Chapter 10

Orderings and valuations

10.1 Ordered fields and their natural valuations

One of the main examples for group valuations was the natural valuation of an ordered abelian group. Let us upgrade ordered groups. A field K together with a relation $<$ is called an **ordered field** and $<$ is called an **ordering of** K if its additive group together with $<$ is an ordered abelian group and the ordering is compatible with the multiplication:

(OM) $\qquad 0 < x \wedge 0 < y \implies 0 < xy\,.$

For the **positive cone of an ordered field**, the corresponding additional axiom is:

(PC·) $\qquad \mathbf{P}\cdot\mathbf{P} \subset \mathbf{P}\,.$

Since $-\mathbf{P}\cdot -\mathbf{P} = \mathbf{P}\cdot\mathbf{P} \subset \mathbf{P}$, all squares of K and thus also all sums of squares are contained in $\mathbf{P}$. Since $-1 \in -\mathbf{P}$ and $\mathbf{P} \cap -\mathbf{P} = \{0\}$, it follows that $-1 \notin \mathbf{P}$ and in particular, -1 is not a sum of squares. From this, we see that the characteristic of an ordered field must be zero (if it would be $p > 0$, then -1 would be the sum of $p-1$ 1's and hence a sum of squares).

Since the correspondence between orderings and positive cones is bijective, we may identify the ordering with its positive cone. In this sense, $\mathbf{X}_K$ will denote the set of all orderings resp. positive cones of K.

Let us consider the natural valuation of the additive ordered group of the ordered field $(K,<)$. Through the definition $va+vb := vab$, its value set vK becomes an ordered abelian group and v becomes a homomorphism from the multiplicative group of K onto vK. We have obtained the **natural valuation of the ordered field** $(K,<)$. The place associated with the natural valuation will be called **natural place of the ordered field** $(K,<)$. Since the natural valuation of an ordered field is in fact the natural valuation of its ordered additive group, it inherits the properties of the latter. In particular, the natural valuation of an ordered field is trivial if and only if the ordering is archimedean. We have seen in the last section that if we consider a field valuation as a group valuation, then the components in its associated skeleton are all isomorphic to the additive group of the residue field. Since the components of natural group valuation are archimedean, it follows that the residue field of the natural valuation is an archimedean ordered group. We leave it to the reader to show that the so induced ordering is compatible with the multiplication (we will prove this fact later in a more general setting), and that the residue field is thus an archimedean ordered field. By a theorem of D. Hilbert, which is an analogue of the Theorem of Hölder (Theorem 3.55), every archimedean ordered field can be embedded into the reals $\mathbb{R}$, as an ordered field (cf. [HILB]; see also S. Prieß-Crampe [PC1], Chapter II, §3, or L. Fuchs [FU1], Chapter VIII, §1, for the proof in the more general case of archimedean rings). So if P is the natural place of $(K,<)$, then there is an embedding σ of KP in $\mathbb{R}$. Since $\sigma \circ P$ and P are equivalent places (which we usually identify), we can always assume without loss of generality that

The residue field of the natural valuation resp. the natural place) is a subfield of $\mathbb{R}$.

A field K is called **formally real** if it can be ordered. The ordering may not be unique, so there may be a variety of natural valuations on K. An arbitrary place P of K is called a **real place** if its residue field KP is formally real. Every subfield of $\mathbb{R}$ is formally real, so every natural place is a real place. We are going to state some basic facts about these valuations and places.

Remark 10.1 The natural valuation was introduced by R. Baer under the name "Ordnungsbewertung" (cf. [BAER]). For W. Krull, the main reason for the introduction of general valuations was the investigation of the formally real fields, as introduced by E. Artin and O. Schreier [AR–SCHR]. In his far-sighted article "Allgemeine Bewertungstheorie" ([KRU7]), Krull says:

> Nachdem nun durch Artin und Schreier der Begriff der allgemeinen reellen Körper rein algebraisch eingeführt wurde, liegt die Frage nahe, ob es nicht möglich ist, den Bewertungsbegriff so zu verallgemeinern, daß er zu einer — ihrer Natur nach halb algebraischen, halb analytischen — Behandlung der reellen Körper brauchbar wird. ... Im Rahmen dieses Programms erscheint die Weiterentwicklung der Theorie der geordneten Körper als ein — sogar verhältnismäßig einfacher — Spezialfall der Theorie der allgemein bewerteten Körper. Im übrigen können auch die "transzendentesten" geordneten Körper mit ... unsrer allgemeinen Bewertungstheorie behandelt werden. Das erscheint mir, wie bereits zu Beginn der Einleitung hervorgehoben, als das wichtigste Ergebnis und die eigentliche Rechtfertigung der Einführung des verallgemeinerten Bewertungsbegriffes.

For the further development of the valuation theoretical approach to formally real fields, see S. Lang [LANG2] (1953), R. Brown [BRW2] (1971), M. Knebusch [KN1] (1972) and [KN2] (1973), A. Prestel [PR1] (1975), M. Knebusch and M. Wright [KN–WR] (1976), T. Y. Lam [LAM1] (1980) and [LAM2] (1983), S. Prieß-Crampe [PC1] (1983).

The algebraic theory of formally real fields is known as the **Artin-Schreier-Theory**. One of its basic results is the following: *A field is formally real if and only if* -1 *is not a sum of squares in that field, and this holds if and only if non-trivial sums of squares in K never equate to zero* (cf. [PR1] Theorem 1.8 or [JAC], Chapter VI). Using this characterization, the following is easy to prove:

Lemma 10.2 *If the field* K *admits a real place, then it is formally real.*

Proof: Let P be real place of K and $\sum_i a_i^2$ a non-trivial finite sum of squares in K, that is, there is at least one nonzero summand. We may assume that all a_i lie in $\mathcal{O}_P$ and that at least one of them lies in $\mathcal{O}_P^\times$; otherwise, we pick some a_j with $v_P a_j$ minimal, and we divide the whole sum by a_j^2, thereby replacing the a_i by the elements a_i/a_j which satisfy our assumption. Now we apply P and obtain the finite sum $(\sum_i a_i^2)P = \sum_i (a_i P)^2$ of squares in KP. By our assumption on the a_i, this sum has at least one nonzero summand. Since KP is formally real by assumption, the sum is nonzero. Hence, the sum $\sum_i a_i^2$ is nonzero (and it lies in $\mathcal{O}_P^\times$). This proves that K is formally real. $\square$

