ON CERTAIN DEFINABLE COARSENINGS OF VALUATION RINGS AND THEIR APPLICATIONS

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Abstract. We show how Galois extensions $(L|K,v)$ of prime degree of valued fields give rise to definable coarsenings of the valuation rings of L and K. In the case of Artin-Schreier and Kummer extensions with wild ramification, we can also define the ramification ideal. We demonstrate the use of the coarsenings on L, their maximal ideals, and the ramification ideals for the classification of defects and for the presentation of the Kähler differentials of the extension of the valuation rings of $(L|K,v)$, and their annihilators. Finally, we give a construction that realizes predescribed convex subgroups of suitable value groups as those that are associated with Galois extensions of degree p with independent defect, which in turn give rise to definable coarsenings.

1. Introduction

In this paper, we consider Galois extensions $(L|K,v)$ of prime degree, as studied in [2, 3]. We will be interested in definable coarsenings of both the valuation ring $\mathcal{O}_L$ of v on L and the valuation ring $\mathcal{O}_K$ of v on K; however, it is the former that are important for our applications. Under certain additional assumptions the coarsenings of $\mathcal{O}_K$ have already been shown in [8] to be definable in the ring language, for the case of residue characteristic 0. In [9] the latter condition is removed. See also the survey [6]. In contrast to these approaches, we do not work on one single valued field, but our coarsenings are induced by extensions $(L|K,v)$ of prime degree, and we are content with definability in the language $\mathcal{L}_{\mathrm{val},K}$ of valued fields with a predicate for membership in K. Also in contrast to the above approaches, in general we achieve parameter-free definitions.

We will present applications of these definable coarsenings to the presentation of the Kähler differentials $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ of the extension of the valuation rings of $(L|K,v)$, and their annihilators. For this purpose, we will also be concerned with definibility of the ramification ideals of these extensions.

For the definitions of the basic properties of extensions of valued fields, such as the defect, defect extensions, defectless extensions, immediate extensions, and unibranched extensions, see Section 2.1.

Let us start by presenting the motivation for our work in this paper.

1.1. Deeply ramified fields and Kähler differentials.

We call a valued field (K,v) a **deeply ramified field** if it satisfies the conditions

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(DRvr) if $\operatorname{char} Kv = p > 0$, then the homomorphism

$$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K \tag{1}$$

is surjective (i.e., $\mathcal{O}_K/p\mathcal{O}_K$ is **semiperfect**) if $\operatorname{char} K = 0$, and the completion $\widehat{K}$ of (K,v) is perfect if $\operatorname{char} K = p$,

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete).

Every perfect valued field of positive characteristic p and every perfectoid field is a deeply ramified field with p-divisible value group. Note that condition (DRvr) as stated above is a simplified version of the condition (DRvr) given in [21]; for the equivalence of the two, see [21, Theorem 1.2(3) and Lemma 4.1(3)].

In [7] Gabber and Ramero use Kähler differentials, under the name "modules of relative differentials", to define deeply ramified fields. However, it has become more common, and for our purposes more practical, to use the above definition. When A is a ring and B is an A-algebra, then we denote by $\Omega_{B|A}$ the Kähler differentials of $B|A$ (see Section 2.3). Given a valued field (K,v), we denote by $K^{\rm sep}$ the separable algebraic closure and extend v from K to $K^{\rm sep}$. Rewritten with our prefered definition of "deeply ramified field", [7, Theorem 6.6.12(vi)] becomes the following result:

Theorem 1.1. *For a valued field (K,v),*

$$\Omega_{\mathcal{O}_{K^{\rm sep}}|\mathcal{O}_K} \;=\; 0 \tag{2}$$

holds if and only if (K,v) is a deeply ramified field.

The statement does not depend on the choice of the extension of v from K to $K^{\rm sep}$ since all of the possible extensions are conjugate.

A main goal of the papers [2, 3] is to compute the Kähler differentials of extensions $\mathcal{E} = (L|K,v)$ of valued fields that are Artin-Schreier extensions or Kummer extensions of prime degree, and use this in [3] to give an alternative proof of Theorem 1.1. The computation in the case of defect extensions is dealt with in [2]; we will state the results in Section 4. Then [3] settles the case of unibranched defectless extensions; we will state the results in Section 5. We call $(L|K,v)$ **unibranched** if the valuation v has only one extension from K to L. A valued field is **henselian** if and only if all of its algebraic extensions are unibranched.

The following additional theorem shows the particular importance of extensions of prime degree of the valued field (K,v). From a computational (and model theoretic) point of view, these should be more accessible than the extension $(K^{\rm sep}|K,v)$.

Theorem 1.2. [3, Theorem 1.3] *Let (K,v) be a valued field of residue characteristic $p > 0$. If K has characteristic 0, then assume in addition that it contains all p-th roots of unity. Then (K,v) is a deeply ramified field if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all unibranched Galois extensions $(L|K,v)$ of prime degree p.*

This justifies our focus on Artin-Schreier extensions and Kummer extensions of prime degree in the present paper. The condition on the p-th roots of unity is not a serious restriction because of the following result, which follows from [3, Theorem 1.5 and Proposition 6.1 2)]:

Proposition 1.3. *Take a valued field (K,v) of characteristic 0 with residue characteristic $p>0$, and a primitive p-th root of unity ζ_p. Then (K,v) is a deeply ramified field if and only if $(K(\zeta_p),v)$ is, for any exension of v from K to $K(\zeta_p)$.*

One of our main goals is the proof of the following result, which will be proved by Theorems 1.7 and 1.9 below.

Theorem 1.4. *Take a unibranched Galois extension $\mathcal{E}=(L|K,v)$ of prime degree p, not necessarily equal to* $\operatorname{char} Kv$. *If* $\operatorname{char} K=0$, *then assume in addition that K contains a primitive p-th root of unity. Then there is a parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable $\mathcal{O}_L$-ideal $J_{\mathcal{E}}$ such that*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\simeq\; J_{\mathcal{E}}/J_{\mathcal{E}}^p\,. \tag{3}$$

For each unibranched Galois extension $\mathcal{E}=(L|K,v)$ of prime degree p, not necessarily equal to $\operatorname{char} Kv$, with Galois group G, we consider the $\mathcal{O}_L$-ideal

$$I_{\mathcal{E}} \;:=\; \left(\frac{\sigma b-b}{b}\;\Big|\;\sigma\in G\,,\;b\in L^\times\right)\,. \tag{4}$$

For the properties of $I_{\mathcal{E}}$, see Section 2.2. We say that the extension $\mathcal{E}$ is **purely wild** if $I_{\mathcal{E}}$ is contained in $\mathcal{M}_L$. This matches the (more general) definition of "purely wild extension" that we will give in Section 2.2.

If I is any $\mathcal{O}_L$-ideal, its **invariance valuation ring** $\mathcal{O}(I)$ (see the definition in Section 2.5) is the largest of all coarsenings $\mathcal{O}'$ of $\mathcal{O}_L$ such that I is an $\mathcal{O}'$-ideal. The ideal $I_{\mathcal{E}}$ and the notion of invariance valuation ring will be instrumental for the determination of ideals $J_{\mathcal{E}}$ that satisfy (3), and for the proof of the following result at the end of Section 2.5.

Theorem 1.5. *Take a purely wild Galois extension $\mathcal{E}=(L|K,v)$ of prime degree $p=\operatorname{char} Kv$. Then there is a coarsening $\tilde{\mathcal{O}}_{\mathcal{E}}$ of $\mathcal{O}_L$ with associated valuation $\tilde{v}_{\mathcal{E}}$ such that $\tilde{v}_{\mathcal{E}}$ is the coarsest coarsening of v_L such that $(L|K,\tilde{v}_{\mathcal{E}})$ is still a purely wild extension.*

In one case we can show that $\tilde{\mathcal{O}}_{\mathcal{E}}$ is equal to a parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable coarsening of $\mathcal{O}_L$ that we will introduce in the next section (see Proposition 1.8). In all other cases, we do not know whether $\tilde{\mathcal{O}}_{\mathcal{E}}$ has a parameter-free $\mathcal{L}_{\mathrm{val},K}$-definition.

The annihilator of an $\mathcal{O}_L$-module M is the largest among all $\mathcal{O}_L$-ideals J for which $JM=\{0\}$; we denote it by $\operatorname{ann} M$. Hence if $M=0$, then $\operatorname{ann} M=\mathcal{O}_L$, which trivially has a parameter-free $\mathcal{L}_{\mathrm{val}}$-definition. The following result will be proved by Theorems 1.7 and 1.9 below.

Theorem 1.6. *Under the assumptions of Theorem 1.4,* $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ *has a parameter-free $\mathcal{L}_{\mathrm{val},K}$-definition.*

1.2. **Galois defect extensions of prime degree.** These extensions have been studied in [21] and [2]. Take a Galois defect extension $\mathcal{E}=(L|K,v)$ of prime degree p. Then $\operatorname{char} Kv=p$, the extension $\mathcal{E}$ is purely wild, and $I_{\mathcal{E}}$ is its unique ramification ideal (see Section 2.2). We set

$$\mathcal{O}_{\mathcal{E}} \;:=\; \mathcal{O}(I_{\mathcal{E}})$$

and denote its maximal ideal by $\mathcal{M}_{\mathcal{E}}$. However, the reader should note that the ideals $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ will be chosen differently in the case of defectless Galois extensions of prime degree, see Section 1.3 below.

We will prove in Section 4:

Theorem 1.7. *Take a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p.*
1) The ideals $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ are parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v).
2) If $\mathrm{char}\, K = 0$, then assume in addition that K contains a primitive p-th root of unity. Then also the ideals $I_{\mathcal{E}}$ and $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ are parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v), and the ideal $J_{\mathcal{E}} = I_{\mathcal{E}}$ satisfies (3).

This theorem proves Theorems 1.4 and 1.6 for the case of $\mathcal{E}$ a defect extension.

Another example for the importance of the ideals $I_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ is that they allow us to give an elegant reformulation and generalization of a definition given in [21] and [2]: we say that $\mathcal{E}$ has **independent defect** if

$$I_{\mathcal{E}} \;=\; \mathcal{M}_{\mathcal{E}} \quad \text{and } \mathcal{M}_{\mathcal{E}} \text{ is a nonprincipal } \mathcal{O}_{\mathcal{E}}\text{-ideal}, \tag{5}$$

otherwise we say that $\mathcal{E}$ has **dependent defect**. Another, equivalent, definition independent defect will be given in (7) below. We prove in Section 4:

Proposition 1.8. *If $\mathcal{E}$ has independent defect, then $\mathcal{O}_{\mathcal{E}} = \tilde{\mathcal{O}}_{\mathcal{E}}$. Hence in this case, $\tilde{\mathcal{O}}_{\mathcal{E}}$ is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable. In the case of dependent defect, $\mathcal{O}_{\mathcal{E}} \subseteq \tilde{\mathcal{O}}_{\mathcal{E}}$.*

According to part 1) of Theorem 1.7, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \simeq I_{\mathcal{E}}/I_{\mathcal{E}}^p$. By [2, Theorem 1.4], this is 0 if and only if $\mathcal{E}$ has independent defect. Hence in this case, $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$ has a parameter-free $\mathcal{L}_{\mathrm{val}}$-definition. For the case of dependent defect, see Section 4.3 for the computation of annihilator.

1.3. **Defectless Galois extensions of prime degree.** The paper [3] is devoted to the case of unibranched defectless extensions $\mathcal{E} = (L|K, v)$ of prime degree p, not necessarily equal to $\mathrm{char}\, Kv$. Then we either have $[L : K] = (vL : vK)$ or $[L : K] = [Lv : Kv]$. Ideals $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ are not needed in the latter case and will therefore not be defined. The interesting case is the one of $[L : K] = (vL : vK)$, for which we determine in [3] a presentation of $\mathcal{O}_L$ as a union over a chain of simple ring extensions of $\mathcal{O}_K$. As a byproduct we obtain the $\mathcal{O}_L$-ideal

$$I_x \;:=\; (cx \mid c \in K \text{ with } vcx > 0)\,, \tag{6}$$

where x is a generator of the extension $L|K$ as described in Theorem 5.5. In [3, Proposition 3.4] it is shown that I_x is the maximal ideal $\mathcal{M}_{\mathcal{E}}$ of a coarsening $\mathcal{O}_{\mathcal{E}}$ of $\mathcal{O}_L$. It follows that $\mathcal{O}_{\mathcal{E}} = \mathcal{O}(I_x)$.

The reason why in the defectless case $\mathcal{O}(I_{\mathcal{E}})$ cannot be the right definition for $\mathcal{O}_{\mathcal{E}}$ is the fact that in this case, $I_{\mathcal{E}}$ is principal by [18, Proposition 3.1] and therefore, we always have $\mathcal{O}(I_{\mathcal{E}}) = \mathcal{O}_L$.

The following is the analogue of Theorem 1.7; we will prove it in Section 5:

Theorem 1.9. *Take a defectless unibranched Galois extension $\mathcal{E} = (L|K, v)$ of prime degree q, not necessarily equal to the residue characteristic.*
1) If $[L : K] = (vL : vK)$, then the ideals $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ are parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v).

