COUNTING THE NUMBER OF DISTINCT DISTANCES OF ELEMENTS IN VALUED FIELD EXTENSIONS

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Abstract. The defect of valued field extensions is a major obstacle in open problems in resolution of singularities and in the model theory of valued fields, whenever positive characteristic is involved. We continue the detailed study of defect extensions through the tool of distances, which measure how well an element in an immediate extension can be approximated by elements from the base field. We show that in several situations the number of essentially distinct distances in fixed extensions, or even just over a fixed base field, is finite, and we compute upper bounds. We apply this to the special case of valued functions fields over perfect base fields. In particular, this provides important information used in forthcoming research on the ramification theory of two-dimensional valued function fields.

1. Introduction

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. The value group of (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av.

The **defect**, also known as **ramification deficiency**, of finite extensions $(L|K,v)$ of valued fields is a phenomenon that only appears when the residue field Kv has positive characteristic. It is a main obstacle to the solution of deep open problems in positive characteristic, such as:

• local uniformization (the local form of resolution of singularities), which is not known for arbitrary dimension in positive characteristic,

• the model theory of valued fields, in particular the open question whether Laurent series fields over finite fields have a decidable theory.

Both problems are linked through the structure theory of valued function fields, in which it is essential to tame the defect, as well as wild ramification, cf. [9, 12, 14, 15, 16]. While implicitly known through the work of algebraic geometers and model theorists since the 1950s, the connection of the defect with the problem of local uniformization and the model theory of valued fields with positive residue characteristic has been pointed out in detail in the cited works of the second author. Defects also appear in crucial examples, as in the paper [4].

Using tools of ramification theory, the study of extensions of valued fields of residue characteristic $p > 0$ with nontrivial defect can be reduced to the study of

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normal extensions of degree p with nontrivial defect. Such extensions are immediate. An arbitrary extension $(L|K,v)$ of valued fields is **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto. As a consequence, for every $a \in L \setminus K$ the set

$$v(a-K) \;:=\; \{v(a-c) \mid c \in K\}$$

does not have a maximal element; this follows from [8, Theorem 1]. If a is an element of any valued field extension of (K,v) such that $v(a-K)$ has no maximal element, then this set is an initial segment of vK. We associate with it a cut in the divisible hull $\widetilde{vK}$ of vK by taking as the lower cut set the smallest initial segment in $\widetilde{vK}$ which contains $v(a-K)$. This cut is called the **distance of a over K** and denoted by $\operatorname{dist}(a,K)$. For more details, see Section 2.2.

Distances can be used to classify defect extensions. If an extension $L|K$ of degree p is Galois and the field K is itself of characteristic p, then $L|K$ is an **Artin–Schreier extension**, that is, L is generated over K by an element ϑ such that

$$\vartheta^p - \vartheta \;\in\; K\,; \tag{1}$$

we call ϑ an **Artin-Schreier generator** of the extension. If such an extension of a valued field (K,v) has nontrivial defect, then the extension of the valuation v from K to L is unique and $(L|K,v)$ is immediate (see Lemma 2 below); we call it an **Artin–Schreier defect extension**. A classification of Artin–Schreier defect extensions (dependent vs. independent defect) is introduced in [11]. It is then shown that the classification can be read off from the distance $\operatorname{dist}(\vartheta,K)$ of the Artin-Schreier generator.

The classification is important because work by M. Temkin (see e.g. [21]) and by the second author indicates that dependent defect is more harmful to the above cited problems than independent defect. In the paper [4], S. D. Cutkosky and O. Piltant give an example of an extension of two-dimensional valued function fields consisting of a tower of two Artin-Schreier defect extensions where strong monomialization fails. Work of Cutkosky, S. ElHitti and L. Ghezzi shows that both of them have dependent defect (see e.g. [5]); this again lends credibility to the hypothesis that dependent defect is the more harmful one.

In a collaboration of the second author with O. Piltant ([18]) the investigation of two-dimensional valued function fields in the spirit of [4] is continued. Higher ramification groups are employed to gain insights in the nature of the appearing defects. As the value groups are usually not discrete, the classical ramification numbers are replaced by cuts in the value group, which in turn are related to distances. In order to obtain a meaningful characterization of the role the defect plays in a given valued function field, we have to abstract from inessential differences between distances. To this end, equivalence classes of distances have to be introduced as follows.

If $c \in K$, then $v(ca-K) = \{vc + v(a-c) \mid c \in K\} =: vc + v(a-K)$, which means that the cut $\operatorname{dist}(a,K)$ is just shifted by adding vc to all elements of the lower cut set; we then write

$$\operatorname{dist}(ca,K) \;=\; vc + \operatorname{dist}(a,K)\,. \tag{2}$$

We do not regard $\operatorname{dist}(a,K)$ and $\operatorname{dist}(ca,K)$ as essentially distinct, so we rather work with classes of distances that are equivalent modulo vK.

Defects are abundant in valued function fields of positive characteristic. It is even possible to construct two-dimensional valued function fields that allow infinite towers of Artin-Schreier defect extensions (see [1, 10]). Thus the question arises how many essentially distinct distances of generators of Artin-Schreier defect extensions exist over a fixed valued field (K, v) (and in particular, whether this number could be finite at all). In Section 4 we give an answer under certain finiteness assumptions, see Theorem 23. These conditions hold for instance in the following situation:

Theorem 1. *Take a valued function field $(K|K_0, v)$ over a perfect trivially valued base field (K_0, v). Then the number of distinct distances of elements in Artin-Schreier defect extensions modulo vK is bounded by $2 \cdot \operatorname{trdeg} K|K_0$.*

This result is particulary important for the above described investigation of two-dimensional valued function fields in [18]. But the scope of our results in the present paper is not restricted to function fields, so they add significant insight to the general theory of the defect. Also, we do not restrict our interest to the distances appearing in Artin-Schreier defect extensions. More generally, we would like to count all the essentially distinct distances over a valued field (K, v) of all elements a in the algebraic closure $\tilde{K}$ of K for which $v(a - K)$ has no maximal element. But it seems unlikely that we will get a finite number if we allow the elements a to attain arbitrarily large degree over K, so we need again some conditions. The first way to impose suitable conditions is to restrict the scope to all elements $a \in L$ where $L|K$ is a finite extension such that the extension of v from K to L is unique. For this case, we obtain in Section 3 an upper bound in terms of the defect of the extension $(L|K, v)$ and its ramification index $(vL : vK)$, see Theorem 19.

Another approach is to limit the scope to all $a \in \tilde{K}$ of bounded degree over K. It is an open problem whether the number of essentially distinct distances in this case is always finite and to compute an upper bound for it, even under the finiteness conditions of Theorem 23. However, we are able to show that under these finiteness conditions, the number of distances that are distinct modulo $\widetilde{vK}$ is always finite; we give an upper bound in Theorem 24.

Note that there are examples of valued fields of rank 1, but infinite p-degree, where even the number of distances of elements in immediate purely inseparable extensions of degree p (and of elements in Artin-Schreier defect extensions) that are distinct modulo $\widetilde{vK}$ is infinite.