The question arises whether an ordering of K can be found such that the real place P is the natural place with respect to this ordering. But this can not be true if the formally real residue field KP has no archimedean ordering (e.g., if its cardinality is bigger than $|\mathbb{R}|$, then it can not be embedded in $\mathbb{R}$ and thus can not admit an archimedean ordering). Instead of "natural place", we have to use the notions **compatible with the ordering**, **convex valuation** and **v-compatible ordering** which we define as in the group case (cf. page 47). We note:

Lemma 10.3 *Let $(K,<)$ be an ordered field and v a valuation of K. Then v is compatible with the ordering if and only if it satisfies*

(COMP) $\quad \forall x,y\in K:\ \ 0<x\leq y\ \Rightarrow\ vx\geq vy\ .$

This in turn is equivalent to each of the following assertions:
1) the valuation ring $\mathcal{O}_v$ is a convex subset of $(K,<)$,
2) the valuation ideal $\mathcal{M}_v$ (or equivalently, $1+\mathcal{M}_v$) is a convex subset of $(K,<)$,
3) the positive cone $\mathbf{P}$ of $(K,<)$ contains $1+\mathcal{M}_v$,
4) $\mathcal{M}_v<1$.

Proof: If v is compatible with the ordering then by definition, it satisfies (COMP). Now assume (COMP) and let $0\leq a\leq b$ with $b\in\mathcal{O}_v$. Then $va\geq vb$ and hence, $a\in\mathcal{O}_v$. Since $\mathcal{O}_v$ is closed under $x\mapsto -x$, this yields that $\mathcal{O}_v$ is convex.

Assume that $\mathcal{O}_v$ is convex and let $0\leq a\leq b$ with $b\in\mathcal{M}_v$. Then $0\leq b^{-1}\leq a^{-1}$ and $b^{-1}\notin\mathcal{O}_v$. By the convexity of $\mathcal{O}_v$, we find $a^{-1}\notin\mathcal{O}_v$, that is, $a\in\mathcal{M}_v$. Since $\mathcal{M}_v$ is closed under $x\mapsto -x$, this yields that $\mathcal{M}_v$ is convex.

Assume that $\mathcal{M}_v$ is convex. Recall that in an ordered abelian group $(G,<)$ for every $g\in G$, a subset $M\subset G$ is convex if and only if $g+M$ is. Hence, $\mathcal{M}_v$ is convex if and only if $1+\mathcal{M}_v$ is. If the latter is the case, then $1+\mathcal{M}_v\subset\mathbf{P}$ since otherwise, it would contain an element <0 and in view of $1\in 1+\mathcal{M}_v$ and convexity, it would follow that $0\in 1+\mathcal{M}_v$, which is impossible.

Assume that $1+\mathcal{M}_v\subset\mathbf{P}$. Since $1\notin\mathcal{M}_v$ and $\mathcal{M}_v$ is closed under $x\mapsto -x$, this yields that $1-a>0$ for all $a\in\mathcal{M}_v$, that is, $\mathcal{M}_v<1$.

Assume that $\mathcal{M}_v<1$, and let $a,b\in K$ such that $va<vb$. Then $b/a\in\mathcal{M}_v$, hence also $n|b|/|a|\in\mathcal{M}_v$ for every $n\in\mathbb{N}$. Our assumption then yields that $n|b|/|a|<1$, i.e. $n|b|<|a|$. We have thus proved that $|a|\leq n|b|\Rightarrow va\geq vb$, showing that v is compatible with the order, by our original definition given for the group case. □

We let $\mathbf{X}_K^v$ denote the set of all v-compatible orderings (resp. positive cones) of K.

Recall from the group case that a valuation is compatible with the order if and only if it is a coarsening of the natural valuation. If v is the natural valuation of $(K,<)$, then by definition, $\mathcal{M}_v$ is precisely the set of all $a\in K$ which satisfy $|a|<1/n$ for all $n\in\mathbb{N}$; such an element a is called **infinitesimal**. We see that precisely the infinitesimals are sent to 0 by the natural place. The existence of nonzero infinitesimals characterizes the non-archimedean ordered fields (in particular, every nonstandard model of the reals has a non-trivial natural valuation and nonzero infinitesimals). Observe that every convex valuation v of $(K,<)$ will satisfy $\mathcal{M}_v<1/n$ for all $n\in\mathbb{N}$; this follows from $\mathcal{M}_v<1$. We thus find that a valuation is convex if and only if its valuation ideal is contained in the valuation ideal of the natural valuation. Hence, it is a coarsening of the natural valuation (cf. Section **??**).

Let v be a convex valuation of $(K,<)$ with associated place $P = P_v$. Then by the foregoing lemma, $\mathcal{M}_v$ is a convex subset of $(K,<)$. Hence, the ordered additive group $\mathcal{M}_v$ is a convex subgroup of the ordered additive group $\mathcal{O}_v$. By Lemma 2.21, the ordering of $\mathcal{O}_v$ thus induces an ordering of the additive group of $\overline{K} = \mathcal{O}_v/\mathcal{M}_v$. If $\mathbf{P}$ denotes the positive cone of $(K,<)$, then $\mathbf{P}\cap\mathcal{O}_v$ is the positive cone of $(\mathcal{O}_v,<)$, and $\overline{\mathbf{P}} := (\mathbf{P}\cap\mathcal{O}_v)/\mathcal{M}_v = (\mathbf{P}\cap\mathcal{O}_v)P$ is the positive cone of the induced ordering of $\overline{K}$. Observe that for $a\in\mathcal{O}_v$, $a\in\mathbf{P}$ if and only if $\overline{a}\in\overline{\mathbf{P}}$. Since $(ab)P = aP\cdot bP$, we have $\overline{\mathbf{P}}\cdot\overline{\mathbf{P}}\subset\overline{\mathbf{P}}$, which shows that the induced ordering is in fact an ordering of the field $\overline{K}$. In particular, we conclude:

Lemma 10.4 *The place associated with a convex valuation is a real place.*

The ordering (resp. the positive cone $\mathbf{P}$) of $(K,<)$ induces also some structure on the value group of the convex valuation v, which we will describe now. The ordering induces the **character** (that is, a multiplicative group homomorphism) $\mathrm{sign}_{\mathbf{P}} : K^\times \to \{1,-1\}$ sending a to 1 if and only if $a>0$. Since all squares are positive, $K^{\times 2}$ lies in the kernel of this homomorphism, and we obtain a character $\chi_{\mathbf{P}}\colon K^\times/K^{\times 2}\to\{1,-1\}$ (observe that $\mathbf{P}$ is uniquely determined by $\chi_{\mathbf{P}}$). On the other hand, composing the valuation v with the canonical epimorphism $vK\to vK/2vK$, we obtain an epimorphism $K^\times\to vK/2vK$. Again, the squares are in the kernel, so we obtain an epimorphism $h_v\colon K^\times/K^{\times 2}\to vK/2vK$. Both groups $K^\times/K^{\times 2}$ and $vK/2vK$ are $\mathbb{F}_2$-vector spaces. Hence there is an embedding $s: vK/2vK\to K^\times/K^{\times 2}$ such that $h_v\circ s = \mathrm{id}$. Now $\sigma_{\mathbf{P}} := \chi_{\mathbf{P}}\circ s\colon vK/2vK\to\{1,-1\}$ is a character induced by $\mathbf{P}$. Note that it depends on our choice of s. With a fixed s, the following holds:

Theorem 10.5 *Let v be a valuation of the field K. The map*

$$\mathbf{X}_K^v \ni \mathbf{P} \mapsto (\overline{\mathbf{P}},\sigma_{\mathbf{P}}) \in \mathbf{X}_{\overline{K}}\times \mathrm{Hom}(vK/2vK,\{1,-1\}) \tag{10.1}$$

is a bijection.

Proof: We choose $\mathcal{B}\subset K^\times$ such that the elements $bK^{\times 2}$, $b\in\mathcal{B}$, form an $\mathbb{F}_2$-basis of $s(vK/2vK)$. It follows that the elements $h_v(bK^{\times 2}) = vb+2vK$, $b\in\mathcal{B}$, form an $\mathbb{F}_2$-basis of $vK/2vK$. Hence, for every element $a\in K\setminus\{0\}$, there are elements $b_1,\dots,b_n\in\mathcal{B}$ and $c\in K\setminus\{0\}$ such that $va = vb_1+\ldots+vb_n+2vc$, with $b_1,\dots,b_n$ uniquely determined. Consequently, $a = b_1\cdot\ldots\cdot b_nuc^2$ with a unit $u\in\mathcal{O}_v^\times$. Since c^2 is positive for every ordering, the sign of a only depends on the signs of $b_1,\dots,b_n$ and u. By our definition of $\sigma_{\mathbf{P}}$, we have $\sigma_{\mathbf{P}}(vb+2vK)=1 \Leftrightarrow \chi_{\mathbf{P}}(bK^{\times 2}) = \chi_{\mathbf{P}}\circ s(vb+2vK) = 1\Leftrightarrow b\in\mathbf{P}$. For a positive cone $\mathbf{P}$ compatible with v, the sign of u with respect to $\mathbf{P}$ is the same as the sign of $\overline{u}$ with respect to $\overline{\mathbf{P}}$. We conclude that $\mathbf{P}$ is uniquely determined by $(\overline{\mathbf{P}},\sigma_{\mathbf{P}})$.

It remains to show the surjectivity of (10.1). Given a character $\sigma: vK/2vK\to\{1,-1\}$ and a positive cone $\mathbf{P}_{\overline{K}}$ of $\overline{K}$, we define

$$\mathrm{sign}(b_1\cdot\ldots\cdot b_nuc^2) = \sigma(b_1)\cdot\ldots\cdot\sigma(b_n)\cdot\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}).$$

This is well-defined since it does not depend on the choice of u and c. To show that $\mathbf{P}=\{a\in K\mid \mathrm{sign}(a)=1\}\cup\{0\}$ is a positive cone of K, it suffices to show $\mathbf{P}+\mathbf{P}\subset\mathbf{P}$ and $\mathbf{P}\cdot\mathbf{P}\subset\mathbf{P}$. The latter is immediate since the above defined sign function is multiplicative.

To show the former, let $a, a' \in \mathbf{P} \setminus \{0\}$ and let a be represented as above. If $va < va'$, then $a + a' = a(1 + a'/a) = b_1 \cdot \ldots \cdot b_n u(1 + a'/a)c^2$ where $(1 + a'/a)$ is a 1-unit; since u and $u(1 + a'/a)$ have the same residue, it follows that $\mathrm{sign}(a + a') = \mathrm{sign}(a)$, showing that $a + a' \in \mathbf{P}$. Now assume that $va = va'$. Then there is a representation $a' = b_1 \cdot \ldots \cdot b_n u' c^2$, and $\overline{u}$ and $\overline{u'}$ have the same sign with respect to $\mathbf{P}_{\overline{K}}$. Since $\mathbf{P}_{\overline{K}}$ is a positive cone, $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u} + \overline{u'})$ is equal $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u'})$. Consequently, $\mathrm{sign}(u + u') = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u + u'}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u} + \overline{u'}) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = \mathrm{sign}(u)$. Hence again, $\mathrm{sign}(a + a') = \mathrm{sign}(a)$, and $a + a' \in \mathbf{P}$.