2) Now assume in addition that K contains a primitive p-th root of unity if $\operatorname{char} K = 0$. *Set $J_{\mathcal{E}} = I_{\mathcal{E}}$ if $[L:K] = [Lv:Kv]$, and $J_{\mathcal{E}} = I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}$ if $[L:K] = (vL:vK)$. Then , and the ideals $I_{\mathcal{E}}$, $J_{\mathcal{E}}$ and* $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ *are parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v).*

Part 2) proves Theorems 1.4 and 1.6 for the case of $\mathcal{E}$ a defectless extension.

1.4. **Predescribed associated convex subgroups.** For Galois defect extensions $\mathcal{E} = (L|K,v)$ of prime degree, we have already defined in Section 1.2 the valuation rings $\mathcal{O}_{\mathcal{E}}$. They correspond to convex subgroups of vL via the definition

$$H_{\mathcal{E}} := v\mathcal{O}_{\mathcal{E}}^{\times} = v\mathcal{O}_{\mathcal{E}} \cap -v\mathcal{O}_{\mathcal{E}}$$

(cf. Section 2.4). Since the extension $\mathcal{E}$ is immediate, $H_{\mathcal{E}}$ is a convex subgroup of both vK and vL.

Using this notion, we can modify definition (5) for independent defect in the following way: $\mathcal{E}$ has **independent defect** if

(7) $\quad vI_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ and $vK/H_{\mathcal{E}}$ has no smallest positive element;

otherwise we will say that $\mathcal{E}$ has **dependent defect**. We will show in Proposition 4.3 that our definitions (5) and (7) are equivalent to the original definition given in [2, (6)] and in [21, (1.3)] (note that in the latter, the group $H_{\mathcal{E}}$ has not yet been given the definition that we use in the present paper).

If (K,v) has **rank 1** (i.e., its value group is order isomorphic to a subgroup of $\mathbb{R}$), then condition (7) just means that $vI_{\mathcal{E}}$ consists of all positive elements in vK (as vK not having a smallest positive element is automatic in this case). In the case of independent defect, we will call $H_{\mathcal{E}}$ the **convex subgroup associated with $\mathcal{E}$**. In order not to overload our sentences, we will write "associated convex subgroup" for "convex subgroup associated with a Galois defect extension of prime degree".

If H is a convex subgroup of an ordered abelian group Γ that is the smallest among all convex subgroups that contain a given element $\gamma \in \Gamma$, then we call it a **principal convex subgroup**, and if it is largest among all convex subgroups that do not contain a given element $\gamma \in \Gamma$, then we call it a **subprincipal convex subgroup**. A subprincipal convex subgroup may or may not be principal. For an ordered abelian group Γ, we denote by $\mathcal{C}(\Gamma)$ the chain of its convex subgroups, ordered by inclusion, and by $\mathcal{C}_{\mathrm{pr}}(\Gamma) \subseteq \mathcal{C}(\Gamma)$ the chain of its principal convex subgroups.

Now take any totally ordered set I and Γ to be the Hahn sum $\coprod_{i\in I} C_i$ with $C_i = \mathbb{Z}$ for all $i \in I$; for its definition, see Section 6.3. Then Γ is an ordered abelian group with an order reversing bijection from I onto $\mathcal{C}_{\mathrm{pr}}(\Gamma) \setminus \{0\}$. In Section 6 we will prove:

Theorem 1.10. *Let p be a prime and Γ as above. Then for each subset $\mathcal{C}^{\mathrm{sp}} \subseteq \mathcal{C}(\Gamma)$ containing only subprincipal convex subgroups, there is a perfect valued field (K,v) of characteristic p such that:*
a) vK is the p-divisible hull of Γ,
b) the associated convex subgroups of (K,v) are exactly the convex hulls of the elements of $\mathcal{C}^{\mathrm{sp}}$ in vK.

A corresponding result for valued fields of mixed characteristic requires much more technical effort and will thus be postponed to a subsequent paper.

In an ordered abelian group Γ with only finitely many proper convex subgroups (i.e., convex subgroups not equal to Γ), each of them is subprincipal. Therefore, the next result follows immediately from our theorem:

Corollary 1.11. *Let p be a prime and Γ as above, and assume that I is finite. Then for each set $\mathcal{H}$ of proper convex subgroups of Γ, there exists a perfect valued field (K,v) of characteristic p such that statements a) and b) of Theorem 1.10 hold (with $\mathcal{H}$ in place of $\mathcal{C}^{\rm sp}$).*

2. Preliminaries

2.1. **Basic properties of extensions of valued fields.** For a valued field (K,v), we denote the value group by vK, the residue field by Kv, the valuation ring by $\mathcal{O}_K$, and its maximal ideal by $\mathcal{M}_K$. We set $vK^{>0} := \{\alpha \in vK \mid \alpha > 0\}$ and $vK^{<0} := \{\alpha \in vK \mid \alpha < 0\}$. If A is a subset of K, then $vA := \{va \mid a \in A\}$. Throughout, we will use the convention that $v0 = \infty > \alpha$ for all $\alpha \in vK$.

By $(L|K,v)$ we denote a field extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto. In this case, if $a \in L \setminus K$, then

$$v(a-K) \;:=\; \{v(a-c) \mid c \in K\}$$

has no maximal element; this follows from [17, Lemma 2.9 2)] and the fact that each subextension of an immediate extension is immediate. By [17, Lemma 2.9 3)], an extension $(L|K,v)$ is immediate if for all $a \in L \setminus K$, $v(a-K)$ has no maximal element.

A subset S of a totally ordered set Γ is an **initial segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \leq \alpha$, it follows that $\gamma \in S$. Symmetrically, S is a **final segment** of Γ if for every $\alpha \in S$ and every $\gamma \in \Gamma$ with $\gamma \geq \alpha$, it follows that $\gamma \in S$. By [17, Lemma 2.8 1)] the set $v(a-K) \cap vK$ is an initial segment of vK for arbitrary extensions $(L|K,v)$ and $a \in L$. A final segment is called **principal** if it contains a smallest element.

Let Γ be an ordered abelian group. If S is a final segment of Γ and if $\gamma \in \Gamma$, then $\gamma + S := \{\gamma + \alpha \mid \alpha \in S\}$ is again a final segment of Γ. If S is an initial segment of Γ, then $-S := \{-\alpha \mid \alpha \in S\}$ is a final segment of Γ.

If $(L|K,v)$ is a finite unibranched extension, then by the Lemma of Ostrowski ([25, Corollary to Theorem 25, Section G, p. 78]),

$$[L:K] \;=\; \tilde{p}^{\nu} \cdot (vL:vK)[Lv:Kv]\,, \tag{8}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \mathrm{char}\, Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K,v) := \tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. If $d(L|K,v) = 1$, then the extension $(L|K,v)$ is called **defectless**; otherwise we call it a **defect extension**. A henselian field (K,v) is a **defectless field** if every finite unibranched extension of (K,v) is defectless; note that this is always the case if $\mathrm{char}\, Kv = 0$.

Equation (8) implies what is called the Fundamental Inequality

$$[L:K] \;\geq\; (vL:vK)[Lv:Kv]\,.$$

Take a valued field (K,v) and a valuation $\bar{v}$ on Kv. As compositions of functions, neither "$v\circ\bar{v}$" nor "$\bar{v}\circ v$" make sense. However, if P_v and $P_{\bar{v}}$ are the places associated with v and $\bar{v}$, respectively, then composition makes sense. If $a\in K$, then its image under the composition is $a(P_vP_{\bar{v}}) = (aP_v)P_{\bar{v}} = (av)\bar{v}$. Therefore, we will write the composition of v and $\bar{v}$, i.e., the valuation associated with $P_vP_{\bar{v}}$, as $v\circ\bar{v}$.

2.2. **Preliminaries from ramification theory.**

For a crash course in ramification theory, see [10, §5]. For more detail, see [23, §9]. Take a valued field extension $\mathcal{E} = (L|K,v)$. Assume that $L|K$ is a Galois extension, and let G denote its Galois group. We define the **series of upper ramification groups**

$$G_I \;:=\; \left\{\sigma\in G \;\middle|\; \frac{\sigma b-b}{b}\in I \;\text{ for all } b\in L^\times\right\}\,, \tag{9}$$

where I runs through all $\mathcal{O}_L$-ideals (cf. [25, §12]). Every G_I is a normal subgroup of G ([25, (d) on p.79]). Note that $G_{\mathcal{O}_L}$ is the decomposition group of $\mathcal{E}$; this easily follows from its definition as $\{\sigma\in G \mid v\sigma = v \text{ on } L\}$ (see [10, §5 (16)]). This also implies that $\mathcal{E}$ is unibranched if and only if $G = G_{\mathcal{O}_L}$ (cf. [10, Theorem 5.4 b)]).

Further, $G_{\mathcal{M}_L}$ is the ramification group of $\mathcal{E}$ (see [10, §5 (18)]). We call G_I a **higher ramification group** if it is a nontrivial subgroup of $G_{\mathcal{M}_L}$. Every G_I is a higher ramification group if and only if $G = G_{\mathcal{M}_L}$.

The function

$$\varphi:\; I \;\mapsto\; G_I \tag{10}$$

from the set of $\mathcal{O}_L$-ideals to the set of upper ramification groups preserves $\subseteq$, that is, if $I\subseteq J$, then $G_I\subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the series of upper ramification groups is linearly ordered by inclusion. Note that in general, φ will neither be injective nor surjective as a function to the set of normal subgroups of G. This gives rise to the task to determine the smallest ideal that is sent by φ to a group G_I in its image. To this end, for each subgroup H of $G_{\mathcal{O}_L}$ we introduce the $\mathcal{O}_L$-ideal

$$I_H \;:=\; \left(\frac{\sigma b-b}{b} \;\middle|\; \sigma\in H\,,\, b\in L^\times\right) \tag{11}$$

and consider the function

$$\psi:\; H \;\mapsto\; I_H \tag{12}$$

from the set of all subgroups H of $G_{\mathcal{O}_L}$ to the set of all $\mathcal{O}_L$-ideals. Also ψ preserves $\subseteq$ and is in general neither injective nor surjective. However, it is easy to see that $G_{(0)} = \{\mathrm{id}\}$ and $I_{\{\mathrm{id}\}} = (0)$. If I_H is nonzero, then it is called it a **ramification ideal**; if in addition it is contained in $\mathcal{M}_L$, then it is called it a **higher ramification ideal**. Observe that the only ramification ideal that is not a higher ramification ideal is $\mathcal{O}_L$ and therefore, every ramification ideal of an extension satisfying $G = G_{\mathcal{M}_L}$ is a higher ramification ideal.

Remark 2.1. In [21], ramification ideals are defined by use of ramification jumps. For the equivalence of the two definitions, see [18, Proposition 2.7]. #

The following is [18, Proposition 2.6 3),4)]:

Proposition 2.2. *The function φ is an inclusion preserving bijection from the set of all $\mathcal{O}_L$-ideals of the form (11) onto the set of all upper ramification groups, with ψ its inverse. The function φ induces an inclusion preserving bijection from the set of all higher ramification ideals onto the set of all higher ramification groups, with its inverse induced by ψ.*

If $\mathcal{E} = (L|K, v)$ is a unibranched Galois extension of prime degree, then the only subgroups of $G = \mathrm{Gal}\, L|K$ are G and $\{\mathrm{id}\}$, and we set

$$I_{\mathcal{E}} := I_G\,,$$

so that $I_{\mathcal{E}}$ is the unique ramification ideal of $\mathcal{E}$. In general, if the degree of a unibranched Galois extension $\mathcal{E}$ is $p^k n$ where $p = \mathrm{char}\, Kv$ and $(p, n) = 1$, then the number of distinct ramification ideals of $\mathcal{E}$ is at most k. In [18, Proposition 3.22] an example of a unibranched Galois extension of degree p^2 is given which has only one ramification ideal.

For more information on ramification ideals, see [18].

A unibranched algebraic extension $(L|K, v)$ is called **tame** if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK' : vK)$ is not divisible by $\mathrm{char}\, Kv$,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K, v)$ is defectless.

A henselian valued field (K, v) is called a **tame field** if the algebraic closure K^{ac} of K with the unique extension of v is a tame extension of (K, v). It follows from conditions (T1)–(T3) that all tame fields are perfect defectless fields. For the algebra and model theory of tame fields, see [15].

The **ramification field** of a Galois extension $(L|K, v)$ with Galois group G is the fixed field in L of the ramification group $G_{\mathcal{M}_L}$. When dealing with a valued field (K, v), we will tacitly assume v extended to its algebraic closure. Then the **absolute ramification field of** (K, v) (with respect to the chosen extension of v), denoted by (K^r, v), is the ramification field of the Galois extension $(K^{\mathrm{sep}}|K, v)$. If $(K(a)|K, v)$ is a finite defect extension, then $(K^r(a)|K^r, v)$ is a defect extension with the same defect (see [21, Proposition 2.13]). On the other hand, $K^{\mathrm{sep}}|K^r$ is a p-extension (see [4, Theorem (20.18)]), so every finite extension of K^r is a tower of purely inseparable extensions and Galois extensions of degree p. If (K, v) is henselian, then (K^r, v) is its unique maximal tame extension (see [20, Proposition 4.1]). Hence the next fact follows from [21, Proposition 2.13]:

Lemma 2.3. *If (K, v) is henselian, $(K(a)|K, v)$ is a finite defect extension, and $(L|K, v)$ is a tame extension, then $d(L(a)|L, v) = d(K(a)|K, v)$.*

A unibranched algebraic extension $(L|K, v)$ is called **purely wild** if every finite subextension $(L_0|K, v)$ satisfies:

(a) $(vL_0 : vK)$ is a power of the characteristic exponent of Kv,

(b) $L_0v|Kv$ is purely inseparable.

