K^h,v)\,,$$

showing that also $(K^h(a)|K^h,v)$ is immediate.

Further, $(K^r|K^h,v)$ is a tame and hence the union of its finite defectless subextensions. Thus by Lemma 2.4, $(K^r(a)|K^r,v)$ is immediate with $[K^r(a):K^r]=[K^h(a):K^h]$ and $\mathrm{dist}\,(a,K^r)=\mathrm{dist}\,(a,K^h)$. By Proposition 2.7, $d(K^r(a)|K^r,v)=d(K^h(a)|K^h,v)$.

Finally, if $(N|K,v)$ is a subextension of $(K^r|K,v)$, then $N^r=K^r$. Hence by (10), $[N(a):N]=[N^r(a):N^r]=[K^r(a):K^r]=[K(a):K]$, by (11), $d(N(a)|N,v)=d(N^r(a)|N^r,v)=d(K^r(a)|K^r,v)=d(K(a)|K,v)$, and by (12), $\mathrm{dist}\,(a,N)=\mathrm{dist}\,(a,N^r)=\mathrm{dist}\,(a,K^r)=\mathrm{dist}\,(a,K)$. □

For the proof of the following results, see Lemma 2.9 of [16].

Lemma 2.15. *Take any valued field (K,v) and let K^h and K^r be its henselization and its absolute ramification field with respect to any extension of v to the algebraic closure of K. If $\mathrm{char}\,Kv=0$, then K^r is algebraically closed. If $\mathrm{char}\,Kv=p>0$, then every finite extension of K^r is a tower of normal extensions of degree p. Further, if $L|K$ is a finite extension, then there is already a finite tame extension N of K^h such that $L.N|N$ is such a tower.*

The proof of this lemma uses the fact that if $\mathrm{char}\,Kv=p>0$, then $K^{\mathrm{sep}}|K^r$ is a p-extension. From this we can also conclude:

Corollary 2.16. *Every absolute ramification field contains all p-th roots of unity.*

Finally, we will need the following fact:

Lemma 2.17. *Let (K^r,v) be the absolute ramification field of (K,v), and assume that $v=w\circ\overline{w}$. Then the extension $K^rw|Kw$ is separable.*

Proof. For a detailed proof, see [23]. For the convenience of the reader, we give here a sketch of the proof. We use several facts from ramification theory. The assertion is trivial if $\operatorname{char} Kv = 0$, so we let $\operatorname{char} Kv = p > 0$.

i) If (K^h, v) is the henselization of (K, v), then $(K^h w, \overline{w})$ is the henselization of $(Kw, \overline{w})$. In particular, $K^h w | Kw$ is separable.

ii) If $(L_1|L_2, w \circ \overline{w})$ is a finite defectless extension, then so are $(L_1|L_2, w)$ and $(L_1 w|L_2 w, \overline{w})$.

iii) Since $(K^r|K^h, v)$ is a tame extension, also $(K^r w|K^h w, \overline{w})$ is a tame extension. Indeed, if $(L|K^h, v)$ is a finite subextension, then p does not divide $(vL : vK^h)$ and hence also not $(\overline{w}(Lw) : \overline{w}(K^h w))$, the extension $Lv|K^h v = (Lw)\overline{w}|(K^h w)\overline{w}$ is separable, and $(L|K^h, v)$ is defectless, which by ii) implies that $(Lw|K^h w, \overline{w})$ is defectless.

Now as $(K^r w|K^h w, \overline{w})$ is a tame extension, $K^r w|K^h w$ is separable, and in view of i) we obtain that $K^r w|Kw$ is separable. □

2.3. **1-units and p-th roots in valued fields of mixed characteristic.**

Throughout this section, (K, v) will be a valued field of characteristic zero and residue characteristic $p > 0$, with valuation ring $\mathcal{O}$ and valuation ideal $\mathcal{M}$. We assume that v is extended to the algebraic closure $\tilde{K}$ of K.

We will need a few easy observations about the relation of congruences and powers of elements.

Lemma 2.18. *1) If $b_1, \ldots, b_n \in \mathcal{O}$, then*

$$(b_1 + \ldots + b_n)^p \equiv b_1^p + \ldots + b_n^p \mod p\mathcal{O}\,. \tag{14}$$

2) Take elements $b_1, \ldots, b_n \in K$ of values $\geq -\frac{vp}{p}$. Then

$$(b_1 + \cdots + b_n)^p \equiv b_1^p + \cdots + b_n^p \mod \mathcal{O}\,.$$

3) Take $\eta \in \tilde{K}$ such that $\eta^p = a \in \mathcal{O}$. Then for every $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$ we have that $a \equiv c^p \mod p\mathcal{O}$.

Proof. 1): We have:

$$(b_1 + b_2)^p = b_1^p + \sum_{i=1}^{p-1} \binom{p}{i} b_1^{p-i} b_2^i + b_2^p\,. \tag{15}$$

Since the binomial coefficients under the sum are all divisible by p and since $b_1, b_2 \in \mathcal{O}$, all summands on the right hand side for $1 \leq i \leq p-1$ lie in $p\mathcal{O}$, which proves our assertion in the case of $n = 2$. The general case follows by induction on n.

2): If $vb_1 \geq -\frac{vp}{p}$ and $vb_2 \geq -\frac{vp}{p}$, then $vb_1^{p-i} b_2^i \geq -vp$ for $1 \leq i \leq p-1$, so all summands in the sum on the right hand side of (15) have non-negative value. As for part 1), the assertion now follows by induction on n.

3): For $c \in K$ with $v(\eta - c) > 0$ we have that $vc \geq 0$ and, by part 1):

$$(\eta - c)^p \equiv \eta^p - c^p = a - c^p \mod p\mathcal{O}_{K(\eta)}\,.$$

If $v(\eta - c) \geq \frac{vp}{p}$, then $v(\eta - c)^p \geq vp$, i.e., $a - c^p \equiv (\eta - c)^p \equiv 0 \mod p\mathcal{O}_{K(\eta)}$. □

Lemma 2.19. *Take $\eta \in \tilde{K}$ such that $\eta^p \in K$ and $v\eta = 0$. Then for $c \in K$ such that $v(\eta - c) > 0$, $v(\eta - c) < \frac{1}{p-1} vp$ holds if and only if $v(\eta^p - c^p) < \frac{p}{p-1} vp$,* *and if this is the case, then* $v(\eta^p - c^p) = pv(\eta - c)$. *If* $v(\eta - c) > \frac{1}{p-1}vp$, *then* $v(\eta^p - c^p) = vp + v(\eta - c)$.

Proof. Take any $c \in K$ such that $0 < v(\eta - c)$. Then $vc = v\eta = 0$. We have that

$$\eta^p \;=\; (\eta - c + c)^p \;=\; (\eta - c)^p + \sum_{i=1}^{p-1} \binom{p}{i} (\eta - c)^i c^{p-i} \;+\; c^p\;.$$

Since $vc = 0$ and the binomial coefficients under the sum all have value vp, the unique summand with the smallest value is $p(\eta - c)c^{p-1}$. Therefore,

$$v(\eta^p - c^p) \;\geq\; \min\{v(\eta - c)^p, vp(\eta - c)\} \;=\; \min\{pv(\eta - c), vp + v(\eta - c)\}\;, \tag{16}$$

with equality holding if $pv(\eta - c) \neq vp + v(\eta - c)$. We observe that

$$v(\eta - c) < \frac{vp}{p-1} \iff pv(\eta - c) < vp + v(\eta - c)\;, \tag{17}$$

and the same holds for "$>$" in place of "$<$". Assume that $v(\eta - c) < \frac{vp}{p-1}$. Then by (17) and (16),

$$v(\eta^p - c^p) \;=\; pv(\eta - c) \;<\; \frac{p}{p-1}vp\;.$$

Now assume that $v(\eta - c) \geq \frac{1}{p-1}vp$. Then by (17), $pv(\eta - c) \geq vp + v(\eta - c)$, and (16) yields that

$$v(\eta^p - c^p) \;\geq\; vp + v(\eta - c) \;\geq\; vp + \frac{1}{p-1}vp \;=\; \frac{p}{p-1}vp\;.$$

Finally, if $v(\eta - c) > \frac{1}{p-1}vp$, then from (17) and (16) we conclude that

$$v(\eta^p - c^p) \;=\; vp + v(\eta - c)\;.$$

□

A **1-unit** in (K, v) is an element of the form $u = 1 + b$ with $b \in \mathcal{M}$; in other words, u is a unit in $\mathcal{O}$ with residue 1. We will call the value $v(u - 1)$ the **level** of the 1-unit u. Taking η to be a 1-unit u in Lemma 2.19, we obtain:

Corollary 2.20. *Assume that* u *is a* 1*-unit. Then the level of* u *is smaller than* $\frac{1}{p-1}vp$ *if and only if the level of* u^p *is smaller than* $\frac{p}{p-1}vp$, *and if this is the case, then* $v(u^p - 1) = pv(u - 1)$.

Lemma 2.21. *Take* $\eta \in \tilde{K}$ *such that* $\eta^p \in K$. *If there is some* $c \in K$ *such that*

$$v(\eta - c) \;>\; v\eta + \frac{vp}{p-1}\;, \tag{18}$$

then η *lies in the henselization of* (K, v) *within* $(\tilde{K}, v)$.

Proof. If $\eta \in K$, then there is nothing to show, so let us assume that $\eta \notin K$. Every root of $X^p - \eta^p$ is of the form $\eta\zeta_p^i$ with $0 \leq i \leq p - 1$. For $0 \leq i \neq j \leq p - 1$ we have that

$$v(\eta\zeta_p^i - \eta\zeta_p^j) \;=\; v\eta + jv\zeta_p + v(\zeta_p^{i-j} - 1) \;=\; v\eta + \frac{vp}{p-1}\;,$$

where the last equality holds since $v\zeta_p = 0$ and

$$v(\zeta - 1) \;=\; \frac{vp}{p-1} \tag{19}$$

for every primitive p-th root of unity ζ (see e.g. the proof of [18, Lemma 14]). Hence if (18) holds, then it follows from Krasner's Lemma that $\eta \in K(c)^h = K^h$, where K^h denotes the henselization of (K,v) within $(\tilde{K},v)$. $\square$

For our work with 1-units, we will need the following result, which is Lemma 14 of [18].

Lemma 2.22. *A henselian field of characteristic 0 and residue characteristic $p>0$ contains an element C such that $C^{p-1}=-p$ if and only if it contains a primitive p-th root ζ_p of unity.*

The element C satisfies:

$$C^p = -pC \quad \text{and} \quad vC = \frac{vp}{p-1}\,. \tag{20}$$

The following construction will play an important role in Section 3.4. Take a 1-unit $\eta \in \tilde{K}$ such that $\eta^p \in K$. Then also η^p is a 1-unit. Assume that K contains an element C as in Lemma 2.22. Consider the substitution $X = CY+1$ for the polynomial $X^p - \eta^p$. We then obtain the polynomial $(CY+1)^p - \eta^p$. Dividing this polynomial by C^p and using the fact that $C^p = -pC$, we obtain the polynomial

$$f_\eta(Y) = Y^p + g(Y) - Y - \frac{\eta^p - 1}{C^p}, \tag{21}$$

where

$$g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i\,. \tag{22}$$

Note that $g(Y) \in \mathcal{M}_K[Y]$ since $C \in K$ and $vC = \frac{vp}{p-1}$. We see that an element $\tilde{\eta}$ is a root of $X^p - \eta^p$ if and only if the element $\frac{\tilde{\eta}-1}{C}$ is a root of f_η. Thus the roots of f_η are of the form $\frac{\zeta_p^i \eta - 1}{C}$ with $0 \le i \le p-1$.

Set

$$\vartheta_\eta \;:=\; \frac{\eta - 1}{C}\,. \tag{23}$$

Then $K(\eta) = K(\vartheta_\eta)$, with f_η the minimal polynomial of ϑ_η over K.

Lemma 2.23. *In a henselian field (K,v) of mixed characteristic with residue characteristic p which contains a primitive p-th root of unity, every 1-unit of level greater than $\frac{p}{p-1}vp$ is a p-th power.*

Proof. By Lemma 2.22, K contains an element C as in that lemma. Take a 1-unit $u \in K$ of level greater than $\frac{p}{p-1}vp$. Apply the above transformation to the polynomial $X^p - u$ with $\eta^p = u$. By our assumption on u we have that $\frac{\eta^p-1}{C^p} \in \mathcal{M}_K$. Hence $f_\eta(Y)$ is equivalent modulo $\mathcal{M}_K[Y]$ to $Y^p - Y$, which splits in the henselian field K. Therefore, $\eta \in K$. $\square$

2.4. **Higher ramification groups.**

Take a henselian field (K,v). Assume that $L|K$ is a Galois extension, and let $G = \mathrm{Gal}\,(L|K)$ denote its Galois group. For ideals I of $\mathcal{O}_L$ we consider the (upper series of) **higher ramification groups**

$$G_I \;:=\; \left\{ \sigma \in G \;\middle|\; \frac{\sigma b - b}{b} \in I \;\text{ for all } b \in L^\times \right\} \tag{24}$$

(see [29], §12). Note that $G_{\mathcal{M}_L}$ is the ramification group of $(L|K,v)$. For every ideal I of $\mathcal{O}_L$, G_I is a normal subgroup of G ([29] (d) on p.79). The function

$$\varphi:\ I\ \mapsto\ G_I \tag{25}$$

preserves $\subseteq$, that is, if $I\subseteq J$, then $G_I\subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the higher ramification groups are linearly ordered by inclusion. Note that in general, φ will neither be injective, nor surjective.

The function

$$v:\ I\ \mapsto\ \Sigma_I\ :=\ \{vb\mid 0\neq b\in I\} \tag{26}$$

is an order preserving bijection from the set of all ideals of $\mathcal{O}_L$ onto the set of all final segments of the positive part $(vL)^{>0}$ of the value group vL (including the final segment $\emptyset$). The set of these final segments is again linearly ordered by inclusion, and the function (26) is order preserving: $J\subseteq I$ holds if and only if $\Sigma_J\subseteq\Sigma_I$ holds. The inverse of the above function is the order preserving function

$$\Sigma\ \mapsto\ I_\Sigma\ :=\ \{a\in L\mid va\in\Sigma\}\cup\{0\}\,. \tag{27}$$

Now the higher ramification groups can be represented as

$$G_\Sigma\ :=\ G_{I_\Sigma}\ =\ \left\{\sigma\in G\ \left|\ v\,\frac{\sigma b-b}{b}\in\Sigma\cup\{\infty\}\ \text{ for all } b\in L^\times\right.\right\}\,,$$

where Σ runs through all final segments of $(vL)^{>0}$.

Like the function (25), also the function $\Sigma\mapsto G_\Sigma$ is in general not injective. We call Σ a **ramification jump** if

$$\Sigma'\subsetneq\Sigma\ \Rightarrow\ G_{\Sigma'}\subsetneq G_\Sigma\,.$$

If Σ is a ramification jump, then I_Σ is called a **ramification ideal**.

Given any ramification group $H\subseteq G$, we define

$$\Sigma_-(H)\ :=\ \bigcap_{G_\Sigma=H}\Sigma\qquad\text{and}\qquad\Sigma_+(H)\ :=\ \bigcup_{G_\Sigma=H}\Sigma\,. \tag{28}$$

and note that arbitrary unions and intersections of final segments of $(vL)^{>0}$ are again final segments of $(vL)^{>0}$. From its definition it is obvious that $\Sigma_-(H)$ is a ramification jump, $G_{\Sigma_-(H)}=H$, and that

$$I_-(H)\ :=\ I_{\Sigma_-(H)}$$

is a ramification ideal. It is generated by the set

$$\left\{\frac{\sigma b-b}{b}\ \mid\ \sigma\in H\,,\ b\in L^\times\right\}\,. \tag{29}$$

In this paper we are particularly interested in the case where $(L|K,v)$ is a Galois extension of prime degree p. Then $G=\mathrm{Gal}\,(L|K)$ is a cyclic group of order p and thus has only one proper subgroup, namely $\{\mathrm{id}\}$, and this subgroup is equal to G_Σ for $\Sigma=\emptyset$. If in this case G itself is the ramification group of the extension, then there must be a unique ramification jump. As we will show in the next section, this ramification jump carries important information about the extension $(L|K,v)$.

3. Defect extensions of prime degree

We will investigate defect extensions $(L|K,v)$ of prime degree p. By what we have already stated in the Introduction, such extensions are immediate unibranched extensions; moreover, $p = \operatorname{char} Kv > 0$. By Theorem 2.8, for every $a \in L\setminus K$ the set $v(a-K)$ is an initial segment of vK without maximal element, and $\operatorname{dist}(a,K) > va$.

In the following, we distinguish two cases:

- the equal characteristic case where $\operatorname{char} K = p$,
- the mixed characteristic case where $\operatorname{char} K = 0$ and $\operatorname{char} Kv = p$.

We fix an extension of v from L to the algebraic closure $\tilde{K}$ of K.

Note: to shorten expressions, we will often write "independent defect extension" in place of "defect extension with independent defect".

In a first section, we investigate the set Σ_σ defined in (2) for σ in the absolute Galois group $\operatorname{Gal}(K) := \operatorname{Gal}(K^{\mathrm{sep}}|K)$.

3.1. The set Σ_σ.

We start with the following two easy but helpful observations.

Lemma 3.1. *Let $(K(a)|K,v)$ be any algebraic extension of valued fields. If $\sigma \in \operatorname{Gal}(K)$ is such that $\sigma a \neq a$, then*

$$\left\{ v\, \frac{\sigma(a-c)-(a-c)}{a-c} \;\middle|\; c \in K \right\} \;=\; \left\{ v\, \frac{\sigma a - a}{a-c} \;\middle|\; c\in K \right\} \;=\; -v(a-K)+v(\sigma a - a)\,.$$

Proof. The first equality holds since $\sigma c = c$, and the second holds since

$$v\,\frac{\sigma a - a}{a-c} \;=\; -v(a-c)+v(\sigma a-a)\,.$$

□

Lemma 3.2. *Take a nontrivial immediate unibranched extension $(K(a)|K,v)$. Then the following assertions hold.*

1) For each $\sigma \in \operatorname{Gal}(K)$ and $c \in K$,

$$v(a-c) \;<\; v(\sigma a - a)\;.$$

2) For each $\sigma\in\operatorname{Gal}(K)$ such that $\sigma a\neq a$,

$$\operatorname{dist}(a,K) \;\leq\; v(\sigma a-a)^-.$$

Proof. 1): Since the extension is immediate and $a \notin K$, the set $v(a-K)$ has no maximal element. Thus it suffices to prove that $v(a-c)\leq v(\sigma a-a)$. If this were not true, then for some $\sigma\in\operatorname{Gal}(K)$ and $c \in K$, $v(a-c) > v(\sigma a-a)$. But this implies that

$$v\sigma(a-c) \;=\; v(\sigma a - c) \;=\; \min\{v(\sigma a-a)\,,\, v(a-c)\} \;=\; v(\sigma a-a) \;<\; v(a-c)\;,$$

which contradicts our assumption that $K(a)|K$ is a unibranched extension, as $v\sigma$ is also an extension of v from K to $K(a)$.