Further, $\mathbf{P}$ contains all 1-units since $\mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u}) = 1$ for every 1-unit u. This proves that $\mathbf{P}$ is compatible with v. By our definition, $\mathrm{sign}(u) = \mathrm{sign}_{\mathbf{P}_{\overline{K}}}(\overline{u})$, showing that $\overline{\mathbf{P}} = \mathbf{P}_{\overline{K}}$. Finally, $\sigma_{\mathbf{P}} = \sigma$ since $\sigma_{\mathbf{P}}$ and σ are uniquely determined by their action on the basis elements $vb + 2vK$, $b \in \mathcal{B}$. $\square$

There are the following special cases of this theorem. If $\overline{K}$ admits a unique ordering (e.g. if $\overline{K} = \mathbb{R}$), then (10.1) is in fact a bijection between $\mathbf{X}_K^v$ and $\mathrm{Hom}(vK/2vK, \{1, -1\})$. If on the other hand, vK is 2-divisible, then $\mathrm{Hom}(vK/2vK, \{1, -1\})$ contains only the trivial character, and (10.1) is a bijection between $\mathbf{X}_K^v$ and $\mathbf{X}_{\overline{K}}$. If $\overline{K}$ admits a unique ordering and vK is 2-divisible, then K admits precisely one v-compatible ordering. In any case, we have the following answer to our initial question:

Corollary 10.6 *If v is a valuation of the field K such that $\overline{K}$ is formally real, then there exists a v-compatible ordering of K. In other words, if P is a real place of K, then v_P is compatible with some ordering of K.*

A further consequence of our theorem is the following. Assume that (L, v) is an immediate extension of (K, v), that is, value group and residue field remain unchanged. If $\mathcal{B}$ and s are as above, we observe that the values vb, $b \in \mathcal{B}$, remain $\mathbb{F}_2$-independent over $2vK = 2vL$. Hence, the elements $b \in \mathcal{B}$ will also be $\mathbb{F}_2$-independent over $L^{\times 2}$. Consequently, s remains to be an embedding of $vK/2vK$ in $L^\times/L^{\times 2}$. The same construction as in the proof of the theorem then gives a bijection between $\mathbf{X}_L^v$ and $\mathbf{X}_{\overline{K}} \times \mathrm{Hom}(vK/2vK, \{1, -1\})$. This proves:

Corollary 10.7 *Let $(L|K, v)$ be an immediate extension of (K, v). Then every v-compatible ordering of K admits a unique extension to a v-compatible ordering of L.*

The extension can easily be defined: By assumption, for every $a \in L$ there is $a' \in K$ such that $v(a - a') > va$. Then we (have to) set $\mathrm{sign}(a) = \mathrm{sign}(a')$. Indeed, if the order is compatible with v on L, then its positive cone contains $1 + \mathcal{M}_{\mathbf{L}}$ and thus, $\mathrm{sign}(a) = \mathrm{sign}(a(1 + (a' - a)/a)) = \mathrm{sign}(a')$ since $1 + (a' - a)/a$ is a 1-unit.

Example 10.8 Let us see in how many ways a power series field $K = k((\Gamma))$ over a formally real field k can be ordered. Let v be the canonical valuation of K. First, we note that every element of $1 + \mathcal{M}_v$ is a square in K. The reader may prove this by computing explicitly the expansion of the square roots of a given 1-unit. Later, we will see that this is a consequence of the fact that v is henselian and the residue field k has characteristic 0. (In fact, the same arguments show that the group of 1-units is even divisible.) It follows from Lemma 10.3 that v is compatible with every ordering on K, that is, $\mathbf{X}_K = \mathbf{X}_K^v$. It also follows that the sign of a power series $a = \sum_\alpha a_\alpha t^\alpha$ must be the same as that of $a_{va}t^{va}$ (since $a/a_{va}t^{va}$ is a 1-unit). If $\alpha \in 2\Gamma$, then $t^\alpha = (t^{\alpha/2})^2$ is a square and thus positive under every ordering of K. If we choose a maximal set of values $\beta \in \Gamma$, $\mathbb{F}_2$-independent over 2Γ, then the signs of all t^α are already determined by the signs of the elements t^β. On the other hand, we have freedom to choose the signs of all t^β and to determine the sign of the coefficients $a_\alpha \in k$ by choosing an ordering of k. We obtain $\mathbf{X}_K = \mathbf{X}_K^v \leftrightarrow \mathbf{X}_k \times \mathrm{Hom}(vK/2vK, \{1,-1\})$ in accordance with Theorem 10.5. ♢

Loosely speaking, we have shown above how to transfer the property "formally real" upwards (from the residue field to the field) and downwards (from the field to the residue field) by a place. There are many other properties that admit such a treatment, e.g. "real closed", "Rolle field", "half-ordered", "algebraically closed", "perfect". Since we will need certain properties of places and valuations, like "henselian" or "algebraically maximal", we will discuss these cases later. Here, we will give some examples which all use the idea of the proof of Lemma 10.2.

A field is called **euclidean** if it is formally real and every element or its negative is a square. Consequently, *every valuation of a euclidean field has a 2-divisible value group.* Suppose that $(K,<)$ is an ordered field with convex valuation v. Every element of $\overline{K}$ may be written as $\overline{a}$ with $a \in \mathcal{O}_{\mathbf{K}}^\times$. Then a or $-a$ and thus also $\overline{a}$ or $-\overline{a}$ is a square. This proves: *The residue field of a euclidean field with respect to a convex valuation is euclidean.*

The **Pythagoras number** of a field K is the smallest number $n \leq \infty$ such that every sum of squares in K equals a sum of at most n squares. If $n = 1$, that is, if every sum of squares is a square, then K is called **pythagorean**. Assume K to be formally real and the Pythagoras number of K to be n. Let v be any convex valuation of K. Given a sum $\sum_{i=1}^m \overline{a}_i^2$ of squares in $\overline{K}$ with $m \geq n$, we can find $b_1, \ldots, b_n \in K$ such that $\sum_{i=1}^m a_i^2 = \sum_{i=1}^n b_i^2$. Then all b_i are in $\mathcal{O}_v$. Indeed, if this would not be true, we could choose some b_j of minimal value and divide the second sum by b_j to obtain a sum of squares which are all in $\mathcal{O}_v$ but not all in $\mathcal{M}_v$ such that the sum is in $\mathcal{M}_v$. Passing to the residue field, we would obtain a non-trivial sum of squares which sums up to zero. This contradicts the fact that $\overline{K}$ is formally real, being the residue field of a convex valuation. Hence, all b_i are in $\mathcal{O}_v$, and passing to the residue field, we find that $\sum_{i=1}^m \overline{a}_i^2 = \sum_{i=1}^n \overline{b}_i^2$. We have proved: *If v is a convex valuation of the formally real field K, then the Pythagoras number of $\overline{K}$ is smaller or equal to the Pythagoras number of K. In particular, if K is pythagorean, then so is $\overline{K}$.*