Finally, let us give some information on the prerequisites and tools we use in this paper. Distances were introduced in the paper [11], but the definition given there is different from the one we use in this paper. The paper [2] presents a comparison of the two definitions and derives the basic properties of the new notion of distances that we use here. One important foundation of our work is Kaplansky's theory of pseudo Cauchy sequences (see [8]) which was developed further in [19]. But none of our earlier papers has addressed the number of possible distances.

2. Preliminaries

For general facts from valuation theory, we refer the reader to [6, 7, 20, 22].

2.1. **Defect.** Take a finite normal extension $L|K$ and a valuation v on K. Then v has finitely many distinct extensions $v_1, \dots, v_g$ to L. All of them have the same ramification index $(v_iL : vK)$, which we will denote by e, and all of them have the same inertia degree $[Lv_i : Kv]$, which we will denote by f. Then we have the fundamental equality

$$[L : K] \;=\; d \cdot e \cdot f \cdot g \,, \tag{3}$$

where by the Lemma of Ostrowski (cf. [20, Théorème 2, p. 236]) or [22, Corollary to Theorem 25, Section G, p. 78]), d is a power ≥ 1 of the residue characteristic $\operatorname{char} Kv$ if this is positive, and equal to 1 otherwise. If $d > 1$, then we speak of **nontrivial defect**. If in addition $L|K$ is an extension of prime degree, then it follows from (3) that $[L : K] = d = \operatorname{char} Kv > 0$ and $e = f = g = 1$, that is, there is a unique extension of v from K to L and $(L|K, v)$ is immediate. We have proved:

Lemma 2. *If L is a normal extension of prime degree p of (K, v) with nontrivial defect, then the extension of v from K to L is unique and $(L|K, v)$ is immediate.*

We will almost always consider extensions $(L|K, v)$ for which the extension of v from K to L is unique. We will call such extensions **uv–extensions** in short; they are necessarily algebraic extensions. Note that every purely inseparable algebraic extension is a uv–extension.

For a finite uv–extension $(L|K, v)$, we can define its **defect** even if the extension is not normal:

$$d(L|K, v) \;:=\; \frac{[L : K]}{(vL : vK)[Lv : Kv]} \,.$$

By the Lemma of Ostrowski, this is a power of p (including $p^0 = 1$), where $p = \operatorname{char} Kv$ if this is positive, and $p = 1$ otherwise (this is called the **characteristic exponent** of Kv). The extension is called **defectless** if $d(L|K, v) = 1$; otherwise, we call it a **defect extension**. Note that if $(L|K, v)$ is a defect extension of prime degree p, then $p = \operatorname{char} Kv$. We note:

Lemma 3. *If $(L|K, v)$ is a nontrivial finite immediate uv–extension, then $[L : K]$ is a power of* $\operatorname{char} Kv$ *and*

$$d(L|K, v) \;=\; [L : K] \,.$$

A valued field (K, v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, if the extension of v to $\tilde{K}$ of K is unique (i.e., $\tilde{K}$ is a uv–extension of (K, v)). In this case, v extends uniquely to each algebraic extension of K. Every algebraically closed valued field is trivially henselian.

Every valued field (K, v) admits a **henselization**, that is, a minimal henselian extension of (K, v), in the sense that it admits a unique valuation preserving embedding over K in every other henselian extension of (K, v). In particular, if w is any extension of v to $\tilde{K}$, then (K, v) has a unique henselization in $(\tilde{K}, w)$, as it is the decomposition field of the normal extension $(K^{\mathrm{sep}}|K, v)$, where $K^{\mathrm{sep}} \subseteq \tilde{K}$ is the separable-algebraic closure of K.

Henselizations of (K, v) are unique up to valuation preserving isomorphism over K. Moreover, they are always immediate separable-algebraic extensions of (K, v) (cf. [6, Theorem 17.19]). A valued field is henselian if and only if it is equal to any (and thus all) of its henselizations.

The following fact is Lemma 2.1 of [3]:

Lemma 4. *An algebraic extension $(L|K, v)$ is a uv–extension if and only if for an arbitrary henselization K^h of (K, v), the extensions $L|K$ and $K^h|K$ are linearly disjoint.*

For the remainder of this paper, we fix an extension of v from K to $\tilde{K}$. This will also fix the henselization of (K,v). Therefore, we will speak of *the* henselization of (K,v), and denote it by (K^h,v).

Since the henselization is an immediate extension and the compositum $L.K^h$ of L and K^h lies in L^h (in fact, it is equal to L^h), this lemma yields:

Lemma 5. *For every finite uv–extension $(L|K,v)$,*

$$d(L|K,v) \;=\; d(L.K^h|K^h,v)\;.$$

2.2. **Distances.** Take an arbitrary extension $(L|K,v)$ of valued fields and $a\in L\backslash K$. There are several possible definitions for the **distance** of a from K that have been used in papers by the authors. We choose the definition that is most suitable for our purposes in this paper.

By $\operatorname{dist}(a,K)$ we denote the cut induced by the set $v(a-K)\cap\widetilde{vK}$ in the divisible hull $\widetilde{vK}$ of vK. Namely, the lower cut set of $\operatorname{dist}(a,K)$ is the smallest initial segment of $\widetilde{vK}$ that contains $v(a-K)\cap\widetilde{vK}$. This definition is slightly different from the one introduced in [11] and [19]. There, we have used the cut in $\widetilde{vK}$ induced by the subset $v(a-K)\cap vK$ to define $\operatorname{dist}(a,K)$. A detailed study of the new notion of distance and a comparison with the former notion can be found in [2]. Note that when $v(a-K)\subseteq vK$, the two notions coincide.

Our definition enables us to compare $\operatorname{dist}(a,K)$ with $\operatorname{dist}(a,L)$ when $(L|K,v)$ is an algebraic extension since then, both $\operatorname{dist}(a,K)$ and $\operatorname{dist}(a,L)$ are cuts in the same ordered abelian group $\widetilde{vK}=\widetilde{vL}$. Then $\operatorname{dist}(a,K)<\operatorname{dist}(a,L)$ will mean that the left cut set of $\operatorname{dist}(a,K)$ is a proper subset of that of $\operatorname{dist}(a,L)$.

The following is Lemma 3.9 of [2].

Lemma 6. *Take algebraic extensions $(L|K,v)$ and $(L(a)|L,v)$. Then* $\operatorname{dist}(a,K)\leq\operatorname{dist}(a,L)$.
If $\operatorname{dist}(a,K)<\operatorname{dist}(a,L)$, *then there is $b\in L$ such that*

$$v(a-b)\;>\;v(a-K)\;=\;v(b-K)\qquad\textit{and}\qquad \operatorname{dist}(b,K)\;=\;\operatorname{dist}(a,K)\;.$$

If $(L|K,v)$ is an arbitrary valued field extension and $a\in L$, then we will say that a is **weakly immediate over** K if $v(a-K)$ has no maximal element. In the language of pseudo Cauchy sequences, this means that a is a pseudo limit of a pseudo Cauchy sequence (also called "pseudo convergent sequence" in [8]) in (K,v) that has no pseudo limit in K. In the language used in [19] it means that the approximation type of a over K is immediate. Note that this does *not* imply that the extension $(K(a)|K,v)$ is immediate (cf. [2, Example 3.17]). But conversely, by what we have already said in the introduction, every element in an immediate extension of (K,v) is weakly immediate over K. Observe that if a is weakly immediate over K then $\infty\notin v(a-K)$, that is, $a\notin K$.