2): This is an immediate consequence of part 1). □

With the help of Lemma 2.12, we prove:

Lemma 3.3. *Take a defect extension $(K(a)|K,v)$ of prime degree and any $b \in K(a)^\times$. Then for all $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma a \neq a$ there is some $c \in K$ such that*

$$v\,\frac{\sigma b - b}{b} \;>\; -v(a-c) \,+\, v(\sigma a - a)\;. \tag{30}$$

Proof. As stated already, $(K(a)|K,v)$ is immediate with $[K(a):K] = p = \mathrm{char}\,Kv$. The element $b \in K(a)^\times$ can be written as $f(a)$ for $f(X) \in K[X]$ of degree smaller than p. By Theorem 2.8, $v(a-K)$ has no maximal element. Hence by [2, Lemma 11], we can choose $\gamma \in v(a-K)$ so large that for all $c \in K$ with $v(a-c) \geq \gamma$, all values $v\partial_i f(c)$ are fixed and equal to $v\partial_i f(a)$ whenever $0 \leq i < p$, and that (7) and (8) hold by Lemma 2.12. It suffices to restrict our attention to those $c \in K$ for which $v(a-c) \geq \gamma$. Then we have that

$$v\partial_1 f(a)(a-c) \;=\; v\partial_1 f(c)(a-c) \;<\; v\partial_i f(c)(a-c)^i \;=\; v\partial_i f(a)(a-c)^i \tag{31}$$

for all $i > 1$. From part 1) of Lemma 3.2 we infer that

$$0 \;<\; v\left(\frac{\sigma a - a}{a-c}\right) \;<\; v\left(\frac{\sigma a - a}{a-c}\right)^i$$

for all $i > 1$. Using this together with (31), we obtain:

$$\begin{aligned} v\partial_1 f(a)(\sigma a - a) \;&=\; v\partial_1 f(a)(a-c)\left(\frac{\sigma a - a}{a-c}\right) \\ &<\; v\partial_i f(a)(a-c)^i\left(\frac{\sigma a - a}{a-c}\right)^i \;=\; v\partial_i f(a)(\sigma a - a)^i\;. \end{aligned}$$

It follows that

$$\begin{aligned} v(\sigma f(a) - f(a)) \;&=\; v(f(\sigma a) - f(a)) \;=\; v\left(\sum_{i=1}^{\deg f} \partial_i f(a)(\sigma a - a)^i\right) \\ &=\; v\partial_1 f(a)(\sigma a - a) \;=\; v\partial_1 f(c) \,+\, v(\sigma a - a)\;. \end{aligned}$$

Now (8) shows that

$$v\partial_1 f(c) + v(a-c) \;=\; v(f(a) - f(c)) \;\geq\; \min\{vf(a), vf(c)\}\;.$$

The value on the right hand side is fixed, but the value of the left hand side increases with $v(a-c)$. Since $v(a-K)$ has no maximal element, we can choose γ so large that the value on the left hand side is larger than the one on the right hand side, which can only be the case if $vf(a) = vf(c)$, whence $vf(a) < v\partial_1 f(c) + v(a-c)$. Consequently,

$$v\,\frac{\sigma f(a) - f(a)}{f(a)} \;=\; \partial_1 f(c) \,+\, v(\sigma a - a) \,-\, vf(a) \;>\; -v(a-c) \,+\, v(\sigma a - a)\;.$$

□

Theorem 3.4. *Take a defect extension $\mathcal{E} = (L|K,v)$ of prime degree. Then the following assertions hold.*

1) For every generator $a \in L$ of the extension and every $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma a \neq a$ we have:

$$\Sigma_\sigma \;=\; -v(a-K) + v(\sigma a - a)\;. \tag{32}$$

2) The set Σ_σ is a final segment of $vK^{>0} = \{\alpha \in vK \mid \alpha > 0\}$.

Proof. 1): The inclusion "⊇" in (32) follows from Lemma 3.1. To show the reverse inclusion, we use Lemma 3.3. Since $v(a-K)$ is an initial segment of vK, $-v(a-K)$ is a final segment of vK. Thus we can infer from (30) that

$$v\,\frac{\sigma b-b}{b}\;\in\;-v(a-K)\,+\,v(\sigma a-a)\,.$$

This proves the inclusion "⊆".

2): Since $\mathcal{E}$ is an immediate unibranched extension, taking $c=0$ in part 1) of Lemma 3.2 yields that $v(\sigma b-b)\geq vb$ for all $b\in L^\times$, showing that $v\frac{\sigma b-b}{b}\in vL^{>0}=vK^{>0}$. Since $-v(a-K)$ is a final segment of vK, the same holds for $\Sigma_\sigma=-v(a-K)+v(\sigma a-a)$. □

3.2. **Galois defect extensions of prime degree.**

A Galois extension of degree p of a field K of characteristic $p>0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** X^p-X-c with $c\in K$. A Galois extension of degree p of a field K of characteristic 0 which contains all p-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^p\in K$. For these facts, see [26, Chapter VIII, §8].

If $(L|K,v)$ is a Galois defect extension of degree p of fields of characteristic 0, then a Kummer generator of $L|K$ can be chosen to be a 1-unit. Indeed, choose any Kummer generator η. since $(L|K,v)$ is immediate, we have that $v\eta\in vK(\eta)=vK$, so there is $c\in K$ such that $vc=-v\eta$. Then $v\eta c=0$, and since $\eta cv\in K(\eta)v=Kv$, there is $d\in K$ such that $dv=(\eta cv)^{-1}$. Then $v(\eta cd)=0$ and $(\eta cd)v=1$. Hence ηcd is a 1-unit. Furthermore, $K(\eta cd)=K(\eta)$ and $(\eta cd)^p=\eta^p c^p d^p\in K$. Thus we can replace η by ηcd and assume from the start that η is a 1-unit. It follows that also $\eta^p\in K$ is a 1-unit.

Throughout this article, whenever we speak of "Artin-Schreier extension" we refer to fields of positive characteristic, and with "Kummer extension" we refer to fields of characteristic 0.

Theorem 3.5. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree with Galois group G. The set Σ_σ does not depend on the choice of the generator σ of G. Writing $\Sigma_{\mathcal{E}}$ for Σ_σ , we have that $\Sigma_{\mathcal{E}}$ is a final segment of $vK^{>0}$ and satisfies*

$$\Sigma_{\mathcal{E}}\;=\;\Sigma_-(G)\;=\;\Sigma_+(\{\mathrm{id}\})\,,$$

showing that $\Sigma_{\mathcal{E}}$ is the unique ramification jump of the extension $\mathcal{E}$. Further, the ramification ideal $I_-(G)$ is equal to the ideal of $\mathcal{O}_L$ generated by the set

$$\left\{\left.\frac{\sigma b-b}{b}\;\right|\;b\in L^\times\right\}\,, \tag{33}$$

for any generator σ of G.

If $(L|K,v)$ is an Artin-Schreier defect extension with any Artin-Schreier generator a, then

$$\Sigma_{\mathcal{E}}\;=\;-v(a-K)\,. \tag{34}$$

If K contains a primitive root of unity and $(L|K,v)$ is a Kummer extension with Kummer generator a of value 0, then

$$\Sigma_{\mathcal{E}}\;=\;\frac{vp}{p-1}\;-\;v(a-K)\,. \tag{35}$$

Proof. Assume first that $(L|K,v)$ is an Artin-Schreier defect extension with Artin-Schreier generator a. Then for every generator σ of G, we have that $\sigma a = a+i$ for some $i \in \mathbb{F}_p$ and thus, $v(\sigma a - a) = vi = 0$. Hence equation (32) shows that Σ_σ does not depend on the choice of σ and that (34) holds.

Now assume that K contains a primitive root of unity and $(L|K,v)$ is a Kummer extension with Kummer generator a which is a 1-unit. Then $\sigma a - a = \zeta - 1$ for some primitive root of unity ζ, and by equation (19),

$$v(\sigma a - a) \;=\; va + v(\zeta - 1) \;=\; \frac{vp}{p-1}\,. \tag{36}$$

Hence by equation (32), Σ_σ does not depend on the choice of σ, and (35) holds.

If $\Sigma \subsetneq \Sigma_\sigma$, then $\sigma \notin G_\Sigma$ and hence $G_\Sigma = \{\mathrm{id}\}$. If $\Sigma_\sigma \subseteq \Sigma$, then $\sigma \in G_\Sigma$ and hence $G_\Sigma = G$. Trivially, $\Sigma_\mathcal{E}$ is the intersection of all final segments that contain it, so

$$\Sigma_\mathcal{E} \;=\; \bigcap_{G_\Sigma = G} \Sigma \;=\; \Sigma_-(G)\,.$$

Since $-v(a-K)$ has no smallest element, equations (34) and (35) show that the same is true for $\Sigma_\mathcal{E}$. Therefore, $\Sigma_\mathcal{E}$ is the union of all final segments properly contained in it, whence

$$\Sigma_\mathcal{E} \;=\; \bigcup_{G_\Sigma = \{\mathrm{id}\}} \Sigma \;=\; \Sigma_+(\{\mathrm{id}\})\,.$$

Finally, from Section 2.4 we know that $I_-(G)$ is generated by the set (29). However, as $\Sigma_\mathcal{E} = \Sigma_\sigma$ for every generator σ of G, it is also generated by the set (33). □

We define the **distance of** $\mathcal{E}$ to be the cut

$$\operatorname{dist} \mathcal{E} \;:=\; (-\Sigma_\mathcal{E})^+$$

in $\widetilde{vK}$. By applying the distance operator to the right hand sides of equations (34) and (35), we obtain:

Corollary 3.6. *If $\mathcal{E}$ is an Artin-Schreier defect extension, then*

$$\operatorname{dist} \mathcal{E} \;=\; \operatorname{dist}(a,K)$$

for every Artin-Schreier generator a of $\mathcal{E}$. Consequently, all Artin-Schreier generators of $\mathcal{E}$ have the same distance.

If $\mathcal{E}$ is a Kummer extension, then

$$\operatorname{dist} \mathcal{E} \;=\; -\frac{vp}{p-1} \;+\; \operatorname{dist}(a,K)$$

for every Kummer generator a of value 0. *Consequently, all Kummer generators of $\mathcal{E}$ of value* 0 *have the same distance.*

Also Kummer defect extensions have canonical generators whose distance is equal to $\operatorname{dist} \mathcal{E}$ and whose minimal polynomials resemble Artin-Schreier polynomials. Details will be worked out in Section 3.4.

Proposition 3.7. *Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p.*

1) We have that

$$\operatorname{dist} \mathcal{E} \;\le\; 0^-\,. \tag{37}$$

If $\mathcal{E}$ is an Artin-Schreier defect extension, then

$$\operatorname{dist}(a,K)\;\leq\;0^- \tag{38}$$

for every Artin-Schreier generator a. If $\mathcal{E}$ is a Kummer defect extension, then

$$0\;<\;\operatorname{dist}(a,K)\;\leq\;\left(\frac{vp}{p-1}\right)^- \tag{39}$$

for every Kummer generator a of value 0.

2) The extension $\mathcal{E}$ has independent defect if and only if

$$\operatorname{dist}\mathcal{E}\;=\;H^- \tag{40}$$

for some proper convex subgroup H of vK. In particular, if the value group of (K,v) is archimedean, then $\mathcal{E}$ has independent defect if and only if $\operatorname{dist}\mathcal{E}=0^-$.

3) An Artin-Schreier defect extension with Artin-Schreier generator a has independent defect if and only if

$$\operatorname{dist}(a,K)\;=\;H^- \tag{41}$$

for some proper convex subgroup H of vK.

A Kummer defect extension of prime degree with Kummer generator a of value 0 has independent defect if and only if

$$\operatorname{dist}(a,K)\;=\;\frac{vp}{p-1}\;+\;H^-\;, \tag{42}$$

for some convex subgroup H of vK that does not contain vp.

Proof. 1): Inequality (37) follows from part 2) of Theorem 3.4 together with the definition of $\Sigma_{\mathcal{E}}$ in Theorem 3.5. From inequality (37) we obtain inequality (38) and the second inequality in (39) by an application of Corollary 3.6. The first inequality in (39) follows from Theorem 2.8 since $va=0$.

2): By definition, the lower cut set of $\operatorname{dist}\mathcal{E}$ is the smallest initial segment of $\widetilde{vK}$ containing $-\Sigma_{\mathcal{E}}$. Since $-\Sigma_{\mathcal{E}}$ is an initial segment of vK, $\operatorname{dist}\mathcal{E}=H^-$ is equivalent to $\Sigma_{\mathcal{E}}=\{\alpha\in vK\mid \alpha>H\}$.

The final assertion of part 2) follows from the fact that the only proper convex subgroup in an archimedean ordered abelian group is $\{0\}$.

3): This follows from part 2) together with Corollary 3.6. In the case of a Kummer extension we have that $\operatorname{dist}(a,K)>va=0$, so H cannot contain vp. □

We choose any extension of v from $K(a)$ to $\tilde{K}$ and take (K^r,v) to be the absolute ramification field of (K,v).

Proposition 3.8. *Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree p with an Artin-Schreier or Kummer generator a. Further, take an absolute ramification field as above, and an intermediate field N of $K^r|K$. Then also $\mathcal{E}_N:=(L.N|N,v)$ is a Galois defect extension of degree p,*

$$\operatorname{dist}\mathcal{E}_N\;=\;\operatorname{dist}\mathcal{E}\;,$$

and $\mathcal{E}_N$ has independent defect if and only $\mathcal{E}$ has. Further, if (N,v) is an independent defect field, then so is (K,v).

Proof. We may assume that a is a generator of $\mathcal{E}$ as in Theorem 3.5. By equation (13) of Proposition 2.14, also $\mathcal{E}_N$ is a Galois defect extension of prime degree p, and $\operatorname{dist}(a,N) = \operatorname{dist}(a,K)$. In view of Corollary 3.6, we obtain that $\operatorname{dist}\mathcal{E}_N = \operatorname{dist}\mathcal{E}$. From this, the third assertion follows by part 2) of Proposition 3.7.

In order to prove the final assertion, assume that (N,v) is an independent defect field. Take a p-th root of unity ζ_p. Then by definition, also $N(\zeta_p) \subseteq K^r = K(\zeta_p)^r$ is an independent defect field with respect to v. Take any Galois defect extension of degree p of $K(\zeta_p)$ with a generator a as above. Then $(N(\zeta_p)(a)|N(\zeta_p),v)$ has independent defect, and by what we have proved already, the same is true for $(K(\zeta_p)(a)|K(\zeta_p),v)$. This shows that (K,v) is an independent defect field. □

We will now prove the equivalence of assertions a), b), c) and d) in Theorem 1.14. The equivalence of assertions a) and b) follows from the fact that $\Sigma_{\mathcal{E}} = \Sigma_-(G)$, proven in Theorem 3.5. Further, the equivalence of assertions b) and c) in Theorem 1.14 is valid because $vL = vK$ and an ideal I_Σ of $\mathcal{O}_L$ is prime if and only if $\Sigma = \{\alpha \in vL \mid \alpha > H\}$ for some proper convex subgroup H of vL.

For the proof of the equivalence of assertions a) and d), we observe that by Lemma 2.1, idempotence of $\operatorname{dist}\mathcal{E}$ is equivalent to it being equal to H^- or H^+ for some convex subgroup H of vK. By Proposition 3.7, $\operatorname{dist}\mathcal{E} = H^+$ is not possible. Now the equivalence follows from part 2) of Proposition 3.7. □

For Artin-Schreier defect extensions, a different definition was given for dependent and independent defect in [16]. We will show in the next section that our new definition is consistent with the previous one.

3.3. **Artin-Schreier defect extensions.**

In this section, we consider the case of a valued field (K,v) of positive characteristic p and an Artin-Schreier defect extension $(L|K,v)$ with Artin-Schreier generator ϑ, that is, $\vartheta^p - \vartheta \in K$. The following definition was introduced in [16]: if there is an immediate purely inseparable extension $(K(\eta)|K,v)$ of degree p such that

$$\vartheta \sim_K \eta\,, \tag{43}$$

then we say that the Artin-Schreier defect extension has **dependent defect**; otherwise it has **independent defect**. Note that (43) implies that $\operatorname{dist}(\eta,K) < \infty$, that is, η does not lie in the completion of (K,v), since otherwise it would follow that $\vartheta = \eta$.

The above definition does not depend on the Artin-Schreier generator of the extension $L|K$. Indeed, by [16, Lemma 2.26], $\vartheta' \in L$ is another Artin-Schreier generator of $L|K$ if and only if $\vartheta' = i\vartheta + c$ for some $i \in \mathbb{F}_p^\times$ and $c \in K$. If we set $\eta' = i\eta + c$, then $K(\eta) = K(\eta')$ and $v(\vartheta' - \eta') = v(i(\vartheta - \eta)) = v(\vartheta - \eta) > \operatorname{dist}(\vartheta, K)$, that is, $\vartheta' \sim_K \eta'$.

Our above definition is consistent with the new one given in the introduction, as follows from the equivalence of assertions a) and d) of Theorem 1.14 together with the following fact, which is [16, Proposition 4.2].

Proposition 3.9. *An Artin-Schreier defect extension is independent (in the sense as defined in* [16]*) if and only if its distance (defined as the distance of any Artin-Schreier generator) is idempotent.*

The name “dependent defect” was chosen because the existence of Artin-Schreier defect extensions with dependent defect depends on the existence of purely inseparable defect extensions of degree p. The following proposition makes this dependence more precise:

Proposition 3.10. *Take a valued field (K,v) of positive characteristic p.*

1) If (K,v) admits a purely inseparable defect extension not contained in its completion, then it also admits one of degree p.

2) If $(K(\eta)|K,v)$ is a purely inseparable defect extension of degree p not contained in the completion of (K,v), then for every $b\in K$ of high enough value and every root ϑ of the polynomial

$$Y^p - Y - \left(\frac{\eta}{b}\right)^p ,$$

the extension $(K(\vartheta)|K,v)$ is an Artin-Schreier extension with dependent defect and Artin-Schreier generator ϑ such that $\vartheta\sim_K \eta/b$.

3) Take an Artin-Schreier extension $(L|K,v)$ with dependent defect. Then there exists a purely inseparable defect extension $(K(\eta)|K,v)$ of degree p not contained in the completion of (K,v) and an Artin-Schreier generator ϑ of $L|K$ such that $\eta^p=\vartheta^p-\vartheta$ and $\vartheta\sim_K\eta$.

4) (K,v) is an independent defect field if and only if every immediate purely inseparable extension of (K,v) lies in its completion.

Proof. Assertion 1) is proved in the beginning of Section 4.3 of [16], assertion 2) follows from [16, Proposition 4.3], and assertion 4) follows from assertions 1), 2) and 3).

