CORRECTION AND NOTES TO THE PAPER "A CLASSIFICATION OF ARTIN-SCHREIER DEFECT EXTENSIONS AND CHARACTERIZATIONS OF DEFECTLESS FIELDS"

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Abstract. We correct a mistake in a lemma in the paper cited in the title and show that it did not affect any of the other results of the paper. To this end we prove results on linearly disjoint field extensions that do not seem to be commonly known. We give an example to show that a separability assumption in one of these results cannot be dropped (doing so had led to the mistake). Further, we discuss recent generalizations of the original classification of defect extensions.

1. Introduction

In the paper [4] the author introduced a classification of Artin-Schreier defect extensions. Defect extensions of a valued field (K, v) can only appear when the characteristic of the residue field Kv is positive. They constitute a major obstacle to the solution of the following open problems in positive characteristic:

1) local uniformization (the local form of resolution of singularities) in arbitrary dimension,

2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over any finite field $\mathbb{F}_q$, and of its perfect hull.

Both problems are closely connected with the structure theory of valued function fields of positive characteristic p.

Since the classification was introduced, several indications have been found that one of the two types of defects is not as harmful as the other. But in [4] it was only introduced for valued fields (K, v) of equal positive characteristic (i.e., $\mathrm{char}\, K = \mathrm{char}\, Kv = p > 0$). Recently, it was extended in [1] to all defect extensions of prime degree, including the case of valued fields (K, v) of mixed characteristic (i.e., $\mathrm{char}\, K = 0$, $\mathrm{char}\, Kv = p > 0$). In the process of generalizing results to the mixed characteristic case (see Section 4 of [1]), a mistake was found in the proof of Lemma 4.12 of [4]. The following claim had been stated without a reference (with a slightly different notation):

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Claim: *If a field K is relatively algebraically closed in an extension field F and L is an algebraic extension of K linearly disjoint from F over K, then L is relatively algebraically closed in $L.F$.*

(Here, the compositum $L.F$ of L and F is taken in a fixed algebraic closure of F). But this claim is not true in general if $L|K$ is not separable. We show this by Example 2.2 below. It is worth mentioning that this example was implicitly used by F. Delon in [2] to show that an algebraically maximal valued field is not necessarily defectless; this is worked out in detail in Example 3.25 of [5]. For the definitions of these notions and others used but not explained in these notes, and for further background, see [5, 4, 1].

A correct version of the above claim reads as follows:

If a field K is relatively separable-algebraically closed in an extension field F and L is an algebraic extension of K, then L is relatively separable-algebraically closed in $L.F$.

We prove this assertion in Lemma 2.1 in Section 2. We prove more than this, in order to clarify the situation, but also because these results are hard to find in the literature.

In Section 3 we state and prove a corrected version of the faulty Lemma 4.12. Its statement is slightly weaker than in the original version, as we only obtain that K is relatively separable-algebraically closed in M_w. But this suffices for the proof of the crucial Proposition 4.13 of [4].

Finally, let us mention that one purpose of introducing the classification of defect extensions was to prove Theorem 1.2 of [4], which states:

A valued field of positive characteristic is henselian and defectless if and only if it is separable-algebraically maximal and inseparably defectless.

This fact in turn was used in [3] to construct henselian defectless fields for a crucial example. It was hoped that the generalization of the classification to the mixed characteristic case would result in the proof of some analogue of Theorem 1.2 for this case. Unfortunately, so far we were only able to prove a partial analogue (Theorem 1.7 of [1]). The problem is that it is still not entirely clear what the analogue of purely inseparable defect extensions may be in mixed characteristic.

2. A lemma about linearly disjoint extensions of fields

Lemma 2.1. *Let $F|K$ be an arbitrary field extension and $L|K$ an algebraic extension.*

1) Assume that K is relatively algebraically closed in F and b is algebraic over K, or that K is relatively separable-algebraically closed in F and b is separable-algebraic over K. Then F and $K(b)$ are linearly disjoint over K.

2) Assume that K is relatively separable-algebraically closed in F and $L|K$ is separable-algebraic. Then F and L are linearly disjoint over K.

3) Assume that K is relatively algebraically closed in F and $L|K$ is separable-algebraic. Then L is relatively algebraically closed in $L.F$.

4) Assume that K is relatively separable-algebraically closed in F and $L|K$ is algebraic. Then L is relatively separable-algebraically closed in $L.F$.