Lemma 7. *Take a finite defectless uv–extension $(L|K,v)$. Then the following assertions hold.*

a) For every $b\in L\setminus K$, the set $v(b-K)$ has a maximal element.

b) Every $a\in\tilde{L}=\tilde{K}$ that is weakly immediate over K is also weakly immediate over L, and

$$\operatorname{dist}(a,L)\;=\;\operatorname{dist}(a,K)\;.$$

Proof. a): This follows from Proposition 3.12 and Lemma 3.10 of [2].
b): This is Corollary 3.11 of [2]. □

To obtain another important distance equality, we need the following theorem from [13]:

Theorem 8. *Take K^h to be the henselization of K in $(\tilde{K}, v)$. Take $a \in \tilde{K} \setminus K$ and assume that for some $b \in K^h$,*

$$v(a-b) \; > \; v(a-K)\,.$$

Then K^h and $K(a)$ are not linearly disjoint over K.

Lemma 9. *Take an algebraic uv–extension $(L|K, v)$. Then for all $a \in L \setminus K$ which are weakly immediate over K,*

$$v(a-K^h) \; = \; v(a-K) \quad \textit{and} \quad \operatorname{dist}(a, K^h) \; = \; \operatorname{dist}(a, K)\,. \tag{4}$$

Proof. Take $a \in L \setminus K$ and suppose that $v(a-K) \subsetneq v(a-K^h)$. Then there is an element $b \in K^h$ such that $v(a-b) > v(a-K)$. But then by Theorem 8, $K(a)|K$ and hence also $L|K$ is not linearly disjoint from $K^h|K$, a contradiction to Lemma 4. So we have that $v(a-K) = v(a-K^h)$, which implies the equality of the distances. □

2.3. **Weakly and strongly immediate elements.** We have already defined what it means for an element in an extension of (K, v) to be weakly immediate over (K, v). A useful stronger property is the following. Take any extension $(L|K, v)$ of valued fields and an element $a \in L \setminus K$. Then we will say that a is **strongly immediate over** K if $v(a-K)$ has no maximal element and in addition, for every polynomial $g \in K[X]$ of degree $< [K(a) : K]$ there is $\alpha \in v(a-K)$ such that for all $c \in K$ with $v(a-c) \geq \alpha$, the value $vg(c)$ is fixed.

Lemma 10. *If the element a is strongly immediate over K, then $(K(a)|K, v)$ is immediate. If in addition, $(K(a)|K, v)$ is a uv–extension, then $[K(a) : K] = d((K(a)|K, v) = p^k$ for some $k \geq 1$, with p the characteristic exponent of Kv.*

Proof. For the first assertion, see [19, Lemma 5.3]. The second assertion follows from the first together with Lemma 3. □

In general, even if $(K(a)|K, v)$ is a uv–extension and a is weakly immediate over K, the extension may not be immediate and a may not be strongly immediate over K. But this holds if the degree $[K(a) : K]$ is a prime:

Lemma 11. *Take a uv–extension $(K(a)|K, v)$ of prime degree p with its generator a weakly immediate over K. Then $(K(a)|K, v)$ is immediate and a is strongly immediate over K.*

Proof. By [11, Lemma 9], $(K(a)|K, v)$ is immediate. Note that by Lemma 3, $p = \operatorname{char} Kv > 0$.

Suppose that there is a polynomial $g \in K[X]$ of degree $< p$ for which there is no $\alpha \in v(a-K)$ such that the value $vg(c)$ is fixed for all $c \in K$ with $v(a-c) \geq \alpha$. Since $v(a-K) = v(a-K^h)$ by (4), there is no $\alpha \in v(a-K^h)$ such that the value $vg(c)$ is fixed for all $c \in K^h$ with $v(a-c) \geq \alpha$. Take $f \in K^h[X]$ to be of minimal degree with this property. As $\deg f \leq \deg g < p$, it follows from [19, Proposition 6.5] that $\deg f = 1$. Hence $f(X) = X - b$ for some $b \in K^h$.

Since $v(a-K^h) = v(a-K)$ has no maximal element, we can choose some $\alpha \in v(a-K^h)$ with $\alpha > v(a-b)$. Take any $c \in K^h$ such that $v(a-c) \geq \alpha$. Then $v(c-b) = \min\{v(c-a), v(a-b)\} = v(a-b)$, so the value $vf(c)$ is fixed for all such c. This contradicts our choice of f and shows that a polynomial g as chosen in the beginning cannot exist. $\square$

Lemma 12. *Take a henselian field (K,v) and an element $a \in \tilde{K}$ which is weakly immediate over K. If a is not strongly immediate over K, then there is an immediate extension $(K(b)|K,v)$ with* $\mathrm{dist}\,(b,K) = \mathrm{dist}\,(a,K)$ *and* $[K(b):K] < [K(a):K]$.

Proof. Using the notions of [19], we argue as follows. Since $v(a-K)$ has no maximal element, the approximation type $\mathrm{appr}\,(a,K)$ is immediate by [19, Lemma 4.1 a)]. Take g to be an associated minimal polynomial for $\mathrm{appr}\,(a,K)$. Since the extension $(K(a)|K,v)$ is not strongly immediate, we have that $\deg g < [K(a):K]$. Take $b \in \tilde{K}$ to be a root of g. Then [19, Theorem 6.4] shows that there is an extension w of v from K to $K(b)$ such that $(K(b)|K,w)$ is immediate and $\mathrm{appr}\,(b,K) = \mathrm{appr}\,(a,K)$. Since (K,v) is henselian, w and v must agree on $K(b)$, showing that $(K(b)|K,v)$ is immediate. The equality of the approximation types implies that $v(b-K) = v(a-K)$, which in turn implies that $\mathrm{dist}\,(b,K) = \mathrm{dist}\,(a,K)$. $\square$

2.4. **The ramification field.** For general ramification theory, see [7] or [17]. For information on tame valued fields, see [16]. We will summarise here the main properties of the ramification field that we will use.

Let $(N|K,v)$ be a normal algebraic extension of henselian fields. We take the **ramification field** V of this extension to be the fixed field of the ramification group $\{\sigma \in \mathrm{Aut}(N|K) \mid 0 \neq x \in \mathcal{O}_L \Rightarrow v(\sigma x - x) > vx\}$ of the automorphism group of $N|K$ *in the maximal separable subextension of* $N|K$.

The **absolute ramification field** of a henselian field (K,v) is the ramification field of the normal algebraic extension $(K^{\mathrm{sep}}|K,v)$, where K^{sep} denotes the separable-algebraic closure of K.