In order to prove assertion 3), take an Artin-Schreier extension $(L|K,v)$ with dependent defect and an arbitrary Artin-Schreier generator ϑ_0. Then by Proposition 3.9, $\operatorname{dist}(\vartheta_0,K)$ is not idempotent, i.e., $p\operatorname{dist}(\vartheta_0,K)<\operatorname{dist}(\vartheta_0,K)$ in view of part 1) of Proposition 3.7. This means that there is some $c\in K$ such that $v(\vartheta_0-c)>p\operatorname{dist}(\vartheta_0,K)$. Set $a:=(\vartheta_0-c)^p-(\vartheta_0-c)\in K$ so that $\vartheta:=\vartheta_0-c$ becomes a root of the Artin-Schreier polynomial X^p-X-a. Then by [16, Theorem 4.5 (c)]), the root η of the polynomial X^p-a generates an immediate extension which does not lie in the completion of (K,v), and $\vartheta\sim_K\eta$ holds. □

3.4. **Kummer defect extensions of prime degree p.**

In this section we will study the case of Kummer defect extensions of prime degree p of a valued field (K,v) of characteristic 0 and residue characteristic $p>0$. We will use the construction from Section 2.3 that associates to a Kummer generator η an element ϑ_η whose minimal polynomial f_η given in (21) bears some resemblance with an Artin-Schreier polynomial. To this end, we assume that K contains an element C as in (20). For the construction we do not need that the extension $\mathcal{E}$ is Galois, but if (K,v) is henselian then by Lemma 2.22 it contains a primitive p-th root of unity as it contains C, which then yields that the extension is indeed Galois.

Theorem 3.11. *Take a valued field (K,v) of mixed characteristic containing a primitive p-th root of unity ζ_p and an element C as in (20), and a Kummer defect extension $\mathcal{E}=(K(\eta)|K,v)$ of prime degree p with η a 1-unit such that $\eta^p\in K$. Define ϑ_η by (23). Then*

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta_\eta-K) \tag{44}$$

and

$$-\frac{vp}{p-1} \;<\; -\frac{vp}{p-1} \;+\; \mathrm{dist}\,(\eta,K) \;=\; \mathrm{dist}\,(\vartheta_\eta,K) \;=\; \mathrm{dist}\,\mathcal{E} \;\leq\; 0^- \,. \tag{45}$$

The following assertions are equivalent:

a) $\mathcal{E}$ *has independent defect,*

b) $\mathrm{dist}\,(\vartheta_\eta,K) = H^-$ *for some proper convex subgroup* H *of* vK*,*

c) $\mathrm{dist}\,(\eta,K) = \frac{vp}{p-1} + H^-$ *for some proper convex subgroup* H *of* vK*.*

If H *satisfies assertions b) or c), then* $vp \notin H$*.*

Proof. Take $\sigma \in \mathrm{Gal}\,(K)$ such that $\sigma(\eta) = \zeta_p\eta$. From equations (20) and (36), we deduce:

$$v(\sigma\vartheta_\eta - \vartheta_\eta) \;=\; v\frac{\sigma\eta-\eta}{C} \;=\; \frac{vp}{p-1} - \frac{vp}{p-1} \;=\; 0\,.$$

By equation (32) of Theorem 3.4 in conjunction with Theorem 3.5, this yields equation (44), which in turn implies that $\mathrm{dist}\,(\vartheta_\eta,K) = \mathrm{dist}\,\mathcal{E}$.

Now we prove (45). By the definition of ϑ_η and Lemma 2.3,

$$\mathrm{dist}\,(\vartheta_\eta,K) \;=\; \mathrm{dist}\,\left(\frac{\eta-1}{C},K\right) \;=\; -vC \;+\; \mathrm{dist}\,(\eta-1,K) \;=\; -vC \;+\; \mathrm{dist}\,(\eta,K)\,.$$

Since $vC = \frac{vp}{p-1}$, Corollary 3.6 gives us:

$$\mathrm{dist}\,(\vartheta_\eta,K) \;=\; -\frac{vp}{p-1} \;+\; \mathrm{dist}\,(\eta,K)\,. \tag{46}$$

The leftmost and the rightmost inequalities of (45) follow from equation (39). This completes the proof of (45).

Finally, we prove the equivalences. From part 2) of Proposition 3.7 we know that $\mathcal{E}$ has independent defect if and only if $\mathrm{dist}\,\mathcal{E} = H^-$ for some proper convex subgroup H of vK. Since $\mathrm{dist}\,\mathcal{E} = \mathrm{dist}\,(\vartheta_\eta,K)$ as we have already proved, the latter is just assertion b). Equation (46) shows that assertios b) and c) are equivalent.

Our last assertion follows from (45). □

4. Properties of Galois extensions of prime degree with independent defect

Throughout this section we will assume that (K,v) is a valued field of residue characteristic $p > 0$. Except in Proposition 4.7, we also assume that K contains a primitive p-th root of unity if $\mathrm{char}\,K = 0$.

The following is Lemma 4.9 of [16]:

Proposition 4.1. *Assume that* $\mathrm{char}\,K = p$ *and* $(K(\vartheta)|K,v)$ *is an independent Artin-Schreier defect extension with Artin-Schreier generator* ϑ *of distance* 0^-*. Then every algebraically maximal immediate extension (and in particular, every maximal immediate extension) of* (K,v) *contains an independent Artin-Schreier defect extension* $(K(\vartheta')|K,v)$ *of distance* 0^- *and such that* $\vartheta \sim_K \vartheta'$*.*

Here is the analogue of this result in the case of mixed characteristic:

Proposition 4.2. *Assume that* $\operatorname{char} K = 0$ *and that* $(K(\eta)|K,v)$ *is an independent defect extension of distance* 0^-*, generated by a 1-unit* η *with* $\eta^p \in K$*. Then every algebraically maximal immediate extension of* (K,v) *contains an independent defect extension* $(K(\eta')|K,v)$ *of prime degree and distance* 0^-*, where* η' *is also a* p*-th root of a* 1*-unit in* K *and* $\eta \sim_K \eta'$*.*

Proof. Take an algebraically maximal immediate extension (M,v) of (K,v). We note that (M,v) is henselian. If $\eta \in M$, then the assertion is trivial.

Assume that $\eta \notin M$. Then $(M(\eta)|M,v)$ is an extension of degree p with $\eta^p \in M$. Since M is algebraically maximal, $(M(\eta)|M,v)$ is defectless. Indeed, otherwise $(M(\eta)|M,v)$ would be a defect extension of degree p, hence a nontrivial immediate extension, a contradiction to the maximality of (M,v). Therefore by Lemma 2.10, the set $v(\eta - M)$ admits a maximal element. Since by Theorem 2.8 the set $v(\eta - K)$ has no maximal element, we have that $v(\eta - K) \subsetneq v(\eta - M)$. Hence there is an element $b \in M$ such that $v(\eta - b) > \operatorname{dist}(\eta, K)$. Since equation (46) yields that $\operatorname{dist}(\eta,K) = \left(\frac{vp}{p-1}\right)^-$ we may deduce that

$$v(a-b) \;\geq\; \frac{vp}{p-1}\,. \tag{47}$$

If $b^p \in K$, we set $\eta' = b$.

Now assume that $b^p \notin K$. Since η is a 1-unit, so is b and thus,

$$v\left(\frac{\eta}{b} - 1\right) \;=\; v(\eta - b) \;\geq\; \frac{vp}{p-1}\,.$$

The element $\frac{\eta}{b}$ is a 1-unit of level $\geq \frac{1}{p-1}vp$, hence by Corollary 2.20, $\frac{\eta^p}{b^p}$ is a 1-unit of level $\geq \frac{p}{p-1}vp$. As $(M|K,v)$ is immediate, there is some $c \in K$ such that

$$v\left(\frac{\eta^p}{b^p} - c\right) > v\left(\frac{\eta^p}{b^p} - 1\right) \;\geq\; \frac{p}{p-1}vp\,. \tag{48}$$

Then c is also a 1-unit, and we have that

$$v\left(\frac{\eta^p}{b^p c} - 1\right) \;=\; v\left(\frac{\eta^p}{b^p} - c\right) \;>\; \frac{p}{p-1}vp\,.$$

Therefore, by Lemma 2.23 the 1-unit $\frac{\eta^p}{b^p c}$ admits a p-th root u in the henselian field M. Then $bu \in M$ with

$$(bu)^p \;=\; b^p\frac{\eta^p}{b^p c} \;=\; \frac{\eta^p}{c} \;\in\; K\,.$$

Since $\frac{\eta^p}{b^p}$ is a 1-unit of level $\geq \frac{p}{p-1}vp$, (48) yields that the same holds for c. Since

$$c \;=\; \frac{\eta^p}{(bu)^p}\,,$$

Corollary 2.20 shows that the level of the 1-unit $\frac{\eta}{bu}$ is $\geq \frac{1}{p-1}vp$. We obtain that

$$v(\eta - bu) \;=\; v\left(\frac{\eta}{bu} - 1\right) \;\geq\; \frac{vp}{p-1}\,,$$

and we set $\eta' = bu$.

In both cases we have now achieved that η' is a 1-unit which is a p-th root of an element in K such that $v(\eta - \eta') \geq \frac{vp}{p-1}$, which by inequality (39) of Proposition 3.7

yields that $v(\eta-\eta') > \mathrm{dist}\,(\eta,K)$. From part 1) of Lemma 2.2 we obtain that

$$\mathrm{dist}\,(\eta',K) \;=\; \mathrm{dist}\,(\eta,K)\;,$$

which by part 3) of Proposition 3.7 shows that like $(K(\eta)|K,v)$, also $(K(\eta')|K,v)$ is an independent defect extension of distance 0^-. □

From Propositions 4.1 and 4.2 we obtain the following result.

Corollary 4.3. *Assume that there is a maximal immediate extension of (K,v) in which K is relatively algebraically closed. Then (K,v) admits no independent Galois defect extension of prime degree and distance 0^-.*

We wish to generalize the previous result to the case of independent defect extensions with arbitrary distance.

Lemma 4.4. *Assume that for every coarsening w of v (including the valuation v itself) such that Kw is of positive characteristic there is a maximal immediate extension (M_w,w) of (K,w) in which K is relatively algebraically closed. Then (K,v) admits no independent Galois defect extension of prime degree.*

Proof. The case of equal positive characteristic has been settled in Lemma 4.11 of [16]. Hence we assume now that (K,v) is of characteristic 0 with residue characteristic $p>0$ and containing a primitive p-th root of unity.

Suppose that $(L|K,v)$ is an independent Galois defect extension of prime degree. By Corollary 4.3, its distance cannot be 0^-. Hence it is equal to H^- for some nontrivial proper convex subgroup H of vK. Denote by v_H the coarsening of v with respect to H, and by $\mathcal{M}_{v_H}$ its valuation ideal. From Theorem 3.11 we know that $vp\notin H$, so we have that $p\in\mathcal{M}_{v_H}$ and therefore, $\mathrm{char}\,Kv_H=p$. By Lemma 2.13, a coarsening of a henselian valuation is again henselian, so (K,v_H) is henselian.

By our assumption, we can write $L=K(\vartheta_\eta)$ with ϑ_η as in Section 3.4. Then $\mathrm{dist}\,(\vartheta_\eta,K)=H^-$, which means that $v(\vartheta_\eta-K)$ is cofinal in $(vK)^{<0}\setminus H$. It follows that $v_H(\vartheta_\eta-K)$ is cofinal in $vK^{<0}/H=(\widetilde{v_HK})^{<0}$. Since $\widetilde{v_HK}$ is divisible, $(\widetilde{v_HK})^{<0}$ has no largest element. Thus in particular, $v_H(\vartheta_\eta-K)$ has no maximal element. Together with Lemma 2.9, this shows that $(L|K,v_H)$ is an immediate extension of henselian fields. Hence, $(L|K,v_H)$ is a Galois defect extension of prime degree and distance 0^-. On the other hand, by assumption (K,v_H) admits a maximal immediate extension in which K is relatively algebraically closed. Therefore, Corollary 4.3 shows that (K,v_H) admits no Galois defect extension of prime degree and distance 0^-, a contradiction. □

Lemma 4.5. *Take a coarsening w of v (possibly the valuation v itself) such that (K,w) admits a maximal immediate extension (M_w,w) in which K is relatively algebraically closed. If $(L|K,v)$ is a finite separable and defectless extension, then $(M_w.L,w)$ is a maximal immediate extension of (L,w) such that L is relatively algebraically closed in $M_w.L$.*

Proof. Since $(L|K,v)$ is defectless by assumption, the same is true for the extension $(L|K,w)$ by Lemma 2.13. We note that (K,w) is henselian since it is assumed to be relatively algebraically closed in the henselian field (M_w,w). Hence we may apply Lemma 2.4: since $(M_w|K,w)$ is immediate and $(L|K,w)$ is defectless, $(M_w.L|L,w)$ is immediate and $M_w|K$ and $L|K$ are linearly disjoint. The latter implies that L is relatively algebraically closed in $M_w.L$ (for the proof of this fact, see [21] or [23, Chapter 24]). On the other hand, [28, Theorem 31.22] shows that $(M_w.L, w)$ is a maximal field, being a finite extension of a maximal field, and it is therefore a maximal immediate extension of (L, w). □

Proposition 4.6. *If (K, v) is algebraically maximal and $(L|K, v)$ is a finite separable and defectless extension, then (L, v) admits no independent Galois defect extension of prime degree.*

Proof. Take a coarsening w of v such that Kw is of positive characteristic. Note that every immediate extension of (K, w) is also immediate under the finer valuation v. Since (K, v) is algebraically maximal, this yields that also (K, w) is algebraically maximal.

Take (M_w, w) to be a maximal immediate extension of (K, w). Then K is relatively algebraically closed in M_w. Lemma 4.5 yields that $(M_w.L, w)$ is a maximal immediate extension of (L, w) such that L is relatively algebraically closed in $M_w.L$.

This shows that for every coarsening w of v such that Lw is of positive characteristic there is a maximal immediate extension of (L, w) in which L is relatively algebraically closed. By Lemma 4.4 this proves that (L, v) admits no independent Galois defect extension of prime degree. □

Proposition 4.7. *Assume that (K, v) is a valued field of positive residue characteristic p. Then the following are equivalent*

a) (K, v) is henselian and defectless,

b) (K, v) is algebraically maximal and in every finite tower of extensions of degree p over K^r every defect extension of degree p is separable and independent.

Proof. Assume first that a) holds. Since K is henselian and defectless, it admits in particular no immediate algebraic extension, that is, (K, v) is algebraically maximal.

Take now a finite tower L of extensions of degree p over K^r. Choose generators $a_1, \ldots, a_s$ of the extension $L|K^r$ and set $K' = K(a_1, \ldots, a_s)$. Then $(K'|K, v)$ is finite, hence by assumption a defectless extension. Since the extension $(K^r|K, v)$ is tame, Proposition 2.7 yields that

$$1 \;=\; d(K'|K, v) \;=\; d(K'.K^r|K^r, v) \;=\; d(L|K^r, v)\,.$$

Hence $L|K^r$ is a defectless extension, and so is every extension of degree p in the tower $L|K^r$. This shows that condition b) holds.

Suppose now that (K, v) satisfies condition b). Since (K, v) is algebraically maximal, it is henselian. Take a finite extension $(L|K, v)$. We wish to show that the extension is defectless. Take K' to be the relative separable-algebraic closure of K in L. By Lemma 2.15, there is a finite tame extension N of K such that $K'.N|N$ is a tower $N = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_m = K'.N$ of Galois extensions $N_i|N_{i-1}$ of degree p. If $\operatorname{char} K = 0$, we can assume in addition that N contains a primitive p-th root of unity, replacing N by $N(\zeta_p)$ if necessary; since p does not divide $[N(\zeta_p) : N]$, this is also a tame extension of (K, v).

We first show that the extension $(K'|K, v)$ is defectless. Proposition 2.7 shows that $d(K'.N|N, v) = d(K'|K, v)$, so it suffices to show that $(K'.N|N, v)$ is defectless. We observe that also $K^r = N_0.K^r \subseteq N_1.K^r \subseteq \ldots N_m.K^r = K'.K^r$ is a tower of Galois extensions $N_i.K^r|N_{i-1}.K^r$ of degree p. Assume that $(N_{i-1}|N, v)$ is a defectless extension for some $i \leq m$ and consider the extension $(N_i|N_{i-1}, v)$. Condition

b) implies that the extension $(N_i.K^r|N_{i-1}.K^r, v)$ is either defectless or an independent Galois defect extension. Since (K, v) is algebraically maximal and $(N_{i-1}|K, v)$ is a finite separable defectless extension, Proposition 4.6 shows that $(N_i|N_{i-1}, v)$ cannot be an independent defect extension. Therefore, also $(N_i.K^r|N_{i-1}.K^r, v)$ cannot be an independent defect extension. Hence by assumption, this extension is defectless. From Proposition 2.7 it thus follows that $(N_i|N_{i-1}, v)$ is defectless. This shows that also $(N_i|N, v)$ is a defectless extension. By induction on i we obtain that $(K'.N|N, v)$ is a defectless extension.

The above conclusion together with Proposition 2.7 yields that

$$d(L|K, v) = d(L.K^r|K^r, v) = d(L.K^r|K'.K^r, v). \tag{49}$$

Since $L|K'$ is purely inseparable, $L.K^r|K'.K^r$ is a tower of purely inseparable extensions of degree p. On the other hand, assumption b) implies that every defect extension of degree p in the tower $L.K^r|K^r$ is separable. Thus every extension in the tower $L.K^r|K'.K^r$ is defectless. This shows that $d(L.K^r|K'.K^r, v) = 1$ and together with equation (49) yields that $(L|K, v)$ is a defectless extension. □

Note that if $\operatorname{char} K = p > 0$, then condition b) holds if and only if (K, v) is separable-algebraically maximal and inseparably defectless. Indeed, assume that (K, v) satisfies b). Then it is separable-algebraically maximal. If (K, v) would admit a purely inseparable defect extension (L, v), then Proposition 2.7 would yield that $(L.K^r|K^r, v)$ were also a purely inseparable defect extension, which contradicts our assumption that every defect extension of degree p in the tower $L.K^r|K^r$ is separable.

Suppose now that (K, v) is separable-algebraically maximal and inseparably defectless. Then (K, v) is algebraically maximal, and by Proposition 2.7, (K^r, v) is inseparably defectless. Take a finite extension $(L|K^r, v)$. By Lemma 2.15, $L|K^r$ is a finite tower of normal extensions of degree p. As (K^r, v) is inseparably defectless, Lemma 2.6 yields that every purely inseparable extension of degree p in this tower is defectless. Moreover, since every finite extension of K^r does not admit purely inseparable defect extensions, it also admits no dependent Artin-Schreier defect extensions. This yields that every defect extension of degree p in the tower $L|K^r$ is independent.

We have now shown that in the case of valued fields of positive characteristic, our above characterization of henselian defectless fields is equivalent to Theorem 1.2 of [16].

5. The trace of defect extensions of prime degree

In this section we will consider the trace on a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree. If $L|K$ is an Artin-Schreier extension, then we write $L = K(\vartheta)$ where ϑ is an Artin-Schreier generator. If $L|K$ is a Kummer extension, then we write $L = K(\eta)$ where η is a Kummer generator, that is, $\eta^p \in K$; as explained at the beginning of Section 3.2, we can assume that η is a 1-unit.

The proof of the following fact can be found in [11, Section 6.3].