An extension $\mathcal{E} = (L|K, v)$ is purely wild if and only if $L|K$ and $K^r|K$ are linearly disjoint (see [20, Lemma 4.2]). If $\mathcal{E}$ is a Galois extension, then the equality $G = G_{\mathcal{M}_L}$ is equivalent to $\mathcal{E}$ being purely wild. Indeed, $G = G_{\mathcal{M}_L}$ holds if and only if K is the ramification field of $(L|K, v)$, which by [4, (20.15), p. 166] is equal to $K^r \cap L$. Now $K^r \cap L = K$ holds if and only if $L|K$ and $K^r|K$ are linearly disjoint, that is, $\mathcal{E}$ is purely wild. Note that every purely wild extension is unibranched since K^r contains the henselization of (K, v) (under the chosen extension of v to the algebraic closure), and by [14, Theorem 2], L being linearly disjoint from the henselization over K implies that $(L|K, v)$ is unibranched.

In Section 1.1 we defined a Galois extension $\mathcal{E}$ of prime degree to be purely wild if $I_{\mathcal{E}} \subseteq \mathcal{M}_L$, that is, $I_{\mathcal{E}}$ is a higher ramification ideal. By Proposition 2.2, this holds if and only if $G_{I_{\mathcal{E}}}$ is a higher ramification group, which implies the equality $G = G_{\mathcal{M}_L}$. So the definition of "purely wild extension" we gave in Section 1.1 is compatible with our above definition.

Lemma 2.4. *A unibranched Galois extension $\mathcal{E} = (L|K, v)$ of prime degree is tame if and only if $I_{\mathcal{E}} = \mathcal{O}_L$.*

Proof. It follows from the definitions that $\mathcal{E}$ is tame if and only if it is not purely wild, that is, the ideal $I_{\mathcal{E}}$ is not contained in $\mathcal{M}_L$ and therefore must be equal to $\mathcal{O}_L$. □

The following statements are consequences of [15, Theorem 3.2] and [24, Proposition 2.5].

Lemma 2.5. *1) Take a valued field (K, v) with perfect residue field characteristic $p > 0$ and p-divisible value group. Then every maximal immediate algebraic extension of (K, v) is a tame field and therefore a defectless field.*

2) Every algebraic extension of a tame field is again a tame field.

2.3. **Kähler differentials.** Assume that A is a ring and B is an A-algebra. Then $\Omega_{B|A}$ denotes the **module of relative differentials** (also called **Kähler differentials**), that is, the B-module for which there is a universal derivation

$$d : B \to \Omega_{B|A}$$

such that for every B-module M and derivation $\delta : B \to M$ there is a unique B-module homomorphism

$$\phi : \Omega_{B|A} \to M$$

such that $\delta = \phi \circ d$.

For background on Kähler differentials, see [22], [7], [2] and [3].

2.4. **Coarsenings of valuation rings and their associated valuations.**

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are valuation rings with associated valuation v_1 and v_2, then we call $\mathcal{O}_2$ a **coarsening of** $\mathcal{O}_1$ and v_2 a **coarsening of** v_1 if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, and we talk of **proper coarsening** if $\mathcal{O}_1 \subsetneq \mathcal{O}_2$.

Take any valued field (L, v). For every coarsening $\mathcal{O}$ of $\mathcal{O}_L$, we set

$$H(\mathcal{O}) := v\mathcal{O} \cap -v\mathcal{O} = v\mathcal{O}^\times .$$

This is a convex subgroup of the value group vL of (L,v). If $\mathcal{M}$ is the maximal ideal of $\mathcal{O}$, then

$$v\mathcal{M} \;=\; \{\alpha \in vL \mid \alpha > v\mathcal{O}^\times\} \;=\; \{\alpha \in vL \mid \alpha > H(\mathcal{O})\}\,. \tag{13}$$

The valuation w associated with $\mathcal{O}$ is (up to equivalence) given by

$$wa \;=\; va + H(\mathcal{O}) \in vL/H(\mathcal{O}) \tag{14}$$

for every $a \in L$, the value group of w is (canonically isomorphic to) $vL/H(\mathcal{O})$, and the value group of the valuation induced by v on the residue field Lw is canonically isomorphic to $H(\mathcal{O})$ (see [5, Section 2.3]). The function $\mathcal{O} \mapsto H(\mathcal{O})$ sends every coarsening $\mathcal{O}$ of $\mathcal{O}_L$ to a convex subgroup of vL. Its inverse is given by sending a convex subgroup H of vL to

$$\mathcal{O}(H) \;:=\; \{b \in L \mid \exists \alpha \in H :\; \alpha \leq vb\}\,. \tag{15}$$

We call this the **coarsening of $\mathcal{O}_L$ associated with H**. Its maximal ideal is

$$\mathcal{M}(H) \;:=\; \{b \in L \mid \forall \alpha \in H :\; \alpha < vb\}\,. \tag{16}$$

2.5. **Invariance group and invariance valuation ring.**

The survey [16] provides the conceptual background and various applications of invariance groups and invariance valuation rings, while the technical results used here are taken from [19].

Take any valued field (L,v) and $\mathcal{O}_L$-ideal I. We set

$$\mathcal{O}(I) \;:=\; \{b \in L \mid bI \subseteq I\} \quad \text{and} \quad \mathcal{M}(I) \;=\; \{b \in L \mid bI \subsetneq I\}\,. \tag{17}$$

We call $\mathcal{O}(I)$ the **invariance valuation ring** of I. The following is part of [19, Theorem 3.6]:

Proposition 2.6. *For every $\mathcal{O}_L$-ideal I, $\mathcal{O}(I)$ is a valuation ring of L containing $\mathcal{O}_L$, with maximal ideal $\mathcal{M}(I)$, which is a prime $\mathcal{O}_L$-ideal. It is the largest of all valuation rings $\mathcal{O}$ of L containing $\mathcal{O}_L$ for which I is an $\mathcal{O}$-ideal.*

If the ideal I is definable (or parameter-free definable) in an expansion $\mathcal{L}$ of $\mathcal{L}_{\rm val}$, then also $\mathcal{O}(I)$ and $\mathcal{M}(I)$ are definable (or parameter-free definable, respectively) in $\mathcal{L}$:

$$\mathcal{O}(I) \;=\; \{b \in L \mid \forall c \in I :\; bc \in I\}, \tag{18}$$

$$\mathcal{M}(I) \;=\; \{b \in \mathcal{O}(I) \mid \exists a \in I\, \forall c \in I :\; bc \neq a\}\,. \tag{19}$$

Note that we could also have used the equivalence

$$x \in \mathcal{M} \iff \frac{1}{x} \notin \mathcal{O}$$

for the maximal ideal $\mathcal{M}$ of a valuation ring $\mathcal{O}$, which shows that $\mathcal{M}$ is $\mathcal{L}$-definable if and only if $\mathcal{O}$ is. Throughout this paper, we can use this principle to obtain a definition of $\mathcal{M}_\mathcal{E}$ from that of $\mathcal{O}_\mathcal{E}$. However, we will frequently not use this if an alternate definition is simpler or more interesting.

Take a subset S of an ordered abelian group Γ. We take its **invariance group** to be

$$\mathcal{G}(S) \;:=\; \{\gamma \in \Gamma \mid S + \gamma = S\}\,.$$

This is a subgroup of Γ, and it is a convex subgroup if S is convex (which in particular is the case if S is an initial or a final segment of Γ). If $\gamma \in \Gamma$, then

$$\mathcal{G}(\gamma + S) \;=\; \mathcal{G}(S) \;=\; \mathcal{G}(-S)\,. \tag{20}$$

For these facts and more results on invariance groups, see [19, Section 2.4].

By [19, Theorem 3.6 3)],

$$H(\mathcal{O}(I)) \;=\; \mathcal{G}(vI) \tag{21}$$

If the ideal I is principal, then so is the final segment vI. This implies that $H(\mathcal{O}(I)) = \mathcal{G}(vI) = \{0\}$, which in turn yields $\mathcal{O}(I) = \mathcal{O}_L$.

The next result is part of [19, Lemma 3.5].

Lemma 2.7. *For every coarsening $\mathcal{O}$ of $\mathcal{O}_L$ with maximal ideal $\mathcal{M}$,*

$$H(\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{M})\,. \tag{22}$$

We leave the straightforward proof of the following result to the reader.

Lemma 2.8. *If I is an $\mathcal{O}_L$-ideal and $J = aI$ with $0 \neq a \in L$, then $\mathcal{O}(J) = \mathcal{O}(I)$ and $\mathcal{M}(J) = \mathcal{M}(I)$.*

We conclude this section with the

Proof of Proposition 1.5: Since $\mathcal{E}$ is purely wild, $I_{\mathcal{E}}$ is contained in $\mathcal{M}_L$, so $vI_{\mathcal{E}} \cap \{0\} = \emptyset$. Take $\tilde{H}_{\mathcal{E}}$ to be the largest of all convex subgroups H' of vL such that $vI_{\mathcal{E}} \cap H' = \emptyset$, and set $\tilde{\mathcal{O}}_{\mathcal{E}} = \mathcal{O}(\tilde{H}_{\mathcal{E}})$. Now consider a proper coarsening $\mathcal{O}'$ of $\tilde{\mathcal{O}}_{\mathcal{E}}$. Then the convex subgroup $H(\mathcal{O}')$ of vL properly contains $\tilde{H}_{\mathcal{E}}$, so $vI_{\mathcal{E}} \cap H(\mathcal{O}') \neq \emptyset$. Hence $0 \in vI_{\mathcal{E}} \cap H(\mathcal{O}')$ and if v' denotes the coarsening of v associated with $\mathcal{O}'$, then this means that $0 \in v'I_{\mathcal{E}}$. Consequently, $I_{\mathcal{E}}$ is not contained in the maximal ideal of $\mathcal{O}'$ and therefore, $(L|K, v')$ is not purely wild.

The inclusion $\mathcal{O}_{\mathcal{E}} \subseteq \tilde{\mathcal{O}}_{\mathcal{E}}$ holds since $\mathcal{E}$ is purely wild. □

3. Immediate elements in arbitrary valued field extensions

Take any valued field extension $(L|K, v)$. We call $a \in L \setminus K$ an **immediate element** in $(L|K, v)$ if the set $v(a - K)$ has no maximal element and is an initial segment of vK. Consequently, $(L|K, v)$ is immediate if and only if every $a \in L \setminus K$ is an immediate element in $(L|K, v)$.

Assume that $a \in L \setminus K$ is an immediate element in $(L|K, v)$, then we set

$$I_{a-K;K} \;:=\; (b \in K \mid vb \in -v(a - K)) \tag{23}$$

and

$$I_{a-K} \;:=\; (b \in L \mid \exists c \in K :\; vb \geq -v(a - c))\,. \tag{24}$$

If $v(a - K) = vK$, or in other words, a lies in the completion of (K, v), then $I_{a-K;K} = K$. If $v(a - K)$ is bounded from above, then $-v(a - K)$ is bounded from below and therefore, $I_{a-K;K}$ is a fractional $\mathcal{O}_K$-ideal and I_{a-K} is a fractional $\mathcal{O}_L$-ideal.

By (24), I_{a-K} is $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v) with parameter a. Hence by (18) and (19), also the invariance valuation ring $\mathcal{O}(I_{a-K})$ and its maximal ideal $\mathcal{M}(I_{a-K})$ are $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v) with parameter a.

Since a is not a parameter in K, we may in general not have a parameter-free $\mathcal{L}_{\mathrm{val},K}$-definition of $\mathcal{O}(I_{a-K;K})$ and $I_{a-K;K}$ in (K,v). For example, we have not even excluded the case where a is transcendental over K. On the other hand, if a is algebraic over K with a suitable minimal polynomial, then the situation may change, as we will see in Sections 4.1 and 4.2.

4. Defect extensions of prime degree

Throughout this section, we let $\mathcal{E} = (L|K,v)$ be a Galois defect extension of prime degree p with Galois group G. For every $\sigma \in G$ with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma b - b}{b}\right) \,\middle|\, b \in L^\times,\ \sigma b \neq b \right\} . \tag{25}$$

This set is a final segment of $vL = vK$ and independent of the choice of σ (see [21, Theorems 3.4 and 3.5]); we denote it by $\Sigma_{\mathcal{E}}$. As a consequence, we find that

$$\Sigma_{\mathcal{E}} \;=\; vI_G \;=\; vI_{\mathcal{E}} . \tag{26}$$

Recall that the definitions of $\mathcal{O}_{\mathcal{E}}$ and its maximal ideal $\mathcal{M}_{\mathcal{E}}$ were given in Section 1.2, and the definition of $H_{\mathcal{E}}$ was given in Section 1.4. The following result is part of [21, Theorem 3.4].