Lemma 13. *Take a normal extension $(N|K,v)$ of henselian fields with residue characteristic $p > 0$. Then its ramification field V has the following properties:*

a) The extension $V|K$ is separable.

b) Every subextension of $N|V$ is a tower of normal extensions of degree p.

c) The valued field extension $(V|K,v)$ is tame and hence every finite subextension $(E|K,v)$ of $(V|K,v)$ is defectless.

d) For every finite subextension $L|K$ of $N|K$,

$$d(L|K,v) \;=\; d(L.V|V,v)\,.$$

e) For all $a \in N \setminus K$ weakly immediate over K,

$$\mathrm{dist}\,(a,V) \;=\; \mathrm{dist}\,(a,K)\,.$$

Proof. Assertion a) follows from our definition.

Assertion b) follows from the fact that the ramification group is a p-group (cf. [7, Theorem 5.3.3] and the proof of [11, Lemma 2.9]).

For assertion c), note that V is a subfield of the absolute ramification field K^r of (K,v), which by part b) of [16, Lemma 2.13] is a tame extension of (K,v). Hence by part a) of the same lemma, also V is a tame extension of (K,v). Thus every finite subextension $(E|K,v)$ of the tame extension $(V|K,v)$ is defectless. In view of this, the equality of the defects follows from [11, Proposition 2.8].

For the proof of d) suppose that $\mathrm{dist}\,(a,V) > \mathrm{dist}\,(a,K)$. Then by Lemma 6 there is an element $b\in V$ such that $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(b,K)$. On the other hand, $(K(b)|K,v)$ is a defectless uv–extension, by part c). Together with part a) of Lemma 7 this contradicts the fact that a is weakly immediate over K. $\square$

3. The number of distinct distances in a given valued field extension

Take a finite (not necessarily immediate) uv–extension $(L|K,v)$. We wish to count the number of distances appearing in this extension that are distinct modulo vK. We define

$$\mathrm{ndd}\,(L|K,v)$$

to be the minimal $m\geq 0$ such that there are elements $a_1,\ldots,a_m\in L\setminus K$ so that each a_i is weakly immediate over K and for every $b\in L\setminus K$ for which $v(b-K)$ has no maximal element, there is $i\in\{1,\ldots,m\}$ and $\alpha\in vK$ with

$$\mathrm{dist}\,(b,K) \;=\; \alpha+\mathrm{dist}\,(a_i,K)\;,$$

that is, $\mathrm{dist}\,(b,K)$ and $\mathrm{dist}\,(a_i,K)$ are equal modulo vK. If there is no such b (which in particular is the case when $(L|K,v)$ is defectless, according to part a) of Lemma 7), then we set $\mathrm{ndd}\,(L|K,v)=0$. We will see that such a number m always exists.

Similarly,

$$\mathrm{ndd}^*(L|K,v)$$

shall denote the number of distances appearing in $(L|K,v)$ that are distinct modulo $\widetilde{vK}$. Observe that

$$\mathrm{ndd}^*(L|K,v)\;\leq\;\mathrm{ndd}\,(L|K,v)\;\text{ and }\;\mathrm{ndd}^*(L|K,v)=0\;\Leftrightarrow\;\mathrm{ndd}\,(L|K,v)=0\,. \tag{5}$$

We note:

Lemma 14. *Take any algebraic extension $(L|K,v)$ of valued fields and a subextension $(L_0|K,v)$. If $\mathrm{ndd}\,(L_0|K,v)=0$, then $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(a,L_0)$ for every $a\in L$ which is weakly immediate over K. The converse holds if no element of L_0 lies in the completion of (K,v).*

Proof. Assume first that $\mathrm{ndd}\,(L_0|K,v)=0$ and take $a\in L\setminus K$ weakly immediate over K. If $\mathrm{dist}\,(a,K)\neq\mathrm{dist}\,(a,L_0)$, then $\mathrm{dist}\,(a,L_0)>\mathrm{dist}\,(a,K)$ and by Lemma 6 there is $b\in L_0$ such that $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(b,K)$. But then $v(a-K)$ has no maximal element, contradicting our assumption that $\mathrm{ndd}\,(L_0|K,v)=0$.

Now assume that $\mathrm{ndd}\,(L_0|K,v)>0$ and that no element of L_0 lies in the completion of (K,v). Take $a\in L_0\setminus K$ weakly immediate over K. Since a does not lie in the completion of (K,v), $\mathrm{dist}\,(a,K)$ is a proper cut in $\widetilde{vK}$. But as $a\in L_0$, the lower cut set of $\mathrm{dist}\,(a,L_0)$ is $\widetilde{vK}$, so $\mathrm{dist}\,(a,L_0)>\mathrm{dist}\,(a,K)$. $\square$

Lemma 15. *Take a finite uv–extension $(L|K,v)$ and an algebraic extension $(K'|K,v)$ such that $(vK':vK)<\infty$ and $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(a,K')$ for all $a\in L$. Then*

$$\begin{aligned}\mathrm{ndd}\,(L|K,v) &\leq \mathrm{ndd}\,(L.K'|K',v)\cdot(vK':vK)\,,\\ \mathrm{ndd}^*(L|K,v) &\leq \mathrm{ndd}^*(L.K'|K',v)\,.\end{aligned}$$

Proof. Set $n = (vK' : vK)$ and choose representatives $\beta_1, \dots, \beta_n \in vK'$ of the distinct cosets in vK'/vK. If two distances $\mathrm{dist}\,(a_1, K)$ and $\mathrm{dist}\,(a_2, K)$ are equal modulo vK' then there is $i \in \{1, \dots, n\}$ and $\alpha \in vK$ such that $\mathrm{dist}\,(a_1, K) = \alpha + \beta_i + \mathrm{dist}\,(a_2, K)$, where the latter is equal to $\beta_i + \mathrm{dist}\,(a_2, K)$ modulo vK. This shows that the maximum number of distances that are distinct modulo vK but equal modulo vK' is n, which proves the first inequality.

The second inequality follows from the fact that all β_i lie in $\widetilde{vK}$. □

The next lemma computes $\mathrm{ndd}\,(K(a)|K, v)$ for uv–extensions $(K(a)|K, v)$ with a strongly immediate over K. We derive it from [8, Lemma 8] and [19, Lemma 5.2]. We use the Taylor expansion

$$f(X) = \sum_{i=0}^{n} f_i(c)(X - c)^i \tag{6}$$

where f_i denotes the i-th formal derivative of f (also called Hasse derivative or Hasse-Schmidt derivation); these are the polynomials derived from f that allow the above characteristic blind Taylor expansion.