If the field K is not formally real, then -1 is a sum of squares in K. The **level** of K is the smallest number n such that -1 is a sum of n squares. If K is formally real, then we define the level of K to be ∞. Let $-1 = \sum_{i=1}^n a_i^2$ in K, and let v be any valuation of K. Since $-1 \notin \mathcal{M}_v$, not all a_i can lie in $\mathcal{M}_v$. If all a_i lie in $\mathcal{O}_v$, then $-1 = \sum_{i=1}^n \overline{a}_i^2$. If there are some a_i of value < 0, then by the same procedure as in our last argument, we find a non-trivial sum of at most n squares in $\overline{K}$ which sums up to zero. Dividing by an arbitrary nonzero square in this sum, we find that -1 is a sum of less than n squares in $\overline{K}$. This proves: *If v is an arbitrary valuation of the field K, then the level of $\overline{K}$ is smaller or equal to the level of K.*

Remark 10.9 There is a version of Theorem 10.5 for valuations such that Kv is a subfield of $\mathbb{R}$ (for every v-compatible ordering, v will thus be the natural valuation); see [BRW2] and [LAM1] for details. Further, Theorem 10.5 can be generalized in order to treat simultaneously orderings and semiorderings; see [PR1], Chapter 7. For an exemplary application of real places in the theory of quadratic forms (namely, the characterization of fields which satisfy the Weak Hasse Principle), see Theorem 9.1 of [PR1].

Exercise 10.1 *Let $(K,<)$ be an ordered field. Then its prime field is $\mathbb{Q}$. Let v be a valuation of K and show that v is compatible with $<$ if and only if $\mathcal{M}_v < r$ for every positive $r \in \mathbb{Q}$, and that this is the case if and only if $\mathcal{M}_v < r$ for some positive $r \in \mathbb{Q}$.*

Exercise 10.2 *Let $(K,<)$ be an ordered field with natural valuation v and $a \in K$. Show that $va = 0$ (that is, a is archimedean equivalent to 1) if and only if there exists some $n \in \mathbb{N}$ such that $1/n < |a| < n$. What happens if we replace v by an arbitrary convex valuation?*

Exercise 10.3 *Compare Theorem 10.5 and Corollary 10.7 with Corollary 2.25. Explain why in Corollary 2.25, an ordering is uniquely determined by the skeleton, whereas in Theorem 10.5, the valuation v and the ordering of $\overline{K}$ do in general not suffice to fix a unique ordering of the field. Prove a generalization of Corollary 2.25 for arbitrary valuations, analogous to our above approach.*

10.2 Real places and henselian places

Remark 10.10 Apart from the paper "Allgemeine Bewertungstheorie" of W. Krull [KRU7], the paper "The theory of real places" by S. Lang [LANG2] was the first attempt to study systematically the relation between formally real fields and their places. Since then, this approach has been very fruitful (the list of papers at the beginning of Section 10.1 is certainly not exhaustive). For an interesting historical survey, see the notes on the literature of §5 in T. Y. Lam [LAM1]. For further reading, we refer the reader to §7 and §8 in A. Prestel [PR1], §2 and §3 of Chapter III in S. Prieß-Crampe [PC1], and §5 of [LAM1].

The choice of the material in the present section and the last part of Section 10.1 is inspired by the paper [RIB28] of P. Ribenboim, in which the results concerning euclidean fields, the level and the Pythagoras number were proved for generalized power series fields.

In this section, we will give examples for properties that are transferred through henselian places between valued fields and their residue fields. In particular, we will consider properties that have been introduced in Section 10.1, and complement them in the case of henselian places.

Recall that if P is a real place of the field K, then K is formally real by virtue of Lemma 10.2, and Corollary 10.6 shows that v_P is compatible with some ordering of K. Recall further that a field K is called **euclidean** if it is formally real and a or $-a$ is a square for every $a \in K$. In view of the condition that K be formally real, the latter condition is equivalent to the condition that *either a or* $-a$ is a square whenever $0 \neq a \in K$. The standard example for a euclidean field is $\mathbb{R}$. We leave it to the reader to prove that every relatively algebraically closed subfield of a euclidean field is again euclidean.

Lemma 10.11 *Let P be a real place of the field K. If K is euclidean, then $v_P K$ is 2-divisible and KP is euclidean. The converse holds if (K,P) is henselian.*

Proof: The first assertion was already stated for convex valuations in Section 10.1. Cf. also Corollary 9.38. Now assume that (K,P) is henselian, $v_P K$ is 2-divisible and KP is euclidean. Since KP is formally real, it is of characteristic 0. Let $a \in K$. Since $v_P K$ is 2-divisible, there is some $b \in K$ such that $v_P ab^2 = 0$. Since KP is euclidean, either $(ab^2)P$ or $-(ab^2)P$ is a square in KP. Since 2 is not divisible by the residue characteristic of (K,P), it follows from Lemma 9.32 that ab^2 or $-ab^2$ and thus also a or $-a$ is a square in K. Since K is also formally real by Lemma 10.2, we have thus proved that it is euclidean. $\square$

Let us observe that a euclidean field admits precisely one ordering. Indeed, since a euclidean field is formally real, it admits at least one ordering. On the other hand, every positive cone contains all squares. So it contains precisely all squares since it can not contain $a \neq 0$ and $-a$ at the same time. Hence, the only possible positive cone on K is the set of all squares of K.

A field K is called **real closed** if K is formally real but no proper algebraic extension of K is formally real. E. Artin and O. Schreier have proved that K is real closed if and only if $K(\sqrt{-1}) = \tilde{K}$, and this is the case if and only if the absolute Galois group of K is finite (in this case, it is $\mathbb{Z}/2\mathbb{Z}$); cf. [AR–SCHR]. We wish to characterize real closed by their convex places.