Proposition 4.1. *For every $a \in L \setminus K$ and every generator σ of G,*

$$\Sigma_{\mathcal{E}} \;=\; -v(a-K) + v(\sigma a - a) . \tag{27}$$

Proposition 4.2. *Under the above assumptions on $\mathcal{E}$,*

$$\mathcal{O}_{\mathcal{E}} = \mathcal{O}(I_{a-K}) \;=\; \{b \in L \mid bI_{a-K} \subseteq I_{a-K}\} \tag{28}$$

for every $a \in L \setminus K$.

Proof. By definition, $\mathcal{O}_{\mathcal{E}} = \mathcal{O}(I_{\mathcal{E}})$. Equation (26) implies that

$$I_{\mathcal{E}} \;=\; \{b \in L \mid vb \in -v(a-K) + v(\sigma a - a)\} \;=\; (\sigma a - a)I_{a-K} .$$

By Lemma 2.8,

$$\mathcal{O}_{\mathcal{E}} \;=\; \mathcal{O}(I_{\mathcal{E}}) \;=\; \mathcal{O}((\sigma a - a)I_{a-K}) \;=\; \mathcal{O}(I_{a-K}) ,$$

which proves (28). □

Proof of part 1) of Theorem 1.7:
Based on (28), $\mathcal{O}_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in L \setminus K\ \forall c \in K\ \exists c' \in K :\ vb(x-c) = v(x-c')\} \cup \{0\} , \tag{29}$$

and the quantifier "$\forall x \in L \setminus K$" can also be replaced by "$\exists x \in L \setminus K$". Further, $\mathcal{M}_{\mathcal{E}} = \{b \in L \mid bI_{a-K} \subsetneq I_{a-K}\}$ has the following parameter free definition in the language $\mathcal{L}_{\mathrm{val},K}$:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b \in \mathcal{O}_{\mathcal{E}} \mid \exists x \in L \setminus K\ \exists c \in K\ \forall c' \in K :\ vb(x-c) \neq v(x-c')\} . \tag{30}$$

□

For Artin-Schreier and Kummer extensions with indepedent defect, we give alternative definitions of $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ in Sections 4.1 and 4.2.

We will now show that the three definitions of independent defect which we use are equivalent. By definition, $H_{\mathcal{E}} = H(\mathcal{O}_{\mathcal{E}})$. Hence by Lemma 2.7, it is the invariance group of $v\mathcal{O}_{\mathcal{E}}$ and of $v\mathcal{M}_{\mathcal{E}}$. From (21) together with (26) we obtain:

$$H_{\mathcal{E}} \;=\; \mathcal{G}(vI_{\mathcal{E}}) \;=\; \mathcal{G}(\Sigma_{\mathcal{E}})\,. \tag{31}$$

Proposition 4.3. *The following statements are equivalent:*

a) $I_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal (definition of "independent defect" in (5));

b) $vI_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ and $vK/H_{\mathcal{E}}$ has no smallest positive element (definition of "independent defect" in (7));

c) $\Sigma_{\mathcal{E}} = \{\alpha \in vL \mid \alpha > H\}$ for some proper convex subgroup H of vL such that vL/H has no smallest positive element (definition of "independent defect" in [2]*).*

Proof. a)⇔b): By (13), $v\mathcal{M}_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H(\mathcal{O}_{\mathcal{E}})\} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$. Hence the equality $vI_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$ in (7) is equivalent to $vI_{\mathcal{E}} = v\mathcal{M}_{\mathcal{E}}$. Since the function $M \mapsto vM := \{va \mid a \in M\}$ that sends every $\mathcal{O}_L$-module $M \subseteq L$ to a corresponding final segment in vL is bijective, the latter equality is equivalent to the equality $I_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$. Further, as $vK/H_{\mathcal{E}}$ is the value group of $\mathcal{O}_{\mathcal{E}}$, $vK/H_{\mathcal{E}}$ having no smallest positive element is equivalent to $\mathcal{M}_{\mathcal{E}}$ being a nonprincipal $\mathcal{O}_{\mathcal{E}}$-module.

b)⇒ c): This holds since $vI_{\mathcal{E}} = \Sigma_{\mathcal{E}}$ by equation (26).

c)⇒ b): Assume that $\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H\}$ for some proper convex subgroup H of vL such that vL/H has no smallest positive element. Then by [19, Lemma 2.13 5)], H is the invariance group of $\Sigma_{\mathcal{E}}$, hence by (31), it is equal to $H_{\mathcal{E}}$. Further, $vI_{\mathcal{E}} = \Sigma_{\mathcal{E}}$ by (26). □

While we have given parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$ of $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$, the problem with doing the same for $I_{\mathcal{E}}$ is that we may not have enough information on the factor $\sigma a - a$. We will now show that this changes when we know that the extension is an Artin-Schreier or a Kummer extension of prime degree. For such extensions, we will also give alternative parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$ in the case of independent defect.

4.1. **The equal characteristic case.** Let us first discuss the case where (K, v) is of **equal positive characteristic**, that is, $\operatorname{char} K = \operatorname{char} Kv = p > 0$. Then every Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p is an Artin-Schreier extension, that is, generated by an **Artin-Schreier generator** $\vartheta \in L \setminus K$ with $\vartheta^p - \vartheta \in K$. By [21, Theorem 3.5],

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta - K)\,. \tag{32}$$

for every such ϑ. Further, $v(\sigma\vartheta - \vartheta) = 0$, hence

$$I_{\mathcal{E}} \;=\; I_{\vartheta - K} \;=\; \{b \in L \mid \exists c \in K :\; vb \geq -v(\vartheta - c)\}$$

in this case. Equation (32) shows that the set $v(\vartheta - K)$ does not depend on the choice of the Artin-Schreier generator of $L|K$, hence $I_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K\ \exists c \in K :\; x^p - x \in K \;\wedge\; vb \geq -v(x - c)\} \tag{33}$$

and

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in L \setminus K \; \exists c \in K : \; x^p - x \in K \;\to\; vb \geq -v(x-c)\} \,. \tag{34}$$

Now assume that $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$. Then equation (11) and the equivalence a)⇔c) from [2, Theorem 1.7] show that

$$pv(\vartheta - K) \;=\; v(\vartheta^p - \vartheta - \wp(K)) \;=\; \{\alpha \in pvK \mid \alpha < H_{\mathcal{E}}\} \,, \tag{35}$$

where $\wp(K) := \{a^p - a \mid a \in K\}$. Since $vL/H_{\mathcal{E}}$ has no smallest positive element, equation (35) is equivalent to

$$H_{\mathcal{E}} \;=\; \{\beta \in vK \mid \beta > v(\vartheta^p - \vartheta - \wp(K)) \text{ and } -\beta > v(\vartheta^p - \vartheta - \wp(K))\} \,. \tag{36}$$

In other words,

$$H_{\mathcal{E}} \;=\; \{\pm\beta \in vK \mid \forall c \in K : v(\vartheta^p - \vartheta - c^p + c) < \beta \leq 0\} \,. \tag{37}$$

The convex subgroup $H_{\mathcal{E}}$ gives rise to an $\mathcal{L}_{\rm val}$-definition with parameter $\vartheta^p - \vartheta$ of the coarsening $\mathcal{O}_{\mathcal{E};K} = \mathcal{O}(H_{\mathcal{E}}) = \{b \in K \mid \exists \alpha \in H_{\mathcal{E}} : \; \alpha \leq vb\}$ (taken in K) of the valuation ring $\mathcal{O}_K$, namely

$$\mathcal{O}_{\mathcal{E};K} \;=\; \{b \in K \mid \forall c \in K : v(\vartheta^p - \vartheta - c^p + c) < vb\} \,, \tag{38}$$

whose value group is $vK/H_{\mathcal{E}}$.

For our applications, we are more interested in the coarsening of $\mathcal{O}_L$ corresponding to $H_{\mathcal{E}}$. By (15),

$$\mathcal{O}_{\mathcal{E}} \;=\; \mathcal{O}(H_{\mathcal{E}}) \;=\; \{b \in L \mid \exists \alpha \in H_{\mathcal{E}} : \; \alpha \leq vb\} \,. \tag{39}$$

Since equation (35) does not depend on the choice of the Artin-Schreier generator of $L|K$, $\mathcal{O}_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{{\rm val},K}$:

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in L \setminus K \;\; \forall c \in K : \; x^p - x \in K \;\to\; v(x-c) < vb\} \tag{40}$$

and

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K \;\; \forall c \in K : \; x^p - x \in K \;\wedge\; v(x-c) < vb\} \,. \tag{41}$$

For the maximal ideal $\mathcal{M}_{\mathcal{E}}$ of $\mathcal{O}_{\mathcal{E}}$ we can give the following parameter free definition in the language $\mathcal{L}_{{\rm val},K}$:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K \; \exists c \in K : \; x^p - x \in K \;\wedge\; -v(x-c) \leq vb\} \,. \tag{42}$$

4.2. **The mixed characteristic case.** Now we discuss the case where (K, v) is of **mixed characteristic**, that is, $\operatorname{char} K = 0$ and $\operatorname{char} Kv = p > 0$. We assume in addition that K contains a primitive p-th root of unity ζ_p. Then every Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p is a **Kummer extension**, that is, generated by a **Kummer generator** $\eta \in L \setminus K$ with $\eta^p \in K$. We can choose η to be a 1-unit, i.e., $v(\eta - 1) > 0$ (see [2, Section 2.5]). Then by [21, Theorem 3.5 and Lemma 2.5],

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p - 1) - v(\eta - K) \;=\; \frac{1}{p-1} vp - v(\eta - K) \tag{43}$$

for every such η. Further, $\sigma\eta - \eta = \zeta_p - 1$ for suitable ζ_p, hence

$$I_{\mathcal{E}} \;=\; \{b \in L \mid \exists c \in K : \; vb \geq \frac{1}{p-1} vp - v(\vartheta - c)\}$$

in this case. Similarly as in the equal characteristic case, $I_{\mathcal{E}}$ has the following parameter free definitions in the language $\mathcal{L}_{\mathrm{val},K}$:

$$(44)\quad I_{\mathcal{E}} \;=\; \{b\in L \mid \exists x\in L\setminus K\ \exists c\in K:\ x^p-x\in K \,\wedge\, vb\geq \frac{1}{p-1}vp - v(x-c)\}$$

and

$$(45)\quad I_{\mathcal{E}} \;=\; \{b\in L \mid \forall x\in L\setminus K\ \exists c\in K:\ x^p-x\in K \,\rightarrow\, vb\geq \frac{1}{p-1}vp - v(x-c)\}\,.$$

Now assume that $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$. Then equation (12) and the equivalence a)⇔c) from [2, Theorem 1.8] show that

$$(46)\qquad v(\eta^p-K^p)-v(\zeta_p-1)^p \;=\; \{\alpha\in pvK \mid \alpha<H_{\mathcal{E}}\}\,.$$

Similarly as in the equal characteristic case, we obtain that

$$(47)\qquad H_{\mathcal{E}} \;=\; \{\pm\beta\in vK \mid \forall c\in K: v(\eta^p-c^p)-\frac{p}{p-1}vp<\beta\leq 0\}\,,$$

and we have the following $\mathcal{L}_{\mathrm{val}}$-definition with parameter η^p:

$$(48)\qquad \mathcal{O}_{\mathcal{E};K} \;=\; \{b\in K \mid \forall c\in K: v(\eta^p-c^p)-\frac{p}{p-1}vp<vb\}\,.$$

Note that for this definition and definition (38) it is not needed that (K,v) be henselian, and that in fact, they can be applied to deeply ramified fields, which are not required to be henselian. For the case of henselian fields (K,v), these definitions are used in [9, Theorem 4.11] to define corresponding henselian valuations on K that are definable in the language of rings.

