Chapter 4

Valued fields

4.1 Valuations, valuation rings and places

4.1.1 Field valuations

Let K be a field and $v:\ K \ni x \mapsto vx \in \Gamma\infty$ a valuation of its additive group $(K,+,0)$. Suppose that Γ also admits an addition $+$ such that $(\Gamma,+,0,<)$ is an ordered abelian group. We extend the connective $+$ to $\Gamma\infty$ by setting $\infty+\alpha=\alpha+\infty=\infty+\infty=\infty$, and the ordering by $\alpha<\infty$, for every $\alpha\in\Gamma$. Then the group valuation v is called a **valuation of the field** K if it is a homomorphism from the multiplicative group of K onto the additive group Γ. Hence, a map $v:\ K \ni x \mapsto vx \in \Gamma\infty$ is a valuation of K if it satisfies

(V0) $\quad vx=\infty \Longleftrightarrow x=0\,,$

(VT) $\quad v(x-y)\geq \min\{vx,vy\}\,,$

(VH) $\quad v(xy)=vx+vy\,.$

for all $x,y\in K$. In this case, (K,v) is called a **valued field**, and $vK:=\{vc\mid 0\neq c\in K\}$ is called the **value group of** (K,v) (it follows from (VH) that vK is a subgroup of Γ, hence it is itself an ordered abelian group, and $\infty>vK$). Henceforth, we will write $vK\infty$ in the place of $\Gamma\infty$.

We occasionally use $\mathbf{K}$ and $\mathbf{L}$ to denote valued fields (K,v) and (L,v), respectively.

From (VH), we obtain $v1=v(1\cdot 1)=v1+v1$, which implies $v1=0$. With this, (VH) gives $vx+vx^{-1}=0$ as a rule for inverses. So every field valuation satisfies

(V1) $\quad v1=0\,,$

(VI) $\quad vx^{-1}=-vx\,.$

Recall also the properties (VS), (V$\neq$), (V=) and (VM) that v has since it is a group valuation. Equivalence of valuations is defined as in the case of group valuations, but here the isomorphism between the value sets is required to be an isomorphism of ordered abelian groups. A valuation is said to be **trivial on** K if $vK=\{0\}$. Further, we define the **rank of** (K,v) to be the rank of the value group vK. So (K,v) is of rank 1 if and only if vK is archimedean.

It follows from (VT) and (VH) that

$$\mathcal{O}_v\ :=\ \{x\in K\mid vx\geq 0\}$$

is a subring of K, called the **valuation ring of** (K,v) (or: the **valuation ring of** v). If several fields are around, we will also write $\mathcal{O}_K$ or $\mathcal{O}_{\mathbf{K}}$.

Again by (VT) and (VH),

$$\mathcal{M}_v \;:=\; \{x \in K \mid vx > 0\}$$

is an ideal of $\mathcal{O}$, called the **valuation ideal of** (K,v) (or: **valuation ideal of** v). We will also write $\mathcal{M}_K$ or $\mathcal{M}_{\mathbf{K}}$. Later, when there is no danger of confusion, we sill drop subscripts and write $\mathcal{O}$ and $\mathcal{M}$.

Now $\mathcal{O}_v \setminus \mathcal{M}_v$ is the set of elements of value 0, and in view of (VI), these are precisely the elements which are invertible in $\mathcal{O}_v$. So we find that $\mathcal{M}_v$ is the unique maximal ideal of $\mathcal{O}_v$, i.e., $\mathcal{O}_v$ is a **local ring**. Since $\mathcal{M}_v$ is maximal, $\mathcal{O}_v/\mathcal{M}_v$ is a field, called the **residue field of** (K,v). We denote it by Kv or $\overline{K}$. The canonical epimorphism

$$\mathcal{O}_v \longrightarrow \mathcal{O}_v/\mathcal{M}_v \;=\; Kv$$

is called the **residue map of** (K,v). The image $a+\mathcal{M}_v$ of an element $a \in \mathcal{O}_v$ will be denoted by av or $\overline{a}$ and called the **residue of** a. We observe that

$$av \;=\; 0 \iff a \in \mathcal{M}_v \iff va > 0\;.$$

Two valuations v_1 and v_2 on a field K are called **equivalent** if there is an order preserving isomorphism $\varphi : v_1K \to v_2K$ such that $v_2a = \varphi(v_1a)$ for all $a \in K^\times$. Since $\varphi(\alpha) \geq 0 \Leftrightarrow \alpha \geq 0$, we see that equivalent valuations v_1, v_2 have the same valuation ring $\mathcal{O}_{v_1} = \mathcal{O}_{v_1}$. If there is no danger of confusion, we will identify equivalent valuations.

The characteristic of the field Kv is called the **residue characteristic of** (K,v). If $\operatorname{char} K = p > 0$, then the residue characteristic is also equal to p. Indeed, if $p \cdot 1$ equals 0 in K, then the same is true in $\mathcal{O}_v$ and thus also in the residue field $\mathcal{O}_v/\mathcal{M}_v$. In other words: if $\operatorname{char} K > 0$ then $\operatorname{char} Kv = \operatorname{char} K$. But there is also the **mixed characteristic case** where $\operatorname{char} K = 0$ and $\operatorname{char} Kv > 0$, as we will see in the next section.

By (VI), an element $a \in \mathcal{O}_v$ is a **unit** in $\mathcal{O}_v$ if and only if $va = 0$ (and this in turn is equivalent to $av \neq 0$). Thus, the kernel the group homomorphism $v : K^\times \to vK$ is the multiplicative group $\mathcal{O}_v^\times$ of all units in $\mathcal{O}_v$. So we have an isomorphism

$$K^\times/\mathcal{O}_v^\times \;\cong\; vK\;.$$

An element with residue 1 is called a **1-unit**, and we observe that the inverse of a 1-unit is again a 1-unit. Note that $1+\mathcal{M}_v$ is the multiplicative group of all 1-units. The elements $a, a' \in \mathcal{O}_v^\times$ have the same residue if and only if a/a' is a 1-unit. Consequently, the residue map, being a homomorphism from the multiplicative group $\mathcal{O}_v^\times$ onto the multiplicative group $Kv^\times$ with kernel $1+\mathcal{M}_v$, induces an isomorphism

$$\mathcal{O}_v^\times/1+\mathcal{M}_v \;\cong\; Kv^\times\;.$$

From (VI), we see that the elements of negative value are precisely the inverses of the elements in $\mathcal{M}_v$. Since the residue map sends every element in $\mathcal{M}_v$ to 0, an extension of it to all of K should send every element of negative value to "$\frac{1}{0}$", an element not contained in Kv which we will denote by ∞ (there is not too much danger of a confusion with the maximal element ∞ which we adjoin to value sets). Now this extension is a map P_v from K onto $Kv \cup \{\infty\}$. It is called the **place associated with** v. We will also denote it by P_K, $P_{\mathbf{K}}$ or just P. It is customary to write the application of a place from the right, that is, for $a \in K$, its image under P_v is denoted by aP_v. In this spirit, the residue field Kv can also be written as KP_v. Note that as we did with value sets, we exclude ∞ from KP_v.

4.1.2 Places

Suppose that R is a subring of its quotient field K and f is a homomorphism from R into any field L. Then f can be extended to a homomorphism from K into L if and only if f is injective. If f is not injective, then there is $0 \neq r \in R$ such that $f(r) = 0$, so $1/r$ cannot be sent to any element in L. We have seen this case already above, where we chose to send $1/r$ to ∞. This led to the notion of a place.

We axiomatize places as follows. A map $P: K \to KP \cup \{\infty\}$ is a **place of** K if KP is a field and there exists a subring $\mathcal{O}_P$ of K such that

(PH) P is a homomorphism from $\mathcal{O}_P$ onto KP, and $(K \setminus \mathcal{O}_P)P = \{\infty\}$,

(PI) $\forall x \in K^\times : \; xP = 0 \iff \frac{1}{x}P = \infty$.

The ring $\mathcal{O}_P$ is called the **valuation ring of** P. Note that axiom (PH) yields that there is some element $a \in K$ such that $aP \neq 0, \infty$. Axiom (PH) implies that

$$\mathcal{O}_P = \{a \in K \mid aP \neq \infty\} \;;$$

moreover, we deduce:

$$aP \neq 0, \infty \implies a, \tfrac{1}{a} \in \mathcal{O}_P \wedge \tfrac{1}{a}P \neq 0, \infty \,. \tag{4.1}$$

If (K, v) is a valued field and P_v is the place associated with v, then with $\mathcal{O}_{P_v} = \mathcal{O}_v$, it satisfies axioms (PH) and (PI).

Two places P_1 and P_2 of K are called **equivalent** if there is a field isomorphism $\sigma : KP_1 \to KP_2$ such that $aP_2 = \sigma(aP_1)$ if $aP_1 \neq \infty$ and $aP_2 = \infty$ if $aP_1 = \infty$. It then follows that $aP_1 = \infty \Leftrightarrow aP_1 = \infty$, that is, equivalent places P_1 and P_2 have the same valuation ring $\mathcal{O}_{P_1} = \mathcal{O}_{P_2}$. As we do for valuations, we will identify equivalent places if there is no danger of confusion. A place is said to be **trivial on** K if it is an isomorphism on K, that is, if it is equivalent to the identity on K.

4.1.3 Valuation rings

We have already met the concept of a valuation ring in connection with valuations and places, so we will axiomatize it now. A subring $\mathcal{O}$ of a field K is called a **valuation ring of** K if it is a non-trivial and satisfies

(VR) $\forall x \neq 0 : \; x \in \mathcal{O} \vee \frac{1}{x} \in \mathcal{O}$.

Suppose that (K, v) is a valued field and $\mathcal{O}_P$ is the valuation ring associated with v. For every $x \in K^\times$ we have that $vx \geq 0$ or $vx^{-1} = -vx \geq 0$. Hence $x \in \mathcal{O}_P$ or $x^{-1} \in \mathcal{O}_P$, showing that $\mathcal{O}_P$ satisfies axiom (VR).