Lemma 5.1. *Take a separable field extension $K(a)|K$ of degree n and let $f(X) \in K[X]$ be the minimal polynomial of a over K. Then*

$$\mathrm{Tr}_{\,K(a)|K}\left(\frac{a^m}{f'(a)}\right) \;=\; \begin{cases} 0 & \text{if } 1 \leq m \leq n-2 \\ 1 & \text{if } m = n-1\,. \end{cases} \tag{50}$$

□

For arbitrary $d \in K$, we note:

$$d(a-c)^{p-1} \in \mathcal{M}_{K(a)} \iff vd \;>\; -(p-1)v(a-c)\,. \tag{51}$$

Take Λ to be the smallest final segment of $\widetilde{vK}$ containing $-(p-1)v(a-K)$. Then the above equation yields that

$$vd \in \Lambda \iff \exists c \in K:\; d(a-c)^{p-1} \in \mathcal{M}_{K(a)}\,. \tag{52}$$

First we consider the equal characteristic case. By Lemma 5.1,

$$\mathrm{Tr}_{\,K(\vartheta)|K}\left(\vartheta^i\right) \;=\; \begin{cases} 0 & \text{if } 1 \leq i \leq p-2 \\ -1 & \text{if } i = p-1\,. \end{cases} \tag{53}$$

This also holds for $\vartheta - c$ for arbitrary $c \in K$ in place of ϑ since it is also an Artin-Schreier generator. In particular,

$$\mathrm{Tr}_{\,K(\vartheta)|K}\left(d(\vartheta-c)^{p-1}\right) \;=\; -d$$

By (52) and the first equation of Corollary 3.6 it follows that

$$\begin{aligned} \mathrm{Tr}_{\,K(\vartheta)|K}\left(\mathcal{M}_{K(\vartheta)}\right) \;&\supseteq\; \{d \in K \mid vd > -(p-1)\,\mathrm{dist}\,(\vartheta,K)\} \\ &=\; \{d \in K \mid vd > -(p-1)\,\mathrm{dist}\,\mathcal{E}\}\,. \end{aligned} \tag{54}$$

Now we consider the mixed characteristic case. Since $\eta^p \in K$, we have that

$$\mathrm{Tr}_{\,K(\eta)|K}(\eta^i) \;=\; 0$$

for $1 \leq i \leq p-1$. For $c \in K$ and $0 \leq j \leq p-1$, we compute:

$$(\eta-c)^j \;=\; \sum_{i=1}^{j}\binom{j}{i}\eta^i(-c)^{j-i} \;+\; (-c)^j\,.$$

Thus for every $d \in K$,

$$\mathrm{Tr}_{\,K(\eta)|K}(d(\eta-c)^j) \;=\; pd(-c)^j\,. \tag{55}$$

If $vd > -(p-1)\mathrm{dist}\,(\eta,K)$, then we may choose $c \in K$ with $vd > -(p-1)v(\eta-c)$; this remains true if we make $v(\eta-c)$ even larger. Since η is a 1-unit, there is $c \in K$ such that $v(\eta-c) > 0$, which implies that $vc = 0$. Hence we may choose $c \in K$ with $vd > -(p-1)v(\eta-c)$ *and* $vc = 0$. Applying (55) with $j = p-1$, we find that $\mathrm{Tr}_{\,K(\eta)|K}(d(-c)^{-(p-1)}(\eta-c)^{p-1}) = pd$. We obtain, using the second equation of Corollary 3.6:

$$\begin{aligned} \mathrm{Tr}_{\,K(\eta)|K}\left(\mathcal{M}_{K(\eta)}\right) \;&\supseteq\; \{pd \mid d \in K \text{ and } vd > -(p-1)\,\mathrm{dist}\,(\eta,K)\} \\ &=\; \{d \in K \mid vd > -(p-1)\,\mathrm{dist}\,(\eta,K) + vp\} \\ &=\; \{d \in K \mid vd > -(p-1)\,\mathrm{dist}\,\mathcal{E}\}\,. \end{aligned} \tag{56}$$

In order to prove the opposite inclusions in (54) and (56), we have to find out enough information about the elements $g(a) \in K(a)$ that lie in $\mathcal{M}_{K(a)}$. Using the Taylor expansion, we write

$$g(a) \;=\; \sum_{i=0}^{p-1} \partial_i g(c)(a-c)^i \,.$$

By Lemma 2.12 there is $c \in K$ such that among the values $v\partial_i g(c)(a-c)^i$, $0 \le i \le p-1$, there is precisely one of minimal value, and the same holds for all $c' \in K$ with $v(a-c') \ge v(a-c)$. In particular, we may assume that $v(a-c) > va$. For all such c, we have:

$$vg(a) \;=\; \min_{0\le i\le p-1} v\partial_i g(c)(a-c)^i \,.$$

Hence for $g(a)$ to lie in $\mathcal{M}_{K(a)}$ it is necessary that $v\partial_i g(c)(a-c)^i > 0$, or equivalently,

$$v\partial_i g(c) \;>\; -iv(a-c) \tag{57}$$

for $0 \le i \le p-1$ and $c \in K$ as above.

In the equal characteristic case, for $g(\vartheta) \in \mathcal{M}_{K(\vartheta)}$ and $c \in K$ as above, we find:

$$\mathrm{Tr}_{\,K(\vartheta)|K}(g(\vartheta)) \;=\; \sum_{i=0}^{p-1} \mathrm{Tr}_{\,K(\vartheta)|K}(\partial_i g(c)(\vartheta-c)^i) \;=\; -\partial_{p-1} g(c) \,.$$

Since $\partial_{p-1} g(c) > -(p-1)v(\vartheta-c)$ by (57), this proves the desired equality in (54).

In the mixed characteristic case, for $g(\eta) \in \mathcal{M}_{K(\eta)}$ and $c \in K$ as above, we find:

$$\mathrm{Tr}_{\,K(\eta)|K}(g(\eta)) \;=\; \sum_{j=0}^{p-1} \mathrm{Tr}_{\,K(\eta)|K}(g_j(c)(\eta-c)^j) \;=\; p\sum_{j=0}^{p-1} \partial_j g(c)(-c)^j \,.$$

As we assume that $v(\eta-c) > 0$, we have that $vc = 0$ and $-iv(\eta-c) \ge -(p-1)v(\eta-c)$ for $0 \le i \le p-1$. Hence by (57), $v\sum_{i=0}^{p-1} \partial_i g(c)(-c)^i \ge -(p-1)v(\eta-c)$. This proves the desired equality in (56). We have now proved the first assertion of the following theorem:

Theorem 5.2. *Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. Then*

$$\mathrm{Tr}_{\,L|K}\,(\mathcal{M}_L) \;=\; \{d \in K \mid vd > -(p-1)\,\mathrm{dist}\,\mathcal{E}\} \,. \tag{58}$$

The extension $\mathcal{E}$ has independent defect if and only if for some proper convex subgroup H of vK,

$$\mathrm{Tr}_{\,L|K}\,(\mathcal{M}_L) \;=\; \{d \in K \mid vd > \alpha \text{ for all } \alpha \in H\} \;=\; \mathcal{M}_{v_H} \,, \tag{59}$$

where $\mathcal{M}_{v_H}$ is the valuation ideal of the coarsening v_H of v on K whose value group is $vK/(H\cap vK)$. In particular, if $H=\{0\}$ (which is always the case if the rank of (K,v) is 1), then this means that

$$\mathrm{Tr}_{\,L|K}\,(\mathcal{M}_L) \;=\; \mathcal{M}_K \,.$$

In the mixed characteristic case, $\mathcal{M}_{v_H}$ will always contain p, so that $\mathrm{char}\, Kv_H = p$.

Proof. Equation (58) is already proven. By the equivalence of assertions a) and d) in Theorem 1.14, which we have already proved, the extension $\mathcal{E}$ has independent defect if and only if $\operatorname{dist}\mathcal{E}$ is idempotent. This in turn is equivalent to $(p-1)\operatorname{dist}\mathcal{E} = \operatorname{dist}\mathcal{E}$ and $\operatorname{dist}\mathcal{E} = H^-$ for some proper convex subgroup H of vK, or in other words,

$$-(p-1)\operatorname{dist}\mathcal{E} \;=\; H^+ \,,$$

which turns (58) into the first equation of (59). This equation means that d is an element of the valuation ideal $\mathcal{M}_{v_H}$ of the coarsening v_H of v whose value group has divisible hull $\widetilde{vK/H}$. Hence the second equation in (59) holds.

The last statement of the theorem follows from Theorem 3.11. □

From this theorem we obtain the equivalence of assertions a) and e) in Theorem 1.14.

6. Semitame, deeply ramified and gdr fields

Throughout this section, we will consider a valued field (K,v) of residue characteristic $p>0$. All statements we will prove are trivial for valued fields of residue characteristic 0.

When we deal with valued fields (K,v) of mixed characteristic with residue characteristic p, we will write $v = v_0\circ v_p\circ\overline{v}$ as in the paragraph before Proposition 1.3, set $\operatorname{crf}(K,v) := Kv_0v_p$ and denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp. Further, $\frac{1}{p^\infty}\mathbb{Z}vp$ will denote the p-divisible hull of the subgroup $\mathbb{Z}vp$ of vK generated by vp. If K has positive characteristic p, then we set $\operatorname{crf}(K,v) := Kv$ and $(vK)_{vp} = Kv$.

6.1. Some basic results.

To start with, we state a few simple observations.

Lemma 6.1. *1) If* $\operatorname{char} K = p > 0$*, then*

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K \tag{60}$$

is surjective if and only if K is perfect; in particular, (DRvr) holds if and only if $\hat{K}$ is perfect.

2) If (60) is surjective, then (DRvr) holds.

3) If $\operatorname{char} K = 0$*, then the following assertions are equivalent:*

a) (60) is surjective,

b) for every $a\in\mathcal{O}_K$ there is $c\in\mathcal{O}_K$ such that $a\equiv c^p \mod p\mathcal{O}_K$,

c) for every $\hat{a}\in\mathcal{O}_{\hat{K}}$ there is $c\in\mathcal{O}_K$ such that $\hat{a}\equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$,

d) (DRvr) holds.

4) If (K,v) satisfies (DRvr), then so does every extension of (K,v) within its completion.

Proof. 1): From $\operatorname{char} K = p > 0$ it follows that $p\mathcal{O}_K = \{0\}$, hence the surjectivity of the homomorphism in (5) means that every element in $\mathcal{O}_K$ is a p-th power. Hence the same is true for every element in K, i.e., K is perfect. Replacing K by $\hat{K}$ in (60), we thus obtain that $\hat{K}$ is perfect.

2): Assume first that $\mathrm{char}\, K = p > 0$. Then by part 1) the surjectivity of (60) implies that K is perfect. Since the completion of a perfect field is again perfect, it follows that $\hat{K}$ is perfect. Hence again by part 1), (DRvr) holds.

Now assume that $\mathrm{char}\, K = 0$. Take $\hat{a} \in \mathcal{O}_{\hat{K}}$. Then there exists $a \in K$ such that $\hat{a} \equiv a \mod p\mathcal{O}_{\hat{K}}$. By assumption, there is some $c \in \mathcal{O}_K$ such that $a \equiv c^p \mod p\mathcal{O}_K$. It follows that $\hat{a} \equiv a \equiv c^p \mod p\mathcal{O}_{\hat{K}}$, showing that (DRvr) also holds in this case.

3): Assume that $\mathrm{char}\, K = 0$. The proof of the equivalence of a) and b) is straightforward. Trivially, c) implies b), and part 2) of our lemma shows that a) implies d). To show that d) implies c), take $\hat{a} \in \mathcal{O}_{\hat{K}}$. Then by (DRvr), using the equivalence of a) and b) with $\hat{K}$ in place of K, there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $\hat{a} \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. We take $c \in \mathcal{O}_K$ such that $c \equiv \hat{c} \mod p\mathcal{O}_{\hat{K}}$. Then $\hat{a} \equiv \hat{c}^p \equiv c^p \mod p\mathcal{O}_{\hat{K}}$, whence $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$.

4): Take $(L|K,v)$ to be a subextension of $(\hat{K}|K,v)$. Then $\hat{L} = \hat{K}$, and in the case of $\mathrm{char}\, K = p > 0$ our assertion follows from part 1).

Now assume that (K,v) is of mixed characteristic and satisfies (DRvr). Then by the implication d)⇒c) of part 3), for every $\hat{a} \in \mathcal{O}_{\hat{K}} = \mathcal{O}_{\hat{L}}$ there is $c \in \mathcal{O}_K \subseteq \mathcal{O}_L$ such that $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$. Hence (60) is surjective in (L,v), and the implication a)⇒d) of part 3) shows that (L,v) satisfies (DRvr). □

Lemma 6.2. *If (K,v) satisfies (DRvr), then the following assertions hold:*

1) The residue fields Kv and $\mathrm{crf}\,(K,v)$ are perfect.

2) If $\mathrm{char}\, K = p > 0$, then vK is p-divisible and (K,v) is a semitame field.

Proof. To prove part 1), take any $a \in \mathcal{O}$. By assumption, there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $a \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. From this we obtain that $av = \hat{c}^p v = (\hat{c}v)^p \in \hat{K}v = Kv$. Hence Kv is perfect. If (K,v) is of mixed characteristic, then the same holds with $v_0 \circ v_p$ in place of v, which shows that $\mathrm{crf}\,(K,v)$ is perfect.

To prove part 2), assume that $\mathrm{char}\, K = p > 0$. Then by part 1) of Lemma 6.1, (DRvr) implies that $\hat{K}$ is perfect, so $vK = v\hat{K}$ is p-divisible and (DRst) holds, showing that (K,v) is a semitame field. □

Take any $d \in \mathcal{M}_K$. If for every $a \in \mathcal{O}_K$ there is $c \in \mathcal{O}_K$ such that $a \equiv c^p \mod d\mathcal{O}_K$, we will say that the function

$$\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K \tag{61}$$

is **surjective modulo** $d\mathcal{O}_K$. This implies that the function

$$\mathcal{O}_K^\times \ni x \mapsto x^p \in \mathcal{O}_K^\times \tag{62}$$

is **surjective modulo** $d\mathcal{O}_K$ (with the obvious modification of the above definition).

Lemma 6.3. *For a valued field (K,v) of mixed characteristic, the following assertions hold:*

1) If (K,v) is a gdr field, then $(vK)_{vp}$ is p-divisible; in particular, vK contains $\frac{1}{p^\infty}\mathbb{Z}vp$. If in addition $(vK)_{vp} = vK$, then (K,v) is a semitame field.

2) Assume that there is $d \in \mathcal{M}_K$ such that the function (62) is surjective modulo $d\mathcal{O}_K$. Then for every $a \in K$ with p-divisible value va there is $c \in K$ such that

$$v(a - c^p) \;\geq\; va + vd\,. \tag{63}$$

If in addition $vd \in (vK)_{vp}$ and $(vK)_{vp}$ is p-divisible, then the function (61) is surjective modulo $d\mathcal{O}_K$.

Proof. 1): First, let us show that every $\alpha \in vK$ with $0 \le \alpha < vp$ is divisible by p. Take $a \in \mathcal{O}$ such that $va = \alpha$. From (DRvr) we obtain that there is $\hat{c} \in \mathcal{O}_{\hat{K}}$ such that $a \equiv \hat{c}^p \mod p\mathcal{O}_{\hat{K}}$. Since $va < vp$, this yields that $va = v\hat{c}^p = pv\hat{c}$, showing that $\alpha = va$ is divisible by p in $v\hat{K} = vK$.

By assumption, vp is not the smallest positive element in vK, hence there is $\alpha \in vK$ such that $0 < \alpha < vp$, and we know that α is divisible by p. We may assume that $2\alpha \ge vp$ since otherwise we replace α by $vp - \alpha$. In this way we make sure that $(vK)_{vp}$ is equal to the smallest convex subgroup containing α. This implies that for every $\beta \in (vK)_{vp}$ there is some $n \in \mathbb{Z}$ such that $0 \le \beta - n\alpha < vp$. Then by what we have already shown, $\beta - n\alpha$ is divisible by p. Since also α is divisible by p, the same is consequently true for β.

If in addition $(vK)_{vp} = vK$, then vK is p-divisible, and since (DRvr) holds by assumption, (K, v) is a semitame field.

2): Take $a \in K$ with p-divisible value. Then there is $b \in K$ such that $pvb = va$. Hence $vb^{-p}a = 0$ and by assumption, there is $c_0 \in K$ such that $v(b^{-p}a - c_0^p) \ge vd$, whence

$$v(a - (bc_0)^p) \;=\; pvb + v(b^{-p}a - c_0^p) \;\ge\; va + vd\,.$$

With $c := bc_0$, this yields (63).

Now assume in addition that $vd \in (vK)_{vp}$ and $(vK)_{vp}$ is p-divisible, and take $a \in \mathcal{O}_K$. If $va > (vK)_{vp}$, then $a \equiv 0^p \mod d\mathcal{O}_K$. If $va \in (vK)_{vp}$, then va is p-divisible and by what we have already shown there is $c \in K$ such that $a \equiv c^p \mod d\mathcal{O}_K$. This proves that (61) is surjective modulo $d\mathcal{O}_K$. □

Proposition 6.4. *Take a valued field (K, v) of mixed characteristic such that $(vK)_{vp}$ is p-divisible. Further, take $d \in \mathcal{M}_K$ such that $vd < vp$ and $nvd \ge vp$ for some $n \in \mathbb{N}$. Then the following assertions are equivalent:*

a) the function (60) is surjective, so (K, v) is a gdr field,

b) the function (61) is surjective modulo $d\mathcal{O}_K$,

c) the function (62) is surjective modulo $d\mathcal{O}_K$.

Proof. Since $vd < vp$, we have that $p\mathcal{O}_K \subset d\mathcal{O}_K$. Hence the proof of implication a)⇒c) is straightforward. Implication c)⇒b) is the content of part 2) of Lemma 6.3.

b)⇒a): Assume that assertion b) holds, and take any $a \in \mathcal{O}_K$. By part 2) of Lemma 6.3, there is $c_1 \in K$ such that $v(a - c_1^p) \ge vd$. Now we proceed by induction: having chosen c_k such that

$$v(a - c_1^p - \ldots - c_k^p) \;\ge\; kvd\,,$$

we can employ part 2) of Lemma 6.3 again to find $c_{k+1} \in K$ such that

$$v(a - c_1^p - \ldots - c_k^p - c_{k+1}^p) \ge kvd + vd\,.$$

After n many steps we have:

$$v(a - c_1^p - \ldots - c_n^p) \;\ge\; nvd \ge vp\,.$$

Using part 1) of Lemma 2.18, we obtain:

$$a \;\equiv\; c_1^p + \ldots + c_n^p \;\equiv\; (c_1^p + \ldots + c_n)^p \quad \mod p\mathcal{O}_K\,.$$

This proves that the function (60) is surjective. By part 2) of Lemma 6.1, (DRvr) holds. By assumption, $(vK)_{vp}$ is p-divisible, hence also (DRvp) holds. This proves that (K,v) is a gdr field. □

Lemma 6.5. *Assume that (K,v) is of mixed characteristic with $(vK)_{vp}$ p-divisible and Kv perfect, and take $\eta \in \tilde{K}$ such that $\eta^p \in \mathcal{O}_K$. Then either $v(\eta - K)$ does not admit a maximal element, or its maximal element is not smaller than $\frac{vp}{p}$.*

Proof. Take $c \in K$ such that $0 \leq v(\eta - c) < \frac{vp}{p}$. Then by use of (14) it follows that $v(\eta^p - c^p) = v(\eta - c)^p < vp$. Since $(vK)_{vp}$ is p-divisible, there is some $d_1 \in K$ such that $vd_1^p(\eta^p - c^p) = 0$, and since Kv is perfect, there is some $d_2 \in K$ such that $v(d^{-p}(\eta^p - c^p) - 1) > 0$, whence $v(\eta^p - c^p - d^p) > v(\eta^p - c^p)$. Again by (14), we obtain that $(\eta - c - d)^p \equiv \eta^p - c^p - d^p \mod p\mathcal{O}$, and it follows that $v(\eta - c - d) > v(\eta - c)$. □

6.2. **Proof of Theorem 1.2.**

1): Assume that (K,v) is nontrivially valued. The implication tame field ⇒ separably tame field is obvious, and so is the implication semitame field ⇒ deeply ramified field. To prove the implication deeply ramified field ⇒ gdr field, we first observe that if $\operatorname{char} K = p > 0$, then $vp = \infty$ which is not the smallest positive element of vK. If $\operatorname{char} K = 0$, then vp is not the smallest positive element of vK since otherwise, if Γ is the largest convex subgroup of vK not containing vp, then $(vK)_{vp}/\Gamma \simeq \mathbb{Z}$ in contradiction to (DRvg).