Using again (39) together with the fact that equation (46) is independent of the choice of a 1-unit $\eta\in L\setminus K$ satisfying $\eta^p\in K$, we can give the following parameter free $\mathcal{L}_{\mathrm{val},K}$-definitions of $\mathcal{O}_{\mathcal{E}}$:

$$(49)\qquad \begin{aligned}\mathcal{O}_{\mathcal{E}} \;=\; \{b\in L \;\mid\; &\forall x\in L\setminus K\ \ \forall c\in K:\ (x^p\in K\,\wedge\, v(x-1)>0)\\ &\rightarrow\ v(x-c)-\frac{1}{p-1}vp<vb\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_{\mathcal{E}} \;=\; \{b\in L \;\mid\; &\exists x\in L\setminus K: x^p\in K\,\wedge\, v(x-1)>0\\ &\wedge\,\forall c\in K: v(x-c)-\frac{1}{p-1}vp<vb\}\,.\end{aligned}$$

For the maximal ideal of $\mathcal{O}_{\mathcal{E}}$ we can give the following parameterfree $\mathcal{L}_{\mathrm{val},K}$-definition:

$$(50)\qquad \begin{aligned}\mathcal{M}_{\mathcal{E}} \;=\; \{b\in L \;\mid\; &\exists x\in L\setminus K: x^p\in K\,\wedge\, v(x-1)>0\\ &\wedge\,\exists c\in K: -v(x-c)+\frac{1}{p-1}vp\leq vb\}\,.\end{aligned}$$

4.3. **Proof of Theorem 1.7.** We have proven the assertion of part 1) of Theorem 1.7 already in (29) and (30). The assertions about $I_{\mathcal{E}}$ in part 2) follow from equations (33), (34), (44) and (45), together with [2, Theorem 1.2], which shows that

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}/I_{\mathcal{E}}^p\,.$$

By [2, Theorem 1.4], $I_{\mathcal{E}}/I_{\mathcal{E}}^p$ is zero if and only if $\mathcal{E}$ has independent defect. In this case, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$. For the proof of the assertion about the annihilator in the case of $\mathcal{E}$ having dependent defect, we use the following reformulation of [2, Proposition 4.7 2)], denoting by $v_{\mathcal{E}}$ the valuation on L having valuation ring $\mathcal{O}_{\mathcal{E}}$:

Proposition 4.4. *If there is $a \in K \setminus I_{\mathcal{E}}^{p-1}$ such that $v_{\mathcal{E}}I_{\mathcal{E}}^{p-1}$ has infimum $v_{\mathcal{E}}a$ in $v_{\mathcal{E}}L$, then*

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;=\; a\,\mathcal{O}(I_{\mathcal{E}})\,, \tag{51}$$

which properly contains $I_{\mathcal{E}}^{p-1}$. In all other cases, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$.

We note that since $I_{\mathcal{E}}$ is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v), so are $I_{\mathcal{E}}^{p-1}$ and $\mathcal{O}(I_{\mathcal{E}})$. In view of the above proposition, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ has the following parameter-free $\mathcal{L}_{\mathrm{val},K}$-definition:

if $I_{\mathcal{E}} = I_{\mathcal{E}}^{p-1}$, then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$; else
if there exists $a \in K \setminus I_{\mathcal{E}}^{p-1}$ such that $b \in I_{\mathcal{E}}^{p-1}$ if and only if $v_{\mathcal{E}}b > v_{\mathcal{E}}a$,
then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = a\,\mathcal{O}(I_{\mathcal{E}})$; else $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$. □

Let us add one more result, which is of interest since by [2, Theorem 1.4] an Artin-Schreier or Kummer extension $\mathcal{E} = (L|K,v)$ has independent defect if and only if $\operatorname{Tr}_{L|K}(\mathcal{M}_L)$ is equal to $\mathcal{M}_{\mathcal{E}} \cap K$, which is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (K,v).

Proposition 4.5. *Take a Galois extension $\mathcal{E} = (L|K,v)$ of prime degree p with independent defect. If* $\operatorname{char} K = 0$*, then assume in addition that K contains a primitive p-th root of unity. Then the trace $\operatorname{Tr}_{L|K}(\mathcal{M}_L)$ is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (K,v).*

We do not know whether the last assertion also holds in the case of dependent defect.

5. Defectless extensions

Throughout this section, we will consider defectless unibranched extensions $\mathcal{E} = (L|K,v)$ of prime degree p, not necessarily equal to $\operatorname{char} Kv$. Then either $[L:K] = [Lv:Kv]$ or $[L:K] = (vL:vK)$. The first case will be discussed in Section 5.2, and the second in Section 5.3. As a preparation, we present the unique ramification ideals for both cases.

5.1. **The ideal $I_{\mathcal{E}}$.** Take a unibranched defectless Galois extension $\mathcal{E} = (L|K,v)$ of prime degree p, not necessarily equal to $\operatorname{char} Kv$. From [18, Theorem 3.19] we obtain:

Theorem 5.1. *1) If $\mathcal{E}$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ of value $v\vartheta \leq 0$ such that $1, \vartheta, \ldots, \vartheta^{p-1}$ form a valuation basis for $(L|K, v)$. For every such ϑ,*

$$I_{\mathcal{E}} \;=\; \left(\frac{1}{\vartheta}\right) . \tag{52}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v\vartheta = 0$, and this holds if and only if $Lv|Kv$ is separable of degree p.

2) Let $\mathcal{E}$ be a Kummer extension. Then there are two cases:

a) $\mathcal{E}$ admits a Kummer generator η such that $v\eta \geq 0$ and $1, \eta, \ldots, \eta^{p-1}$ form a valuation basis for $(L|K, v)$. For every such η,

$$I_{\mathcal{E}} \;=\; (\zeta_p - 1) . \tag{53}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $p \neq \operatorname{char} Kv$.

b) $\mathcal{E}$ admits a Kummer generator η such that η is a 1-unit with $v(\zeta_p - 1) \geq v(\eta - 1)$, $p = \operatorname{char} Kv$, and $1, \eta - 1, \ldots, (\eta - 1)^{p-1}$ form a valuation basis for $(L|K, v)$. For every such η,

$$I_{\mathcal{E}} \;=\; \left(\frac{\zeta_p - 1}{\eta - 1}\right) . \tag{54}$$

We have $I_{\mathcal{E}} = \mathcal{O}_L$ if and only if $v(\eta - 1) = v(\zeta_p - 1)$, and this holds if and only if $Lv|Kv$ is separable of degree p.

Let us show that under the assumptions of the theorem, $I_{\mathcal{E}}$ always has a parameter free $\mathcal{L}_{\mathrm{val},K}$-definition. If $\mathcal{E}$ is an Artin-Schreier extension, then we can use

$$\begin{aligned} I_{\mathcal{E}} \;=\; \{\, b \in L \;\;|\;\; & \exists x \in L :\; x^p - x \in K \;\wedge\; vx \leq 0 \\ & \wedge\, (\forall y \in K :\; vx \neq vy) \;\wedge\; vb \geq -vx \,\} . \end{aligned}$$

If $\mathcal{E}$ is a Kummer extension, then in case 2)a) of the theorem, we have $v\eta > 0$ and therefore, $v(\eta - 1) = 0$. Thus, we can also in this case use (54) for a parameter free $\mathcal{L}_{\mathrm{val},K}$-definition of $I_{\mathcal{E}}$:

$$\begin{aligned} I_{\mathcal{E}} \;=\; \{b \in L \;\;|\;\; & \exists x \in L :\; x^p \in K \;\wedge\; vp \geq (p-1)v(x-1) \\ & \wedge\, (\forall y \in K :\; 0 < vx \neq vy \;\vee\; 0 < v(x-1) \neq vy) \\ & \wedge\, (p-1)vb \geq vp - (p-1)v(x-1) \,\} . \end{aligned}$$

Towards the proof of Theorem 1.9, we need to gather information about $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

5.2. Galois extensions $\mathcal{E} = (L|K, v)$ with $[L : K] = [Lv : Kv]$.

Theorem 5.2. [3, Theorem 4.4] *Take a finite extension $\mathcal{E} = (L|K, v)$ with $Lv|Kv$ separable of degree $[Lv : Kv] = [L : K]$. Then $\mathcal{O}_L = \mathcal{O}_K[x]$ for some $x \in L$ with $vx = 0$ and $Lv = Kv(xv)$, and we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0 .$$

Corollary 5.3. *Take a Kummer extension $\mathcal{E} = (L|K, v)$ with $Lv|Kv$ separable of prime degree $[Lv : Kv] = [L : K]$. Then we have*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}/I_{\mathcal{E}}^p$$

as $\mathcal{O}_L$-modules.

Proof. Since $\mathcal{E}$ is a tame extension, we know from Lemma 2.4 that $I_{\mathcal{E}} = \mathcal{O}_L$. Hence $I_{\mathcal{E}}/I_{\mathcal{E}}^p = 0$ and our assertion follows from the previous theorem. □

The next theorem combines [3, Theorems 4.5 and 4.7]:

Theorem 5.4. *Take an Artin-Schreier extension or a Kummer extension $\mathcal{E} = (L|K, v)$ of degree $p = [Lv : Kv]$ with $Lv|Kv$ inseparable. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}/I_{\mathcal{E}}^p \tag{55}$$

as $\mathcal{O}_L$-modules, and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

The previous corollary and theorem prove Theorem 1.4 with $J_{\mathcal{E}} = I_{\mathcal{E}}$ for the case where $\mathcal{E}$ of degree $p = [Lv : Kv]$, not necessarily equal to $\operatorname{char} K$.

5.3. **Galois extensions $\mathcal{E} = (L|K, v)$ with $[L : K] = (vL : vK)$.** In this case, it is essential for the computation of the Kähler differentials to find an element $x \in L$ with $vx \notin vK$ such that

$$\mathcal{O}_L \;=\; \bigcup_{c\in K \text{ with } vcx>0} \mathcal{O}_K[cx]\,. \tag{56}$$

Note that the rings $\mathcal{O}_K[cx]$ form a chain; indeed, if $c, c' \in K$ with $vc \geq vc'$, then $cx = \frac{c}{c'}c'x \in \mathcal{O}_K[c'x]$, hence $\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x]$.

We define $H_{\mathcal{E}}$ to be the largest convex subgroup of vL which is also a convex subgroup of vK; it exists since $\{0\}$ is such a subgroup and unions over arbitrary collections of convex subgroups are again convex subgroups. We distinguish three mutually exclusive cases describing how vK extends to vL; for convenience, we use the notation of [3]:

(DL2a): there is no smallest convex subgroup of vL that properly contains $H_{\mathcal{E}}$;

(DL2b): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is dense;

(DL2c): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is discrete.

Theorem 5.5. [3, Theorem 3.3] *Assume that $x_0 \in L$ is such that $vx_0 \notin vK$. Then $pvx_0 \in vK$, and the following assertions hold.*

1) If $\mathcal{E}$ is of type (DL2a) or (DL2b), then (56) holds for $x = x_0$.

2) If $\mathcal{E}$ is of type (DL2c), then (56) holds for $x = x_0^j$ with suitable $j \in \{1, \ldots, p-1\}$. If in addition $H_{\mathcal{E}} = \{0\}$, then $\mathcal{O}_L = \mathcal{O}_K[cx]$ for suitable $c \in K$.

The assumption of part 1) holds in particular when every archimedean component of vK is dense, and this in turn holds for every deeply ramified field (K, v).

We set

$$\mathcal{O}_{\mathcal{E}} \;:=\; \mathcal{O}(H_{\mathcal{E}})\,;$$

recall that this is the coarsening of $\mathcal{O}_L$ associated with $H_{\mathcal{E}}$ so that its value group is $vL/H_{\mathcal{E}}$. Denote the maximal ideal of $\mathcal{O}_{\mathcal{E}}$ by $\mathcal{M}_{\mathcal{E}}$. With x as in Theorem 5.5, we have:

Proposition 5.6. [3, Proposition 3.4] *The $\mathcal{O}_L$-ideal $\mathcal{M}_{\mathcal{E}}$ is equal to the $\mathcal{O}_L$-ideal $I_x = (cx \mid c \in K$ with $vcx > 0)$ introduced in (6).*

Corollary 5.7. *The set $\{vcx \mid c \in K$ with $vcx > 0\}$ is coinitial in $vL^{>0} \setminus H_{\mathcal{E}}$.*

Since $\mathcal{O}(\mathcal{M}_{\mathcal{E}}) = \mathcal{O}_{\mathcal{E}}$ by [19, (45)], we also obtain that

$$\mathcal{O}_{\mathcal{E}} \;=\; \mathcal{O}(I_x)\,. \tag{57}$$

As before, we denote by $v_{\mathcal{E}}$ the valuation on L having valuation ring $\mathcal{O}_{\mathcal{E}}$. In the case of Galois defect extensions with independent defect, Proposition 1.8 shows that $v_{\mathcal{E}}$ is the coarsest of all coarsenings v' of v such that $(L|K, v')$ is still purely wild. Here is an analogue in the defectless case:

Proposition 5.8. *The valuation $v_{\mathcal{E}}$ is the coarsest of all coarsenings v' of v such that $[L : K] = (v'L : v'K)$.*

Proof. Take a proper coarsening $\mathcal{O}'$ of $\mathcal{O}_{\mathcal{E}}$, with associated valuation v'. Then $H(\mathcal{O}')$ properly contains $H_{\mathcal{E}}$, hence by Corollary 5.7 its intersection with the set $\{vcx \mid c \in K$ with $vcx > 0\}$ is nonempty. This means that $H(\mathcal{O}')$ contains an element that does not lie in vK, so that the valuation $\bar{v}'$ induced by v on Lv' satisfies $(\bar{v}'(Lv') : \bar{v}'(Kv')) = p$. By the Fundamental Inequalities $[L : K] \geq (v'L : v'K)[Lv' : Kv']$ and $[Lv' : Kv'] \geq (\bar{v}'(Lv') : \bar{v}'(Kv'))$, this implies $(v'L : v'K) = 1 \neq [L : K]$. □

To show that $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ have parameter free $\mathcal{L}_{\mathrm{val},K}$-definitions in (L, v), we need the following preparation:

Lemma 5.9. *For every x_0 with $vx_0 \notin vK$ we have $I_{x_0} \subseteq I_x$.*

Proof. Take $c_0 \in K$ such that $vc_0x_0 > 0$, so that $c_0x_0 \in \mathcal{O}_L$. If x is as in Theorem 5.5, then there is $c \in K$ such that $c_0x_0 \in \mathcal{O}_K[cx]$. Consequently, $c_0x_0 \in I_x$. This proves that $I_{x_0} \subseteq I_x$. □

From this lemma together with Proposition 5.6 we obtain the following parameter free $\mathcal{L}_{\mathrm{val},K}$-definition of $\mathcal{M}_{\mathcal{E}}$:

$$\mathcal{M}_{\mathcal{E}} \;=\; \{b \in L \mid \exists x \in L \setminus K : (\forall y \in K : vx \neq vy) \wedge \exists c \in K :\; vb \geq vcx > 0\}\,. \tag{58}$$

From this, we can define $\mathcal{O}_{\mathcal{E}}$ by including the units of $\mathcal{O}_{\mathcal{E}}$:

$$\mathcal{O}_{\mathcal{E}} \;=\; \{b \in L \mid \forall x \in \mathcal{M}_{\mathcal{E}} :\; -vx < vb\}\,. \tag{59}$$

We now turn to the results for $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in the present case.