Lemma 16. *Take a finite uv–extension $(K(a)|K, v)$ such that a is strongly immediate over K. Following Lemma 10, we write $[K(a) : K] = p^k$ for some $k \geq 1$. Then for every nonconstant polynomial $f \in K[X]$ of degree $< p^k$ there are $\gamma \in v(a - K)$ and $\mathbf{h} = p^\ell$ with $0 \leq \ell < k$ such that for all $c \in K$ with $v(a - c) \geq \gamma$, the value $vf_i(c)$ is fixed for each $i \geq 0$,*

$$v(f(a) - f(c)) \;=\; vf_{\mathbf{h}}(c) + \mathbf{h} \cdot v(a - c)\,, \tag{7}$$

and

$$\mathrm{dist}\,(f(a), K) \;=\; vf_{\mathbf{h}}(c) + \mathbf{h} \cdot \mathrm{dist}\,(a, K)\,. \tag{8}$$

Therefore, $\mathrm{ndd}\,(K(a)|K, v) \leq k$ *and, modulo vK, all distances are multiples of* $\mathrm{dist}\,(a, K)$ *by powers of p.*

Proof. Using the notions of [19], the assumption that a is strongly immediate over K is equivalent to the approximation type of a over K being of degree $[K(a) : K]$. Hence all assertions except for the last one follow from [19, Lemma 5.2, Proposition 7.4 and Lemma 8.2] (see also [8, Lemma 8]). For the proof of the last assertion we use the fact that every element $b \in K(a) \setminus K$ can be written as $f(a)$ with a nonconstant polynomial $f \in K[X]$ of degree smaller than $[K(a) : K] = p^k$. Since there are exactly k many distinct $\mathbf{h} = p^\ell$ with $0 \leq \ell < k$, equation (8) yields that $\mathrm{ndd}\,(K(a)|K, v) \leq k$. □

The following corollary shows that a uv–extension of prime degree generated by a weakly immediate element admits exactly one distance modulo vK. It follows from the previous lemma together with Lemma 11.

Corollary 17. *Take a uv–extension $(K(a)|K, v)$ of prime degree p such that a is weakly immediate over K. Then for every nonconstant polynomial $f \in K[X]$ of degree smaller than p there is $\gamma \in v(a - K)$ such that for all $c \in K$ with $v(a - c) \geq \gamma$, the value $vf_i(c)$ is fixed for each $i \geq 0$, and*

$$v(f(a) - f(c)) \;=\; vf_1(c) + v(a - c)\,. \tag{9}$$

Hence for any $b \in K(a) \setminus K$,

$$\mathrm{dist}\,(b, K) \;=\; \alpha + \mathrm{dist}\,(a, K) \qquad \textit{for some } \alpha \in vK\,.$$

Therefore, $\mathrm{ndd}^*(K(a)|K,v) = \mathrm{ndd}\,(K(a)|K,v) = 1$.

Proposition 18. *Assume that $(L|K,v)$ is a finite uv–extension which is a tower of extensions of degree p. If $d(L|K,v) = p^m$ with $m \geq 0$, then*

$$\mathrm{ndd}\,(L|K,v) \leq m \cdot (vL : vK) \quad \textit{and} \quad \mathrm{ndd}^*(L|K,v) \leq m\,.$$

Proof. We consider a tower $K = L_0 \subset L_1 \subset \ldots \subset L_n = L$ of uv–extensions of degree p. We write $d(L_i|K,v) = p^{m_i}$, with $m_n = m$. We proceed by induction on $i \leq n$.

The induction start is covered by Corollary 17 if $(L_1|K,v)$ is immediate. In this case, we have $(vL_1 : vK) = 1$, $m_1 = 1$ and $\mathrm{ndd}\,(L_1|K,v) = 1 = m_1 \cdot (vL_1 : vK)$. Also, $\mathrm{ndd}^*(L_1|K,v) = 1 = m_1$. If the extension is not immediate, then it is defectless (as it is of prime degree). Hence $m_1 = 0$ and

$$\mathrm{ndd}\,(L_1|K,v) \;=\; 0 \;=\; m_1 \cdot (vL_1 : vK)\,.$$

Also, $\mathrm{ndd}^*(L_1|K,v) = 0 = m_1$.

Now we assume that for some $i < n$ we have already shown that $\mathrm{ndd}\,(L_i|K,v) \leq m_i \cdot (vL_i : vK)$ and $\mathrm{ndd}\,(L_i|K,v) \leq m_i$. Take any $a \in L_{i+1} \setminus L_i$ which is weakly immediate over K. Since $[L_{i+1} : L_i]$ is prime, we have that $L_{i+1} = L_i(a)$. By Lemma 6, either $\mathrm{dist}\,(a,K) = \mathrm{dist}\,(b,K)$ holds for some $b \in L_i$, or $\mathrm{dist}\,(a,K) = \mathrm{dist}\,(a,L_i)$.

Suppose that there is such an element a for which the latter holds. Then a is weakly immediate over L_i and by Lemma 11, the uv–extension $(L_{i+1}|L_i,v)$ is immediate. Hence, $d(L_{i+1}|K,v) = d(L_i|K,v) \cdot p$, so $m_{i+1} = m_i + 1$. By Lemma 17, $\mathrm{ndd}\,(L_{i+1}|L_i,v) = 1$. This says that modulo vL_i, all distances $\mathrm{dist}\,(a,K)$ arising in this way must be equal. Consequently, there can be at most $(vL_i : vK)$ many that are distinct modulo vK, and only one modulo $\widetilde{vK}$. This is in addition to the number of distinct distances arising from elements in L_i. So we obtain that

$$\begin{aligned}
\mathrm{ndd}\,(L_{i+1}|K,v) &\leq (vL_i : vK) + m_i \cdot (vL_i : vK) \;=\; m_{i+1} \cdot (vL_{i+1} : vK)\,,\\
\mathrm{ndd}^*(L_{i+1}|K,v) &\leq 1 + m_i \;=\; m_{i+1}\,.
\end{aligned}$$

Suppose now that there is no such element a. Then

$$\begin{aligned}
\mathrm{ndd}\,(L_{i+1}|K,v) &= \mathrm{ndd}\,(L_i|K,v) \;=\; m_i \cdot (vL_i : vK) \;\leq\; m_{i+1} \cdot (vL_{i+1} : vK)\,,\\
\mathrm{ndd}^*(L_{i+1}|K,v) &= \mathrm{ndd}^*(L_i|K,v) \;=\; m_i \;\leq\; m_{i+1}\,.
\end{aligned}$$

This completes our induction. □

We will now generalize this result to arbitrary finite, not necessarily immediate, uv–extensions.

Theorem 19. *Take a finite uv–extension $(L|K,v)$ and write $d(L|K,v) = p^m$ with $m \geq 0$. Then*

$$\mathrm{ndd}\,(L|K,v) \leq m \cdot [L : K]!/p^m \quad \textit{and} \quad \mathrm{ndd}^*(L|K,v) \leq m\,.$$

If in addition $L|K$ is a normal extension, then $\mathrm{ndd}\,(L|K,v) \leq m \cdot (vL : vK)$.