Lemma 10.12 *Let K be a real closed field with a real place P. Then (K, P) is henselian, $v_P K$ is divisible and KP is real closed.*

Proof: Since P is assumed to be a real place, KP is formally real. Since the henselization $(K, P)^h$ is an immediate extension of (K, P), its residue field is still KP. Hence also K^h is formally real. By our assumption on K we must have that $K^h = K$, that is, (K, P) is henselian.

Assume that k is a formally real algebraic extension of KP. Further, let $\widetilde{v_P K}$ denote the divisible hull of $v_P K$. Then by Theorem 6.42 there exists an algebraic extension $(L|K, P)$ such that $LP = k$ and $v_P L = \widetilde{v_P K}$. But then, also L is formally real and by assumption on K we must have that $L = K$. Hence, $v_P K = \widetilde{vK}$ is divisible, and $KP = k$ shows that KP is real closed. □

Corollary 10.13 *Every real closed field is euclidean. Consequently, every real closed field admits precisely one ordering, the positive cone of it being the set of all squares.*

Proof: Let K be a real closed field and P the natural place with respect to some ordering of K. Then KP is a subfield of $\mathbb{R}$. Let k be the relative algebraic closure of KP in $\mathbb{R}$. Then k is euclidean. In particular, k is formally real, so the foregoing lemma shows that $k = KP$. Again by the foregoing lemma, (K, P) is henselian and $v_P K$ is 2-divisible. Now it follows from Lemma 10.11 that K is euclidean. □

Theorem 10.14 *Let P be a henselian place of K. If* $\mathrm{char}\, KP \neq 2$, *then the level of K is equal to the level of KP.*

Proof: We have already shown in Section 10.1 that the level of KP is smaller or equal to the level of K. It remains to show that the level of K is smaller or equal to the level of KP. If the latter is ∞, then there is nothing to show. Now assume that $-1 = \sum_{i=1}^n \zeta_i^2$ with $\zeta_i \in KP$. We choose elements $a_i \in K$ such that $a_i P = \zeta_i$. Then $(a_1 P)^2 = -1 - \sum_{i=2}^n (a_i P)^2$. Since (K, P) is assumed to be henselian, we can infer from Lemma 9.32 that $-1 - \sum_{i=2}^n a_i^2$ is a square, say b_1^2 with $b_1 \in K$. Then $-1 = b_1^2 + a_2^2 + \ldots + a_n^2$, showing that the level of K is at most n. This proves that the level of K is smaller or equal to the level of KP. □

Since a field is formally real if and only if its level is ∞, we obtain: *If (K, P) is henselian and KP is of characteristic $\neq 2$ and not formally real, then K is not formally real.* This suggests that the residue field of a henselian place of a formally real field is again formally real. The only disturbing point is the condition on the characteristic. To overcome it, we prove a more informative theorem which is due to M. Knebusch and M. Wright [KN–WR].

Theorem 10.15 *A henselian valuation v of a formally real field K is compatible with every ordering of K.*

Proof: We consider the polynomial $X^2 + X + a$ for every $a \in \mathcal{M}_v$. Its reduction $X^2 + X$ has two distinct roots (independently of the characteristic of $\overline{K}$!). Since (K, v) is assumed to be henselian, it follows that $X^2 + X + a$ admits a root b in K. We have that $0 = b^2 + b + a = (b + \frac{1}{2})^2 + a - \frac{1}{4}$, hence $\frac{1}{4} - a$ is a square in K. So $a \leq \frac{1}{4} < 1$ for every ordering of K and every $a \in \mathcal{M}_v$. By Lemma 10.3, this shows that v is compatible with every ordering of K. □

In view of Lemma 10.2 and Lemma 10.4, we conclude:

Corollary 10.16 *If (K, P) is henselian, then K is formally real if and only if KP is formally real.*

If an ordered field admits a non-trivial compatible valuation, then the order is not archimedean. Thus, we can conclude from the foregoing theorem that

Corollary 10.17 *There is no non-trivial henselian valuation on $\mathbb{R}$.*

The following theorem was proved by A. Prestel [PR1], Theorem (8.6).

Theorem 10.18 *Let P be a real place of the field K. Then K is real closed if and only if*

1) $v_P K$ is divisible,

2) KP is real closed,

3) (K, v_P) is henselian.

Proof: Implication "$\Rightarrow$" is the assertion of Lemma 10.12. To prove "$\Leftarrow$", assume that 1), 2) and 3) hold. Since KP is formally real, its characteristic is 0. Hence by Theorem 11.23, (K, v_P) is defectless. By assumption 3), it is also henselian, and by assumption 1), every algebraic extension of $v_P K$ is trivial. So for every finite extension $L|K$, the unique extension of P from K to L satisfies $[L : K] = [LP : KP]$. That is, if $L|K$ is non-trivial, then $LP|KP$ is non-trivial and it follows from assumption 2) that LP is not formally real. Since (L, v_P) is again henselian (cf. Lemma 7.33), we obtain from the foregoing corollary that L is not formally real. We have thus proved that K is real closed. □

In view of our convention that the residue field of the natural valuation is a subfield of $\mathbb{R}$, we can deduce from this theorem the following assertion:

Let P be the natural place of the ordered field $(K, <)$. Then K is real closed if and only if 1) $v_P K$ is divisible, 2) KP is relatively algebraically closed in $\mathbb{R}$, 3) (K, P) is henselian.

For the proof of implication "$\Leftarrow$" of the above theorem, we can also employ the characterization of real closed fields by their Galois groups. Let us assume that (K, P) is a henselian field with divisible value group $v_P K$. If we also know that KP has characteristic 0, then we can infer from Lemma 11.25 that $\mathrm{Gal}\, K \cong \mathrm{Gal}\, KP$. Hence $\mathrm{Gal}\, K \cong \mathbb{Z}/2\mathbb{Z}$ if and only if $\mathrm{Gal}\, KP \cong \mathbb{Z}/2\mathbb{Z}$, showing that K is real closed if and only if KP is real closed.