Note that (VR) implies that $1 \in \mathcal{O}$ and that K is the quotient field of $\mathcal{O}$. Let $\mathcal{M}$ be the subset of all non-units in $\mathcal{O}$. We show that $\mathcal{M}$ is a maximal ideal of $\mathcal{O}$. Let a and $b \neq 0$ be non-units in $\mathcal{O}$. By (VR), we can assume w.l.o.g. that $\frac{a}{b} \in \mathcal{O}$, hence also $\frac{a+b}{b} = 1 + \frac{a}{b} \in \mathcal{O}$. If $a + b$ were a unit in $\mathcal{O}$ then it would follow that $\frac{1}{b} \in \mathcal{O}$, contradicting the assumption that b is not a unit. Further, for every $c \in \mathcal{O}$, also ca is a non-unit because if ca were invertible, the same would hold for a. We have shown that $\mathcal{M}$ is an ideal of $\mathcal{O}$. Since

$\mathcal{O}\setminus\mathcal{M}$ contains only units, every proper ideal of $\mathcal{O}$ must be contained in $\mathcal{M}$, showing that $\mathcal{M}$ is the unique maximal ideal of $\mathcal{O}$. That is, $\mathcal{O}$ is a local ring. By definition of $\mathcal{M}$,

$$\forall x \neq 0:\ x \in K\setminus\mathcal{O} \iff \frac{1}{x} \in \mathcal{M}\,. \tag{4.2}$$

Every valuation ring $\mathcal{O}$ is already uniquely determined by its maximal ideal $\mathcal{M}$ since $\mathcal{O} = K\setminus\{x^{-1}\mid 0\neq x\in\mathcal{M}\}$.

Let us state a few properties of valuation rings:

Lemma 4.1 *Take a valuation ring $\mathcal{O}$ of a field K. Then the following facts hold.*

a) $\mathcal{O}$ is integrally closed in K.

b) Every overring of $\mathcal{O}$ in K is again a valuation ring.

c) The set of overrings of $\mathcal{O}$ in K is linearly ordered by inclusion.

d) The set of ideals and the set of fractional ideals of $\mathcal{O}$ are linearly ordered by inclusion.

e) The ordering defined by $a\mathcal{O}^\times \leq b\mathcal{O}^\times :\Leftrightarrow b\mathcal{O}\subseteq a\mathcal{O}$ turns $K^\times/\mathcal{O}^\times$ into an ordered abelian group. Its positive cone is $\{a\mathcal{O}^\times \mid a\in\mathcal{O}\}$.

f) The map

$$v_{\mathcal{O}}: K^\times \longrightarrow K^\times/\mathcal{O}^\times$$

becomes a valuation on K when we set $v_{\mathcal{O}}0 := \infty$, which is taken to be an element larger than all elements in $K^\times/\mathcal{O}^\times$.

Proof: a): Let $\mathcal{O}$ be a valuation ring of K and P the associated place. Further, let $x\in K$ be an element which is **integrally dependent on** $\mathcal{O}$, that is, there are elements $a_0,\ldots,a_{n-1}\in\mathcal{O}$ such that $x^n+\ldots+a_0=0$. Dividing by x^n, we obtain

$$1 = -a_{n-1}\left(\frac{1}{x}\right) - a_{n-2}\left(\frac{1}{x}\right)^2 - \ldots - a_0\left(\frac{1}{x}\right)^n .$$

If $x\notin\mathcal{O}$, then $\frac{1}{x}\in\mathcal{M}$ and thus $\frac{1}{x}P=0$. Applying P to the above equation, we find $1=1P=0$, a contradiction. This shows $x\in\mathcal{O}$ and proves that $\mathcal{O}$ is integrally closed.

b): If (VR) holds for $\mathcal{O}$, then it holds for every subset of K that contains $\mathcal{O}$.

c): Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are overrings of $\mathcal{O}$ in K, and that $a\in\mathcal{O}_1\setminus\mathcal{O}_2$. We wish to show that $\mathcal{O}_2\subset\mathcal{O}_1$. Take $b\in\mathcal{O}_2$. Then $a/b\notin\mathcal{O}$ because otherwise, $a/b\in\mathcal{O}\subseteq\mathcal{O}_2$ and $a=b\cdot(a/b)\in\mathcal{O}_2$, a contradiction. Since $\mathcal{O}$ is a valuation ring, it follows that $b/a\in\mathcal{O}\subseteq\mathcal{O}_1$ and $b=a\cdot(b/a)\in\mathcal{O}_1$.

d): Take two (possibly fractional) ideals I_1 and I_2 of $\mathcal{O}$ and suppose that $a\in I_1\setminus I_2$. We wish to show that $I_2\subset I_1$. Take $b\in I_2$. Then $a/b\notin\mathcal{O}$ since otherwise, $a=b\cdot(a/b)\in I_2$. Hence, $b/a\in\mathcal{O}$, which yields that $b=a\cdot(b/a)\in I_1$.

e): The fact that this is a linear ordering follows from part d). The compatibility of the ordering with the group operation follows from the fact that $b\mathcal{O}\subset a\mathcal{O}$ implies that $cb\mathcal{O}\subset ca\mathcal{O}$. The neutral element in $K^\times/\mathcal{O}^\times$ is $1\mathcal{O}^\times$, and $1\mathcal{O}^\times\leq a\mathcal{O}^\times\Leftrightarrow a\mathcal{O}\subseteq 1\mathcal{O}=\mathcal{O}\Leftrightarrow a\in\mathcal{O}$, which proves our assertion on the positive cone.

f): Axioms (V0) and (VH) hold by definition. If $b\mathcal{O}\subset a\mathcal{O}$ then $(a-b)\mathcal{O}\subset a\mathcal{O}-b\mathcal{O}\subset a\mathcal{O}$. That is, $b\mathcal{O}\geq a\mathcal{O}$ implies that $(a-b)\mathcal{O}\geq a\mathcal{O}$, proving that $v_{\mathcal{O}}$ satisfies (VT). $\square$

4.1.4 Valuation — valuation ring — place

We have already seen how to associate a valuation ring and a place to every valuation, and a valuation to a valuation ring. Further, we have seen that equivalent valuations have the same valuation ring. Further, we have associated valuation rings to places and observed that equivalent places have the same valuation ring. In fact, the three concepts of valuation, valuation ring and place are completely interchangeable, up to equivalence. We will now fill in the missing details.

First, we shall show that two valuations v_1 and v_2 on a field K are equivalent if their valuation rings $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$ are equal. The idea is to set $\varphi(v_1a) = v_2a$ for $a \in K^\times$; we have to show that this is well-defined. So pick $b \in K^\times$ such that $v_1a = v_1b$. Then a/b and b/a are both elements of $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$, whence $v_2a = v_2b$. By definition, we have that $v_2a = \varphi(v_1a)$ for all $a \in K^\times$. We have to show that $\varphi : v_1K \to v_2K$ is an isomorphism of ordered abelian groups. We have $\varphi(v_1c + v_1d) = \varphi(v_1cd) = v_2cd = v_2c + v_2d = \varphi(v_1c) + \varphi(v_1d)$, hence φ is a group homomorphism. If $v_1c < v_1d$, then $d/c \in \mathcal{O}_{v_1} = \mathcal{O}_{v_2}$ but $c/d \notin \mathcal{O}_{v_1} = \mathcal{O}_{v_2}$, i.e., $v_2c < v_2d$. This shows that φ preserves the ordering and is injective. Clearly, φ is also surjective.

In part f) of Lemma 4.1 we have associated a valuation $v_{\mathcal{O}}$ to a given valuation ring $\mathcal{O}$ with value group $K^\times/\mathcal{O}^\times$. According to part f) of the same lemma, the non-negative elements in this value group are precisely the images of the elements in $\mathcal{O}$. This shows that the map $v \mapsto \mathcal{O}_v$ is the left inverse of the map $\mathcal{O} \mapsto v_{\mathcal{O}}$. Modulo equivalence, it is also its right inverse because two valuations with the same valuation ring are equivalent, as we have already shown. So we can assign to every equivalence class of valuations the unique valuation ring of all the valuations contained in the class, and to every valuation ring $\mathcal{O}$ the equivalence class of all valuations equivalent to $v_{\mathcal{O}}$. Then these two maps are inverses of each other.

Turning to places, we first wish to show that two places P_1 and P_2 are equivalent if they have the same valuation ring. Note that $\mathcal{O}_{P_1} = \mathcal{O}_{P_2}$ implies $\mathcal{M}_{P_1} = \mathcal{M}_{P_2}$ because the maximal ideals are unique. The places P_1 and P_2 induce isomorphisms $\sigma_i : \mathcal{O}_{P_i}/\mathcal{M}_{P_i} \to KP_i$, $i = 1, 2$. We set $\sigma := \sigma_2\sigma_1^{-1}$. Then $aP_2 = \sigma(aP_1)$ if $aP_1 \neq \infty$. If $aP_1 = \infty$ then $a \notin \mathcal{O}_{P_1} = \mathcal{O}_{P_2}$ and hence $aP_2 = \infty$. This proves that P_1 and P_2 are equivalent.

Given a place P on a field K, how do we associate a valuation? One way is to take its valuation ring $\mathcal{O}_P$ and construct a valuation from it, as we have done above. An alternative way is to associate a relation first. Define

$$a \, v \, b \;:\Longleftrightarrow\; a \neq 0 \wedge \tfrac{b}{a}P \neq \infty \,. \tag{4.3}$$

This relation is called **valuation divisibility relation**. Instead of $a\,v\,b$ we will prefer to write $va \leq vb$, even if we formally work with the relation. We derive an equivalence relation by setting $a \sim_P b :\Leftrightarrow a\,v\,b \wedge b\,v\,a$. The set of equivalence classes can be ordered by setting $a/{\sim_P} \leq b/{\sim_P}$ if $a\,v\,b$. Observe that $\infty := 0/{\sim_P}$ is the maximal element of the so obtained ordered set. We introduce addition on the set of all equivalence classes $\neq 0/{\sim_P}$ by $a/{\sim_P} + b/{\sim_P} := ab/{\sim_P}$. We leave it to the reader to show that we have obtained an ordered abelian group Γ and a valuation $v : K \ni a \mapsto a/{\sim_P} \in \Gamma\infty$.