Now assume that (K,v) is a separably tame field. If $\operatorname{char} K > 0$, then by [19, Corollary 3.12], (K,v) is dense in its perfect hull. Then the completion of the perfect hull is also the completion of (K,v). Since the completion of a perfect valued field is again perfect, we obtain that the completion of (K,v) is perfect. Now part 1) of Lemma 6.1 shows that (K,v) is a semitame field.

Assume that $\operatorname{char} K = 0$. Then the separably tame field (K,v) is a tame field. By [19, Lemma 3.1 and Theorem 3.2], (K,v) is henselian and defectless, vK is p-divisible and Kv is perfect. Take any $b \in K$ that is not a p-th power, and take $\eta \in \tilde{K}$ with $\eta^p = b$. The unibranched extension $(K(\eta)|K,v)$ is defectless, hence by Lemma 2.9, $v(\eta - K)$ has a maximal element. By Lemma 6.5, this maximal element is not smaller than $\frac{vp}{p}$. Now part 3) of Lemma 2.18 shows the existence of $c \in K$ such that $b \equiv c^p \mod p\mathcal{O}_K$. This proves that (K,v) is a semitame field.

2): Assume that (K,v) is a gdr field of rank 1 and mixed characteristic. Since the rank is 1, we have that $(vK)_{vp} = vK$. Hence by part 1) of Lemma 6.3, (K,v) is a semitame field. This together with part 1) of our theorem shows the required equivalence in the case of mixed characteristic. For the case of equal characteristic, it will be shown in the proof of part 3).

3): Assume that (K,v) is a nontrivially valued field of characteristic $p > 0$.

The implications a)⇒b)⇒c) have already been shown in part 1).

c)⇒d): This holds by definition.

d)⇒e): This holds by part 1) of Lemma 6.1.

e)⇒f): If the completion of (K,v) is perfect, then it contains the perfect hull of K; since (K,v) is dense in its completion, it is then also dense in its perfect hull.

f)⇒g): If (K,v) is dense in its perfect hull, then in particular it is dense in $K^{1/p} = \{a^{1/p} \mid a \in K\}$. Since $x \mapsto x^p$ is an isomorphism which preserves valuation divisibility, the latter holds if and only if (K^p,v) is dense in (K,v).

g)⇒f): Assume that (K^p,v) is dense in (K,v). Since for each $i \in \mathbb{N}$, $x \mapsto x^{p^i}$ is an isomorphism which preserves valuation divisibility, it follows that $(K^{1/p^{i-1}},v)$ is dense in $(K^{1/p^i},v)$. By transitivity of density we obtain that (K,v) is dense in $(K^{1/p^i},v)$ for each $i \in \mathbb{N}$, and hence also in its perfect hull.

f)⇒e): This implication was already shown in the proof of part 1) of our theorem.

e)⇒a): Assume that $\hat{K}$ is perfect. The extension $(\hat{K}|K,v)$ is immediate, so $vK = v\hat{K}$, which is p-divisible. Hence (DRst) holds. By part 1) of Lemma 6.1, also (DRvr) holds.

4): The assertion follows from the implication f)⇒a) of part 3) as a perfect field is equal to its perfect hull. □

6.3. **Proof of Propositions 1.3 and 1.4.**

For the proof of Propositions 1.3 and 1.4, we will need some preparation.

Lemma 6.6. *Assume that (K,v) is of mixed characteristic, and set $w := v_p \circ \overline{v}$. Then (K,v) is a gdr field if and only if (Kv_0,w) is a gdr field.*

Proof. First assume that (K,v) is a gdr field. Then vp is not the smallest positive element in vK, which implies that wp is not the smallest element in $w(Kv_0)$. Take any $b \in \mathcal{O}_{Kv_0}$. Then choose $a \in \mathcal{O}_K$ such that $av_0 = b$. Since (K,v) is a gdr field, there is some $c \in \mathcal{O}_K$ such that $a - c^p \in p\mathcal{O}_K$. It follows that $cv_0 \in \mathcal{O}_{Kv_0}$ with $b - (cv_0)^p = (a-c^p)v_0 \in p\mathcal{O}_{Kv_0}$, showing that (Kv_0,w) satisfies (DRvr) by part 3) of Lemma 6.1. Hence (Kv_0,w) is a gdr field.

Now assume that (Kv_0,w) is a gdr field. Then wp is not the smallest element in $w(Kv_0)$, which implies that vp is not the smallest positive element in vK. Take any $a \in \mathcal{O}_K$. Then $av_0 \in \mathcal{O}_{Kv_0}$ and there is some $d \in \mathcal{O}_{Kv_0}$ such that $av_0 - d^p \in p\mathcal{O}_{Kv_0}$. Choose $c \in \mathcal{O}_K$ such that $cv_0 = d$. It follows that $a - c^p \in p\mathcal{O}_K$. Again using part 3) of Lemma 6.1, we conclude that (K,v) is a gdr field. □

Proof of Proposition 1.3.
In view of Lemma 6.6 it suffices to prove the proposition under the additional assumption that v_0 is trivial, that is, $vK = (vK)_{vp}$. Then the assertion is trivial if $\overline{v}$ is trivial, so we assume that it is not. This implies that vp is not the smallest positive element in vK.

Let us first assume that (K,v) is a gdr field. Then $\frac{vp}{p} \in vK$ by part 1) of Lemma 6.3, so $\frac{v_p p}{p} \in v_pK$, showing that v_pp is not the smallest positive element in v_pK. It remains to show that (K,v_p) satisfies (DRvr); by part 3) of Lemma 6.1 it suffices to prove that (60) is surjective in (K,v_p). Take any $a \in \mathcal{O}_{v_p}$. Since (K,v) is a gdr field, by part 2) of Lemma 6.3 there is $c \in K$ such that $v(a-c^p) \geq va + vp$, whence $v_p(a-c^p) \geq v_pa + v_pp \geq v_pp$.

Now assume that (K,v_p) is a gdr field. As $\overline{v}$ is not trivial, we know already that (DRvp) holds in (K,v), so it remains to show that (60) holds. Since (K,v_p) is a gdr field, for every $a \in \mathcal{O}_v \subseteq \mathcal{O}_{v_p}$ there is some $c \in K$ such that $v_p(a-c^p) \geq v_pp$, whence $v(a-c^p) > \frac{vp}{p}$. Choosing $d \in K$ with $vd = \frac{vp}{p}$ and applying Proposition 6.4, we conclude that (K,v) is a gdr field. □

Proof of Proposition 1.4.
Take an arbitrary valued field (K, v) and assume that $v = w \circ \overline{w}$ with w and $\overline{w}$ nontrivial. Assume first that $\operatorname{char} K > 0$. Then by part 3) of Theorem 1.2, the properties "semitame", "deeply ramified" and "gdr" are equivalent, so we have to prove the assertions of the lemma only for "gdr".

As w is nontrivial and a coarsening of v, the topologies generated by v and w are equal, and it follows that (K, v) is dense in its perfect hull if and only if the same holds for (K, w). By the equivalence of assertions c) and f) in part 3) of Theorem 1.2, it follows that (K, v) is a gdr field if and only if (K, w) is a gdr field. If the latter is the case, then from Lemma 6.2 we see that Kw is perfect, and as it is of positive characteristic like K, we obtain from part 3) of Theorem 1.2 that $(Kw, \overline{w})$ is also a gdr field.

Now we assume that $\operatorname{char} K = 0$ and prove the assertions for the property "gdr". First we discuss the case where $\operatorname{char} Kw > 0$ and write w in the same way as we do for v: $w = w_0 \circ w_p \circ \overline{w}$. Then $v_0 = w_0$, $v_p = w_p$, and $\overline{w}$ is a (possibly trivial) coarsening of $\overline{v}$. Hence it follows from Proposition 1.3 that (K, v) is a gdr field if and only if (K, w) is a gdr field. If the latter is the case, then because of $\operatorname{char} Kw > 0$ it follows as before that $(Kw, \overline{w})$ is also a gdr field.

Now we discuss the case where $\operatorname{char} Kw = 0$. Then (K, w) is trivially a gdr field, and w_0 is a coarsening of v_0. We write $\overline{w} = \overline{w}_0 \circ \overline{w}_p \circ \overline{\overline{w}}$ as for v. We obtain that $\overline{w}_p = v_p$, $\overline{\overline{w}} = \overline{v}$, and $\overline{w}_0$ is possibly trivial, with $w \circ \overline{w}_0 = v_0$. It follows that $(Kv_0, v_p) = ((Kw)\overline{w}_0, \overline{w}_p)$. Using Proposition 1.3, we conclude that (K, v) is a gdr field if and only if $(Kw, \overline{w})$ is a gdr field.

It remains to consider the properties "semitame" and "deeply ramified". We observe that if $\operatorname{char} Kv = p > 0$, then vK is p-divisible if and only if the same is true for wK and $\overline{w}(Kw)$. Likewise, all archimedean components of vK are densely ordered if and only if the same is true for wK and $\overline{w}(Kw)$. From what we have proved before, it thus follows that (K, v) is a semitame (or deeply ramified) field if and only if both (K, v) and $(Kw, \overline{w})$ are semitame (or deeply ramified, respectively).

Further, we recall that in the case of $\operatorname{char} Kw > 0$, (K, w) being a gdr field implies that Kw is perfect, and so $\overline{w}(Kw)$ is p-divisible and thus all of its archimedean components are densely ordered. This proves that (K, v) is a semitame (or deeply ramified) field already if (K, w) is. □

6.4. **Proof of Theorems 1.5 and 1.8 for the equal characteristic case.**

Proposition 6.7. *Take an algebraic extension $(L|K, v)$ of valued fields of positive characteristic. If (K, v) is a gdr field, then so is (L, v). If $L|K$ is finite and (L, v) is a gdr field, then so is (K, v). The same holds for "deeply ramified" and "semitame" in place of "gdr".*

Proof. In view of part 3) of Theorem 1.2, our assertions only need to be proved for gdr fields. By part 3) of Theorem 1.2, a valued field (K, v) of positive characteristic is a gdr field if and only if its completion $(\hat{K}, v)$ is perfect.

Assume that (K, v) is a gdr field. Then the completion $(\hat{L}, v)$ of (L, v) contains $(\hat{K}, v)$. Since $\hat{K}$ is perfect, so is $L.\hat{K}$. Since $(\hat{L}, v)$ is also the completion of $(L.\hat{K}, v)$, it is perfect too. Hence (L, v) is a gdr field.

Now assume that $L|K$ is finite and (L, v) is a gdr field. Then $\hat{L} = L.\hat{K}$ is perfect. As $L.\hat{K}|\hat{K}$ is finite, it follows that $\hat{K}$ is perfect. Thus (K, v) is a gdr field. □

6.5. **Proof of Theorem 1.6 and Corollary 1.7.**

Our next goal is the proof of Theorem 1.6. First, we prove the upward direction. By Proposition 6.7, we only need to prove it in the mixed characteristic case.

Lemma 6.8. *Assume that (K,v) is a henselian gdr field of mixed characteristic with residue characteristic $p>0$, and that $(L|K,v)$ is a finite extension. Then the following assertions hold.*

1) If $[L:K]=[Lv:Kv]$, then also (L,v) is a gdr field.

2) Take a prime q different from p. Assume that $L=K(a)$ with $a^q\in K$ and $va\notin vK$. Then also (L,v) is a gdr field.

Proof. Like (K,v), also (L,v) satisfies (DRvp). Hence by part 3) of Lemma 6.1, (L,v) will be a gdr field once (60) is surjective.

In order to prove part 1), we take a finite extension $(L|K,v)$ such that $[L:K]=[Lv:Kv]$. Since Kv is perfect by Lemma 6.2, $Lv|Kv$ is separable and we write $Lv=Kv(\xi)$ with $\xi\in Lv$. Since also Lv is perfect, there are $\xi_0,\dots,\xi_n\in Kv$ with $n=[Lv:Kv]-1$ such that $\xi=(\xi_n\xi^n+\ldots+\xi_1\xi+\xi_0)^p$. Let F be the extension of $\mathbb{F}_p$ generated by the coefficients of the minimal polynomial of ξ over Kv and the elements $\xi_0,\dots,\xi_n$. As a finitely generated extension of the perfect field $\mathbb{F}_p$, F is separably generated, that is, it admits a transcendence basis $t_1,\dots,t_k$ such that $F|\mathbb{F}_p(t_1,\dots,t_k)$ is separable-algebraic. We have that $F\subseteq Kv$, so we may choose $x_1,\dots,x_k\in K$ such that $x_iv=t_i$. Then $v\mathbb{Q}(x_1,\dots,x_k)=v\mathbb{Q}=\mathbb{Z}vp$ and $\mathbb{Q}(x_1,\dots,x_k)v=\mathbb{F}_p(t_1,\dots,t_k)$ (cf. [3, chapter VI, §10.3, Theorem 1]). Using Hensel's Lemma, we find an extension K_0 of $\mathbb{Q}(x_1,\dots,x_k)$ within the henselian field K such that $K_0v=F$ and $vK_0=v\mathbb{Q}(x_1,\dots,x_k)=\mathbb{Z}vp$.

Using Hensel's Lemma again, we find $a\in L$ such that $av=\xi$, $[K_0(a):K_0]=[F(\xi):F]$ and $vK_0(a)=vK_0=\mathbb{Z}vp$. By construction, $\xi^{1/p}\in F(\xi)$, so we can choose $b\in K_0(a)$ such that $bv=\xi^{1/p}$. Then $av=(bv)^p=b^pv$, so $v(a-b^p)>0$ and thus $v(a-b^p)\geq vp$.

We observe that since F contains all coefficients of the minimal polynomial of ξ over Kv,

$$[Kv(\xi):Kv]\;=\;[F(\xi):F]\;=\;[K_0(a):K_0]\;\geq\;[K(a):K]\;\geq\;[Kv(\xi):Kv]\,.$$

Hence equality holds everywhere; in particular, $K(a)=L$. Also, we obtain that $1,a,\dots,a^n$ is a basis of $K(a)|K$ with the residues $1,av,\dots,a^nv$ linearly independent over Kv. Hence if we write an arbitrary element of $K(a)$ as $\sum_{i=0}^n c_ia^i$ with $c_i\in K$, then

$$v\sum_{i=0}^n c_ia^i\;=\;\min_{0\leq i\leq n} vc_i\,.$$

Thus, for the sum to have non-negative value, all c_i must have non-negative value. Since (K,v) is a gdr field, we then have $d_i\in K$ such that $c_i\equiv d_i^p \mod p\mathcal{O}_K$. Consequently,

$$\sum_{i=0}^n c_ia^i\;\equiv\;\sum_{i=0}^n d_i^p(b^p)^i\;\equiv\;\left(\sum_{i=0}^n d_ib^i\right)^p\mod p\mathcal{O}_L\,,$$

where the last equivalence holds by part 1) of Lemma 2.18. This shows that (60) is surjective, which proves that (L,v) is a gdr field.

In order to prove part 2), we take a prime q different from p and a finite extension $(L|K,v)$ such that $L=K(a)$ with $a^q\in K$ and $va\notin vK$. We obtain that $[K(a):K]=q=(vK(a):vK)$. As p and q are coprime, also $pva=va^p$ generates $vK(a)$ over vK, and $K(a)=K(a^p)$. Therefore, $1,a^p,\ldots,a^{p(q-1)}$ is a basis of $K(a)|K$ with the values $v1,va^p,\ldots,va^{p(q-1)}$ belonging to distinct cosets of vK. Hence if we write an arbitrary element b of $K(a)$ as $b=\sum_{i=0}^{q-1}c_ia^{pi}$ with $c_i\in K$, then

$$vb\;=\;v\sum_{i=0}^{q-1}c_ia^{pi}\;=\;\min_{0\le i<q} vc_i+iva^p\,.$$

Assume that $vb\geq 0$. Then all c_ia^{pi} must have non-negative value. However, for $i>0$ this does not imply that $vc_i\geq 0$; we only know that $vc_ia^{pi}>0$ since $iva^p\notin vK$, whence $va^{pi}>-vc_i$.

Suppose that va is not equivalent to an element in vK modulo $(vL)_{vp}$. Then the same holds for vc_i+piva in place of va, for $1\leq i<q$, so that $vc_ia^{pi}\notin(vL)_{vp}$. In this case, b is equivalent to c_0 modulo $p\mathcal{O}_L$. Since (K,v) is a gdr field, there is $d_0\in K$ such that $b\equiv c_0\equiv d_0^p \mod p\mathcal{O}_L$. Hence we may now assume that va is equivalent to an element $\delta\in vK$ modulo $(vL)_{vp}$. We choose $d\in K$ with $vd=\delta$ and replace a by a/d, so from now on we can assume that $va\in(vL)_{vp}$.

As (K,v) is a gdr field, $(vK)_{vp}$ is p-divisible by part 1) of Lemma 6.3. It follows that $p(vK)_{vp}$ lies dense in $(vL)_{vp}$ and thus there is $b_i\in K$ such that $-vc_i\leq pvb_i\leq va^{pi}$, whence $vc_ib_i^p\geq 0$ and $vb_i^{-p}a^{pi}\geq 0$. Again since (K,v) is a gdr field, there are $d_i\in K$ such that $c_ib_i^p\equiv d_i^p \mod p\mathcal{O}_K$. Hence we obtain that

$$\sum_{i=0}^{q-1}c_ia^{pi}\;=\;\sum_{i=0}^{q-1}(c_ib_i^p)(b_i^{-p}a^{pi})\;\equiv\;\sum_{i=0}^{q-1}d_i^pb_i^{-p}a^{pi}\;\equiv\;\left(\sum_{i=0}^{q-1}d_ib_ia^i\right)^p \mod p\mathcal{O}_L\,,$$

where the last equivalence holds by part 1) of Lemma 2.18. Again, this shows that (60) is surjective, which proves that (L,v) is a gdr field. □

Proposition 6.9. *Take a valued field (K,v) of mixed characteristic, fix any extension of v to $\tilde{K}$, and let (K^r,v) be the respective absolute ramification field of (K,v). If (K,v) is a gdr field, then so is (K^r,v).*

Proof. In this proof we will freely make use of facts from ramification theory; for details, see [7, 8, 19].

We let L be a maximal extension of K inside of K^r that is again a gdr field; since the union over an ascending chain of gdr fields is again a gdr field, L exists by Zorn's Lemma.