Proof. First, we show that we may assume (K,v) to be henselian. For every $a \in L \setminus K$, Lemma 9 shows that $\mathrm{dist}\,(a,K^h) = \mathrm{dist}\,(a,K)$. By Lemma 15 we obtain that

$$\mathrm{ndd}\,(L|K,v) \leq \mathrm{ndd}\,(L.K^h|K^h,v) \cdot (vK^h : vK) = \mathrm{ndd}\,(L.K^h|K^h,v)$$

and $\mathrm{ndd}^*(L|K,v) \leq \mathrm{ndd}^*(L.K^h|K^h,v)$. Lemma 5 shows that $d(L|K,v) = d(L.K^h|K^h,v)$, and Lemma 4 yields that $[L.K^h : K^h] = [L : K]$. Since the henselization of a valued field is an immediate extension of the field, $(vL.K^h : vK^h) = (vL^h : vK^h) = (vL : vK)$. Thus, we may replace K by its henselization.

We denote the normal hull of L over K by N. Since (K,v) is henselian, there is a unique extension of v from L to N and $(N|K,v)$ is again a uv–extension. Now we take V to be the ramification field of $(N|K,v)$. From Lemma 13 we obtain that $(V|K,v)$ is a defectless uv-extension such that $d(L.V|V,v) = d(L|K,v) = p^m$ and that $\mathrm{dist}\,(a,V) = \mathrm{dist}\,(a,K)$ for every $a \in N \setminus K$ which is weakly immediate over K. From Lemma 15 we thus obtain that $\mathrm{ndd}\,(L|K,v) \leq \mathrm{ndd}\,(L.V|V,v)\cdot(vV : vK)$ and $\mathrm{ndd}^*(L|K,v) \leq \mathrm{ndd}^*(L.V|V,v)$. By part b) of Lemma 13 we know that the subextension $L.V|V$ of $N|V$ is a tower of normal extensions of degree p. Hence Proposition 18 shows that $\mathrm{ndd}\,(L.V|V,v) \leq m\cdot(v(L.V) : vV)$ and $\mathrm{ndd}^*(L.V|V,v) \leq m$. Altogether we have that $\mathrm{ndd}^*(L|K,v) \leq \mathrm{ndd}^*(L.V|V,v) \leq m$ and that

$$\begin{aligned}\mathrm{ndd}\,(L|K,v) &\leq \mathrm{ndd}\,(L.V|V,v)\cdot(vV : vK)\\ &\leq m\cdot(v(L.V) : vV)\cdot(vV : vK) \;=\; m\cdot(v(L.V) : vK)\,.\end{aligned}$$

If $L|K$ is a normal extension, then $N = L$ and $V \subseteq L$. From this we get that $(v(L.V) : vK) = (vL : vK)$, which yields the second assertion of our theorem.

In the general case, we have that $d(N|K,v) \geq d(L|K,v) = p^m$ and

$$(v(L.V) : vK) \;\leq\; (vN : vK) \;\leq\; [N : K]/d(N|K,v) \;\leq\; [N : K]/p^m\,.$$

Since $[N : K] \leq [L : K]!$, this yields the first assertion of our theorem. $\square$

4. The number of distinct distances in all Artin-Schreier defect extensions

Throughout this section, let (K,v) be a field of positive characteristic p. As before, we assume that v is extended to the algebraic closure $\tilde{K}$ of K. By Zorn's Lemma, there always exists a maximal immediate subextension $(K'|K,v)$ of the purely inseparable extension $(K^{1/p}|K,v)$, where $K^{1/p} = \{c^{1/p} \mid c \in K\}$. Throughout the present and the final section of this paper, we will assume that $K'|K$ is finite, so that its degree is p^m for some $m \geq 0$. If K has finite p-degree k, that is, $[K^{1/p} : K] = [K : K^p] = p^k$ with $k \geq 0$, then $m \leq k$.

We will now apply our previous results to consider the possible distances (modulo vK) of all elements that are contained in any Artin-Schreier defect extension of (K,v). In view of Corollary 17, we only have to determine the distance of one generator of such an extension. The Artin-Schreier defect extension $(K(\vartheta)|K,v)$ with Artin-Schreier generator ϑ is called **dependent** if there is a purely inseparable immediate extension $(K(\eta)|K,v)$ of degree p such that

$$v(\vartheta-\eta) \;>\; v(\vartheta-c) \quad \text{for all } c \in K\,.$$

This implies that $v(\vartheta - c) = v(\eta - c)$ for all $c \in K$ and that

$$\mathrm{dist}\,(\vartheta,K) \;=\; \mathrm{dist}\,(\eta,K)\,.$$

We note that by assumption, $\eta \in K^{1/p}$.

Proposition 20. *Under the assumptions on (K,v) outlined above,*

$$\mathrm{ndd}\,(K^{1/p}|K,v) \;=\; \mathrm{ndd}\,(K'|K,v) \leq m\,.$$

Moreover, if (K,v) is of finite p-degree and $d(K^{1/p}|K,v)=p^s$, then

$$\mathrm{ndd}\,(K^{1/p}|K,v) \;\leq\; s\,.$$

Proof. For every $a\in K^{1/p}$ which is weakly immediate over K, there must be some $b\in K'$ with $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(b,K)$. Otherwise, we would obtain that $\mathrm{dist}\,(a,K')=\mathrm{dist}\,(a,K)$ which yields that a is weakly immediate over K'; since $[K'(a):K']=p$, this would show by Lemma 11 that $(K'(a)|K',v)$ and hence also $(K'(a)|K,v)$ are immediate extensions, contradicting the maximality of K'. So we have that $\mathrm{ndd}\,(K^{1/p}|K,v)=\mathrm{ndd}\,(K'|K,v)$.

Since $(K'|K,v)$ is immediate, we have that $d(K'|K,v)=[K':K]=p^m$. Hence by Proposition 18, $\mathrm{ndd}\,(K'|K,v)\leq m$, because $(vL:vK)=1$.

For the proof of the last assertion, note that if (K,v) is of finite p-degree, then

$$p^m=[K':K]=d(K'|K,v)\leq d(K^{1/p}|K,v)=p^s.$$

Thus $m\leq s$. □

From Proposition 20 together with Corollary 17 we obtain the following result:

Proposition 21. *Under the assumptions on (K,v) outlined in the beginning of this section, there are elements $c_1,\ldots,c_m\in K$ such that for every dependent Artin-Schreier defect extension $(K(a)|K,v)$ there is $i\in\{1,\ldots,m\}$ such that for every $b\in K(a)\setminus K$ there is some $\alpha\in vK$ with*

$$\mathrm{dist}\,(b,K) \;=\; \alpha+\mathrm{dist}\,(c_i^{1/p},K)\,.$$

Hence all distinct distances modulo vK of elements in dependent Artin-Schreier defect extensions of (K,v) are already among the distinct distances modulo vK of elements in purely inseparable defect extensions of degree p of (K,v), and their number is bounded by m.