Theorem 5.10. [3, Theorem 4.6] *Take an Artin-Schreier extension $\mathcal{E} = (L|K, v)$ of degree $p = (vL : vK)$. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^p \tag{60}$$

as $\mathcal{O}_L$-modules, and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

The next result follows from [3, Theorem 4.8].

Theorem 5.11. *Let $\mathcal{E} = (L|K, v)$ be a Kummer extension of prime degree $p = (vL : vK)$. If $p \neq \operatorname{char} Kv$, then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{M}_\mathcal{E}/\mathcal{M}_\mathcal{E}^p = I_\mathcal{E}\mathcal{M}_\mathcal{E}/(I_\mathcal{E}\mathcal{M}_\mathcal{E})^p \tag{61}$$

as $\mathcal{O}_L$-modules.

If $p = \operatorname{char} Kv$, then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_\mathcal{E}\mathcal{M}_\mathcal{E}/(I_\mathcal{E}\mathcal{M}_\mathcal{E})^p \tag{62}$$

as $\mathcal{O}_L$-modules.

In case 2)a) of Theorem 5.1 (the case of a Kummer generator η with $v\eta \notin vK$), we have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $p \notin \mathcal{M}_\mathcal{E}$ and $\mathcal{M}_\mathcal{E}$ is a nonprincipal $\mathcal{O}_\mathcal{E}$-ideal. This case and condition $p \notin \mathcal{M}_\mathcal{E}$ always hold when $p \neq \operatorname{char} Kv$.

In case 2)b) of Theorem 5.1 (the case of a Kummer generator η which is a 1-unit), we always have that $p = \operatorname{char} Kv$ and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

Proof. (61) and (62) follow directly from [3, Theorem 4.8]. For the case of $p \neq \operatorname{char} Kv$, note that $v(\zeta_p - 1) = 0$ and it follows from part 2) of Theorem 5.1 that $I_\mathcal{E} = \mathcal{O}_L$, so that $\mathcal{M}_\mathcal{E} = I_\mathcal{E}\mathcal{M}_\mathcal{E}$.

The cases 2)a) and 2)b) of Theorem 5.1 correspond to cases i) and ii) of [3, Lemma 3.11], respectively. Note that by [3, Lemma 3.11], case i), and hence case 2)a) of Theorem 5.1, always holds when $p \neq \operatorname{char} Kv$. □

5.4. **Annihilators.** Let us compute the annihilators of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in the above cases whenever it is nonzero. The following is Proposition 3.21 of [19], adapted to our current notation.

Proposition 5.12. *Take $n \geq 2$, $a \in \mathcal{O}_L$ and $\mathcal{O}$ a valuation ring containing $\mathcal{O}_L$ with maximal ideal $\mathcal{M}$. Assume that $(a\mathcal{M})^n \neq a\mathcal{M}$. Then the following statements hold.*

1) We have that

$$\operatorname{ann} a\mathcal{M}/(a\mathcal{M})^n = \begin{cases} (a\mathcal{M})^{n-1} & \text{if } \mathcal{M} \text{ is a principal } \mathcal{O}\text{-ideal,} \\ (a\mathcal{O})^{n-1} & \text{if } \mathcal{M} \text{ is a nonprincipal } \mathcal{O}\text{-ideal.} \end{cases}$$

2) The annihilator is equal to $\mathcal{M}_L$ if and only if $n = 2$, $a \notin \mathcal{M}_L = \mathcal{M}$ and $\mathcal{M}_L$ is a principal $\mathcal{O}_L$-ideal.

Since $I_\mathcal{E}$ is a principal $\mathcal{O}_L$-ideal, we can choose $a \in \mathcal{O}_L$ such that $I_\mathcal{E} = a\mathcal{O}_L$ to obtain that $I_\mathcal{E}\mathcal{M}_\mathcal{E} = a\mathcal{M}_\mathcal{E}$. Now we apply Proposition 5.12.

Proposition 5.13. *Let $\mathcal{E}$ be an Artin-Schreier extension or a Kummer extension of degree $p = \operatorname{char} Kv$. Assume that $[L : K] = (vL : vK)$ and that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$. Then*

$$\operatorname{ann} \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \begin{cases} (I_\mathcal{E}\mathcal{M}_\mathcal{E})^{p-1} & \text{if } \mathcal{M}_\mathcal{E} \text{ is a principal } \mathcal{O}_\mathcal{E}\text{-ideal,} \\ I_\mathcal{E}^{p-1} & \text{if } \mathcal{M}_\mathcal{E} \text{ is a nonprincipal } \mathcal{O}_\mathcal{E}\text{-ideal.} \end{cases}$$

Further, $\operatorname{ann} \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ if and only if $p = 2$, $a \notin \mathcal{M}_L = \mathcal{M}_\mathcal{E}$ and $\mathcal{M}_L$ is a principal $\mathcal{O}_L$-ideal.

Let us note that if (DRvp) holds (and in particular, if (K,v) is a deeply ramified field), then the maximal ideal of any coarsening of $\mathcal{O}_L$ is always nonprincipal. In this case, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$ always holds, and $\mathcal{M}_L$ is never the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

In view of Theorem 5.2, the case of defectless Kummer extensions of prime degree not equal to the residue characteristic satisfying $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ is covered by the following result.

Proposition 5.14. *Let $\mathcal{E}$ be a Kummer extension of degree $p = (vL : vK) \neq \operatorname{char} Kv$. Assume that $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is nonzero. Then $\mathcal{M}_{\mathcal{E}}$ is a principal $\mathcal{O}_{\mathcal{E}}$-ideal, and*

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;=\; \mathcal{M}_{\mathcal{E}}^{p-1}\,.$$

Further, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ if and only if $p = 2$ and $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$.

Proof. We apply Theorem 5.11. Since $p \neq \operatorname{char} Kv$, (61) holds, and we set $a = 1$. For the same reason, case 2)a) of Theorem 5.1 and condition $p \notin \mathcal{M}_{\mathcal{E}}$ hold. Then by Theorem 5.11, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$ implies that $\mathcal{M}_{\mathcal{E}}$ is a principal $\mathcal{O}_{\mathcal{E}}$-ideal. Now all further statements follow from Proposition 5.12. □

Proof of Theorem 1.9:

1): This holds by (59) and (58).

2): The ideal $I_{\mathcal{E}}$ is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v) by Theorem 5.1. If $[L:K] = (vL:vK)$, then part 1) shows that also $(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})$ is parameter-free $\mathcal{L}_{\mathrm{val},K}$-definable in (L,v). Hence the same is true for the ideal $J_{\mathcal{E}}$ defined in part 2). It follows from Corollary 5.3, Theorem 5.4 and Theorem 5.10 that $J_{\mathcal{E}}$ satisfies (3).

In view of Propositions 5.13 an 5.14, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ has the following parameter-free $\mathcal{L}_{\mathrm{val},K}$-definition in the case of $[L:K] = (vL:vK)$:

if $I_{\mathcal{E}} = I_{\mathcal{E}}^{p-1}$, then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$; else
  if there exists $a \in L$ such that $\mathcal{M}_{\mathcal{E}} = a\mathcal{O}_{\mathcal{E}}$,
  then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (aI_{\mathcal{E}})^{p-1}$; else $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$.

In the case of $[L:K] = [Lv:Kv]$, since then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}/I_{\mathcal{E}}^p$ and $I_{\mathcal{E}}$ is principal, the definition is simpler:

if $I_{\mathcal{E}} = I_{\mathcal{E}}^{p-1}$, then $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$; else $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$. □

6. Valued fields with prescribed associated convex subgroups

6.1. **Technical preliminaries for the proof of Theorem 1.10.** The following result is part of [18, Corollary 3.20 1)]:

Lemma 6.1. *An Artin-Schreier extension with Artin-Schreier generator ϑ is a defect extension if and only if $v(\vartheta - K)$ has no maximal element and is contained in $vL^{<0}$.*

Lemma 6.2. *1) Let (K,v) be a valued field of characteristic $p > 0$ whose value group is not p-divisible. Take $a \in K$ such that $va < 0$ is not divisible by p. Let ϑ be a root of the Artin–Schreier polynomial $X^p - X - a$. Then $(K^{1/p^\infty}(\vartheta)|K^{1/p^\infty}, v)$ is an immediate defect extension with independent defect, and $v(\vartheta - K^{1/p^\infty}) \subseteq (vK^{1/p^\infty})^{<0}$.*

2) Take a perfect field k of characteristic $p>0$ and K to be $k(t)$, $k(t)^h$ or $k((t))$, equipped with the t-adic valuation $v=v_t$. Let ϑ be a root of the Artin–Schreier polynomial X^p-X-1/t. Then the assertion of part 1) holds, and

$$v(\vartheta - K^{1/p^\infty}) \;=\; (vK^{1/p^\infty})^{<0}\;. \tag{63}$$

Proof. 1): We have that $v\vartheta = va/p$ and $[K(\vartheta):K]=p=(vK(\vartheta):vK)$. The Fundamental Inequality shows that $K(\vartheta)v=Kv$ and that the extension $(K(\vartheta)|K,v)$ is unibranched. The further extension of v to the perfect hull

$$K(\vartheta)^{1/p^\infty} = K^{1/p^\infty}(\vartheta)$$

is unique, as the extension is purely inseparable. It follows that also the extension $(K^{1/p^\infty}(\vartheta)|K^{1/p^\infty},v)$ is unibranched. On the other hand, $[K^{1/p^\infty}(\vartheta):K^{1/p^\infty}]=p$ since the separable extension $K(\vartheta)|K$ is linearly disjoint from $K^{1/p^\infty}|K$. The value group $vK^{1/p^\infty}(\vartheta)=vK(\vartheta)^{1/p^\infty}$ is the p-divisible hull of $vK(\vartheta)=vK+\mathbb{Z}v\vartheta$. Since $pv\vartheta\in vK$, this is the same as the p-divisible hull of vK, which in turn is equal to vK^{1/p^∞}. The residue field of $K^{1/p^\infty}(\vartheta)$ is the perfect hull of $K(\vartheta)v=Kv$. Hence it is equal to the residue field of K^{1/p^∞}. It follows that the extension $(K^{1/p^\infty}(\vartheta)|K^{1/p^\infty},v)$ is immediate and that its defect is p, equal to its degree. Since K^{1/p^∞} is perfect, it is deeply ramified and hence according to [21, part (1) of Theorem 1.10] the extension must have independent defect. The inclusion $v(\vartheta - K^{1/p^\infty}) \subseteq (vK^{1/p^\infty})^{<0}$ follows from [12, Corollary 2.30].

2): From part 1) we know that $(K^{1/p^\infty}(\vartheta)|K^{1/p^\infty},v)$ is an immediate defect extension with independent defect. Since the value group is archimedean ordered, its only proper convex subgroup is $\{0\}$. By the definition of independent defect given in part c) of Proposition 4.3 and by (32), it follows that $-v(\vartheta-K^{1/p^\infty})=(vK^{1/p^\infty})^{>0}$, which proves (63). □

When we take $K_0=\mathbb{F}_p((t))$ in part 2) of this lemma, where $\mathbb{F}_p$ is the field with p elements, we obtain "Abhyankar's Example", see [13, Example 3.12].

As a preparation for the next lemma, take a valued field (K,v) with $v=w\circ\bar{w}$ and denote the valuation ring of w by $\mathcal{O}_w$. If $c\in K$ with $wc=0$, then $vc>0$ if an only $\bar{w}(cw)>0$. Switching to inverses, we find that the residue map $a\mapsto aw$ sends the set $\{a\in\mathcal{O}_w^\times \mid va<0\}$ onto the set $\{b\in Kw \mid \bar{w}b<0\}$.

Now observe that every convex subgroup H of $\bar{w}(Kw)$ is also a convex subgroup of vK. Choose H to be a proper convex subgroup of $\bar{w}(Kw)$ and denote the coarsenings of v and $\bar{w}$ with respect to H by v_H and $\bar{w}_H$, respectively. Then $v_H=w\circ\bar{w}_H$. By what we have shown, the residue map $a\mapsto aw$ sends the set $\{a\in\mathcal{O}_w^\times \mid v_Ha<0\}$ onto the set $\{b\in Kw \mid \bar{w}_Hb<0\}$. Consequently, it sends the set $A:=\{a\in\mathcal{O}_w^\times \mid va<H\}$ onto the set $B:=\{b\in Kw \mid \bar{w}b<H\}$. Since for every $a\in\mathcal{O}_w^\times$. $va=\bar{w}(aw)$ (up to equivalence of valuations), we find that $vA=\{\alpha\in v\mathcal{O}_w^\times \mid \alpha<H\}$ if and only if $\bar{w}B=\{\beta\in\bar{w}(Kw)\mid\beta<H\}$.