Proof. 1): Take an algebraic extension $K(b)|K$. The minimal polynomial $f \in F[X]$ of b over F is a divisor of the minimal polynomial of b over K, so all roots of f are algebraic over K and so are the coefficients of f since they are symmetric functions in these roots. If K is assumed to be relatively algebraically closed in F, it follows that $f \in K[X]$. If in addition b is separable over K, then also the coefficients of f are separable over K and it suffices to assume that K is relatively separable-algebraically closed in F to obtain that $f \in K[X]$. In both cases, f is also the minimal polynomial of b over K. Thus, $[F(b) : F] = [K(b) : K]$, showing that F and $K(b)$ are linearly disjoint over K.

2) Now let $L|K$ be a separable-algebraic extension. Then L is a union of simple subextensions of $L|K$; if K is relatively separable-algebraically closed in F, then by part 1), these are linearly disjoint from F over K. It then follows that L itself is linearly disjoint from F over K.

3) Assume that K is relatively algebraically closed in F and $L|K$ is separable-algebraic. Then also $L.F|F$ is separable algebraic (since the minimal polynomial of any $b \in L$ over F is a divisor of its minimal polynomial over K).

Let $a \in L.F$ be algebraic over L; hence, a is also algebraic over K, and by what we have just shown, it is separable-algebraic over F. By part a), the minimal polynomial of a over F coincides with that over K, so we know that a is separable-algebraic over K. Consequently, $L(a)|K$ is a separable-algebraic extension. From part 2) we infer that $L(a)$ is linearly disjoint from F over K. By [6, Chapter VIII, Proposition 3.1], $L(a)$ is linearly disjoint from $L.F$ over L. In particular, $a \in L.F$ implies $a \in L$. This proves that L is relatively algebraically closed in $L.F$.

4) Assume that K is relatively separable-algebraically closed in F and $L|K$ is algebraic. If K' denotes the relative algebraic closure of K in F, then $K'|K$ is purely inseparable, and consequently, the same is true for the algebraic subextension $L.K'|L$ of $L.F|L$. Therefore, if we are able to show that $L.K'$ is relatively separable-algebraically closed in $L.F$, then the same holds for L. We may thus assume from the start that K is relatively algebraically closed in F, and we need to show that L is relatively separable-algebraically closed in $L.F$.

Let $L_0|K$ be the maximal separable subextension of $L|K$, so $L|L_0$ is purely inseparable. By part 3), L_0 is relatively algebraically closed in $L_0.F$. Suppose that L is not relatively separable-algebraically closed in $L.F$. Then the relative algebraic closure of L in $L.F$ contains a nontrivial separable-algebraic subextension L_1 of L_0 . By part 2), L_1 is linearly disjoint from $L_0.F$ over L_0 . This shows that $L_1.F|L_0.F$ is a nontrivial separable subextension of $L.F|L_0.F$. But as $L|L_0$ is purely inseparable, so is $L.F|L_0.F$. This contradiction shows that L is relatively separable-algebraically closed in $L.F$. □

Assertion 2) of the lemma fails when $L|K$ is algebraic but neither separable nor simple, even when K is relatively algebraically closed in F. Likewise, assertion 3) fails when $L|K$ is algebraic but not separable. This will be shown in the following example.

Example 2.2. We take elements t, x, y which are algebraically independent over $\mathbb{F}_p$. We choose any prime p, set

$$F \;:=\; \mathbb{F}_p(t,x,y)$$

and define

$$s \;:=\; x^p + ty^p \quad \text{and} \quad K \;:=\; \mathbb{F}_p(t,s)\;.$$

Then s is transcendental over $\mathbb{F}_p(t)$, so K has p-degree 2, that is, $[K : K^p] = p^2$.