In order to make a statement about *all* possible Artin-Schreier defect extensions $(K(a)|K,v)$, we also have to consider the **independent** ones, that is, the ones that are not dependent. It is shown in [11] that if a is an Artin-Schreier generator of the extension, then $\mathrm{dist}\,(a,K)$ is the lower edge of some proper convex subgroup H of $\widetilde{vK}$, that is, the lower cut set of $\mathrm{dist}\,(a,K)$ is the largest initial segment of $\widetilde{vK}$ that does not meet H. We summarize:

Lemma 22. *The distances of all elements in Artin-Schreier defect extensions $(K(a)|K,v)$ modulo vK are among the lower edges of convex subgroups of the value group vK together with the distances of the elements in $K^{1/p}$.*

The **rank** of (K,v), if finite, is the number of proper convex subgroups of the value group vK. Putting the previous results together, we obtain:

Theorem 23. *Take a valued field (K,v) of finite rank r, satisfying the assumptions outlined in the beginning of this section. Then the number of distinct distances modulo vK of elements in all normal defect extensions of prime degree of (K,v) as well as the number of distinct distances modulo vK of elements in Artin-Schreier defect extensions of (K,v) are bounded by $r+m$. In particular, if K has finite p-degree k, then this number is bounded by $r+k$.*

For a function field K over a perfect base field K_0, the p-degree k is equal to the transcendence degree $\mathrm{trdeg}\, K|K_0$. For a valued function field $(K|K_0,v)$ over a trivially valued base field (K_0,v), the rank is bounded by $\mathrm{trdeg}\, K|K_0$. This proves Theorem 1.

5. The number of distinct distances of all elements of bounded degree

Throughout this section we shall work under the following assumptions, unless indicated otherwise. We take (K,v) to be a valued field of positive characteristic p and finite rank r.

For every natural number i we denote by $\mathrm{ndd}\,^*_i(K,v)$ the number of distinct distances modulo $\widetilde{vK}$ of elements $a\in\widetilde{K}\setminus K$ satisfying the following conditions:

$$\left\{\begin{array}{l}[K(a):K]\leq p^i,\\ (K(a)|K,v) \text{ is a uv–extension,}\\ a \text{ is weakly immediate over } K.\end{array}\right. \tag{10}$$

We will show now that for every $i\in\mathbb{N}$ the number $\mathrm{ndd}\,^*_i(K,v)$ is finite.

Theorem 24. *Assume in addition that (K,v) has finite p-degree and that*

$$d(K^{1/p}|K,v) \;=\; p^m\;.$$

Then $\mathrm{ndd}\,^*_i(K,v)$ is finite for every natural number i. More precisely,

$$\mathrm{ndd}\,^*_i(K,v)\leq r+im.$$

Proof. In what follows, let $a\in\widetilde{K}$ satisfy the assumptions (10). Lemma 9 shows that $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(a,K^h)$. This implies in particular that a is weakly immediate over K^h. Furthermore, the assumptions (10) together with Lemma 4 yield that $[K^h(a):K^h]=[K(a):K]$. Hence, for every natural number i we have that $\mathrm{ndd}\,^*_i(K,v)\leq\mathrm{ndd}\,^*_i(K^h,v)$.

We wish to show that also (K^h,v) satisfies the assumptions stated at the beginning of this section. Since $K^h|K$ is a separable algebraic extension, K^h has the same p-degree as K, so $[(K^h)^{1/p}:K^h]=p^k$. It follows that $(K^h)^{1/p}=K^{1/p}.K^h$. Since $K^{1/p}|K$ is finite and v extends uniquely from K to $K^{1/p}$, Lemma 5 yields that

$$p^m \;=\; d(K^{1/p}|K,v) \;=\; d(K^{1/p}.K^h|K^h,v) \;=\; d((K^h)^{1/p}|K^h,v)\;.$$

Furthermore, $vK^h=vK$ is again of rank r. Hence we can assume that (K,v) is henselian.

Take K^r to be the absolute ramification field of K with respect to the fixed extension of v to $\tilde{K}$. Lemma 13 shows that $\mathrm{dist}\,(a,K)=\mathrm{dist}\,(a,K^r)$. This implies in particular that a is weakly immediate over K^r. Moreover, $[K^r(a):K^r]\leq[K(a):K]$ and $\widetilde{vK}=\widetilde{vK^r}$. Therefore, $\mathrm{ndd}\,^*_i(K,v)\leq\mathrm{ndd}\,^*_i(K^r,v)$ for every $i\in\mathbb{N}$.

We wish to show that also (K^r,v) satisfies the assumptions stated at the beginning of this section. Since $K^r|K$ is a separable algebraic extension, K^r has the same p-degree as K, so $[(K^r)^{1/p}:K^r]=p^k$. It follows that $(K^r)^{1/p}=K^{1/p}.K^r$. Lemma 13 yields that

$$p^m=d(K^{1/p}|K,v)=d(K^{1/p}.K^r|K^r,v)=d((K^r)^{1/p}|K^r,v).$$

Furthermore, vK^r/vK is a torsion group, hence vK^r is again of rank r. Hence we can assume that $K^r = K$. Note that by Lemma 13 this means that the extension $K(a)|K$ is a tower of normal extensions of degree p. In particular, it is of degree p^t for some $t \in \{0, \ldots, i\}$.

We proceed by induction on i. The case of $i = 1$ is covered by Theorem 23. Now assume that $i \geq 2$ and

$$\mathrm{ndd}\,^*_{i-1}(K, v) \leq r + (i-1)m\,.$$

To give an upper bound for $\mathrm{ndd}\,^*_i(K, v)$, it is enough to consider elements of degree p^i over K which are weakly immediate over K. Indeed, the distances of elements a of degree at most p^{i-1} are already counted in $\mathrm{ndd}\,^*_{i-1}(K, v)$. Hence we assume that $[K(a) : K] = p^i$.

If a is not strongly immediate over K, then by Lemma 12 there is an immediate extension $(K(b)|K, v)$ with $\mathrm{dist}\,(b, K) = \mathrm{dist}\,(a, K)$ and $[K(b) : K] < [K(a) : K]$. By Lemma 13 the degree $[K(b) : K]$ must be a power of p. We conclude that $[K(b) : K] \leq p^{i-1}$, showing that $\mathrm{dist}\,(a, K)$ is already counted in $\mathrm{ndd}\,^*_{i-1}(K, v)$. Hence we assume that a is strongly immediate over K. By Lemma 10 this implies that the extension $(K(a)|K, v)$ is immediate.

Assume first that $K(a)|K$ is purely inseparable. Then from Lemma 6 we deduce that $\mathrm{dist}\,(a, K) = \mathrm{dist}\,(a, K^{1/p^{i-1}})$ or $\mathrm{dist}\,(a, K) = \mathrm{dist}\,(d, K)$ for some $d \in K^{1/p^{i-1}}$. If the latter holds, then d is weakly immediate over K and therefore, $\mathrm{dist}\,(b, K)$ appears already as a distance of some weakly immediate element of degree $\leq p^{i-1}$. So we may assume that the former holds. Then $K^{1/p^{i-1}}(a)|K^{1/p^{i-1}}$ is a purely inseparable extension of degree p and the element a is weakly immediate over $K^{1/p^{i-1}}$. Since $d(K^{1/p^i}|K^{1/p^{i-1}}, v) = d(K^{1/p}|K, v) = p^m$, Proposition 20 shows that there are at most m distinct distances of elements of K^{1/p^i} weakly immediate over $K^{1/p^{i-1}}$, modulo $vK^{1/p^{i-1}} = \frac{1}{p^{i-1}}vK$, hence also modulo $\widetilde{vK}$. This renders at most m additional distinct distances $\mathrm{dist}\,(a, K)$ modulo $\widetilde{vK}$.