First we will show that (L,v) is henselian. The decomposition $v=v_0\circ v_p\circ\overline{v}$ for v on K carries over to L with extensions of the respective valuations v_0, v_p and $\overline{v}$. We note that v is henselian on L if and only if v_0, v_p and $\overline{v}$ are.

Suppose that v_0 is not henselian on L. As (K^r,v) is henselian, so is (K^r,v_0) which therefore contains a henselization $L^{h(v_0)}$ of L with respect to v_0. As henselizations are immediate extensions, we know that $L^{h(v_0)}v_0=Lv_0$; by Proposition 1.3, (Lv_0,v_p) is a gdr field. Using the same proposition again, we find that also $(L^{h(v_0)},v_0)$ is a gdr field. By the maximality of L we conclude that $L^{h(v_0)}=L$, so v_0 is henselian on L.

Next, suppose that v_p is not henselian on Lv_0. As (K^r,v) is henselian, so is (K^rv_0,v_p) which therefore contains a henselization $Lv_0^{h(v_p)}$ of Lv_0 with respect to

v_p. We know already that (Lv_0, v_p) is a gdr field. As its rank is 1, its henselization lies in its completion. Hence by part 4) of Lemma 6.1, $(Lv_0^{h(v_p)}, v_p)$ satisfies (DRvr). Since (DRvp) holds in (Lv_0, v_p), it also holds in $(Lv_0^{h(v_p)}, v_p)$, so the latter is a gdr field. The extension $Lv_0^{h(v_p)}|Lv_0$ is separable-algebraic, so we can use Hensel's Lemma to find an extension L' of L within K^r such that $L'v_0 = Lv_0^{h(v_p)}$. Using Proposition 1.3 again, we find that (L', v) is a gdr field. Hence $L' = L$ by the maximality of L, that is, $Lv_0 = Lv_0^{h(v_p)}$, showing that (Lv_0, v_p) is henselian.

Finally, suppose that $\overline{v}$ is not henselian on Lv_0v_p. As (K^r, v) is henselian, so is $(K^r v_0 v_p, \overline{v})$ which therefore contains a henselization $Lv_0v_p^{h(\overline{v})}$ of Lv_0v_p with respect to $\overline{v}$. Suppose that $Lv_0v_p^{h(\overline{v})}|Lv_0v_p$ is nontrivial, so it contains a finite separable subextension. Using Hensel's Lemma, we lift it to a subextension $F|L$ of $K^r|L$ such that $[F : L] = [Fv_0v_p : Lv_0v_p]$. By what we have shown already, (L, v_0v_p) is henselian, and by definition it is of mixed characteristic. Therefore, we can employ part 1) of Lemma 6.8 to deduce that (F, v_0v_p) is a gdr field. By Proposition 1.3, also (F, v) is a gdr field. This contradiction to the maximality of L shows that $Lv_0v_p^{h(\overline{v})} = Lv_0v_p$, that is, $(Lv_0v_p, \overline{v})$ is henselian. Altogether, we have now shown that (L, v) is henselian.

The residue field of K^r is the separable-algebraic closure of Kv. Suppose that Lv is not separable-algebraically closed, so it admits a finite separable-algebraic extension. Using Hensel's Lemma, we lift it to a subextension $F|L$ of $K^r|L$ such that $[F : L] = [Fv : Lv]$. Again by part 1) of Lemma 6.8, (F, v) is a gdr field, contradicting the maximality of L. Hence Lv is separable-algebraically closed.

The value group of K^r is the closure of vK under division by all primes other than p. Suppose that $vL \neq vK^r$. Then there is some prime $q \neq p$ and $\alpha \in vK^r \setminus vL$ with $q\alpha \in vL$. Take $a \in \tilde{K}$ such that $a^q \in L$ with $va^q = q\alpha$. It follows that $(L(a)|L, v)$ is a tame extension, hence a lies in the maximal tame extension L^r of L. Since $K \subseteq L \subset K^r$, we know that $K^r = L^r$, so $a \in K^r$. By part 2) of Lemma 6.8, also $(L(a), v)$ is a gdr field, which again contradicts the maximality of (L, v). We conclude that $vL = vK^r$.

By what we have shown, $Lv = K^r v$ and $vL = vK^r$. As $K^r = L^r$, we see that $(K^r|L, v)$ is a tame extension. Together with the equality of the value groups and residue fields, this implies that $L = K^r$. Thus (K^r, v) is a gdr field. □

Proposition 6.10. *Assume that (K, v) is a gdr field of mixed characteristic, and take $a \in \mathcal{O}_K$.*

1) Assume that $va = 0$. Then for every $c \in \mathcal{O}_K$ with $0 < v(a - c^p) \in (vK)_{vp}$ there is $c_1 \in \mathcal{O}_K$ such that

$$v(a - c_1^p) \;=\; vp + \frac{1}{p} v(a - c^p)\,.$$

2) Assume that $va \in (vK)_{vp}$ and that $\mathrm{dist}\,(a, K^p) < va + \frac{p}{p-1}vp$. *Then*

$$va + vp \;<\; \mathrm{dist}\,(a, K^p) \;=\; va + \frac{p}{p-1} vp \;+\; H^- \;,$$

where H is a convex subgroup of vK not containing vp.

Proof. 1) Set $\alpha := v(a - c^p) > 0$. Since (K, v) is a gdr field, part 2) of Lemma 6.3 shows that there is $\tilde{c} \in K$ such that:

$$v(a - c^p - \tilde{c}^p) \;\geq\; vp + \alpha\,. \tag{64}$$

It follows that $v\tilde{c}^p = \alpha > 0$. Since $vc = va = 0$,

$$v((c + \tilde{c})^p - c^p - \tilde{c}^p) \;=\; v\sum_{i=1}^{p-1}\binom{p}{i}c^{p-i}\tilde{c}^i \;=\; vp + v\tilde{c} \;=\; vp + \frac{\alpha}{p}\,. \tag{65}$$

From (64) and (65), we obtain for $c_1 := c + \tilde{c}$:

$$v(a - c_1^p) \;=\; \min\left\{vp + \alpha, vp + \frac{\alpha}{p}\right\} \;=\; vp + \frac{\alpha}{p}\,.$$

2) First we prove the assertion in the case of $va = 0$. Since (K, v) is a gdr field, there is some $c \in K$ such that $v(a - c^p) \geq vp$, so $\mathrm{dist}\,(a, K^p) \geq vp$.

We will use the following observation. If $(vK)_{vp} \ni v(a - c^p) \geq \frac{p}{p-1}vp - \varepsilon > 0$ for some $c \in K$ and positive $\varepsilon \in vK$, then by part 1) there is $d \in \mathcal{O}_K$ such that

$$v(a - d^p) \;=\; vp + \frac{v(a - c^p)}{p} \;\geq\; vp + \frac{vp}{p-1} - \frac{1}{p}\,\varepsilon \;=\; \frac{p}{p-1}vp - \frac{1}{p}\,\varepsilon\,.$$

By assumption, $\mathrm{dist}\,(a, K^p) < \frac{p}{p-1}vp$. Hence the set of all convex subgroups H' of $\widetilde{vK}$ such that $v(a - K^p) < \frac{p}{p-1}vp + H'$ is nonempty as it contains $\{0\}$. The set is closed under arbitrary unions, so it contains a maximal subgroup H. Since $0 \in v(a - K^p)$, we see that H cannot contain vp.

Take any positive $\delta \notin H$. Then by the definition of H, there is some $n \in \mathbb{N}$ such that $v(a - K^p)$ contains a value $\geq \frac{p}{p-1}vp - n\delta$. We set $\varepsilon := \min\left\{\frac{p}{p-1}vp - vp, n\delta\right\}$ and observe that there is $c \in K$ such that

$$v(a - c^p) \;\geq\; \frac{p}{p-1}vp - \varepsilon \;\geq\; vp \;>\; 0\,.$$

Note that $v(a - c^p) \in (vK)_{vp}$ since $\mathrm{dist}\,(a, K^p) < \frac{p}{p-1}vp$. Using our above observation, by induction starting from $c_0 = c$ we find $c_i \in K$ such that

$$v(a - c_i^p) \;\geq\; \frac{p}{p-1}vp - \frac{1}{p^i}\varepsilon\,.$$

We choose some $j \in \mathbb{N}$ such that $\frac{n}{p^j} < 1$. Then

$$\frac{1}{p^j}\varepsilon \leq \frac{n}{p^j}\delta < \delta$$

and consequently,

$$v(a - c_j^p) \;>\; \frac{p}{p-1}vp - \delta\,.$$

This together with the definition of H shows that

$$vp \;<\; \mathrm{dist}\,(a, K^p) \;=\; \frac{p}{p-1}vp \;+\; H^-\,. \tag{66}$$

If $0 \neq va \in (vK)_{vp}$, then since (K, v) is a gdr field, part 1) of Lemma 6.3 shows that there is $b \in K$ such that $vb^p = va$. By what we have already shown, (66) holds for $b^{-p}a$ in place of a. We have that

$$v(a - (bc)^p) \;=\; vb^p + v(b^{-p}a - c^p) \;=\; va + v(b^{-p}a - c^p)\,,$$

whence

$$\operatorname{dist}(a,K^p) \;=\; va+\operatorname{dist}(b^{-p}a,K^p)\,,$$

which together with (66) for $b^{-p}a$ in place of a proves assertion 2) of our lemma. $\square$

We pause to note the following consequence of Proposition 6.10 which was mentioned in the Introduction, but will not be needed any further in this paper.

Proposition 6.11. *Take a valued field (K,v) of mixed characteristic such that $(vK)_{vp}$ is p-divisible. Further, take $d'\in\mathcal{M}_K$ such that $vp\leq vd'<\frac{p}{p-1}vp+H_0^-$ for the largest convex subgroup H_0 of vK not containing vp. Then the following assertions are equivalent:*

a) the function (60) is surjective, so (K,v) is a gdr field,

b) the function (61) is surjective modulo $d'\mathcal{O}_K$.

Proof. Since $d'\mathcal{O}_K\subseteq p\mathcal{O}_K$, and in view of the equivalence of a) and b) in part 3) of Lemma 6.1, the implication b)$\Rightarrow$a) is trivial.

a)$\Rightarrow$b): Assume that assertion a) holds, and take any $a\in\mathcal{O}_K$. We may assume that $va\in(vK)_{vp}$ since otherwise, $va>vd'$ and there is nothing to show. By our choice of H_0 and part 2) of Proposition 6.10, we now obtain:

$$\operatorname{dist}(a,K^p)\;\geq\; va+\frac{p}{p-1}vp\,+\,H_0^-\;\geq\;\frac{p}{p-1}vp\,+\,H_0^-\;.$$

Therefore, there is $c\in K$ such that $v(a-c^p)\geq vd'$. This proves assertion b). $\square$

The next two propositions will describe the relation between gdr and independent defect fields.

Proposition 6.12. *Every gdr field containg all p-th roots of unity is an independent defect field.*

Proof. Assume first that $\operatorname{char}K>0$. Then by part 3) of Theorem 1.2, the perfect hull of (K,v) lies in its completion. Now part 4) of Proposition 3.10 shows that (K,v) is an independent defect field.

Now assume that $\operatorname{char}K=0$, and take a Galois defect extension $(L|K,v)$ of prime degree. As shown in the beginning of Section 3.2, we can assume that $L=K(\eta)$ with a Kummer generator η which is a 1-unit.

Suppose that there is some $c\in K$ such that $v(\eta-c)\geq\frac{vp}{p-1}$. Since the defect extension $(K(\eta)|K,v)$ is immediate, $v(\eta-c)$ has no maximal element, and so there will also be some element $c\in K$ such that $v(\eta-c)>\frac{vp}{p-1}$. Then by Lemma 2.21, η lies in some henselization K^h. But this is impossible since by Lemma 2.5, the unibranched extension $(K(\eta)|K,v)$ is linearly disjoint from $K^h|K$. We conclude that $v(\eta-K)<\frac{vp}{p-1}$. By Lemma 2.19, this is equivalent to $v(\eta^p-K^p)<\frac{p}{p-1}vp$. Therefore, we can apply part 2) of Proposition 6.10 to $a=\eta^p$. We find that

$$\operatorname{dist}(\eta^p,K^p)\;=\;\frac{p}{p-1}vp\,+\,H^-\,,\tag{67}$$

where H is a convex subgroup of vK not containing vp. By part 1) of Lemma 6.3, $(vK)_{vp}$ is p-divisible. Since $H\subset(vK)_{vp}$, we can again apply Lemma 2.19 to obtain that (67) is equivalent to (42). By part 3) of Proposition 3.7 it follows that $(K(\eta)|K,v)$ has independent defect. This proves that (K,v) is an independent defect field. $\square$

Proposition 6.13. *Assume that $(vK)_{vp}$ is p-divisible and* $\mathrm{crf}\,(K,v)$ *is perfect. If (K,v) is an independent defect field, then it is a gdr field.*

Proof. From our assumption that $(vK)_{vp}$ is p-divisible it follows that (DRvp) holds. It remains to show that (K,v) satisfies (DRvr).

Assume first that $\mathrm{char}\,K>0$. Then by assumption, vK is p-divisible and Kv is perfect, hence the perfect hull of K is an immediate extension of (K,v). Thus by part 4) of Proposition 3.10, our assumption that (K,v) is an independent defect field implies that the perfect hull of K lies in its completion. This means that (K,v) lies dense in its perfect hull. Now part 3) of Theorem 1.2 shows that (K,v) is a gdr field.

Now assume that $\mathrm{char}\,K=0$, and set $w:=v_0\circ v_p$. By Proposition 1.4 it suffices to prove that (K,w) is a gdr field. Assume that $b\in K$ is not a p-th power, and take $\eta\in\tilde{K}$ with $\eta^p=b$. Then from Lemma 6.5 with w in place of v we infer that either $w(\eta-K)$ has a maximal element $\geq\frac{wp}{p}$, or it has no maximal element at all. In the first case, part 3) of Lemma 2.18 shows the existence of $c\in K$ such that $b\equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Assume that $w(\eta-K)$ has no maximal element. If it is not bounded from above in $(wK)_{wp}$, then there is some $c\in K$ such that $w(\eta-c)\geq\frac{wp}{p}$, which by part 3) of Lemma 2.18 gives us that $b\equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Now assume that $w(\eta-K)$ is bounded from above in $(wK)_{wp}$. Then in particular, $w\eta\in(wK)_{wp}$. It follows that $(\eta v_0)^p=bv_0\in Kv_0$ and that $v_p(\eta v_0-Kv_0)$ has no maximal element but is bounded from above in $v_p(Kv_0)=(wK)_{wp}$. Hence by Lemma 2.11, $(Kv_0(\eta v_0)|Kv_0,v_p)$ is a defect extension of degree p. From this it follows that also $(K(\eta)|K,v)$ is a defect extension of degree p. We set $K':=K(\zeta_p)$ where ζ_p is a primitive p-th root of unity. Then by (13) of Lemma 2.14, also $(K'(\eta)|K',v)$ is a defect extension of degree p, with $\mathrm{dist}\,(\eta,K')=\mathrm{dist}\,(\eta,K)$. By assumption, this defect extension is independent, so

$$\mathrm{dist}\,(\eta,K)\;=\;\mathrm{dist}\,(\eta,K')\;=\;\frac{vp}{p-1}\;+\;H^-$$

for some proper convex subgroup H of vK with $vp\notin H$. Hence there is some $c\in K$ such that $v(\eta-c)\geq\frac{vp}{p}$, thus as before, $b\equiv c^p \mod p\mathcal{O}_K$. This implies that $b\equiv c^p \mod p\mathcal{O}_{(K,w)}$.

Altogether, we have shown that (60) is surjective. Hence by part 2) of Lemma 6.1, (DRvr) holds. □

Lemma 6.14. *Fix any extension of v from K to $\tilde{K}$, and let (K^r,v) be the respective absolute ramification field of (K,v). If (K^r,v) is a gdr field, then so is (K,v), and if (K^r,v) is a semitame field, then so is (K,v).*

Proof. Assume that (K^r,v) is a gdr field and hence an independent defect field by Proposition 6.12. By Lemmas 6.2 and 6.3, $(vK^r)_{vp}$ is p-divisible and $\mathrm{crf}\,(K^r,v)$ is perfect. Since vK^r/vK has no p-torsion, it follows that $(vK)_{vp}$ is p-divisible. From Lemma 2.17 we infer that the extension $\mathrm{crf}\,(K^r,v)\,|\,\mathrm{crf}\,(K,v)$ is separable, so $\mathrm{crf}\,(K,v)$ is perfect. We set $K':=K(\zeta_p)\subseteq K^r$ as before. From Proposition 3.8 we conclude that (K',v) is an independent defect field. Hence by definition, the same holds for (K,v). Proposition 6.13 now shows that (K,v) is a gdr field.

Now assume that (K^r,v) is a semitame field. Then by Theorem 1.2, (K^r,v) is a gdr field, hence so is (K,v). Since vK^r is p-divisible and vK^r/vK has no p-torsion, also vK is p-divisible. Hence by definition, (K^r,v) is a semitame field. □

Proof of Theorem 1.6:
It has been proven already in Lemma 6.14 that if (K^r,v) is a gdr field, then so is (K,v), and if (K^r,v) is a semitame field, then so is (K,v). Let us now assume that (K,v) is a gdr field. If $\mathrm{char}\, K>0$, then (K,v) is a gdr field by Proposition 6.7. The case of gdr fields of mixed characteristic has been settled in Proposition 6.9. Being a gdr field, (K^r,v) is also deeply ramified, as its value group is divisible by every prime $q\neq \mathrm{char}\, Kv$ and thus satisfies (DRvg).

Assume now that (K,v) is a semitame field. Then by part 1) of Theorem 1.2, (K,v) is a gdr field. As shown above, it follows that the same is true for (K^r,v). Since vK is p-divisible, vK^r is p-divisible too. Hence (K^r,v) is a semitame field. □

Proof of Corollary 1.7:
Part 1) is an immediate consequence of both the upward and the downward direction of Theorem 1.6. As (K^h,v) is a subextension of (K^r,v), the assertions of part 2) for gdr and semitame fields follow immediately from part 1). Also the assertion for the case of deeply ramified fields follows since the extension $(K^h|K,v)$ is immediate, so (K^h,v) satisfies (DRvg) if and only if (K,v) does. □

6.6. **Proof of Theorem 1.5.**

We will need some preparations.

Proposition 6.15. *Assume that (K,v) is a gdr field of mixed characteristic containing all p-th roots of unity, and that $(L|K,v)$ is a Galois defect extension of prime degree. Then also (L,v) is a gdr field.*

Proof. Let p be the residue characteristic of (K,v). By part 1) of Lemma 6.3, vK contains $\frac{1}{p^\infty}\mathbb{Z}vp$. We choose $d\in K$ such that

$$vd \;=\; \frac{vp}{p}\,.$$

By Proposition 6.4 with L in place of K, in order to show that (L,v) is a gdr field, it suffices to show that the function $\mathcal{O}_L\ni x\mapsto x^p\in\mathcal{O}_L$ is surjective modulo $d\mathcal{O}_L$.