Given a valuation ring $\mathcal{O}$ of the field K, we now wish to show how to obtain a place to which it is the associated valuation ring. Since $\mathcal{M}$ is a maximal ideal, $\mathcal{O}/\mathcal{M}$ is a field. The canonical epimorphism $P_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}/\mathcal{M} =: KP_{\mathcal{O}}$, extended to K by setting

$(K\setminus\mathcal{O})P_{\mathcal{O}}=\{\infty\}$, satisfies (PH) with $\mathcal{O}_{P_{\mathcal{O}}}=\mathcal{O}$. By definition of $\mathcal{M}$ and (VR) we have $x\in\mathcal{M}\Leftrightarrow\frac{1}{x}\in K\setminus\mathcal{O}$, showing that $P_{\mathcal{O}}$ satisfies (PI). We have obtained a map from the set of all valuation rings to the set of all places, with the map $P\mapsto\mathcal{O}_{P_{\mathcal{O}}}$ as its left inverse. Modulo equivalence, it is also its right inverse because two places with the same valuation ring are equivalent, as we have already shown. So we can assign to every equivalence class of places the unique valuation ring of all the places contained in the class, and to every valuation ring $\mathcal{O}$ the equivalence class of all places equivalent to $P_{\mathcal{O}}$. Then these two maps are inverses of each other.

We have now shown that the three concepts "valuation", "place" and "valuation ring" are interchangeable. Every valuation ring corresponds to precisely one equivalence class of valuations and precisely one equivalence class of places. The valuation ring is a proper subring of K if and only if the associated valuation is non-trivial on K, which is the case if and only if the associated place is non-trivial on K.

The arrows in the following picture are bijections:

$$\left\{\begin{array}{c}\text{equivalence classes}\\ \text{of valuations}\end{array}\right\}\longleftrightarrow\{\text{valuation rings}\}\longleftrightarrow\left\{\begin{array}{c}\text{equivalence classes}\\ \text{of places}\end{array}\right\}$$

Exercise 4.1 *In order to define an ordering on $K^\times/\mathcal{O}^\times$ in an alternate way, take $\mathbf{P}$ to be the image of $\mathcal{O}\setminus\{0\}$ under the epimorphism $v:K^\times\longrightarrow K^\times/\mathcal{O}^\times$ and show that $\mathbf{P}$ is a positive cone.*

4.1.5 Embeddings and extensions

As for valued groups, we define an **embedding of valued fields** to be a field embedding which preserves the valuation divisibility relation $vx<vy$. That is, $\iota:\ (K,v)\rightarrow(K',v')$ is an embedding of valued fields if it is an embedding of K in K' and satisfies

$$vx<vy\ \Leftrightarrow\ v'\iota x<v'\iota y\ .$$

In analogy to the case of abelian groups, the reader may show that such an embedding induces an embedding $\rho:vK\rightarrow vK'$ and an embedding $\sigma:\overline{K}\rightarrow\overline{K'}$ satisfying

$$v\iota a=\rho va\quad\text{and}\quad\overline{\iota a}=\sigma\overline{a}\qquad\text{for all } a\in K\ .$$

If this holds, then we say that ι **preserves** ρ and σ (or **induces** ρ and σ). If ι is an isomorphism, then so are ρ and σ.

If ι is a field embedding of K in K' and v is a valuation of K', then $v\iota:\ K\rightarrow vK'\infty$, $x\mapsto v(\iota x)$, is a valuation of K. Its valuation ring is $\iota^{-1}\mathcal{O}_v$ and its valuation ideal is $\iota^{-1}\mathcal{M}_v$. Value group and residue field of v and $v\iota$ coincide. Moreover, $\iota:\ (K,v\iota)\rightarrow(K',v)$ is an embedding of valued fields. If P is a place of K', then the map $P^\iota:\ K\rightarrow K'P\infty$, $x\mapsto(\iota x)P$, is a place of K'. Its associated valuation is $v_{P^\iota}=(v_P)\iota$. Note that $P^{\iota\iota'}=(P^\iota)^{\iota'}$. Let k be a common subfield of K and K' and ι an embedding of K in K' over k. If v is an extension of a valuation v' from k to K', then $v\iota$ is an extension of v' from k to K. If P is a place of $K|k$ and ι is an automorphism of $K|k$, then P^ι is again a place of $K|k$. The proof of these facts is left to the reader. See also Exercise **??** below.

If $L|K$ is a field extension, w is a valuation on L and v is a valuation on K, then we call $(K,v)\subseteq(L,w)$ (also written as $(L,w)|(K,v)$) an **extension of valued fields** and w an **extension of v from K to L** if the **restriction of w to K**, denoted by $w|_K$, coincides with v. In this case, vK is naturally a subgroup of wL, and we leave it to the reader to prove that $Kv \ni av \mapsto aw$ is an embedding of Kv in Lw. Later, whenever we are dealing with only one extension we will denote the valuation on L also by v. Then, the extension will also be denoted as $(L|K,v)$.

Since the axioms for a valuation are universal, the restriction of a valuation v of L to a subfield K of L is again a valuation. Hence, $(L|K,v)$ is a valued field extension once we take the valuation on K to be the restriction of v to K.

If Q is a place on L and P a place on K, then we write $(K,P) \subseteq (L,Q)$ or $(L,Q)|(K,P)$ and say that Q is an **extension of** P **from** K **to** L if $Q|_K = P$. In this case, KP is naturally a subfield of LQ, and we leave it to the reader to prove that $vK \ni va \mapsto wa$ is an embedding of vK in wL.

As to the valuation rings, it is not true that $(L,w)|(K,v)$ is a valued field extension if $\mathcal{O}_w \supset \mathcal{O}_v$. Analyzing our construction of places from valuation rings, we see that it is a crucial point to set $xP = \infty$ for *all* $x \notin \mathcal{O}$. So the required condition is $\mathcal{O}_w \cap K = \mathcal{O}_v$. In view of (4.2), this condition is equivalent to $\mathcal{M}_w \cap \mathcal{O}_v = \mathcal{M}_v$. This gives rise to the following definition. Let $\mathcal{O}, \mathcal{O}'$ be local rings with maximal ideals $\mathcal{M}, \mathcal{M}'$ respectively. Then we will say that $\mathcal{O}$ **lies above** $\mathcal{O}'$ if $\mathcal{O} \supseteq \mathcal{O}'$ and $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$. The latter condition can as well be replaced by the condition $\mathcal{M} \supseteq \mathcal{M}'$. Indeed, $\mathcal{O}' \setminus \mathcal{M}'$ consists only of units, and these cannot be elements of $\mathcal{M}$. This shows that $\mathcal{O} \supset \mathcal{O}'$ together with $\mathcal{M} \supset \mathcal{M}'$ will imply $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$. We have: $(L,w)|(K,v)$ *is a valued field extension if and only if* $\mathcal{O}_w$ *lies above* $\mathcal{O}_v$.

Take an arbitrary extension $(L|K,v)$ of valued fields. Then associated with it we have two other extensions: the one of their value groups $vL \supset vK$, and the one of their residue fields $\overline{L}|\overline{K}$. The study of valued field extensions consists to a great extent of the study of the relation between these three extensions.

The same is true for the restriction of places and valuation rings, but some further remarks are necessary. Given a field extension $L|K$ and places Q on L and P on K, then we will say that Q **is an extension of** P **from** K **to** L if the restriction of Q to K coincides with P. The reader may show that this is the case already when the restriction of Q to $\mathcal{O}_P$ coincides with P. If a place Q on L is given, then the restriction P of Q to K is always a place on K. Indeed, if we set $\mathcal{O}_P := \mathcal{O}_Q \cap K$, then $\mathcal{O}_P$ is a subring of K and P is a homomorphismon $\mathcal{O}_P$. Since $K \setminus \mathcal{O}_P \subseteq L \setminus \mathcal{O}_Q$, we also have that $(K \setminus \mathcal{O}_P)P = \{\infty\}$. Further, it is clear that (PI) is satisfied.

As it is the case with (PI), it is also clear that if $\mathcal{O}_L$ is a valuationring on L, then the subring $\mathcal{O}_L \cap K$ of K will satisfy (VR) and will thus be a valuation ring of K. We say that a valuation ring $\mathcal{O}_L$ is an **extension** of the valuation ring $\mathcal{O}_K$ of K if $\mathcal{O}_L \cap K = \mathcal{O}_K$. For this to hold it does not suffice that $\mathcal{O}_L \supseteq \mathcal{O}_K$. Indeed, we could have $\mathcal{O}_L = L$ but $\mathcal{O}_K \subsetneq K$, in which case $\mathcal{O}_L \cap K = K \neq \mathcal{O}_K$.

From the theory of local rings the following notion is well-known. Let $\mathcal{O}, \mathcal{O}'$ be local rings with maximal ideals $\mathcal{M}, \mathcal{M}'$ respectively. Then we will say that $\mathcal{O}$ **lies above** $\mathcal{O}'$ if $\mathcal{O} \supset \mathcal{O}'$ and $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$. The latter condition can as well be replaced by the condition $\mathcal{M} \supset \mathcal{M}'$. Indeed, $\mathcal{O}' \setminus \mathcal{M}'$ consists only of units, and these can not be elements of $\mathcal{M}$. This shows that $\mathcal{O} \supset \mathcal{O}'$ together with $\mathcal{M} \supset \mathcal{M}'$ will imply $\mathcal{M} \cap \mathcal{O}' = \mathcal{M}'$.

Once we have that $\mathcal{O}_L \supseteq \mathcal{O}_K$, the condition $\mathcal{O}_L \cap K = \mathcal{O}_K$ becomes equivalent to $K \setminus \mathcal{O}_K \subseteq L \setminus \mathcal{O}_L$, which by (4.2) is equivalent to $\mathcal{M}_K \subseteq \mathcal{M}_L$. So we see that $\mathcal{O}_L$ is an extension of $\mathcal{O}_K$ if and only if it lies above $\mathcal{O}_K$.

We will study the relation between extensions of valuations, places and valuation rings in more detail in Section 6.1.

4.2 Examples of valued fields

In this section, we give several basic examples of field valuations.