Using the characterization by Galois groups, we are able to classify the valuations of a real closed field. The following theorem is due to M. Knebusch and M. Wright [KN–WR].

Theorem 10.19 *Let v be a valuation of the real closed field K. Then $\overline{K}$ is real closed or algebraically closed. $\overline{K}$ is real closed if and only if (K, v) is henselian, and this in turn holds if and only if v is compatible with the unique ordering of K.*

Proof: Since K is assumed to be real closed, we have that $[\tilde{K} : K] = 2$. Hence if (K, v) is not henselian, that is, if there is more than one extension of v from K of $\tilde{K}$, then by the fundamental inequality (7.26), there are precisely two extensions v_1, v_2, and both are immediate. Then $\overline{K} = \tilde{K}v_1$ is algebraically closed (cf. Lemma 6.44). On the other hand, if (K, v) is henselian, then we know from Theorem 10.15 that v is compatible with the unique ordering of K. If the latter holds, then the place associated with v is a real place by Lemma 10.4, and it follows from Theorem 10.18 that $\overline{K}$ is real closed. If the latter holds, then $[\tilde{K} : K] = 2 = [\overline{\tilde{K}} : \overline{K}]$, and the fundamental inequality shows that v admits a unique extension from K to $\tilde{K}$, i.e. (K, v) is henselian. □

Now we consider the Pythagoras number.

Theorem 10.20 *Let P be a real henselian place of the field K. Then the Pythagoras number of K is equal to that of KP.*

Proof: We have already shown in Section 10.1 that the Pythagoras number of KP is smaller or equal to that of K. It remains to show that the Pythagoras number of K is smaller or equal to that of KP. Let n be the Pythagoras number of KP and $a_1, \ldots, a_m \in K$ with $m > n$. We have to show that $\sum_{i=1}^{m} a_i^2$ is the sum of at most n squares in K. Without loss of generality we may assume that $v_P a_1 \leq v_P a_i$ for all i. It suffices to show that $s := 1 + \sum_{i=2}^{m} (a_i/a_1)^2$ is the sum of at most n squares in K. By assumption on a_1, we have that $v_P(a_i/a_1) \geq 0$ for all i, and $sP \neq 0$ is thus a sum of squares in KP. So it is the sum of at most n squares in KP. It follows as in the proof of Theorem 10.14, where we replace -1 by s, that also s is a sum of at most n squares in K. □

Finally, let us mention the following analogue to Theorem 10.18. Its proof is left to the reader.

Theorem 10.21 *Let P be a place of K and assume that* $\operatorname{char} KP = 0$. *Then K is algebraically closed if and only if*

1) $v_P K$ is divisible,

2) KP is algebraically closed,

3) (K, v_P) is henselian.

For $\operatorname{char} KP \neq 0$, *the assertion remains valid if condition 3) is replaced by*

3′) (K, v_P) is algebraically maximal.

Exercise 10.4 *Suppose that it was already shown that a subfield of $\mathbb{R}$ is real closed if and only if its Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Deduce the same for every formally real field, using Theorem 10.18 and Lemma 11.25.*

10.3 Non-archimedean exponential fields

Remark 10.22 A special class of formally real fields are those which are equipped with an exponential. The basic example is $(\mathbb{R}, \exp)$, the reals with the usual exponential function. In the last years, model theoretic algebra has witnessed the solution of one of its most renowned problems: A. Wilkie [WIL] showed that the theory of $(\mathbb{R}, \exp)$ is model complete. However, this solution left open a lot of questions, among them the request for a general structure theory comparable to that of formally real fields. While the existence of non-archimedean ordered exponential fields is guaranteed by model theory (cf. section 20.3), the explicit construction of such fields is of interest since it gives a deeper insight in the structure theory and the possible axiomatizations of exponentials. We devote this section to the structure theory of non-archimedean exponential fields since to our opinion, it shows a very convincing application of the theory of valued vector spaces to field theory. This application was worked out in detail by S. Kuhlmann in her thesis [KUS1], and the first part of this section will follow closely her paper [KUS2]. Her approach has extended and refined the methods and results of N. Alling [ALL4] who studied already in 1962 the structure which an exponential induces on the value group and residue field of the natural valuation. Apparently, Alling gave the first (to some extent explicit) construction of non-archimedean exponential fields.

The second part of this section will sketch some ideas from the paper [KU–KUS1]. This paper gives valuation theoretical interpretations of growth and Taylor axioms (as satisfied by the usual exponential function). One of them will be discussed here since it plays a crucial role for the construction of non-archimedean exponential fields which have the same elementary properties as $(\mathbb{R}, \exp)$. Finally, we will discuss a map which is induced on the natural value group by an exponential.

For the background and the recent progress in the theory of exponential fields and exponential functions, see N. Alling [ALL4], L. van den Dries [VDD 2,3,7], L. van den Dries and Levitz [VDD–LEV], C. W. Henson and L. A. Rubel [HEN–RUB], A. Macintyre [MAC4], D. Richardson [RICS], B. I. Dahn and H. Wolter [DAH–WOL1,2] H. Wolter [WOL1–5], B. I. Dahn [DAH], B. I. Dahn and P. Göring [DAH–GÖ], A. Wilkie [WIL], J.-P. Ressayre [RES], M. H. Mourgues [MOU], L. van den Dries, A. Macintyre and D. Marker [VDD–MAC–MAR1,2], C. Miller [MIL1,2], L. van den Dries and C. Miller [VDD–MIL]. For a study of the theory of all analytic functions of $\mathbb{R}$, restricted to the interval $[0, 1]$, see L. van den Dries [VDD8], J. Denef and L. van den Dries [DEN–VDD]. Some of these papers make explicit use of the natural valuation for the study of non-archimedean exponential fields or ordered power series fields.

Throughout this section, let $(K, <)$ be an ordered field with natural valuation v, and define $K^{>0} := \{a \in K \mid a > 0\}$. The additive group $(K, +, 0, <)$ and the multiplicative group $(K^{>0}, \cdot, 1, <)$ are ordered groups. We shall ask for the order isomorphisms between these two groups, in the case where v is non-trivial, i.e., $(K, <)$ is a non-archimedean ordered field.