From Section 3.2 we know that the extension admits a Kummer generator which is a 1-unit $1+a$ with $a\in\mathcal{M}_L$. Proposition 6.12 shows that (K,v) is an independent defect field. By Proposition 3.7, $\mathrm{dist}\,(1+a,K)=\frac{vp}{p-1}+H^-$ for some convex subgroup H of vK that does not contain vp. Hence for every positive $\alpha<\frac{vp}{p-1}$ in $\frac{1}{p^\infty}\mathbb{Z}vp$ there is some $b\in K$ such that $v(1+a-b)\geq\alpha$. Then b must itself be a 1-unit, say $b=1+c$. Now $v(1+a-(1+c))=v(a-c)$ and the 1-unit

$$1+a_c \;:=\; \frac{1+a}{1+c} \;=\; 1+(a-c)-\frac{c(a-c)}{1+c}$$

satisfies $va_c=v(a-c)\geq\alpha$. Since $b\in K$, $1+a_c$ is also a Kummer generator of the extension.

We note that $\frac{1}{p^\infty}\mathbb{Z}vp$ is dense in $\mathbb{Q}vp$. Thus we can choose α so close to $\frac{vp}{p-1}$ that

$$vd \;=\; \frac{vp}{p} \;\leq\; \alpha-2p\left(\frac{vp}{p-1}-\alpha\right) \;<\; \alpha \;<\; \frac{vp}{p-1} \;\leq\; vp\,. \tag{68}$$

By what we have shown above, we may from now on assume that $L|K$ admits a Kummer generator which is a 1-unit $\eta = 1 + a$ with $va \geq \alpha$.

Take an element in $\mathcal{O}_L$ and write it as $f(\eta)$ where $f(X) = \sum_{i=0}^{p-1} c_i X^i$ with $c_i \in K$. The problem is that even though $f(\eta)$ lies in $\mathcal{O}_L$, the coefficients c_i do not necessarily lie in $\mathcal{O}_K$. (This is in contrast to the case of defectless extensions, such as extensions within the absolute ramification field, where for a suitably chosen η, the value of $f(\eta)$ is equal to the minimum of the values of the summands $c_i\eta^i$.) Since $v(\eta - K)$ has no maximal element, Lemma 2.12 shows that there must be some $\gamma \in v(\eta - K)$ such that for all $b \in K$ with $v(\eta - b) \geq \gamma$, the monomials $\partial_i f(b)(\eta - b)^i$ in

$$f(\eta) \;=\; \sum_{i=0}^{p-1} \partial_i f(b)(\eta - b)^i$$

have distinct and thus non-negative values, and that for each i, the values $v\partial_i f(b)$ are constant, say equal to β_i. Consequently,

$$\beta_i + i\gamma \;\geq\; 0 \;\text{ for } 0 \leq i \leq p-1\,.$$

As all of this will remain true if we replace γ by any larger value in $v(\eta - K)$, we can assume that $\gamma > va \geq \alpha > 0$. We fix one b with $v(\eta - b) \geq \gamma$. Then also b must be a 1-unit, and we write $b = 1 + c$ with $c \in \mathcal{M}_K$. Thus, $v(a - c) = v(\eta - b) \geq \gamma$, and it follows that $vc = va \geq \alpha$. We set

$$\eta_c \;:=\; \frac{1+a}{1+c}$$

and observe that

$$a - c \;=\; \eta_c - 1 + \frac{c(a-c)}{1+c} \;\equiv\; \eta_c - 1 \quad \operatorname{mod}\, c(a-c)\mathcal{O}_L\,. \tag{69}$$

We choose some $z \in K$ with value $vz = v(a - c)$. Then

$$f(\eta) \;=\; \sum_{i=0}^{p-1} \partial_i f(b)(a-c)^i \;=\; \sum_{i=0}^{p-1} \partial_i f(b) z^i \left(\frac{\eta_c - 1}{z} + \frac{c(a-c)}{z(1+c)}\right)^i\,. \tag{70}$$

Now $\partial_i f(b) z^i \in \mathcal{O}_K$ for all i. Further, $v\frac{\eta_c - 1}{z} = 0$ and $\frac{c(a-c)}{z(1+c)} \in c\mathcal{O}_L$. Consequently,

$$f(\eta) \;\equiv\; \sum_{i=0}^{p-1} \partial_i f(b) z^i \left(\frac{\eta_c - 1}{z}\right)^i \quad \operatorname{mod}\, c\mathcal{O}_L\,. \tag{71}$$

Since $vc = va \geq \alpha > vd$, this congruence also holds modulo $d\mathcal{O}_L$. Hence in order to show that $f(\eta)$ is a p-th power in $\mathcal{O}_L$ modulo $d\mathcal{O}_L$, it suffices to show that this is true for the polynomial on the right hand side of (71). With the element C as in Lemma 2.22, this polynomial is equal to

$$\sum_{i=0}^{p-1} \partial_i f(b) C^i \left(\frac{\eta_c - 1}{C}\right)^i \;=\; \sum_{i=0}^{p-1} \partial_i f(b) C^i \vartheta^i\,, \tag{72}$$

where

$$\vartheta \;:=\; \vartheta_{\eta_c} \;=\; \frac{\eta_c - 1}{C}\,.$$

Since

$$vC \;=\; \frac{vp}{p-1} \;>\; v(\eta - b) \;=\; v(a-c) \;=\; vz\,,$$

the coefficients $\partial_i f(b)C^i$ still lie in $\mathcal{O}_K$. We note that $v\vartheta = v(\eta_c - 1) - vC = v(a-c) - vC < 0$. Further,

$$0 \;>\; v\vartheta \;\geq\; \gamma - vC \;>\; \alpha - vC\,. \tag{73}$$

Since $vC > v(a-c) = vz$, the coefficients $\partial_i f(b)C^i$ still lie in $\mathcal{O}_K$.

From (21) and (22) we know that ϑ satisfies the equation

$$\vartheta \;=\; \vartheta^p - \frac{\eta_c^p - 1}{C^p} + g(\vartheta)\,,$$

where

$$g(\vartheta) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p}\vartheta^i\,.$$

We compute for $2 \leq i \leq p-1$:

$$\begin{aligned} v\binom{p}{i}C^{i-p}\vartheta^i &= vp + (i-p)vC + iv\vartheta \;=\; (i-1)vC + iv\vartheta \\ &\geq vC + pv\vartheta \;>\; \alpha + pv\vartheta \;=\; \alpha + 2pv\vartheta - pv\vartheta \\ &> \alpha - 2p(vC - \alpha) - pv\vartheta \;\geq\; vd - pv\vartheta\,, \end{aligned}$$

where the last inequality holds by (68). Hence $g(\vartheta) \in d\vartheta^{-p}\mathcal{O}_L$ and

$$\vartheta \;\equiv\; \vartheta^p - \frac{\eta_c^p - 1}{C^p} \mod d\vartheta^{-p}\mathcal{O}_L\,. \tag{74}$$

As $v\vartheta < 0$, we have that $v\vartheta > v\vartheta^p$, so that

$$v\frac{\eta_c^p - 1}{C^p} \;=\; v\vartheta^p\,.$$

Since (K,v) is a gdr field, using part 2) of Lemma 6.3 we can find elements $t, t_i \in \mathcal{O}_K$ such that

$$t^p \;\equiv\; \frac{\eta_c^p - 1}{C^p} \mod pt^p\mathcal{O}_K \;=\; p\vartheta^p\mathcal{O}_L \tag{75}$$

and for $0 \leq i \leq p-1$,

$$t_i^p \;\equiv\; \partial_i f(b)C^i \mod p\mathcal{O}_K \;\subseteq\; d\mathcal{O}_L\,. \tag{76}$$

We have that

$$\vartheta^p - t^p \;\equiv\; (\vartheta - t)^p \mod p\vartheta^p\mathcal{O}_L$$

and consequently,

$$\vartheta^p - \frac{\eta_c^p - 1}{C^p} \;\equiv\; (\vartheta - t)^p \mod p\vartheta^p\mathcal{O}_L\,. \tag{77}$$

From (68) and (73) we derive that

$$vd\vartheta^{-2p} \;=\; vd - 2pv\vartheta \;<\; vd + 2p(vC - \alpha) \;\leq\; \alpha \;<\; vp\,,$$

so that $p\vartheta^p\mathcal{O}_L \subseteq d\vartheta^{-p}\mathcal{O}_L$. Hence by (74) and (77),

$$\vartheta \;\equiv\; (\vartheta - t)^p \mod d\vartheta^{-p}\mathcal{O}_L\,.$$

We write $(\vartheta - t)^p = \vartheta + d\vartheta^{-p}s$ with $s \in \mathcal{O}_L$. Then for $0 \leq i \leq p-1$,

$$(\vartheta - t)^{ip} \;=\; \vartheta^i \;+\; \sum_{j=1}^{i} \binom{i}{j} \vartheta^{i-j}(d\vartheta^{-p}s)^j\,.$$

Since $v\vartheta < 0 < vd\vartheta^{-p}s$, the summand of least value in the sum on the right hand side is the one for $j = 1$. This shows that

$$\vartheta^i \equiv (\vartheta - t)^{ip} \mod d\vartheta^{-p+i-1}\mathcal{O}_L .$$

Here, each $d\vartheta^{-p+i-1}\mathcal{O}_L$ can be replaced by the larger ideal $d\mathcal{O}_L$. Combining this with (76), we obtain:

$$\sum_{i=0}^{p-1} \partial_i f(b) C^i \vartheta^i \equiv \sum_{i=0}^{p-1} t_i^p (\vartheta - t)^{ip} \mod d\mathcal{O}_L . \tag{78}$$

We observe that the corresponding summands in the sums on the right hand sides of (70), (71), (72) and (78) all have the same non-negative value. Consequently,

$$\sum_{i=0}^{p-1} t_i^p (\vartheta - t)^p \equiv \left(\sum_{i=0}^{p-1} t_i (\vartheta - t) \right)^p \mod p\mathcal{O}_L .$$

Together with (78), this leads to

$$f(\eta) \equiv \sum_{i=0}^{p-1} \partial_i f(b) C^i \left(\frac{\eta_c - 1}{C} \right)^i \equiv \left(\sum_{i=0}^{p-1} t_i (\vartheta - t) \right)^p \mod d\mathcal{O}_L ,$$

which completes our proof. □

Proposition 6.16. *Assume that (K,v) is a gdr field of mixed characteristic with algebraically closed residue field. Take a defectless unibranched Galois extension $(L|K,v)$ of degree $p = \operatorname{char} Kv$. Then also (L,v) is a gdr field.*

Proof. Since $(L|K,v)$ is unibranched and defectless, equation (1) shows that $p = [L : K] = (vL : vK)[Lv : Kv]$. However, as Kv is algebraically closed, $[Lv : Kv] = 1$. Hence $(vL : vK) = p$. By part 1) of Lemma 6.3, $v_p \circ \overline{v}(Kv_0) = (vK)_{vp}$ is p-divisible. It follows that $(v_0 L : v_0 K) = p$ and therefore, $Lv_0 = Kv_0$. Applying Proposition 1.3 to (K,v), we find that $(Kv_0, v_p) = (Lv_0, v_p)$ is a gdr field, and applying the proposition again, we conclude that (L,v) is a gdr field. □

Proof of Theorem 1.5. For the case of deeply ramified fields of positive characteristic we have given the proof already in Proposition 6.7, so let us assume that (K,v) is a gdr field of mixed characteristic and $(L|K,v)$ an algebraic extension. By Theorem 1.6, (K^r,v) is a deeply ramified field. Hence $K^r v$ is perfect by Lemma 6.2, but as it is also separable-algebraically closed, it must be algebraically closed.

We let L' be a maximal extension of K^r inside of $L.K^r$ that is again a gdr field; since the union over an ascending chain of gdr fields is again a gdr field, L' exists by Zorn's Lemma. Since K^r contains all p-th roots of unity, so does L', and since $K^r v$ is algebraically closed, so is $L'v$.

Suppose that $L' \neq L.K^r$. Since $\tilde{K}|K^r$ is a p-extension, the same holds for $\tilde{K}|L'$. Consequently, $L.K^r|L'$ contains a Galois subextension $(L''|L',v)$ of degree p. If this is a defect extension, then it follows from Proposition 6.15 that (L'',v) is a gdr field. If the extension is defectless, then it follows from Proposition 6.16 that (L'',v) is a gdr field. In both cases we have obtained a contradiction to the maximality of L'. This proves that $(L.K^r,v)$ is a gdr field. Since $L.K^r = L^r$ by [7, (20.15) b)], we now obtain from Theorem 1.6 that (L,v) is a gdr field.

It remains to deal with deeply ramified fields and with semitame fields. For them the proof follows immediately from what we have already shown, since deeply ramified fields are just the gdr fields that satisfy (DRvg), and semitame fields are just the gdr fields with p-divisible value groups. All of these properties are preserved under algebraic extensions. □

6.7. **Proof of Theorem 1.8.**

The equal characteristic case has already been settled in Proposition 6.7. Thus we assume now that $(L|K,v)$ is a finite extension of valued fields of mixed characteristic and that (L,v) is a gdr field. We wish to show that (K,v) is a gdr field. In order to derive a contradiction, we suppose that this is not the case.

We take an extension of v to $\tilde{K}=\tilde{L}$. This determines the absolute ramification field (K^r,v) of (K,v). By [7, (20.15) b)], $(L.K^r,v)$ is the absolute ramification field (L^r,v) of (L,v). By Theorem 1.6, (L^r,v) is a gdr field. From Lemma 2.15 we know that $L.K^r|K^r$ is a finite tower of Galois extensions of degree p. By our assumption and Theorem 1.6, (K^r,v) is not a gdr field. Then there is a maximal field (N,v) in the tower that is not a gdr field, and a Galois extension (N',v) of (N,v) of degree p that is a gdr field.

By part 1) of Lemma 6.3, vN' contains $\frac{1}{p^\infty}\mathbb{Z}vp$. Since $(N'|N,v)$ is a finite extension, also vN contains $\frac{1}{p^\infty}\mathbb{Z}vp$. By part 1) of Lemma 6.2, $\mathrm{crf}\,(N',v)$ is perfect. As $\mathrm{crf}\,(N',v)|\mathrm{crf}\,(N,v)$ is a finite extension, also $\mathrm{crf}\,(N,v)=Nv_0v_p$ is perfect. Hence the same holds for Nv.

Since (N,v) is not a gdr field, Proposition 6.4 shows that for every $d\in N$ with $0<vd\in\frac{1}{p^\infty}\mathbb{Z}vp$ and $0<vd\leq vp$ there must be some $b_d\in\mathcal{O}_N^\times$ such that there is no $c\in N$ with $b_d-c^p\in d\mathcal{O}_N$. We choose $\eta_d\in\tilde{N}$ such that $\eta_d^p=b_d$. Then there is no $c\in N$ such that $v(\eta_d-c)\geq\frac{vd}{p}$ since this would imply $v(b_d-c^p)=v(\eta_d^p-c^p)\geq vd$ as $\eta_d^p-c^p\equiv(\eta_d-c)^p \mod p\mathcal{O}_N$. Lemma 6.5 shows that $v(\eta_d-N)$ has no maximal element. Hence by Lemma 2.9, $(N(\eta_d)|N,v)$ is a Galois defect extension, and by Proposition 3.7, it has dependent defect.

We distinguish two cases. First, let us assume that $(N'|N,v)$ is not a defect extension. Then by Lemma 2.4, $(N'(\eta_d)|N',v)$ is a Galois defect extension with $\mathrm{dist}\,(\eta_d,N')=\mathrm{dist}\,(\eta_d,N)$, which shows that also this extension has dependent defect. Therefore, (N',v) is not an independent defect field and thus by Proposition 6.12, it is not a gdr field. This contradicts our assumption.

Now let us assume that $(N'|N,v)$ is a defect extension. Since K^r contains all p-th roots of unity, the same holds for N. Therefore, the extension $N'|N$ admits a Kummer generator η, and we can assume that it is a 1-unit. Since Nv_0v_p is perfect, it follows that there is some $c\in N$ such that $v_0\circ v_p(\eta-c)>0$, and thus we can choose some $d\in N$ as above such that $\frac{vd}{p}\in v(\eta-N)$. It follows that

$$v(\eta_d-N)\subsetneq v(\eta-N)\,. \tag{79}$$

This means that $\mathrm{dist}\,(\eta_d,N)<\mathrm{dist}\,(\eta,N)$. Note that $v(\sigma\eta_d-\eta_d)=v(\sigma\eta-\eta)$ as both η_d and η are Kummer generators of value 0 of the extensions $N(\eta_d)|N$ and $N'|N$, respectively.

If $v(\eta_d-N')=v(\eta_d-N)$, then again by Lemma 2.4, $(N'(\eta_d)|N',v)$ is a Galois defect extension with $\mathrm{dist}\,(\eta_d,N')=\mathrm{dist}\,(\eta_d,N)$, yielding a contradiction as before.

Suppose that $\eta_d \in N'$. Then inequality (79) leads to

$$-v(\eta_d - N) + v(\sigma\eta_d - \eta_d) \;\neq\; -v(\eta - N) + v(\sigma\eta - \eta)\,, \tag{80}$$

which in view of equation (32) together with Theorem 3.5 is a contradiction. Hence we can assume that $\eta_d \notin N'$.

Now our proof will be complete once we show that $v(\eta_d - N') \neq v(\eta_d - N)$ is impossible. In order to derive a contradiction, suppose that the two sets are not equal. Then there is some $\tilde{\eta} \in N'$ such that $v(\eta_d - \tilde{\eta}) \notin v(\eta_d - N)$. Since $v(\eta_d - N)$ is an initial segment of $vN' = vN$, it follows that $v(\eta_d - \tilde{\eta}) > v(\eta_d - N)$. By part 1) of Lemma 2.2,

$$v(\eta_d - N) \;=\; v(\tilde{\eta} - N)$$

holds for all $\tilde{\eta} \in N'$ with $v(\eta_d - \tilde{\eta}) > v(\eta_d - N)$. As $\tilde{\eta} \in N' \setminus N$ and $[N' : N] = p$, also $\tilde{\eta}$ is a generator of $N'|N$.