4.2.1 Valuations on a rational function field $k(x)$

Let k be a field and $k(x)$ the rational function field in one variable over k. Many mathematicians will remember that in their schooldays they were (more or less) tortured with the discussion of the properties of suitably chosen rational functions (with real coefficients). One task was to determine the zeros, poles and their orders. Now for every $a \in k$ we define a map $v_{x-a} : k(x) \to \mathbb{Z}$ by taking $v_{x-a}r(x)$ to be the zero order of $r(x)$ at a if a is not a pole of $r(x)$, or otherwise to be the *negative* pole order of $r(x)$ at a (take a representation of $r(x)$ with relatively prime numerator and denominator so that removable singularities do not interfere). For example, $v_{x-a}(x-a) = 1$, $v_{x-a}(\frac{1}{(x-a)^2}) = -2$, $v_{x-b}(x-a) = 0$ if $b \neq a$. Hence, the valuation ring is $\mathcal{O} = \{r(x) \mid r(x)$ has no pole at $a\}$ and its maximal ideal is $\mathcal{M} = \{r(x) \in \mathcal{O} \mid r(a) = 0\}$, which is just the ideal generated by the polynomial $x-a$. Consequently, the residue map sends x to the same residue class as a. Further, k is contained in $\mathcal{O}$, which yields that $k^\times \subset \mathcal{O}^\times$ and $v_{x-a}(k^\times) = \{0\}$ and that the residue map acts on k as an isomorphism (which is induced on k by the natural epimorphism $\mathcal{O} \to \mathcal{O}/\mathcal{M}$). Composition of the residue map with the inverse of this isomorphism shows that modulo equivalence, the associated place P_{x-a} can be chosen to act as the identity on k. Hence, P_{x-a} can be understood to be the **evaluation map** which sends $r(x)$ to $r(a)$, and we have $k(x)P_{x-a} = k$. Note that P_{x-a} sends $x-a$ to 0. If $r(x)$ has a pole at a, the place P_{x-a} will send it to ∞.

Traditionally, the application of a place P is written in the form $g \mapsto gP$, where instead of gP also $g(P)$ was used in the beginning, reminding of the fact that P originated from an evaluation map. If one translates the German "g an der Stelle a auswerten" literally, one gets "evaluate g at the place a", which explains the origin of the word "place".

Let $p(x)$ be any irreducible polynomial in $k[x]$. Since $k[x]$ admits unique factorization, we can write every polynomial $f(x) \neq 0$ in the form $p^n(x)g(x)$ with $g(x)$ prime to $p(x)$. Define $v_{p(x)}f(x) = n$ and $v0 = \infty$. In view of (VH), this determines the value of every rational function. Then $v_{p(x)}$ is a valuation: (V 0) holds by our definition, and (VH) is an easy consequence. To show (VT), let $r(x), s(x) \in k(x)$. After multiplication by a suitable polynomial, it suffices to show (VT) under the assumption that r, s are polynomials. Now we only have to observe that if r, s are both divisible by $p^n(x)$, then so is $r-s$. We have shown that $v_{p(x)}$ is a valuation. It is called the **$p(x)$-adic valuation**. The elements of value ≥ 0 are those rational functions which can be written as a quotient of two polynomials, the one in the denominator being prime to $p(x)$. Consequently, the valuation ring of $v_{p(x)}$ is the localization $k[x]_{(p(x))} = \{\frac{f(x)}{g(x)} \mid f(x), g(x) \in k[x]$ and $g(x)$ is prime to $p(x)\}$. The valuation ideal is the ideal generated by $p(x)$. The associated place $P_{p(x)}$ sends $p(x)$ to 0. The residue field is $k(x)P_{p(x)} = k[x]_{(p(x))}/(p(x)) = k[x]/(p(x))$ (for the last equality, see the corresponding argument in the next example).

Besides these valuations $v_{p(x)}$ there exists v_∞ with associated place P_∞ which sends $\frac{1}{x}$ to 0 (do you remember from your schooldays how to compute the asymptotic behaviour of $r(x)$ for $x \to \infty$?). P_∞ is called the **place at infinity**. To compute $v_\infty r(x)$, write $r(x) = \frac{1}{x}^n \frac{f_0}{g_0}$ where f_0 and g_0 are polynomials in $\frac{1}{x}$ with non-vanishing constant terms (hence P_∞ sends them neither to 0 nor to ∞). Then $v_\infty r(x) = n$. If $r(x) = \frac{f(x)}{g(x)}$ with suitable polynomials f, g, then you will find that $n = \deg g - \deg f$.

Let us determine all valuations v of $k(x)$ which are trivial on k, or equivalently, all places P of $k(x)$ which are isomorphisms on k. Assume that P is non-trivial and $xP \neq \infty$. Then $k[x] \subset \mathcal{O}$, and $\mathcal{M} \cap k[x]$ will be a non-trivial prime ideal of $k[x]$. Since $k[x]$ is a principle ideal domain, this ideal is generated by one element $p(x) \in k[x]$, and since the ideal is prime, $p(x)$ is irreducible. By virtue of $p(x) \in \mathcal{M}$, we have $p(x)P = 0$. Since a place is (up to equivalence) determined by the valuation ideal, it follows that $P = P_{p(x)}$ and thus, $v_P = v_{p(x)}$.

Now assume that let $xP = \infty$. Then by (VI), $\frac{1}{x}P = 0$. As we have already seen above, this determines the valuation and thus also the place on the field $K(x) = K(\frac{1}{x})$ completely (up to equivalence, as always), and we have $P = P_\infty$. We have shown:

If a valuation v of $k(x)$ is non-trivial, but trivial on k, then $v = v_\infty$ or $v = v_{p(x)}$ for some irreducible $p(x) \in k[x]$. If a place P of $k(x)$ is the identity on k but not on $k(x)$, then $P = P_\infty$ or $P = P_{p(x)}$ for some irreducible $p(x) \in k[x]$.

If k is algebraically closed, then every irreducible polynomial is linear, i.e., of the form $x - a$ with $a \in k$. Hence, all places are of the form P_{x-a} or P_∞. Their residue field is always k.

If k is not algebraically closed, then there will exist irreducible polynomials $p(x)$ of degree $\deg p(x) > 1$. For every such $p(x)$, the residue field will be a proper algebraic extension of k of degree $\deg p(x)$ since xP must be a zero of $p(x)$ (because P is a homomorphism on $\mathcal{O}$).

In all cases, the value group is $\mathbb{Z}$. Valuations with value group (isomorphic to) $\mathbb{Z}$ are called **discrete valuations**.

4.2.2 Valuations on $\mathbb{Q}$

In the preceding example, for every prime element (i.e. irreducible polynomial) $p(x)$ in the ring $k[x]$ there was a valuation $v_{p(x)}$ of the quotient field $k(x)$ with $vp(x) = 1$ and residue field $k[x]/(p(x))$. The same is true for the ring $\mathbb{Z}$ and its quotient field $\mathbb{Q}$. For every prime number p there exists a valuation v_p that counts how often the factor p appears in an integer m, i.e. $v_p m = e$ if $m = p^e m'$ where $(p, m') = 1$. For a rational $r = \frac{m}{n}$ we put $v_p r = v_p m - v_p n$ according to (VH). v_p is called the **p-adic valuation**. Note that on $\mathbb{Z}$ it coincides with the p-valuation given by the height function; but on $\mathbb{Q}$, the height of every element is ∞, so we have to distinguish well between p-valuation and p-adic valuation. Valuation ring and valuation ideal of v_p are $\mathcal{O}_p = \{\frac{m}{n} \in \mathbb{Q} \mid (p, n) = 1\}$ and $\mathcal{M}_p = \{\frac{m}{n} \mid (p, n) = 1 \text{ and } (p, m) = p\} = p\mathcal{O}_p$. The residue field is $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ (for every $\frac{m}{n} \in \mathbb{Q}$ with $(p, n) = 1$ the residue $\overline{n}$ of n modulo p is invertible in $\mathbb{Z}/p\mathbb{Z}$, hence the residue of $\frac{m}{n}$ is just $\overline{m} \cdot \overline{n}^{-1} \in \mathbb{Z}/p\mathbb{Z}$), and the value group is $\mathbb{Z}$. Every v_p yields a metric $d_p(x, y) = p^{-v_p(x-y)}$ on $\mathbb{Q}$ and thus a completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to this (ultra)metric, called the **field of p–adic numbers**. Contrary to the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the metric induced by the absolute value $|x|$, in $\mathbb{Q}_p$ two integers are near to each other if their difference is divisible by a high power of p. This reminds us of the ultrametric associated to a valuation. In fact, we will define a completion for every valued field using this ultrametric (cf. Section **??**). In the case of $\mathbb{Q}$ with the p-adic valuation, this definition just gives $\mathbb{Q}_p$.

Observe that in our first example, the characteristic of the residue field $k(x)P$ was the same as that of the valued field $k(x)$, since it contained k. It can be 0 or a prime number p. This is the **equal characteristic case**. In the second example, the valued field $\mathbb{Q}$ has characteristic 0 whereas its residue field $\mathbb{F}_p$ with respect to the p-adic valuation has characteristic p. This is the **mixed characteristic case**. These are all possible cases because $\operatorname{char} K = p > 0$ implies $\operatorname{char} \overline{K} = p$, as we have already seen.

All examples that we gave so far were discrete valuations. Later, we will use the theory of extensions and compositions of valuations to give examples of non-discrete valuations. Another way of giving examples is to equip Hahn products with a multiplication, cf. Section **??**. From the model theoretic point of view, being discrete is not quite the property of a valuation that we want to investigate (cf. section 20.1). Rather, it is the property of admitting a prime element (which is also shared by certain non-discrete valuations). An element π in a valued field (K, v) is called a **prime element** (also: **local parameter** or **uniformizing parameter**) if its value $v\pi$ is the least positive element in the value group vK, or equivalently, if $\mathcal{M} = \pi\mathcal{O}$. For instance, $(k(x), v_x)$ admits x as a prime element, and $(\mathbb{Q}, v_p)$ admits p as a prime element.

4.2.3 Power series fields

Can we explicitly upgrade certain valued abelian groups by introducing a multiplication on them? In a very important special case, this is indeed possible. Let us fix an arbitrary ordered abelian group Γ and a field k. Viewing k as an abelian group and Γ as an ordered set, we have thus given a skeleton with $B_\gamma = k$ for all $\gamma \in \Gamma$, and we can form the Hahn product $\mathbf{H}_{\gamma\in\Gamma}\, k$ of all maps f from Γ to k with well-ordered support $\{\gamma \in \Gamma \mid f(\gamma) \neq 0\}$. In Section 2.3 we have endowed this set with componentwise addition: $(f + g)(\gamma) := f(\gamma) + g(\gamma)$. We define the product of f and g as follows: for every $\gamma \in \Gamma$, we set

$$(f \cdot g)(\gamma) \;:=\; \sum_{\alpha+\beta=\gamma} f(\alpha) \cdot g(\beta)\;. \tag{4.4}$$

Since the supports of f and g are well-ordered, these sums are finite for all γ. Indeed, if $\alpha_1 + \beta_1 = \gamma = \alpha_2 + \beta_2$ and $\alpha_1 < \alpha_2$, then $\beta_2 < \beta_1$, but a well-ordered set only allows finite descending sequences.