Lemma 6.3. *Take a valued field (K,v) of characteristic $p>0$, a decomposition $v=w\circ\bar{w}$, and an Artin-Schreier extension of K with Artin-Schreier generator ϑ. Then the following assertions hold.*

1) $(K(\vartheta)|K,v)$ is a defect extension with $v(\vartheta-K)=\{\alpha\in vK(\vartheta)\mid\alpha<\bar{w}(K(\vartheta)w)\}$ if and only if $(K(\vartheta)|K,w)$ is a defect extension with $w(\vartheta-K)=(wK(\vartheta))^{<0}$.

2) Take a convex subgroup H of $\bar{v}(Kv)$. If $w\vartheta = 0$ and $\bar{w}(\vartheta w - Kw) = \{\beta \in \bar{w}(Kw) \mid \beta < H\}$, then $(K(\vartheta)|K, v)$ is a defect extension with $v(\vartheta - K) = \{\alpha \in vK \mid \alpha < H\}$.

Proof. 1): We can write $wK(\vartheta) = vK(\vartheta)/\bar{w}(K(\vartheta)w)$ and $wa = va + \bar{w}(K(\vartheta)w)$ for each $a \in K(\vartheta)$. This implies that $v(\vartheta - K) = \{\alpha \in vK \mid \alpha < \bar{w}(K(\vartheta)w)\}$ if and only if $w(\vartheta - K) = (wK(\vartheta))^{<0}$. By Lemma 6.1, $(K(\vartheta)|K, v)$ is a defect extension if and only if $v(\vartheta - K)$ is a subset of $vK(\vartheta)^{<0}$ without maximal element, and similarly for w in place of v. These facts together with the equivalence we have already shown proves the assertion of part 1) of our lemma.

2): As above, we denote the valuation ring of w by $\mathcal{O}_w$. Since $\vartheta \in \mathcal{O}_w^\times$, the residue map $a \mapsto aw$ sends $\vartheta - \mathcal{O}_w$ onto $\vartheta w - Kw$. By assumption, we can take $\vartheta w - Kw$ to be the set B from our preparation for the lemma because then $\bar{w}B = \{\beta \in \bar{w}(Kw) \mid \beta < H\}$. We conclude that $v(\vartheta - \mathcal{O}_w) = \{\alpha \in v\mathcal{O}_w^\times \mid \alpha < H\}$.

If $c \in K \setminus \mathcal{O}_w$, then $wc < 0$ and therefore, $w(\vartheta - c) = wc < w\vartheta \leq w(\vartheta - \mathcal{O}_w)$, which implies $v(\vartheta - c) < v(\vartheta - \mathcal{O}_w)$. This shows that $v(\vartheta - \mathcal{O}_w)$ is a final segment of $v(\vartheta - K)$. This together with the equality $v(\vartheta - \mathcal{O}_w) = \{\alpha \in v\mathcal{O}_w^\times \mid \alpha < H\}$ proves that $v(\vartheta - K) = \{\alpha \in vK \mid \alpha < H\}$, as desired. This in turn implies by Lemma 6.1 that $(K(\vartheta)|K, v)$ is a defect extension. □

Remark 6.4. Note that with the respective assumptions, the extensions $(K(\vartheta)|K, v)$, $(K(\vartheta)|K, w)$ and $(K(\vartheta)w|Kw, \bar{w})$ in the above lemma all have independent defect. #

6.2. **Some examples.**

Example 6.5. Take $K_0 = \mathbb{F}_p((t))$. Then $K := K_0^{1/p^\infty} = \mathbb{F}_p((t))^{1/p^\infty}$ is a perfect field, so under the canonical t-adic valuation v_t it is a deeply ramified field. Moreover, (K, v_t) is henselian and of rank 1. Now set $L_0 := K((x))$ and $L := L_0^{1/p^\infty}$, and equip L with the canonical x-adic valuation v_x. Both v_x and v_t are henselian, hence so is their composition $v := v_x \circ v_t$ on L. We note that $K = Lv_x$ as K is perfect. As L is perfect, (L, v) is a deeply ramified field. Its value group vL has rank 2, i.e., it has two proper convex subgroups, namely, v_tK and $\{0\}$. Let ϑ_t be a root of $X^p - X - \frac{1}{t}$, and ϑ_x a root of $X^p - X - \frac{1}{x}$.

By part 1) of Lemma 6.2, both extensions $(L(\vartheta_x)|L, v)$ and $(L(\vartheta_x)|L, v_x)$ are defect extensions with independent defect. By part 2) of Lemma 6.2, $v_x(\vartheta_x - L) = v_xL^{<0}$, which by part 1) of Lemma 6.3 implies that $v(\vartheta_x - L) = \{\alpha \in vL \mid \alpha < v_tK\}$. Hence for $\mathcal{E}_x = (L(\vartheta_x)|L, v)$, its associated convex subgroup $H_{\mathcal{E}_x}$ is the convex subgroup v_tK of vL.

Again by part 1) of Lemma 6.2, the extension $(K(\vartheta_t)|K, v_t)$ has independent defect, and by part 2) of Lemma 6.2, $v_t(\vartheta_t - K) = v_tK^{<0}$. By part 2) of Lemma 6.3 applied with $H = \{0\}$, $(L(\vartheta_t)|L, v)$ is a defect extension with $v(\vartheta_t - L) = (vL(\vartheta_t))^{<0}$. Therefore, $\{0\}$ is the convex subgroup associated with the defect extension $(L(\vartheta_t)|L, v)$.

We have shown that both proper convex subgroups of vL, v_tK and $\{0\}$, appear as the convex subgroups associated with Galois defect extensions of (L, v). ♢

Let us present two modifications of this example.

Example 6.6. In the previous example, we replace K by some algebraically closed (or just henselian defectless) field with an arbitrary nontrivial valuation v_t. Then

(K,v_t) has no defect extensions, and $H=v_tK$ will be the only convex subgroup of vL associated with Galois defect extensions. This is seen as follows. As in Example 6.5 we have that $(L(\vartheta_x)|L,v)$ is a defect extension with $v(\vartheta_x-L)=\{\alpha\in vL\mid \alpha<v_tK\}$. On the other hand, suppose that there is a Galois defect extension $(L(\vartheta)|L,v)$ with $v(\vartheta-L)=vL^{<0}$. Then there is $b\in L$ such that $v(\vartheta-b)\in v_tK$. Set $\bar\vartheta:=(\vartheta-b)v_x$. Note that $\vartheta-b$ is a root of an Artin-Schreier polynomial over L since $(\vartheta-b)^p-(\vartheta-b)=\vartheta^p-\vartheta+b-b^p\in L$. Hence $\bar\vartheta$ is a root of an Artin-Schreier polynomial over K. By construction, $(K(\bar\vartheta)|K,v_t)$ cannot be a defect extension, so there is $\bar c\in K$ such that $v_t(\bar\vartheta-\bar c)$ is the maximum of $v_t(\bar\vartheta-K)$. Choose $c\in L$ such that $cv_x=\bar c$. Then $v(\vartheta-b-c)$ is the maximum of $v(\vartheta-L)$, contradicting our assumption that $(L(\vartheta)|L,v)$ is a defect extension.

At the other extreme, we can keep (K,v_t) and (L_0,v_x) as in the previous example, but now take (L,v_x) to be a maximal purely wild extension of (L_0,v_x). As $L_0v_x=K$ is perfect, we have $Lv_x=L_0v_x=K$. By part 1) of Lemma 2.5, (L,v_x) is a tame field and thus has no defect extensions. Therefore v_tK cannot appear as a convex subgroup associated with any Galois defect extension; indeed, if $v(\vartheta-L)=\{\alpha\in vL\mid\alpha<v_tK\}$, then by part 1) of Lemma 6.3, $(L(\vartheta)|L,v_x)$ would be a defect extension. However, as in the previous example one shows that $\mathcal{E}_t=(L(\vartheta_t)|L,v)$ is a defect extension with $H_{\mathcal{E}_t}=\{0\}$. $\diamondsuit$

Note that these two examples prove part 1) of Theorem 1.10 for the case of I being the ordered set with two elements.

6.3. **A general construction.** We will now present a much more general construction. Take any ordered index set I and for every $i\in I$ an arbitrary ordered abelian group C_i. Then we can form the **Hahn sum** $\coprod_{i\in I}C_i$. As an abelian group, this is the direct sum of the groups C_i, represented as the set of all tuples $(\alpha_i)_{i\in I}$ with only finitely many of the $\alpha_i\in C_i$ nonzero. An ordering on $\Gamma_0:=\coprod_{i\in I}C_i$ is introduced as follows. For $(\alpha_i)_{i\in I}\in\coprod_{i\in I}C_i$, set $i_{\min}:=\min\{i\in I\mid\alpha_i\neq0\}$. Then define $(\alpha_i)_{i\in I}>0$ if $\alpha_{i_{\min}}>0$. If all C_i are archimedean ordered, then the nonzero principal convex subgroups are exactly the subgroups

$$H^i\ :=\ \{(\alpha_j)_{j\in I}\in\coprod_{j\in I}C_j\mid\alpha_j=0\text{ for all }j<i\}$$

for $i\in I$. Likewise, the subprincipal convex subgroups are exactly the subgroups

$$H_i\ :=\ \{(\alpha_j)_{j\in I}\in\coprod_{j\in I}C_j\mid\alpha_j=0\text{ for all }j\leq i\}$$

for $i\in I$. Note that $H_i=\{0\}$ if and only if i is the maximal element of I, and $H^i=\Gamma_0$ if i is the minimal element of I. If $i\in I$ has an immediate successor i^+, then $H^{i^+}=H_i$ is a subprincipal convex subgroup.

Observe that for $i_1,i_2\in I$,

$$H^{i_1}\ \subsetneq\ H^{i_2}\iff i_1\ >\ i_2\,. \tag{64}$$

This shows that the function

$$i\in I\ \mapsto\ H^i$$

is an order reversing bijection from the ordered set I onto the ordered set of nonzero principal convex subgroups.

Now set $C_i = \mathbb{Z}$ for all $i \in I$ and let Γ be the Hahn sum $\coprod_{i\in I} C_i$. For each $i \in I$ we denote by 1_i the element $(\alpha_j)_{j\in I}$ with $\alpha_j = 1 \in \mathbb{Z}$ if $j = i$ and $\alpha_j = 0$ otherwise. Then the principal convex subgroup H^i is the convex hull of the subgroup $1_i\mathbb{Z}$ of Γ.

Take a field k of characteristic $p > 0$ and a set $\{t_i \mid i \in I\}$ of elements algebraically independent over k and define a valuation v on the field $K_\Gamma := k(t_i \mid i \in I)$ by setting $vt_i = 1_i$ for each $i \in I$. Then $vK_\Gamma = \Gamma$.

For each $i \in I$ there are:

- a decomposition $v = v_i \circ \bar{v}_i$, where v_i is the finest coarsening of v on K_Γ that is trivial on t_i and $\bar{v}_i$ is the valuation induced by v on the residue field $K_\Gamma v_i$, which can be identified with $k(t_j \mid i \leq j \in I)$, and
- a decomposition $\bar{v}_i = w_i \circ \bar{w}_i$, where w_i is the t_i-adic valuation on $K_\Gamma v_i$ and $\bar{w}_i$ is the valuation induced by $\bar{v}_i$ on the residue field $K_\Gamma v_i w_i$, which can be identified with $k(t_j \mid i < j \in I)$.

Then v_iK_Γ, $i \in I$, are all subprincipal convex subgroups of Γ, and $v_i \circ w_iK_\Gamma$, $i \in I$, are all nonzero convex subgroups of Γ. Note that v_j is strictly coarser than v_i if $j < i$.

We take K_0 to be the perfect hull of K_Γ, that is, $K_0 = k(t_i^{1/p^n} \mid i \in I, n \in \mathbb{N})$. The valuations v and v_i, $i \in I$, have unique extensions to K_0, and vK_0 is the p-divisible hull of Γ. Further, K_0v_i is the perfect hull $k(t_j^{1/p^n} \mid i \leq j \in I, n \in \mathbb{N})$ of $k(t_j \mid i \leq j \in I)$, so the valuations $\bar{v}_i$ and w_i have unique extensions to K_0v_i and the decompositions $v = v_i \circ \bar{v}_i$ again hold on K_0. Likewise, $K_0v_iw_i$ is the perfect hull $k(t_j^{1/p^n} \mid i < j \in I, n \in \mathbb{N})$ of $k(t_j \mid i < j \in I)$, so also the valuations $\bar{w}_i$ have unique extensions to $K_0v_iw_i$ and the decompositions $\bar{v}_i = w_i \circ \bar{w}_i$ again hold on K_0v_i.