Assume now that $K(a)|K$ is not purely inseparable. Take E to be a maximal separable subextension of $K(a)|K$; we have that $E|K$ is nontrivial. Furthermore, $E|K$ is a tower of Galois extensions of degree p, as $K(a)|K$ is a tower of normal extensions of degree p. This shows that K admits an Artin-Schreier extension $K(\vartheta) \subseteq K(a)$, where ϑ is an Artin-Schreier generator. Since $K(a)|K$ is an immediate extension of henselian fields, the same holds for $K(\vartheta)|K$ and thus $K(\vartheta)|K$ is an Artin-Schreier defect extension. Take a polynomial $f \in K[X]$ such that $\vartheta = f(a)$ with $\deg f < p^i$. Since a is strongly immediate by assumption, we can apply Lemma 16 to obtain that

$$\mathrm{dist}\,(\vartheta, K) \;=\; \mathrm{dist}\,(f(a), K) \;=\; \alpha + p^s \mathrm{dist}\,(a, K) \tag{11}$$

for some $\alpha \in vK$ and $s < i$. Take $c \in K$ such that $vc = \alpha$.

Assume that the Artin-Schreier defect extension $(K(\vartheta)|K, v)$ is dependent. Then $\mathrm{dist}\,(\vartheta, K) = \mathrm{dist}\,(\eta, K)$ for some $\eta \in K^{1/p}$ such that the extension $K(\eta)|K$ is immediate. Hence,

$$p^s \mathrm{dist}\,(a, K) \;=\; \mathrm{dist}\,(\vartheta, K) - vc \;=\; \mathrm{dist}\,(\eta, K) - vc \;=\; \mathrm{dist}\,\left(\frac{\eta}{c}, K\right)\,,$$

where the last equation holds by (2). Since $\frac{1}{p^s}v(\frac{\eta}{c}-K)=v\left((\frac{\eta}{c})^{1/p^s}-K^{1/p^s}\right)$, we obtain that

$$\operatorname{dist}(a,K) \;=\; \operatorname{dist}\left(\left(\frac{\eta}{c}\right)^{1/p^s},\, K^{1/p^s}\right). \tag{12}$$

Since $v(a-K)$ has no maximal element, it follows from equation (12) that also $v\left((\frac{\eta}{c})^{1/p^s}-K^{1/p^s}\right)$ has no maximal element, so $(\frac{\eta}{c})^{1/p^s}$ is weakly immediate over K^{1/p^s}. Moreover, $K^{1/p^s}((\frac{\eta}{c})^{1/p^s})|K^{1/p^s}$ is a purely inseparable extension of degree p. Hence, dist $\left((\frac{\eta}{c})^{1/p^s},K^{1/p^s}\right)$ has already been counted under the purely inseparable case in this or an earlier induction step (depending on the value of $s<i$).

Assume now that $K(\vartheta)|K$ is an independent Artin-Schreier defect extension. Then [15, Proposition 4.2] together with Equation (11) shows that

$$p^s\operatorname{dist}(\vartheta,K) \;=\; \operatorname{dist}(\vartheta,K) \;=\; vc+p^s\operatorname{dist}(a,K)$$

and consequently,

$$\operatorname{dist}(a,K)=-\frac{1}{p^s}vc+\operatorname{dist}(\vartheta,K).$$

This shows that dist (a,K) is equal modulo $\widetilde{vK}$ to the distance of some weakly immediate element of degree p over K, which has already been counted in $\operatorname{ndd}^*_1(K,v)$.

Consequently, we obtain that

$$\operatorname{ndd}^*_i(K,v) \;\le\; \operatorname{ndd}^*_{i-1}(K,v)+m.$$

By induction hypothesis, it follows that

$$\operatorname{ndd}^*_i(K,v) \;\le\; r+mi.$$

□

An interesting special case is covered by the following result. Here the assumptions on the finiteness of the extension $(K^{1/p}|K,v)$ and its defect are not needed.

Proposition 25. *Assume that (K,v) has finite rank r and that the perfect hull of K is contained in the completion of (K,v). Then*

$$\operatorname{ndd}^*_i(K,v)\le r+1$$

for every natural number i. Therefore, there are at most $r+1$ distances distinct modulo $\widetilde{vK}$ of elements satisfying (10) for arbitrary $i\in\mathbb{N}$.

Proof. Similar to the proof of Theorem 24, except that in all purely inseparable cases the only possible distance is ∞. In particular, there are no dependent Artin-Schreier defect extensions. Indeed, if ϑ is an Artin-Schreier generator of an Artin-Schreier defect extension, then [11, Corollary 2.30] yields that $v(\eta-c)<0$ for all $c\in K$. Hence there is no $\eta\in K^{1/p}$ such that $v(\eta-c)=v(\vartheta-c)$ for all $c\in K$. □

We can generalize the previous proposition by dropping the condition that for each considered algebraic element a, $(K(a)|K,v)$ is a uv–extension. If H is a proper convex subgroup of $\widetilde{vK}$, then H^+ denotes the cut at the upper edge of H, that is, its upper cut set is the largest final segment of $\widetilde{vK}$ which does not meet H.

Corollary 26. *Under the assumptions of Proposition 25, there are at most $2r$ distances distinct modulo $\widetilde{vK}$ of elements in $\tilde{K}$ that are weakly immediate over K.*

Proof. Assume that a is weakly immediate over K. Then $\operatorname{dist}(a,K) = \operatorname{dist}(a,K^h)$ or $\operatorname{dist}(a,K) = \operatorname{dist}(d,K)$ for some $d \in K^h$.

In the first case, we obtain that a is weakly immediate over K^h. Hence a satisfies conditions (10) for some $i \in \mathbb{N}$ with K^h in place of K. Now if (K,v) satisfies the assumptions of Proposition 25, then so does its henselization: first of all, they have the same rank, and secondly, $(K^h)^{1/p^\infty} = K^h.K^{1/p^\infty} \subseteq K^h.K^c \subseteq (K^h)^c$. Applying Proposition 25, we see that the number of distances distinct modulo $\widetilde{vK}$ of such elements a is bounded by $r+1$.

In the second case, a is weakly distinguished over K, that is,

$$\operatorname{dist}(a,K) = \alpha + H^+$$

for some $\alpha \in vK$ and a nontrivial convex subgroup H of vK by [13, Theorem 1]. Note that if $H = vK$, we have that $\operatorname{dist}(a,K) = \infty$ and this distance has already been counted above. This gives $r-1$ additional possible distances modulo $\widetilde{vK}$.

Hence we have at most $(r+1)+(r-1) = 2r$ distances distinct modulo $\widetilde{vK}$ of weakly immediate algebraic elements over K. □

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