To generalize the construction, one can introduce a **factor set** $\{\zeta_{\alpha,\beta} \mid \alpha, \beta \in \Gamma\}$ and define $(a \cdot b)_\gamma := \sum_{\alpha+\beta=\gamma} \zeta_{\alpha,\beta} a_\alpha b_\beta$; these factor sets have to satisfy certain compatability conditions. Since we will not need these constructions using factor sets, we omit the details and refer the reader to [KOC2].

We will write the elements $f \in \mathbf{H}_{\gamma\in\Gamma}\, k$ in the form

$$\sum_{\gamma\in\Gamma} c_\gamma t^\gamma$$

where $c_\gamma = f(\gamma)$, and call them **formal power series** or just **power series**. Instead of $\mathbf{H}_{\gamma\in\Gamma}\, k$ we will write $k((t^\Gamma))$ or simply $k((\Gamma))$. The product of two power series now reads as

$$\left(\sum_{\gamma\in\Gamma} c_\gamma t^\gamma\right)\cdot\left(\sum_{\gamma\in\Gamma} d_\gamma t^\gamma\right) \;=\; \sum_{\gamma\in\Gamma}\left(\sum_{\alpha+\beta=\gamma} c_\alpha d_\beta\right) t^\gamma\;,$$

or, if a non-trivial factor set is used, as

$$\left(\sum_{\gamma\in\Gamma} c_\gamma t^\gamma\right)\cdot\left(\sum_{\gamma\in\Gamma} d_\gamma t^\gamma\right) \;=\; \sum_{\gamma\in\Gamma}\left(\sum_{\alpha+\beta=\gamma} \zeta_{\alpha,\beta} c_\alpha d_\beta\right) t^\gamma\;, \tag{4.5}$$

In Section 2.3 we defined the **minimum support valuation** on a Hahn product to associate to every element the minimum of its support (which exists since the support is well-ordered). For power series, we will frequently denote this valuation by v_t, and v_t will also be called the **canonical valuation** of $k((\Gamma))$. We have:

$$v_t \sum_{\gamma\in\Gamma} c_\gamma t^\gamma \;=\; \min\{\gamma\in\Gamma \mid c_\gamma\neq 0\}\;.$$

Lemma 4.2 *$(k((\Gamma)), v_t)$ is a valued field.*

Proof: We know already from Section 2.3 that v_t is a group valuation. If in (4.5), $\gamma_1 = \min\{\gamma\in\Gamma \mid c_\gamma\neq 0\}$ and $\gamma_2 = \min\{\gamma\in\Gamma \mid d_\gamma\neq 0\}$, then the coefficient of $t^{\alpha+\beta}$ on the right hand side is $\zeta_{\alpha,\beta}c_\alpha d_\beta\neq 0$, hence $\alpha+\beta$ is the minimum of the support of the product. This proves that $v_t(a\cdot b) = v_t a + v_t b$, which is axiom (VH). We leave it to the reader to show that the above definition of multiplication turns the Hahn product into an integral domain.

It is more interesting to show the existence of inverses, for which we will use the Ultrametric Fixed Point Theorem, as was done in [PC2]. We will give the proof for the case of a trivial factor set and leave the general case as an exercise to the reader. Take $a = \sum_{\gamma\in\Gamma} c_\gamma t^\gamma \in k((\Gamma))$ and set $\alpha := v_t a$. Consequently, $c_\alpha\neq 0$ and $a = c_\alpha\cdot t^\alpha + a'$ with $v_t a' > \alpha$. This shows that for $b := a\cdot c_\alpha^{-1}t^{-\alpha} - 1$ we have $v_t b > 0$. Now we consider the map $\Xi :\; k((\Gamma)) \;\to\; k((\Gamma))$ given by $x \;\mapsto\; c_\alpha^{-1}t^{-\alpha} - xb$. This map is contractive since $v_t(\Xi x - \Xi y) = v_t(-xb+yb) = v_t(x-y) + v_t b > v_t(x-y)$. By Lemma 2.14, the valued group $k((\Gamma))$ is spherically complete. By the Ultrametric Fixed Point Theorem (1.12), there is a unique fixed point $x\in k((\Gamma))$ for Ξ. It satisfies $x = a_\alpha^{-1}t^{-\alpha} - xb$, that is, $x(1+b) = a_\alpha^{-1}t^{-\alpha}$, whence $x\cdot a\cdot c_\alpha^{-1}t^{-\alpha} = c_\alpha^{-1}t^{-\alpha}$, which gives $x\cdot a = 1$. □

A field of the form $k((\Gamma))$ is called a **power series field**. It follows directly from the definition of v_t that $v_t k((\Gamma)) = \Gamma$. The elements of the support of a power series are called the **exponents** of the power series. A power series lies in the valuation ring $\mathcal{O}_{v_t}$ if and only if has only non-negative elements of Γ. This valuation ring will be denoted by $k[[\Gamma]]$. A ring of this form is called a **power series ring**. The valuation ideal $\mathcal{M}_{v_t}$ consists of all power series with only positive exponents. Every element $a\in\mathcal{O}_{v_t}$ can be written as $c_0 t^0 + b$ with $b\in\mathcal{M}_{v_t}$. Hence, the map $a\mapsto c_0$ is an epimorphism from $\mathcal{O}_{v_t}$ onto k. This shows that the residue field $\mathcal{O}_{v_t}/\mathcal{M}_{v_t}$ of $(k((\Gamma)), v_t)$ is (isomorphic to) k.

When dealing with power series fields, we will always assume that the factor set is trivial, if not stated otherwise. Let us note that in this case, the valued field $(K, v) = (k((\Gamma))$ admits a **cross-section**, that is, a homomorphism $\varphi : vK \to K$ such that $v\circ\varphi = \mathrm{id}_{vK}$. Indeed, for $\varphi(\alpha) := t^\alpha$ we have that $\varphi(\alpha+\beta) = t^{\alpha+\beta} = t^\alpha\cdot t^\beta = \varphi(\alpha)\cdot\varphi(\beta)$. Note that

$t^0 = 1$ and $1/t^\alpha = t^{-\alpha}$. Later, we will give conditions for arbitrary valued field which imply the existence of a cross-section (cf. Section 22.4).

The field

$$k((t)) \;:=\; k((t^{\mathbb{Z}})) \;=\; k((\mathbb{Z})) \;=\; \{\sum_{i=N}^{\infty} c_i t^i \mid N \in \mathbb{Z}\,,\, c_i \in k\}$$

is called the **field of formal Laurent series**; note that a subset of $\mathbb{Z}$ is well-ordered if and only if it is bounded from below by some $N \in \mathbb{Z}$. The valuation ring of $(k((t)), v_t)$ consists of all formal Laurent series with $N \geq 0$ and is denoted by $k[[t]]$. The valuation ideal can be written as $t \cdot k[[t]]$. The value group is $\mathbb{Z}$, and the residue field is k. Since $k((t))$ is a field and contains $k[t]$, it also contains $k(t)$. The canonical valuation v_t on $k((t))$ is an extension of the **t-adic valuation** v_t in the sense of Section 4.2 1), t being understood as a prime polynomial in $k[t]$. The place associated to this valuation sends t to 0. Further, t is a prime element of both $(k(t), v_t)$ and $(k((t)), v_t)$.

A valued field (K, v) is called **spherically complete** if the underlying valued additive group is spherically complete (or equivalently, if the underlying ultrametric space is spherically complete). Again from Lemma 2.14, we obtain:

Theorem 4.3 *Every power series field with its canonical valuation is spherically complete.*

The construction of $k((\Gamma))$ gives an answer to the question whether for a given ordered abelian group Γ and a given field k there exists a valued field with value group Γ and residue field k. In view of its maximality, $k((\Gamma))$ is a "large field" satisfying our conditions. Inspired by the observation that the Hahn sums appear to be the smallest valued groups with a given skeleton, we may ask for "small" valued fields with given value group and residue field. It turns out that this is indeed an important question, as we will see when we deal with the defect of valued fields. The word "smallest" will be made precise by means of the notion "valuation transcendence basis", cf. Section 6.7.

Note that $k((\Gamma))$ is a valued field which has the same residue characteristic as k. If the characteristic of a given residue field is $p > 0$, then the question arises whether we can also construct a valued field of characteristic 0 having this residue field. The analogue of our above power series field construction is the famous theory of **Witt vectors**. We refer the reader to [HAS] or [JAC]. It is of special importance for the structure theory of maximal fields of mixed characteristic. If one is only interested in the problem of constructing arbitrary fields with given value group and residue field, then one can again follow the approach using valuation transcendence bases. It also works in the mixed characteristic case, starting from the prime field $\mathbb{Q}$ with its p-adic valuation.

4.2.4 Puiseux series fields

In this section, we shall consider a special class of valued fields which can be represented as the union over a countable ascending chain of power series fields. Let k be an arbitrary field and t transcendental over k. We choose elements $t_n \in \widetilde{K(t)}$ for every $n \in \mathbb{N}$ such that

1) $t_1 = t$,

2) $t_n^n = t$,

3) $t_n^\ell = t_m$ whenever $\ell m = n$.

Now we consider the power series fields $K_n := k((t_n))$ for $n \in \mathbb{N}$. Every two fields K_n and K_m are contained in the common extension field K_{nm}. Hence, the union $K := \bigcup_{n\in\mathbb{N}} K_n$ is again a field, and it is endowed with a valuation which extends the t_n-adic valuation of every K_n. (This follows from Lemma 20.7 and the fact that valued fields can be axiomatized by universal existential axioms). Such a field is called a **Puiseux series field over** k. We will denote its canonical valuation by v_t. We leave it to the reader to prove that up to isomorphism, the Puiseux series field over k does not depend on the particular choice of the t_n for $n > 1$ if k contains all roots of unity.

The field K can be viewed as the union over an ascending chain of fields K_{n_i}. Indeed, let p_i denote the i-th prime number and set $n_k := \prod_{i=1}^{k} p_i^k$. If $j \leq k$, then n_j divides n_k and therefore, $K_{n_j} \subset K_{n_k}$. Further, for every $m \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that m divides n_k, whence $K_m \subset K_{n_k}$. Consequently, the fields K_{n_k}, $k \in \mathbb{N}$, form an ascending chain of fields with union K.