For each $i \in I$ we take $\tilde{H}^i$ and $\tilde{H}_i$ to be convex hulls of H^i and H_i, respectively, in vK_0; they are in fact the p-divisible hulls of H^i and H_i. Then $\tilde{H}^i$ are the nonzero principal convex subgroups and $\tilde{H}_i$ are the subprincipal convex subgroups of vK_0.

Take a subset $\mathcal{C}^{\rm sp} \subseteq \mathcal{C}(\Gamma)$ containing only subprincipal convex subgroups, and take J to be the subset of I such that $\mathcal{C}^{\rm sp} = \{H_j \mid j \in J\}$. Then the next theorem proves Theorem 1.10.

Theorem 6.7. *There exists an immediate algebraic extension (K, v) of (K_0, v) which is a perfect field with $Kv = K_0v$ and such that the nonzero convex subgroups associated with Galois defect extensions of prime degree of (K, v) are exactly the convex subgroups $\tilde{H}_j$ of $vK_0 = vK$ with $j \in J$.*

Proof. In a first step, we will construct an immediate algebraic extension (K, v) of (K_0, v) such that

i) if $i \in I \setminus J$, then (Kv_i, w_i) is a tame field with , and

ii) if $j \in J$, then $Kv_j = Kv_jw_j(t_j^{1/p^n})$, where Kv_jw_j is some algebraic extension of $K_0v_jw_j$.

Since also in case ii) we have $w_j(Kv_j) = w_j(K_0v_j)$, we conclude that $vK = vK_0$; note that the valuations w_i determine all archimedean components of vK since if I has a smallest element k, then v_k is trivial. Observe that since K_0 is perfect, so are K and all residue fields of K_0 and K.

In a second step, we will show that for each $j \in J$, the convex subgroup $\tilde{H}_j$ is associated with some Galois defect extension of prime degree of (K,v). In a final step, we will show that for $i \in I \setminus J$, $\tilde{H}_i$ is not associated with any Galois defect extension of prime degree of (K,v).

We construct (K,v) by (possibly transfinite) induction. To this end, we enumerate the elements of $I \setminus J$ as i_ν with $\nu < \kappa = |I|$. Assume that we have already constructed a chain of extensions (K_ν, v), $\nu < \mu$, of (K_0, v) (where $1 \leq \mu \leq \kappa$) with the previous decompositions of v carried over to corresponding decompositions of v on all fields in the chain such that for every $\nu < \mu$, i) and ii) hold with (K,v) replaced by (K_ν, v_ν), I replaced by $I_\nu := \{i_\rho \mid \rho < \nu\}$ and J replaced by $J_\nu := I \setminus I_\nu$. Then we proceed as follows.

First assume that μ is a successor ordinal, so there is some ν such that $\mu = \nu+1$. We take any maximal immediate algebraic extension (L_{i_ν}, w_{i_ν}) of $(K_\nu v_{i_\nu}, w_{i_\nu})$. Since $i_\nu \notin I_\nu$, we know from our induction hypothesis that $K_\nu v_{i_\nu} = K_\nu v_{i_\nu} w_{i_\nu}(t_{i_\nu}^{1/p^n})$. So $(K_\nu v_{i_\nu}, w_{i_\nu})$ has p-divisible value group, equal to $w_{i_\nu}(K_0 v_{i_\nu})$, and perfect residue field. Hence by part 1) of Lemma 2.5, (L_{i_ν}, w_{i_ν}) is a tame field. Note that by construction, $w_{i_\nu}(L_{i_\nu}) = w_{i_\nu}(K_\nu v_{i_\nu}) = w_{i_\nu}(K_0 v_{i_\nu})$ and $L_{i_\nu} w_{i_\nu} = K_\nu v_{i_\nu} w_{i_\nu}$. Now we take (K_μ, v_{i_ν}) to be an extension of (K_ν, v_{i_ν}) such that $v_{i_\nu}(K_\mu) = v_{i_\nu}(K_\nu)$ and $K_\mu v_{i_\nu} = L_{i_\nu}$, and such that for every finite subextension $K'_\nu | K_\nu$ of $K_\mu | K_\nu$ we have $[K'_\nu : K_\nu] = [K'_\nu v_{i_\nu} : K_\nu v_{i_\nu}]$; see [11, Section 2.3] for information on the construction of such extensions. Since $L_{i_\nu} w_{i_\nu} = K_\nu v_{i_\nu} w_{i_\nu}$, we can define the extension of v from K_ν to K_μ by setting $v = v_{i_\nu} w_{i_\nu} \bar{w}_{i_\nu}$ on K_μ. We observe that $K_\mu v_{i_\nu} w_{i_\nu} = L_{i_\nu} w_{i_\nu} = K_\nu v_{i_\nu} w_{i_\nu}$.

We have to show that statements i) and ii) also hold for K_μ in place of K_ν, $I_\mu = I_\nu \cup \{i_\nu\}$ in place of I and $J_\mu = I \setminus I_\mu$ in place of J_ν. Take $i \in I_\mu \setminus J_\mu$. If $i = i_\nu$, then i) holds by our construction. Assume that $i \neq i_\nu$. Since $K_\mu | K_\nu$ is an algebraic extension, so is $K_\mu v_i | K_\nu v_i$. As $(K_\nu v_i, w_i)$ is a tame field by induction hypothesis, so is $(K_\mu v_i, w_i)$ by part 2) of Lemma 2.5. It remains to show that $w_i(K_\mu v_i) = w_i(K_0 v_i)$. By the induction hypothesis, it suffices to show that $w_i(K_\mu v_i) = w_i(K_\nu v_i)$. If $i > i_\nu$, then $K_\mu v_i$ is equal to $K_\mu v_{i_\nu} w_{i_\nu} = K_\nu v_{i_\nu} w_{i_\nu}$ or a residue field thereof, hence $K_\mu v_i = K_\nu v_i$. Assume that $i < i_\nu$. It suffices to show that $w_i(K'_\nu v_i) = w_i(K_\nu v_i)$ for every finite subextension $K'_\nu | K_\nu$ of $K_\mu | K_\nu$. We have

$$[K'_\nu : K_\nu] \;\geq\; [K'_\nu v_i : K_\nu v_i] \;\geq\; [K'_\nu v_i w_i : K_\nu v_i w_i] \;\geq\; [K'_\nu v_{i_\nu} : K_\nu v_{i_\nu}] \;=\; [K'_\nu : K_\nu]$$

since $K'_\nu v_{i_\nu}$ and $K_\nu v_{i_\nu}$ are residue fields of $K'_\nu v_i$ and $K_\nu v_i$, respectively. Hence equality holds everywhere, and from the Fundamental Inequality we obtain that $w_i(K'_\nu v_i) = w_i(K_\nu v_i)$, as desired.

Now take $j \in J_\mu$. We will show that we still have $K_\mu v_j = K_\mu v_j w_j(t_i)^{1/p^\infty}$. First assume that $j > i_\nu$. Then $K_\mu v_j$ and $K_\mu v_j w_j$ are equal to $K_\mu v_{i_\nu} w_{i_\nu}$ or residue fields thereof. Since $K_\mu v_{i_\nu} w_{i_\nu} = L_{i_\nu} w_i = K_\nu v_{i_\nu} w_{i_\nu}$, we thus obtain $K_\mu v_j = K_\nu v_j = K_\nu v_j w_j(t_j)^{1/p^\infty} = K_\mu v_j w_j(t_j)^{1/p^\infty}$.

Next assume that $j < i_\nu$. Suppose that $K_\mu v_j$ properly contains $K_\mu v_j w_j(t_j)^{1/p^\infty}$. Then there is a finite subextension $K'_\nu | K_\nu$ of $K_\mu | K_\nu$ such that $K'_\nu v_j$ properly contains $K'_\nu v_j w_j(t_j)^{1/p^\infty}$. Using that $K_\nu v_j = K_\nu v_j w_j(t_j)^{1/p^\infty}$, that $K'_\nu v_{i_\nu}$ is equal to $K'_\nu v_j w_j$ or a residue field thereof, and that $K_\nu v_{i_\nu}$ is equal to $K_\nu v_j w_j$ or a residue

field thereof, we compute:

$$\begin{aligned}[K'_\nu : K_\nu] &\geq [K'_\nu v_j : K_\nu v_j] > [K'_\nu v_j w_j(t_j)^{1/p^\infty} : K_\nu v_j w_j(t_j)^{1/p^\infty}] \\ &= [K'_\nu v_j w_j : K_\nu v_j w_j] \geq [K'_\nu v_{i_\nu} : K_\nu v_{i_\nu}] = [K'_\nu : K_\nu]\,.\end{aligned}$$

This contradiction proves that $K_\mu v_j = K_\mu v_j w_j(t_j)^{1/p^\infty}$ also holds in this case.

Now assume that μ is a limit ordinal. Then take (K_μ, v) to be the union over the chain of extensions (K_ν, v), $\nu < \mu$. Take $i \in I_\mu \setminus J_\mu$. Then there is $\nu < \mu$ such that $i = i_\nu$. By induction hypothesis, $(K_{\nu+1}v_{i_\nu}, w_{i_\nu})$ is a tame field. Since $K_\mu | K_{\nu+1}$ is algebraic, so is $K_\mu v_{i_\nu} | K_{\nu+1}v_{i_\nu}$, hence also $(K_\mu v_{i_\nu}, w_{i_\nu})$ is a tame field. Further, by induction hypothesis we have $w_i(K_\nu v_i) = w_i(K_0 v_i)$ for all $\nu < \mu$. Consequently, $w_i(K_\mu v_i) = \bigcup_{\nu<\mu} w_i(K_\nu v_i) = w_i(K_0 v_i)$. We have thus proved that statement i) holds for K_μ in place of K and I_μ and J_μ in place of I and J.

Take $j \in J_\mu$. Then $j \in J_\nu$ for all $\nu < \mu$, hence by induction hypothesis, $K_\nu v_j = K_\nu v_j w_j(t_j)^{1/p^\infty}$ for $\nu < \mu$. So

$$K_\mu v_j = \bigcup_{\nu<\mu} K_\nu v_j = (\bigcup_{\nu<\mu} K_\nu v_j w_j)(t_j)^{1/p^\infty} = K_\mu v_j w_j(t_j)^{1/p^\infty}\,,$$

where also $K_\mu v_j w_j$ is algebraic over $K_0 v_j w_j$. This shows that statement ii) must hold for K_μ in place of K and I_μ and J_μ in place of I and J.

Finally, we set $K := K_\kappa$. Observe that in none of the above steps of our construction, the perfect residue field k of K_0 has been changed. Hence $Kv = k$. We are going to show that (K, v) satisfies the statements of Theorem 1.10.

Take any $j \in J$; we have to show that there is a Galois defect extension of prime degree of (K, v) with associate convex subgroup $\tilde{H}_j$. We let ϑ_j be a root of $X^p - X - \frac{1}{t_j}$. Since v_j is trivial on t_j, we can identify ϑ_j with $\vartheta_j v_j$. By statement ii), $Kv_j = Kv_j w_j(t_j)^{1/p^\infty}$, where $Kv_j w_j$ is a perfect field. Hence by part 2) of Lemma 6.2,

$$w_j(\vartheta_j - Kv_j) = (w_j(Kv_j))^{<0}\,, \tag{65}$$

which by part 1) of Lemma 6.3 implies that $\bar{v}_j(\vartheta_j - Kv_j) = \{\alpha \in \bar{v}_j(Kv_j) \mid \alpha < \bar{v}_j(Kv_j w_j)\} = \{\alpha \in \bar{v}_j(Kv_j) \mid \alpha < \tilde{H}_j\}$. From part 2) of Lemma 6.3 we obtain that $v(\vartheta_j - K) = \{\alpha \in vK \mid \alpha < \tilde{H}_j\}$, that is, $\tilde{H}_j$ is the convex subgroup associated with the defect extension $(K(\vartheta_j)|K, v)$. Hence by our construction, all $\tilde{H}_j$ for $j \in J$ appear as the convex subgroups associated with Galois defect extensions of (K, v).

Finally, we show that if $i \in I \setminus J$, then H_i cannot appear as the convex subgroup associated with any Galois defect extension of (K, v). Suppose that the contrary is true, and let $(K(\vartheta)|K, v)$ be an Artin-Schreier defect extension with H_i as its associated convex subgroup. Since $\bar{v}_i(Kv_i)$ properly contains $H_i = \bar{v}_i(K_0 v_i w_i)$, it follows that there is some $b \in K$ such that $v(\vartheta - b) \in \bar{v}_i(Kv_i)$. With $\vartheta' := \vartheta - b$ we obtain that $\bar{v}_i(\vartheta' v_i - Kv_i) = \{\alpha \in \bar{v}_i(Kv_i) \mid \alpha < \bar{v}_i(K_0 v_i w_i)\}$. Hence Lemma 6.1 shows that $(Kv_i(\vartheta' v_i)|Kv_i, \bar{v}_i)$ is a nontrivial defect extension. Thus by part 1) of Lemma 6.3, also $(Kv_i(\vartheta' v_i)|Kv_i, w_i)$ is a nontrivial defect extension. However, this contradicts the fact that by our construction, (Kv_i, w_i) is a tame and thus defectless field. □

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