We prove that K is relatively algebraically closed in F. Take $b \in F$ algebraic over K. The element b^p is algebraic over K and lies in $F^p = \mathbb{F}_p(t^p, x^p, y^p)$ and thus also in $K(x) = \mathbb{F}_p(t, x, y^p)$. Since $\operatorname{trdeg} \mathbb{F}_p(t, x, y^p)|\mathbb{F}_p = 3$ while $\operatorname{trdeg} K|\mathbb{F}_p = 2$, we see that x is transcendental over K. Therefore, K is relatively algebraically closed in $K(x)$ and thus, $b^p \in K$. Consequently, $b \in K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p})$. Write

$$b \;=\; r_0 + r_1 s^{\frac{1}{p}} + \ldots + r_{p-1} s^{\frac{p-1}{p}} \quad \text{with} \;\; r_i \in \mathbb{F}_p(t^{1/p}, s) = K(t^{1/p})\;.$$

Since $s^{1/p} = x + t^{1/p}y$, we have that

$$b \;=\; r_0 \;+\; r_1 x + \ldots + r_{p-1}x^{p-1} \;+ \ldots +\; t^{1/p} r_1 y + \ldots + t^{(p-1)/p} r_{p-1} y^{p-1}$$

(in the middle, we have omitted the summands in which both x and y appear). Since x, y are algebraically independent over $\mathbb{F}_p$, the p-degree of $\mathbb{F}_p(x, y)$ is 2, and the elements $x^i y^j$, $0 \le i < p$, $0 \le j < p$, form a basis of $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$. Since $t^{1/p}$ is transcendental over $\mathbb{F}_p(x^p, y^p)$, we know that $\mathbb{F}_p(x, y)$ is linearly disjoint from $\mathbb{F}_p(t^{1/p}, x^p, y^p)$ and hence also from $\mathbb{F}_p(t, x^p, y^p)$ over $\mathbb{F}_p(x^p, y^p)$. This shows that the elements $x^i y^j$ also form a basis of $F|\mathbb{F}_p(t, x^p, y^p)$ and are still $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linearly independent. Hence, b can also be written as a linear combination of these elements with coefficients in $\mathbb{F}_p(t, x^p, y^p)$, and this must coincide with the above $\mathbb{F}_p(t^{1/p}, x^p, y^p)$-linear combination which represents b. That is, all coefficients r_i *and* $t^{i/p} r_i$, $1 \le i < p$, are in $\mathbb{F}_p(t, x^p, y^p)$. Since $t^{i/p} \notin \mathbb{F}_p(t, x^p, y^p)$, this is impossible unless they are zero. It follows that $b = r_0 \in K(t^{1/p})$. Assume that $b \notin K$. Then $[K(b) : K] = p$ and thus, $K(b) = K(t^{1/p})$ since also $[K(t^{1/p}) : K] = p$. But then $t^{1/p} \in K(b) \subset F$, a contradiction. This proves that K is relatively algebraically closed in F.

We show that

$$K(t^{1/p}, s^{1/p})$$

is not linearly disjoint from F over K. Indeed, $s^{1/p} = x + t^{1/p}y \in K(t^{1/p}, x, y)$, which implies that

$$[K(t^{1/p}, s^{1/p}).F : F] \;=\; [F(t^{1/p}) : F] \;=\; p \;<\; p^2 \;=\; [K(t^{1/p}, s^{1/p}) : K]\,.$$

Further, we see that while $K(t^{1/p})$ is linearly disjoint from F over K, it is not relatively algebraically closed in $F(t^{1/p}) = K(t^{1/p}).F$ since $s^{1/p} \in F(t^{1/p}) \setminus K(t^{1/p})$. We have shown that the separability condition on $L|K$ in parts 2) and 3) of Lemma 2.1 is necessary. $\diamondsuit$

3. Corrected version of Lemma 4.12

We consider a valued field (K_0, v).

Lemma 3.1. *Assume that for every coarsening w of v (including v itself), K_0 admits a maximal immediate extension $(N_w|K_0, w)$ such that K_0 is relatively separable-algebraically closed in N_w. If $(K|K_0, v)$ is a finite and defectless extension, then for every coarsening w of v (including v itself), $(M_w, w) = (N_w.K, w)$ is a maximal immediate extension of (K, w) such that K is relatively separable-algebraically closed in M_w.*

Proof. Since $(K|K_0, v)$ is defectless by hypothesis, the same is true for the extension $(K|K_0, w)$ by Lemma 2.4 of [4]. We note that (K_0, w) is henselian since it is assumed to be separable-algebraically closed in the henselian field (N_w, w). So we may apply Lemma 2.5 of [4]: since $(N_w|K_0, w)$ is immediate and $(K|K_0, w)$ is defectless, $(N_w.K|K, w)$ is immediate. By part 4) of Lemma 2.1, K is relatively separable-algebraically closed in $N_w.K$. On the other hand, $(M_w, w) = (N_w.K, w)$ is a maximal field, being a finite extension of a maximal field. $\square$

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