The value group of every K_n is $\mathbb{Z}vt_{n_i} = \mathbb{Z}\frac{vt}{n_i}$, so the union over all K_n has value group $\bigcup_{n\in\mathbb{N}} \mathbb{Z}\frac{vt}{n} = \mathbb{Q}vt$. This is divisible and isomorphic to $\mathbb{Q}$. The residue field of every K_n is k, hence also the residue field of the union of all K_n is k.

For every $n \in \mathbb{N}$, the element t_n is algebraic over $k((t))$. Hence it follows from Corollary 6.26 that $K_n = k((t_n)) = k((t))(t_n)$ is algebraic over $k((t))$. Consequently, also the union K of all K_n is algebraic over $k((t))$.

4.2.5 Formally $\wp$-adic and finitely ramified fields

A valued field (K, v) is called **finitely ramified** if it admits a prime element π (i.e., $v\pi$ is the least positive element in vK), and there is a prime p and a natural number e such that $vp = ev\pi$. Such a finitely ramified field has residue characteristic p; indeed, $vp > 0$ shows that $p = 0$ in $\overline{K}$. On the other hand, $vp = ev\pi < \infty$ shows that the characteristic of K is $\neq p$ and thus, it must be equal to 0. This shows that every finitely ramified field is of mixed characteristic.

An important subclass of the class of finitely ramified fields is that of formally $\wp$-adic fields. We call a finitely ramified field a **formally $\wp$-adic field** if its residue field is a finite field. If in addition, this residue field is equal to $\mathbb{F}_p$ and p is a prime element of the field, then we call it a **formally p-adic field**.

A formally $\wp$-adic field (K, v) is called a **$\wp$-adically closed field** if no proper algebraic extension has both the same prime element and the same residue field as (K, v).

4.2.6 Places in algebraic geometry

An important problem in algebraic geometry is the **resolution of singularities**: we wish to associate to any given algebraic variety V another variety V' which has no singular points. The two varieties are required to be birationally equivalent, which is equivalent to asking that they have the same function field. For varieties over ground field of characteristic 0, it has been shown by Heisuke Hironaka in 1965 that resolution of singularities is always possible.

The local form of resolution of singularities, called **local uniformization**, is the following task: Assume we have a variety V and a point on this variety. We wish to find a variety V' birationally equivalent to V on which the corresponding point is non-singular.

As varieties will in general not have points in common, we have to set up a correspondence between the given point on V and the new point on V'.

Let us have a closer look at our notion of "point". Assume our variety V is given by polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_\ell]$. Naively, by a point of V we then mean an ℓ-tupel $(a_1, \ldots, a_\ell)$ of elements in an arbitrary extension field L of K such that $f_i(a_1, \ldots, a_\ell) = 0$ for $1 \leq i \leq n$. This means that the kernel of the "evaluation homomorphism" $K[X_1, \ldots, X_\ell] \to L$ defined by $X_i \mapsto a_i$ contains the ideal $(f_1, \ldots, f_n)$. So it induces a homomorphism η from the coordinate ring $K[V] = K[X_1, \ldots, X_\ell]/(f_1, \ldots, f_n)$ into L over K. (The latter means that it leaves the elements of K fixed.) However, if $a'_1, \ldots, a'_\ell \in L'$ are such that $a_i \mapsto a'_i$ induces an isomorphism from $K(a_1, \ldots, a_\ell)$ onto $K(a'_1, \ldots, a'_\ell)$, then we would like to consider $(a_1, \ldots, a_\ell)$ and $(a'_1, \ldots, a'_\ell)$ as the same point of V. That is, we are only interested in η up to composition $\sigma \circ \eta$ with isomorphisms σ. This we can get by considering the kernel of η instead of η. This leads us to the modern approach: to view a point as a prime ideal of the coordinate ring.

In order to set up a correspondence of the points of two varietes, however, it is most convenient to work with the convention that a point of V is a homomorphism of $K[V]$ over K, up to composition with isomorphisms. Recall that $K[V] = K[x_1, \ldots, x_\ell]$, where x_i is the image of X_i under the canonical epimorphism $K[X_1, \ldots, X_\ell] \to K[X_1, \ldots, X_\ell]/(f_1, \ldots, f_n) = K[V]$. The function field $K(V)$ of V is the quotient field $K(x_1, \ldots, x_\ell)$ of $K[V]$. It is generated by $x_1, \ldots, x_\ell$ over K.

Now recall what it means to look for another variety V' having the same function field $F := K(V)$ as V (i.e., being birationally equivalent to V). It just means to look for another set of generators $y_1, \ldots, y_k$ of F over K. Now the points of V' are the homomorphisms of $K[y_1, \ldots, y_k]$ over K, up to composition with isomorphisms. But in general, $y_1, \ldots, y_k$ will not lie in $K[x_1, \ldots, x_\ell]$, hence we do not see how a given homomorphism of $K[x_1, \ldots, x_\ell]$ could determine a homomorphism of $K[y_1, \ldots, y_k]$. But if we could extend the homomorphism of $K[x_1, \ldots, x_\ell]$ to all of $K(x_1, \ldots, x_\ell)$, then this extension would a homomorphism of$K[y_1, \ldots, y_k]$. Let us give a very simple example.

Example 4.4 Consider the coordinate ring $K[x]$ of $V = \mathbb{A}^1_K$. That is, x is transcendental over K, and the function field $K(V)$ is just the rational function field $K(x)$ over K. A homomorphism of the polynomial ring $K[V] = K[x]$ is just given by evaluating every polynomial $g(x)$ at $x = a$. It can be extended to a homomorphism (and hence an isomorphism) of $K(x)$ if and only if a is transcendental over K. Indeed, if a is not transcendental over K, then there will be a polynomial h over K having a as a root. Then a is a pole of $1/h(x)$. So we have to accept that the evaluation will not only render elements in $K(a)$, but also the element ∞. So we can extend our homomorphism to a map P on all of $K(x)$, taking into the bargain that it may not always render finite values. But on the subring $\mathcal{O}_P = \{g(x)/h(x) \mid h(a) \neq 0\}$ of $K(x)$ on which P is finite, it is still a homomorphism.

Observe that in this example, P is uniquely determined, up to equivalence, by the homomorphism on $K[x]$. Indeed, we can always write g/h in a form such that a is not a zero of both g and h. Then if a is not a zero of h, we have that $(g/h)P = g(a)/h(a) \in K(a)$. If a is a zero of h, we have that $(g/h)P = \infty$. Thus, the residue field of P is $K(a)$, and the value group is $\mathbb{Z}$. $\diamondsuit$

At this point, we shall introduce a useful notion. Given a field extension $F|K$ (such as an algebraic function field), we will call P a **place of** $F|K$ if it is a place of F whose restriction to K is the identity. We say that P is **trivial** on K if it induces an isomorphism on K. But then, composing P with the inverse of this isomorphism, we find that P is equivalent to a place of F whose restriction to K is the identity. A place P of $F|K$ is said to be a **rational place** if $FP = K$. The **dimension** of P, denoted by $\dim P$, is the transcendence degree of $FP|K$. Hence, P is **zero-dimensional** if and only if $FP|K$ is algebraic. Note that a place P of F is trivial on K if and only if v_P is **trivial** on K, i.e., $v_P K = \{0\}$. This is also equivalent to $K \subset \mathcal{O}_P$. We can characterize places of $F|K$ as follows (the proof is left to the reader):

Lemma 4.5 *A place P of F is (equivalent to) a place of $F|K$ if and only if K is contained in its valuation ring.*

Let's get back to our problem. The first thing we learn from our example is the following. Clearly, we would like to extend our homomorphism of $K[V]$ to a place of $K(V)$ because then, it will induce a map on $K[V']$. But in order to obtain a corresponding point, this map must be a homomorphism of $K[V']$. So we have to require that

$$K[V'] \subseteq \mathcal{O}_P$$

or equivalently, $y_1, \ldots, y_k \in \mathcal{O}_P$.

The next question coming to mind is whether to every point there corresponds exactly one place (up to equivalence), as it is the case in Example 4.4. To destroy this hope, I give again a very simple example. It will also serve to introduce several types of places and their invariants.

Example 4.6 Consider the coordinate ring $K[x_1, x_2]$ of $V = \mathbb{A}^2_K$. That is, x_1 and x_2 are algebraically independent over K, and the function field $K(V) = K(x_1, x_2)$ is just the rational function field in two variables over K. A homomorphism of the polynomial ring $K[V] = K[x_1, x_2]$ is given by evaluating every polynomial $g(x_1, x_2)$ at $x_1 = a_1$, $x_2 = a_2$. For example, let us take $a_1 = a_2 = 0$ and try to extend the corresponding homomorphism of $K[x_1, x_2]$ to $K(V) = K(x_1, x_2)$. It is clear that $1/x_1$ and $1/x_2$ go to ∞. But what about x_1/x_2 or even x_1^m/x_2^n? Do they go to 0, ∞ or some non-zero element in K? The answer is: all that is possible, and there are infinitely many ways to extend our homomorphism to a place of $K(x_1, x_2)$.

There is one way, however, which seems to be the most well-behaved. It is to construct what we will call a **place of maximal rank**; we will explain this notion later in full generality. The idea is to learn from Example 4.4 where we replace K by $K(x_2)$ and x by x_1, and extend the homomorphism defined on $K(x_2)[x_1]$ by $x_1 \mapsto 0$ to a unique place Q of $K(x_1, x_2)$. Its residue field is $K(x_2)$ since $x_1 Q = 0 \in K(x_2)$, and its value group is $\mathbb{Z}$. Now we do the same for $K(x_2)$, extending the homomorphism given on $K[x_2]$ by $x_2 \mapsto 0$ to a unique place $\overline{Q}$ of $K(x_2)$ with residue field K and value group $\mathbb{Z}$. We compose the two places, in the following way. Take $b \in K(x_1, x_2)$. If $bQ = \infty$, then we set $bQ\overline{Q} = \infty$. If $bQ \neq \infty$, then $bQ \in K(x_2)$, and we know what $bQ\overline{Q} = (bQ)\overline{Q}$ is. In this way, we obtain a place $P = Q\overline{Q}$ on $K(x_1, x_2)$ with residue field K. We observe that for every $g \in K[x_1, x_2]$, we have that $g(x_1, x_2)Q\overline{Q} = g(0, x_2)\overline{Q} = g(0, 0)$, so our place P indeed extends the given homomorphism of $K[x_1, x_2]$. Now what happens to our critical fractions? Clearly, $(1/x_1)P = (1/x_1)Q\overline{Q} = (\infty)\overline{Q} = \infty$, and $(1/x_2)P = (1/x_2)Q\overline{Q} = (1/x_2)\overline{Q} = \infty$. But what interests us most is that for all $m > 0$ and $n \geq 0$, $(x_1^m/x_2^n)P = (x_1^m/x_2^n)Q\overline{Q} = 0\overline{Q} = 0$. We see that "$x_1$ goes more strongly to 0 than every x_2^n". We have achieved this by sending first x_1 to 0, and only afterwards x_2 to 0. We have arranged our action "lexicographically".