For $\sigma \in \mathrm{Gal}\,(\tilde{K}|K)$ with $\sigma\tilde{\eta} \neq \tilde{\eta}$, we compute:

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;\geq\; \min\{v(\sigma\tilde{\eta} - \sigma\eta_d), v(\sigma\eta_d - \eta_d), v(\eta_d - \tilde{\eta})\}\,. \tag{81}$$

As an algebraic extension of (K^r, v), also (N, v) is henselian. Hence we have that $v(\sigma\tilde{\eta} - \sigma\eta_d) = v\sigma(\eta_d - \tilde{\eta}) = v(\eta_d - \tilde{\eta})$. Suppose that

$$v(\eta_d - \tilde{\eta}) \;\geq\; v(\sigma\eta_d - \eta_d)\,.$$

As $(vN')_{vp}$ is p-divisible and $N'v$ is perfect, $v(\eta_d - N')$ does not have a maximum inside of $(vN')_{vp}$, so we may assume that $v(\eta_d - \tilde{\eta}) > v(\sigma\eta_d - \eta_d)$. Thus in all cases, we may assume that $v(\eta_d - \tilde{\eta}) \neq v(\sigma\eta_d - \eta_d)$. Hence by (81),

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;=\; \min\{v(\sigma\eta_d - \eta_d), v(\eta_d - \tilde{\eta})\}\,. \tag{82}$$

If $v(\sigma\tilde{\eta} - \tilde{\eta}) = v(\sigma\eta_d - \eta_d)$, then $v(\sigma\tilde{\eta} - \tilde{\eta}) = v(\sigma\eta - \eta)$ and we obtain a contradiction exactly as in (80) with η_d replaced by $\tilde{\eta}$. Hence we now assume that

$$v(\sigma\tilde{\eta} - \tilde{\eta}) \;=\; v(\eta_d - \tilde{\eta}) \;<\; v(\sigma\eta_d - \eta_d)\,.$$

Again because $v(\eta_d - N')$ does not have a maximum inside of $(vN')_{vp}$, we can choose $\tilde{\eta}_1 \in N'$ such that

$$v(\eta_d - \tilde{\eta}_1) \;>\; v(\eta_d - \tilde{\eta}) \;>\; v(\eta_d - N)\,.$$

Like $\tilde{\eta}$, also $\tilde{\eta}_1$ is a generator of $N'|N$. With the same computations as before, we arrive at (82) with $\tilde{\eta}$ replaced by $\tilde{\eta}_1$. We must have that $v(\eta_d - \tilde{\eta}_1) < v(\sigma\eta_d - \eta_d)$ since otherwise, we would obtain a contradiction as before. Therefore,

$$v(\sigma\tilde{\eta}_1 - \tilde{\eta}_1) \;=\; v(\eta_d - \tilde{\eta}_1)$$

and

$$v(\tilde{\eta}_1 - N) \;=\; v(\eta_d - N) \;=\; v(\tilde{\eta} - N)\,.$$

Combining everything, we find:

$$\begin{aligned} -v(\tilde{\eta}_1 - N) + v(\sigma\tilde{\eta}_1 - \tilde{\eta}_1) \;&=\; -v(\eta_d - N) + v(\eta_d - \tilde{\eta}_1) \\ &\neq\; -v(\eta_d - N) + v(\eta_d - \tilde{\eta}) \\ &=\; -v(\tilde{\eta} - N) + v(\sigma\tilde{\eta} - \tilde{\eta})\,, \end{aligned}$$

which again by equation (32) together with Theorem 3.5 is a contradiction. □

6.8. **Proof of Theorems 1.10 and 1.12, and of Proposition 1.13.**

Proof of Theorem 1.10: As before we set $K' = K(\zeta_p)$. Then for any extension of v to $\tilde{K}$, $(K(\zeta_p), v)$ is contained in the respective absolute ramification field.

1): Assume that (K, v) is a gdr field. The assertions on $(vK)_{vp}$ and $\mathrm{crf}\,(K, v)$ have been proven in Lemmas 6.2 and 6.3. By part 1) of Corollary 1.7, also (K', v) is a gdr field. It follows from Proposition 6.12 that (K', v) is an independent defect field. Thus by definition, (K, v) is an independent defect field. The converse is the content of part 1) of Proposition 6.13.

2): We note that every unibranched Galois extension of prime degree different from the residue characteristic is automatically tame.

First, we assume that (K, v) is a semitame field. Then by part 1) of Corollary 1.7, also (K', v) is a semitame field, so vK' is p-divisible. By Lemma 6.2, $K'v$ is perfect. Therefore, equation (1), with K' in place of K, shows that every unibranched Galois extension $(L|K', v)$ of degree p either has defect p, or satisfies $[Lv : Kv] = p$ with $Lv|K'v$ a separable extension. In the latter case, the extension has no defect and is tame. Otherwise, it is a defect extension of degree p. Then, as (K', v) is a gdr field by Theorem 1.2, part 1) of our theorem shows that it must be an independent defect extension.

For the converse, we first show that our assumptions yield that vK', and hence also $(vK')_{vp}$, is p-divisible, and that $\mathrm{crf}\,(K', v)$ is perfect. Indeed, if $\alpha \in vK'$ is not divisible by p and we take $a \in K'$ with $va = \alpha$, then taking a p-th root of a induces a Galois extension that is neither tame nor immediate. The same holds if $a \in K'$ is such that $va = 0$ and av does not have a p-th root in $K'v$, hence $K'v$ is perfect.

Suppose that $\mathrm{crf}\,(K', v)$ is not perfect. Pick $a \in K'$ such that $av_0 \circ v_p$ has no p-th root and choose some $b \in \tilde{K}$ such that $b^p = a$. Since vK' is p-divisible, the same holds for $\overline{v}(K'v_0 \circ v_p)$. In addition, $(K'v_0 \circ v_p)\overline{v} = K'v$ is perfect. It follows that the extension $(K'v_0 \circ v_p(bv_0 \circ v_p)|K'v_0 \circ v_p, \overline{v})$ is immediate of degree $p = [K'(b) : K']$, which implies that the same holds for $(K'(b)|K', v)$. Further, the former extension is unibranched as it is purely inseparable. Since also the extension $(K'(b)|K, v_0 \circ v_p)$ is unibranched as its inertia degree is p, also $(K'(b)|K', v)$ is unibranched. By assumption, its defect must be independent since defect extensions of degree p are not tame. But then there must be $c \in K'$ such that $v(b - c) > \frac{vp}{p}$, whence $bv_0 \circ v_p \in K'v_0 \circ v_p$, contradiction.

Our assumption yields that every Galois defect extension of (K', v) of degree p is independent. Hence we obtain from part 1) that (DRvr) holds, so (K', v) is a semitame field. By part 1) of Corollary 1.7, also (K, v) is a semitame field. $\square$

Proof of Theorem 1.12. Every algebraically maximal field is henselian, and if its residue characteristic is 0, then it is also defectless. Therefore, we may assume that (K, v) is an algebraically maximal gdr field of positive residue characteristic p. If $\mathrm{char}\, K = p$, then by part 3) of Theorem 1.2, (K, v) is dense in its perfect hull. As it is algebraically maximal, this extension must be trivial, i.e., K is perfect.

Take an absolute ramification field (K^r, v) of (K, v) and a finite tower $K^r = L_0 \subset L_1 \subset \ldots \subset L_n$ of extensions of degree p over K^r. By Theorem 1.5, every (L_i, v) is a gdr field. Hence Theorem 1.10 yields that among the extensions $(L_i|L_{i-1}, v)$, $1 \leq i \leq n$, every defect extension is independent. It is also separable because as K is perfect, so are all L_i. Now Proposition 4.7 shows that (K, v) is henselian and defectless. $\square$

Proof of Proposition 1.13. Part 1) follows from Proposition 3.8. Part 2) is the content of part 4) of Theorem 3.10. □

6.9. **Proof of Proposition 1.1.**

It is well known that first order properties of the value group vK of a valued field (K, v) can be encoded in (K, v) in the language of valued fields. The axiomatization for (DRvp) and (DRst) is straightforward. Further, (DRvg) holds in an ordered abelian group $(G, <)$ if and only if for each positive $\alpha \in G$ there is $\beta \in G$ such that $2\beta \leq \alpha \leq 3\beta$.

If (K, v) is of mixed characteristic, then (DRvr) is equivalent to the surjectivity of (60), and this in turn holds if and only if for each $a \in K$ with $va \geq 0$ there is $b \in K$ such that $v(a - b^p) \geq vp$. Hence the classes of semitame, deeply ramified and gdr fields of mixed characteristic are first order axiomatizable.

If (K, v) is of equal positive characteristic, then part 3) of Theorem 1.2 shows that semitame, deeply ramified and gdr fields form the same class. This class can be axiomatized by saying that (K^p, v) is dense in (K, v), or in other words, for every $\alpha \in vK$ and every $a \in K$ there is $b \in K$ such that $v(a - b^p) \geq \alpha$.

In the case of equal characteristic 0, (DRvp), (DRvr) and (DRst) are trivial and all valued fields are semitame and gdr fields, while the class of deeply ramified fields consists of those which satisfy (DRvg). □

7. Two constructions

In this section we give constructions for independent and dependent defect extensions in mixed characteristic. First, we show how to construct a semitame field with an independent defect extension of degree p.

Theorem 7.1. *Consider the field $\mathbb{Q}_p$ of p-adic numbers together with the p-adic valuation v_p. Set $a_0 := p$ and by induction, choose $a_i \in \widetilde{\mathbb{Q}_p}$ such that $a_i^p = a_{i-1}$ for $i \in \mathbb{N}$. Then $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ together with the unique extension of v is a semitame field and hence a deeply ramified field.*

Further, let ζ_p be a primitive p-th root of unity and take $\vartheta \in \widetilde{\mathbb{Q}_p}$ such that

$$\vartheta^p - \vartheta = \frac{1}{p}.$$

Then $(K(\zeta_p, \vartheta)|K(\zeta_p), v)$ is an independent defect extension of degree p.

Proof. By choice of the a_i, $\frac{vp}{p^i} = va_i \in v\mathbb{Q}_p(a_i)$. Therefore,

$$p^i \leq (v\mathbb{Q}_p(a_i) : v\mathbb{Q}_p) \leq (v\mathbb{Q}_p(a_i) : v\mathbb{Q}_p)[\mathbb{Q}_p(a_i)v : \mathbb{Q}_p v] \leq [\mathbb{Q}_p(a_i) : \mathbb{Q}_p] \leq p^i.$$

Hence equality holds everywhere, and $[\mathbb{Q}_p(a_i)v : \mathbb{Q}_p v] = 1$. We thus obtain that $v\mathbb{Q}_p(a_i) = \frac{1}{p^i} v\mathbb{Q}_p$ and $\mathbb{Q}_p(a_i)v = \mathbb{Q}_p v$. Consequently,

$$vK = \bigcup_{i \in \mathbb{N}} v\mathbb{Q}_p(a_i) = \frac{1}{p^\infty}\mathbb{Z} \quad \text{and} \quad Kv = \mathbb{Q}_p v.$$

This shows that vK is p-divisible and that its only proper convex subgroup is $H = \{0\}$. In order to show that (K, v) is a semitame field it remains to show that it satisfies (DRvr).

Take $b \in \mathcal{O}_K$. Then $b \in \mathbb{Q}_p(a_i)$ for some $i \in \mathbb{N}$ and we can write:

$$b \;\equiv\; \sum_{j=0}^{n} c_j a_i^j \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)}$$

with $n < [\mathbb{Q}_p(a_i) : \mathbb{Q}_p] = p^i$ and $c_j \in \{0, \ldots, p-1\}$. Since $c_j^p \equiv c_j \mod p\mathcal{O}_{\mathbb{Q}_p}$ and $a_{i+1}^p = a_i$, we can compute:

$$\left(\sum_{j=0}^{n} c_j a_{i+1}^j\right)^p \;\equiv\; \sum_{j=0}^{n} c_j^p (a_{i+1}^p)^j \;\equiv\; \sum_{j=0}^{n} c_j a_i^j \;=\; b \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)} \,.$$

In view of part 2) of Lemma 6.1, this proves that (K, v) satisfies (DRvr) and is therefore a semitame field.

Now we take ϑ as in the assertion of our theorem. Our first aim is to show that the extension $(K(\vartheta)|K, v)$ is nontrivial and immediate. For each $i \in \mathbb{N}$, we set

$$b_i \;=\; \sum_{j=1}^{i} \frac{1}{a_j} \;\in\; K(a_i)$$

and compute, using part 2) of Lemma 2.18:

$$\begin{aligned}(\vartheta - b_i)^p - (\vartheta - b_i) \;&\equiv\; \vartheta^p - \sum_{j=1}^{i} \frac{1}{a_j^p} - \vartheta + \sum_{j=1}^{i} \frac{1}{a_j} \\ &=\; \frac{1}{p} - \sum_{j=0}^{i-1} \frac{1}{a_j} + \sum_{j=1}^{i} \frac{1}{a_j} \;=\; \frac{1}{a_i} \mod \mathcal{O}_{\mathbb{Q}_p(a_i)} \,.\end{aligned}$$

It follows that $v(\vartheta - b_i) < 0$ and

$$-\frac{vp}{p^i} \;=\; v\frac{1}{a_i} \;=\; \min\{pv(\vartheta - b_i), v(\vartheta - b_i)\} \;=\; pv(\vartheta - b_i) \,,$$

whence

$$v(\vartheta - b_i) \;=\; -\frac{vp}{p^{i+1}} \,. \tag{83}$$

We have that

$$\begin{aligned} p \;&\le\; (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) \;\le\; (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i))[\mathbb{Q}_p(a_i, \vartheta)v : \mathbb{Q}_p(a_i)v] \\ &\le\; [K(a_i, \vartheta) : K] \;\le\; p \,.\end{aligned}$$

Thus equality holds everywhere and we have that $(v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) = p$ as well as $\mathbb{Q}_p(a_i, \vartheta)v = \mathbb{Q}_p(a_i)v = \mathbb{Q}_p v$. The former shows that $v\mathbb{Q}_p(a_i, \vartheta) = \frac{1}{p^{i+1}} v\mathbb{Q}_p$, which implies that for all $i \in \mathbb{N}$, $\vartheta \notin \mathbb{Q}_p(a_i)$. Hence $\vartheta \notin K$, and we have:

$$vK(\vartheta) \;=\; \bigcup_{i \in \mathbb{N}} v\mathbb{Q}_p(a_i, \vartheta) \;=\; \frac{1}{p^\infty}\mathbb{Z} \;=\; Kv \quad \text{and} \quad K(\vartheta)v \;=\; \mathbb{Q}_p v \;=\; vK \,.$$

This shows that $(K(\vartheta)|K, v)$ is nontrivial and immediate, as asserted. As an algebraic extension of $\mathbb{Q}_p$, also (K, v) is henselian, so the extension is unibranched. Therefore, it is a defect extension of degree p. By Proposition 2.14, the same holds for the extension $(K(\zeta_p, \vartheta)|K(\zeta_p), v)$. As we have already proven that (K, v) is a semitame field, it follows from part 2) of Theorem 1.10 that this extension has independent defect. □

What we have just presented is the mixed characteristic analogue of the following example given in, e.g., [14, Example 12]. Take K to be the perfect hull of $\mathbb{F}_p((t))$, that is, $K = \mathbb{F}_p((t))(t^{1/p^i} \mid i \in \mathbb{N})$. Take v to be the t-adic valuation on $\mathbb{F}_p((t))$; since it is henselian, there is a unique extension to K and (K,v) is again henselian. The Artin-Schreier extension $(K(\vartheta)|K,v)$ generated by a root ϑ of the polynomial $X^p - X - \frac{1}{t}$ is nontrivial and immediate. As K is perfect, it does not admit any dependent Artin-Schreier defect extension, so the extension $(K(\vartheta)|K,v)$ has independent defect. In fact, (K,v) is a semitame field.

We turn to the construction of a dependent defect extension of degree p. The following is an analogue of Example 3.22 of [17].

Theorem 7.2. *Set $a_0 := -\frac{1}{p} \in \mathbb{Q}_p$ and by induction, choose $a_i \in \widetilde{\mathbb{Q}_p}$ such that $a_i^p - a_i = -a_{i-1}$ for $i \in \mathbb{N}$. Consider $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ together with the unique extension of v. Then vK is p-divisible and $Kv = \mathbb{F}_p$. Further, take $\eta \in \widetilde{\mathbb{Q}_p}$ such that*

$$\eta^p \;=\; \frac{1}{p}\,.$$

Then $(K(\zeta_p,\eta)|K(\zeta_p),v)$ is a dependent defect extension of degree p. Consequently, neither $(K(\zeta_p),v)$ nor (K,v) satisfy (DRvr).

Proof. By induction on i, we again obtain that $va_i = \frac{1}{p^i}vp$. As in Theorem 7.1 we deduce that $v\mathbb{Q}_p(a_i) = \frac{1}{p^i}v\mathbb{Q}_p$ and $\mathbb{Q}_p(a_i)v = \mathbb{Q}_p v$, and for $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ we obtain that $vK = \frac{1}{p^\infty}v\mathbb{Q}_p$ and $Kv = \mathbb{Q}_p v$. In particular, the only proper convex subgroup of vK is $H = \{0\}$.

We set

$$b_i \;=\; \sum_{j=1}^{i} a_j \;\in\; K(a_i)$$

and compute, using part 2) of Lemma 2.18:

$$\begin{aligned}(\eta - b_i)^p \;&\equiv\; \eta^p - \sum_{j=1}^{i} a_j^p \;=\; \frac{1}{p} + \sum_{j=1}^{i}(a_{j-1} - a_j)\\ &=\; \frac{1}{p} + \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i} a_j \;=\; -a_i \mod \mathcal{O}_{\mathbb{Q}_p(a_i)}\,.\end{aligned}$$

It follows that

$$\frac{vp}{p^i} \;=\; va_i \;=\; pv(\eta - b_i)\,,$$

whence

$$v(\eta - b_i) \;=\; \frac{vp}{p^{i+1}}\,. \tag{84}$$

As in the proof of Theorem 7.1 we deduce that $(K(\eta)|K,v)$ is nontrivial and immediate. It remains to show that its defect is dependent.

From (84) we see that $\operatorname{dist}(\eta,K) \geq 0^-$. Suppose that $\operatorname{dist}(\eta,K) > 0^-$. Then there is an element $c \in K$ such that $v(\eta - c) > v(\eta - b_i)$ for every $i \in \mathbb{N}$. Hence,

$$v(c - b_i) \;=\; \min\{v(\eta - c), v(\eta - b_i)\} \;=\; v(\eta - b_i) \;=\; \frac{vp}{p^{i+1}}\,. \tag{85}$$

Since $c \in K$, we have that $c \in \mathbb{Q}_p(a_i)$ for some $i \in \mathbb{N}$. Then we obtain that $c - b_i \in \mathbb{Q}_p(a_i)$, but equation (85) shows that $v(c - b_i) = \frac{vp}{p^{i+1}} \notin v\mathbb{Q}_p(a_i)$, a contradiction. Therefore, $\operatorname{dist}(\eta, K) = 0^-$.

Since $(K(\eta)|K, v)$ is immediate, there is $d \in K$ such that $d\eta$ is a 1-unit. We have that $vd = -v\eta = \frac{vp}{p}$ and

$$\operatorname{dist}(d\eta, K) \;=\; vd + 0^- \;=\; \frac{vp}{p} + 0^- \;<\; \frac{vp}{p-1} \;+\; 0^- \,.$$

As $K(\eta) = K(d\eta)$ and $\operatorname{dist}(d\eta, K(\zeta_p)) = \operatorname{dist}(d\eta, K)$ by Proposition 2.14, this shows that $(K(\eta)|K, v)$ is a dependent defect extension of degree p. Hence $(K(\zeta_p), v)$ is not an independent defect field and by part 2) of Theorem 1.10, neither $(K(\zeta_p), v)$ nor (K, v) is semitame. However, as vK and $vK(\zeta_p)$ are p-divisible and Kv and $K(\zeta_p)v$ are perfect, it must be axiom (DRvr) that fails in both fields. □

This second example shows that in order to obtain a semitame field it is not sufficient to just make the value group p-divisible and the residue field perfect, not even if one starts from a discretely valued field.

Since $\mathbb{Q}_p$ is a defectless field and a fortiori an independent defect field, but $(K(\zeta_p), v)$ admits a Kummer extension with dependent defect, this example also shows:

Corollary 7.3. *The property of being an independent defect field is not necessarily preserved under infinite algebraic extensions.*

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