What is the associated value group? The results in Section 4.5 will tell us that for every composition $P = Q\overline{Q}$, the value group $v_{\overline{Q}}(FQ)$ of the place $\overline{Q}$ on FQ is a convex subgroup of the value group $v_P F$, and that the value group $v_Q F$ of P is isomorphic to $v_P F / v_{\overline{Q}}(FQ)$. If the subgroup $v_{\overline{Q}}(FQ)$ is a direct summand of $v_P F$ (as it is the case in our example), then $v_P F$ is the lexicographically ordered direct product $v_Q F \times v_{\overline{Q}}(FQ)$. Hence in our case, $v_P K(x_1, x_2) = \mathbb{Z} \times \mathbb{Z}$, ordered lexicographically. The **rank of** (F, P) is defined to be the rank of the ordered abelian group $v_P F$. In our case, the rank is 2. We will see in Section 6.4 that if P is a place of $F|K$, then the rank cannot exceed the transcendence degree of $F|K$. So our place $P = Q\overline{Q}$ has maximal possible rank.

There are other places of maximal rank which extend our given homomorphism (for example, interchange the role of x_1 and x_2). But there is also an abundance of places of smaller rank. In our case, the rank cannot be 0 because that would mean that the value group is $\{0\}$ and the place is trivial; but then it cannot send x_1 and x_2 to 0. So the only other possibility is rank 1, i.e., there is only one proper convex subgroup of the value group, namely $\{0\}$. For an ordered abelian group G, having rank 1 is equivalent to being archimedean ordered and to being embeddable in the ordered additive group of $\mathbb{R}$. Which subgroups of $\mathbb{R}$ can we get as value groups? To determine them, we use the notion of **rational rank** $\mathrm{rr}\, G := \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} G$ of an abelian group G. Note that $\mathbb{Q} \otimes_{\mathbb{Z}} G$ is (isomorphicto) the **divisible hull** of G. In fact, $\mathrm{rr}\, G$ is the maximal number of rationally independent elements in G. We will see in Section 6.4 that for every place P of $F|K$ we have that

$$\mathrm{rr}\, v_P F \;\leq\; \mathrm{trdeg}\, F|K\ . \tag{4.6}$$

Hence in our case, also the rational rank of P can be at most 2. The subgroups of $\mathbb{R}$ of rational rank 2 are well known: they are the groups of the form $r\mathbb{Z} + s\mathbb{Z}$ where $r > 0$ and $s > 0$ are rationally independent real numbers. Moreover, through multiplication by $1/r$, the group is order isomorphic to $\mathbb{Z} + \frac{s}{r}\mathbb{Z}$. As we identify equivalent valuations, we can assume all rational rank 2 value groups (of a rank 1 place) to be of the form $\mathbb{Z} + r\mathbb{Z}$ with $0 < r \in \mathbb{R} \setminus \mathbb{Q}$. To construct a place P with this value group on $K(x_1, x_2)$, we proceed as follows. We want that $v_P x_1 = 1$ and $v_P x_2 = r$; then it will follow that $v_P K(x_1, x_2) = \mathbb{Z} + r\mathbb{Z}$ (cf. Theorem 5.1 below). We observe that for such P, $v_P(x_1^m / x_2^n) = m - nr$, which is > 0 if $m/n > r$, and < 0 if $m/n < r$. Hence, $(x_1^m / x_2^n)P = 0$ if $m/n > r$, and $(x_1^m / x_2^n)P = \infty$ if $m/n < r$. I leave it to you as an exercise to verify that this defines a unique place P of $K(x_1, x_2)|K$ with the desired value group and extending our given homomorphism.

Observe that so far every value group was finitely generated, namely by two elements. Now we come to the groups of rational rank 1. If such a group is finitely generated, then it is simply isomorphic to $\mathbb{Z}$. How do we get places P on $K(x_1, x_2)$ with value group $\mathbb{Z}$? A place with value group $\mathbb{Z}$ is called a **discrete place**. The idea is to first construct the place on the subfield $K(x_1)$. We know from Example 4.4 that (up to equivalence) there is a unique place of $K(x_1)|K$ which is trivial on K and sends x_1 to 0; it has value group $\mathbb{Z}$ and residue field K. Now we can try to extend this place from $K(x_1)$ to $K(x_1, x_2)$ in such a way that the value group doesn't change.

There are many different ways how this can be done. One possibility is to send the fraction x_1/x_2 to an element z which is transcendental over K. We leace it to the reader to verify that there is a unique place which does this and extends the given homomorphism;

it has value group $\mathbb{Z}$ and residue field $K(z)$. If, as in this case, a place P of $F|K$ has the property that $\operatorname{trdeg} FP|K = \operatorname{trdeg} F|K - 1$, then P is called a **prime divisor** and v_P is called a **divisorial valuation**. The places $Q, \overline{Q}$ were prime divisors, one of F, the other one of FQ.

If we want the residue field to be algebraic over K or even equal to K, then we can employ another approach, using the field $K((t))$ of formal Laurent series. It is known that the transcendence degree of $K((t))|K(t)$ is uncountable. If K is countable, this follows directly from the fact that $K((t))$ then has the cardinality of the continuum. But it is quite easy to show that the transcendence degree is at least one (see Exercise 4.3), and already this suffices for our purposes here. So take any $y \in K((t))$, transcendental over $K(t)$; then $x_1 \mapsto t$, $x_2 \mapsto y$ induces an isomorphism $K(x_1, x_2) \to K(t, y)$. We take the restriction of v_t to $K(t, y)$ and pull it back to $K(x_1, x_2)$ through the isomorphism. What we obtain on $K(x_1, x_2)$ is a valuation v which extends our valuation v_P of $K(x_1)$. As is true for v_t, also this extension still has value group $\mathbb{Z} = v_P K(x_1)$ and residue field $K = K(x_1)P$. The desired place of $K(x_1, x_2)$ is simply the place associated with this valuation v.

We have now constructed essentially all places on $K(x_1, x_2)$ which extend the given homomorphism of $K[x_1, x_2]$ and have a finitely generated value group (up to certain variants, like exchanging the role of x_1 and x_2). The somewhat shocking experience to every "newcomer" is that on this rather simple rational function field, there are also places extending the given homomorphism and having a value group which is not finitely generated. For instance, the value group can be $\mathbb{Q}$. (In fact, it can be any subgroup of $\mathbb{Q}$.) We postpone the construction of such places to Section **??**. $\diamondsuit$

After we have treated two special cases, the question arises whether any given homomorphism of a coordinate ring $K[V]$ (or more generally, any subring of a field) can be extended to a place on its quotient field (or more generally, to any field which contains the ring). A positive answer will be given in the next section.

Exercise 4.2 *Prove: if P is a place of $F|K$ then it is an isomorphism on the relative algebraic closure of K in F (which is called the* **(exact) constant field of** *$F|K$).*

Exercise 4.3 *Prove that*

$$y = \sum_{i=1}^{\infty} t^{i!} \in K((t))$$

is transcendental over K. Use this idea to construct more transcendence elements.

4.3 Existence of places and their extensions

We have to answer two questions: Is every homomorphism of a subring of a given field extendable to a place of that field? Does every valuation of a given field admit an extension to a valuation of a given extension field? These questions are closely connected, and the key to their answers is the following theorem:

Theorem 4.7 *Let $\mathcal{R}$ be a subring of the field K containing 1, and let $\mathcal{I}$ be a proper ideal of $\mathcal{R}$. Then there exists a valuation ring $\mathcal{O} \supset \mathcal{R}$ of K whose maximal ideal $\mathcal{M}$ contains $\mathcal{I}$.*

Proof: We consider the set S of all subrings R of K which contain $\mathcal{R}$ and satisfy $R\mathcal{I} \neq R$. We have $\mathcal{R} \in S$, hence S is a nonempty set which is partially ordered by inclusion. Given a totally ordered subset S_0 of S, the cardinality of its union R_0 is bounded by $|K|$, hence R_0 is a ring (this follows from Lemma 20.7 and the fact that rings can be axiomatized by universal axioms). If we are able to show that $R_0 \in S$, then it will follow by Zorn's Lemma that S contains maximal elements. The condition $R_0 \supset \mathcal{R}$ is trivially satisfied. Now suppose that $R_0\mathcal{I} = R_0$. Then there are elements $r_1, \ldots, r_n \in R_0$ and $a_1, \ldots, a_n \in \mathcal{I}$ such that $\sum_{1\leq i\leq n} r_i a_i = 1$. But there is a ring R_1 in S_0 which contains already all r_i, hence $R_1\mathcal{I} = R_1$ contradicting $R_1 \in S$. We have shown that $R_0\mathcal{I} \neq R_0$ and consequently, $R_0 \in S$.

Now it remains to prove that a maximal element $\mathcal{O}$ of S is a valuation ring with the required properties. Since $\mathcal{O} \in S$, we have $\mathcal{O} \supset \mathcal{R}$ and $\mathcal{O}\mathcal{I} \neq \mathcal{O}$. It suffices to show that $\mathcal{O}$ is a valuation ring of K since then, its unique maximal ideal $\mathcal{M}$ will automatically contain the proper ideal $\mathcal{O}\mathcal{I}$. We have to show that $\mathcal{O}$ satisfies (VR), i.e. that $x \in \mathcal{O} \vee \frac{1}{x} \in \mathcal{O}$ for every nonzero $x \in K$. By the lemma following this proof, it follows that $\mathcal{O}[x]\mathcal{I}$ is a proper ideal of the ring $\mathcal{O}[x]$ or $\mathcal{O}[\frac{1}{x}]\mathcal{I}$ is a proper ideal of the ring $\mathcal{O}[\frac{1}{x}]$. But this contradicts the maximality of $\mathcal{O}$, unless $x \in \mathcal{O}$ or $\frac{1}{x} \in \mathcal{O}$. □

Lemma 4.8 *Let $\mathcal{R}$ be a subring of the field K containing 1, and let $\mathcal{I}$ be a proper ideal of $\mathcal{R}$. Then for every $x \in K$, $\mathcal{R}[x]\mathcal{I}$ is a proper ideal of the ring $\mathcal{R}[x]$ or $\mathcal{R}[\frac{1}{x}]\mathcal{I}$ is a proper ideal of the ring $\mathcal{R}[\frac{1}{x}]$.*

Proof: To deduce a contradiction, assume that $\mathcal{R}[x]\mathcal{I} = \mathcal{R}[x]$ and $\mathcal{R}[\frac{1}{x}]\mathcal{I} = \mathcal{R}[\frac{1}{x}]$. Then there are elements $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathcal{I}$ such that $1 = \sum_{0\leq i\leq n} a_i x^i$ and $1 = \sum_{0\leq j\leq m} b_j x^{-j}$. Assume that m and n are the smallest numbers admitting such equations. Let $m \leq n$; for $m \geq n$, the proof is symmetrical. Multiplying the first equation by $1 - b_0$ and the second equation by $a_n x^n$, we obtain

$$\begin{aligned} 1 - b_0 &= (1 - b_0)a_0 + \ldots + (1 - b_0)a_n x^n \\ (1 - b_0)a_n x^n &= a_n b_1 x^{n-1} + \ldots + a_n b_m x^{n-m} . \end{aligned}$$

It follows that

$$1 = b_0 + (1 - b_0)a_0 + \ldots + (1 - b_0)a_{n-1}x^{n-1} + a_n b_1 x^{n-1} + \ldots + a_n b_m x^{n-m} .$$

The coefficients on the right hand side are all elements of $\mathcal{I}$. Hence, we have found a representation of 1 in $\mathcal{R}[x]\mathcal{I}$ of length $n-1$, contradicting the minimality of n. This is the desired contradiction. □

The following result is often called **Chevalley's Extension Theorem**, while the theorem we will derive from it is also called **Chevalley's Place Extension Theorem**.

Corollary 4.9 *Take a subring $\mathcal{R}$ with 1 of a field K, and let $\mathcal{I}$ be a proper prime ideal of $\mathcal{R}$. Then there exists a valuation ring $\mathcal{O} \supset \mathcal{R}$ of K whose maximal ideal $\mathcal{M}$ satisfies $\mathcal{M} \cap \mathcal{R} = \mathcal{I}$.*

Proof: Take $\mathcal{R}_\mathcal{I}$ to be the localization of $\mathcal{R}$ with respect to $\mathcal{I}$. Then $\mathcal{R}_\mathcal{I}$ has the unique maximal ideal $\mathcal{R}_\mathcal{I}\mathcal{I}$ (it is a proper ideal since $\mathcal{I}$ is). We claim that $\mathcal{R}_\mathcal{I}\mathcal{I} \cap \mathcal{R} = \mathcal{I}$. The inclusion "$\supseteq$" follows from $1 \in \mathcal{R} \subseteq \mathcal{R}_\mathcal{I}$. If "$\subseteq$" were not true, then $\mathcal{R}_\mathcal{I}\mathcal{I}$ would contain an element of $\mathcal{R} \setminus \mathcal{I}$; but this element is invertible in $\mathcal{R}_\mathcal{I}$ which contradicts the fact that $\mathcal{R}_\mathcal{I}\mathcal{I}$ is a maximal ideal.

By Theorem 4.7, there is a valuation ring $\mathcal{O} \supset \mathcal{R}_\mathcal{I}$ whose maximal ideal $\mathcal{M}$ contains $\mathcal{R}_\mathcal{I}\mathcal{I}$. Since $\mathcal{R}_\mathcal{I} \setminus \mathcal{R}_\mathcal{I}\mathcal{I}$ only contains units and since $\mathcal{M}$ does not contain units, we find that $\mathcal{M} \cap \mathcal{R}_\mathcal{I} = \mathcal{R}_\mathcal{I}\mathcal{I}$. Consequently, $\mathcal{M} \cap \mathcal{R} = \mathcal{R}_\mathcal{I}\mathcal{I} \cap \mathcal{R} = \mathcal{I}$. □

The following theorem answers our initial questions.

Theorem 4.10 *Let $\mathcal{R}$ be a subring of the field K containing* 1. *Then every non-trivial homomorphism of $\mathcal{R}$ can be extended to a place of K. In particular, if k is a subfield of K, then every place of k admits an extension to a place of K.*

Proof: Let φ be a non-trivial homomorphism of $\mathcal{R}$ and $\mathcal{I}$ its kernel. Note that the kernel is a prime ideal; it is proper since φ is assumed to be non-trivial. From the foregoing corollary we obtain a valuation ring $\mathcal{O} \supset \mathcal{R}_\mathcal{I}$ whose maximal ideal $\mathcal{M}$ satisfies $\mathcal{M} \cap \mathcal{R} = \mathcal{I}$. This shows that the place P which is given by the canonical epimorphism $P : \mathcal{O} \to \mathcal{O}/\mathcal{M}$ is an extension of the canonical epimorphism $\varphi' : \mathcal{R} \to \mathcal{R}/\mathcal{I}$. But since $\mathcal{I}$ is the kernel of φ on $\mathcal{R}$, we know that $\varphi = \sigma' \circ \varphi'$ for a suitable isomorphism of $\mathcal{R}/\mathcal{I}$ (which we may view in a natural way as a subring of $\mathcal{O}/\mathcal{M}$). We may extend σ' to an isomorphism σ of $\mathcal{O}/\mathcal{M}$. Then the place $\sigma \circ P$ (which is equivalent to P) is an extension of the homomorphism φ.

If φ is a place of k with valuation ring $\mathcal{R}$ and valuation ideal $\mathcal{I}$, then $\mathcal{M} \cap \mathcal{R} = \mathcal{I}$ shows that $\mathcal{O}$ lies above $\mathcal{R}$, and we obtain that P is an extension of φ. □

Using the interchangeability of valuations and places, we obtain:

Corollary 4.11 *Let (K, v) be a valued field and $L|K$ a field extension. Then there is always an extension of v to a valuation of L.*

A further application of Theorem 4.10 is the following

Corollary 4.12 *Let $K|k$ be an arbitrary field extension. There are non-trivial places of $K|k$ if and only if* $\operatorname{trdeg} K|k > 0$.

Proof: If P is a place of $K|k$ then we know that the associated valuation ring contains k. The valuation ring is integrally closed in K by Lemma 4.1. If $\operatorname{trdeg} K|k = 0$ then $K|k$ is algebraic and every element of K is integrally dependent of k. This yields $\mathcal{O} = K$, that is, P is trivial on K.

Now assume that $x \in K$ is transcendental over k. Then the polynomial ring $k[x]$ admits a homomorphism φ onto k which sends x to 0. From Theorem 4.10 we infer the existence of a place P of K which extends φ. Hence $xP = 0$, showing that P is not trivial on K. On the other hand, its valuation ring contains k which yields that P is a place of $K|k$. □

"Almost all" fields admit non-trivial valuations or places:

Corollary 4.13 *The only fields which do not admit non-trivial places are precisely the algebraic extensions of finite fields. In particular, if v is a valuation of the field K of characteristic $p > 0$, then its restriction to the relative algebraic closure of $\mathbb{F}_p$ in K is trivial.*

Proof: Let $\operatorname{char} K = 0$. Then K contains $\mathbb{Q}$. We have shown in Section 4.2 that $\mathbb{Q}$ admits non-trivial places (the ones associated with p-adic valuations), hence by Theorem 4.10, so does K. Now let $\operatorname{char} K = p > 0$. The field $\mathbb{F}_p$ has no proper non-trivial subring, hence by Corollary 4.12, no algebraic extension of $\mathbb{F}_p$ admits a non-trivial place. On the other hand, if $\operatorname{trdeg} K|\mathbb{F}_p > 0$, then Corollary 4.12 shows that K admits a non-trivial place. $\square$

The following is an important characterization of the integral closure:

Theorem 4.14 *Let $\mathcal{R}$ be a subring of the field K. Then the intersection of all valuation rings of K containing $\mathcal{R}$ is the integral closure of $\mathcal{R}$ in K.*

Proof: Since every valuation ring of K is integrally closed (Lemma 4.1), a valuation ring of K containing $\mathcal{R}$ will also contain the integral closure of $\mathcal{R}$. Hence, the intersection of all valuation rings of K containing $\mathcal{R}$ contains the integral closure of $\mathcal{R}$. For the converse, assume that $x \in K$ is not integrally dependent on $\mathcal{R}$; we want to show that there is a valuation ring $\mathcal{O}$ of K containing $\mathcal{R}$ such that $x \notin \mathcal{O}$. We consider the ring $\mathcal{R}[x^{-1}]$. Then x^{-1} is not a unit in this ring. Otherwise, there would be elements $a_0, \ldots, a_n \in \mathcal{R}$ such that $a_n x^{-n} + \ldots + a_1 x^{-1} + a_0 = (x^{-1})^{-1} = x$. Multiplying by x^n, we would find $x^{n+1} - a_0 x^n - \ldots - a_n = 0$ in contradiction to our assumption that x be not integrally dependent on $\mathcal{R}$. Since x^{-1} is not a unit in $\mathcal{R}[x^{-1}]$, it follows that $x^{-1}\mathcal{R}[x^{-1}]$ is a proper ideal of this ring. By Theorem 4.7, there exists a valuation ring $\mathcal{O}$ of K containing $\mathcal{R}[x^{-1}]$, whose maximal ideal $\mathcal{M}$ contains $x^{-1}\mathcal{R}[x^{-1}]$. Consequently, $\mathcal{O}$ contains $\mathcal{R}$ and $\mathcal{M}$ contains x^{-1}, showing that $x \notin \mathcal{O}$. $\square$

Taking $\mathcal{R}$ to be the valuation ring of a given valued field (K, v), we can apply this theorem to an arbitrary field extension L of K, in the place of K. Then we obtain:

Corollary 4.15 *Let (K, v) be a valued field, $L|K$ a field extension and $x \in L$. Then $wx \geq 0$ for every extension w of v to L if and only if x is an element of the integral closure of $\mathcal{O}_{\mathbf{K}}$ in L.*