Let us discuss the structure of these groups. The natural valuation v of K is at the same time the natural valuation of $(K, +, 0, <)$. The valuation ring $\mathcal{O}_v$ is a convex subgroup of $(K, +, 0, <)$, and the valuation ideal $\mathcal{M}_v$ is a convex subgroup of $\mathcal{O}_v$.

The **set of positive units** $\mathcal{U}_v^{>0} := \{a \in \mathcal{O}_v^\times \mid a > 0\}$ is a subgroup of $(K^{>0}, \cdot, 1, <)$. Using (COMP), we see that the map

$$\begin{array}{rcl} (K^{>0}, \cdot, 1, <) & \rightarrow & (vK, +, 0, <) \\ a & \mapsto & -va = va^{-1} \end{array}$$

is a surjective group homomorphism preserving $\leq$, with kernel $\mathcal{U}_v^{>0}$. It follows from Lemma 2.20 that $\mathcal{U}_v^{>0}$ is a convex subgroup of $(K^{>0}, \cdot, 1, <)$, and from Corollary 2.22 we obtain

$$(K^{>0}, \cdot, 1, <)/\mathcal{U}_v^{>0} \cong (vK, +, 0, <) . \tag{10.2}$$

Further, the map

$$\begin{array}{rcl} (\mathcal{U}_v^{>0}, \cdot, 1, <) & \rightarrow & (\overline{K}^{>0}, \cdot, 1, <) \\ a & \mapsto & \overline{a} \end{array}$$

is a surjective group homomorphism preserving $\leq$, with kernel $1+\mathcal{M}_v$, the set of 1-units in (K,v) (recall that all 1-units are positive, by virtue of Lemma 10.3). It follows that $(1+\mathcal{M}_v,\cdot,1,<)$ is a convex subgroup of $(\mathcal{U}_v^{>0},\cdot,1,<)$, and

$$(\mathcal{U}_v^{>0},\cdot,1,<)/1+\mathcal{M}_v \;\cong\; (\overline{K}^{>0},\cdot,1,<)\,. \tag{10.3}$$

Since v is the natural valuation of K, we know that $(\overline{K},<)$ is archimedean. Consequently, the same is true for the multiplicative ordered group $(\overline{K}^{>0},\cdot,1,<)$ (if $b>a>1$ then choose $n\in\mathbb{N}$ such that $n(a-1)\geq b$ to obtain that $a^n=(1+a-1)^n\geq 1+n(a-1)>b$). As the element 2 lies in $\mathcal{U}_v^{>0}$ but not in $1+\mathcal{M}_v$, we see that:

a) $\mathcal{U}_v^{>0}$ is the smallest convex subgroup of $(K^{>0},\cdot,1,<)$ containing 2

b) $1+\mathcal{M}_v$ is the biggest convex subgroup of $(K^{>0},\cdot,1,<)$ not containing 2

c) $(\overline{K}^{>0},\cdot,1,<)$ is the component of 2 in $(K^{>0},\cdot,1,<)$.

Let us use the name **exponential** for an isomorphism

$$f:\;(K,+,0,<)\longrightarrow(K^{>0},\cdot,1,<) \tag{10.4}$$

which is normed by one of the equivalent conditions given in the following lemma:

Lemma 10.23 *The following conditions for an isomorphism (10.4) are equivalent:*
1) $f(\mathcal{O}_v)=\mathcal{U}_v^{>0}$ *and* $f(\mathcal{M}_v)=1+\mathcal{M}_v$,
2) $v(f(1)-1)=0$,
3) there is $n\in\mathbb{N}$ *such that* $1+1/n<f(1)<n$,
4) $f(1)$ *is archimedean equivalent to* 2 *in* $(K^{>0},\cdot,1,<)$.

Proof: Assume that 1) holds. Then for every $a\in\mathcal{O}_v$ we have $vf(a)=0$ and thus, $v(f(a)-1)\geq 0$. By virtue of $f(\mathcal{M}_v)=1+\mathcal{M}_v$, we obtain that $v(f(a)-1)=0$ for all $a\in\mathcal{O}_v\setminus\mathcal{M}_v$. Since $1\in\mathcal{O}_v\setminus\mathcal{M}_v$, we now see that 2) holds.

Condition 2) says that $f(1)-1$ is archimedean equivalent to 1 in $(K,+,0,<)$. Now we leave it to the reader to show that this is equivalent to condition 3) (cf. Exercise 10.2).

Assume condition 2), that is, $v(f(1)-1)=0=v2$. Choose $n\in\mathbb{N}$ such that $n|f(1)-1|\geq 2$ and $n\cdot 2\geq|f(1)-1|$; w.l.o.g., we assume $n>2$. Observe that $f(1)>f(0)=1$. Hence, we have $f(1)^n=(1+f(1)-1)^n\geq n(f(1)-1)\geq 2$ and $2^n\geq 2n+1\geq f(1)$, showing that 4) holds.

Assume that 4) holds. Then the order isomorphism f must send the smallest convex subgroup of $(K,+,0,<)$ containing 1 onto the smallest convex subgroup of $(K^{>0},\cdot,1,<)$ containing $f(1)$. The former is just $\mathcal{O}_v$, the latter is $(\mathcal{U}_v^{>0},\cdot,1,<)$ since $f(1)$ is archimedean equivalent to 2 by assumption. Further, f must send the biggest convex subgroup of $(K,+,0,<)$ not containing 1 onto the biggest convex subgroup of $(K^{>0},\cdot,1,<)$ not containing $f(1)$ (or 2). Hence, $f(\mathcal{M}_v)=1+\mathcal{M}_v$. We have proved that 1) holds. □

Note that every isomorphism (10.4) gives rise to an exponential. Indeed, if f is any such isomorphism, then we may take $a\in K^{>0}$ such that $f(a)=2$; setting $e(x):=f(ax)$, we thus obtain an isomorphism e which satisfies $e(1)=2$. So it makes sense to call an ordered field an **exponential field** as soon as it admits an isomorphism (10.4). Instead of "exponential", one may also speak of "weak exponential" since the usual exponential exp on $\mathbb{R}$ has many more properties that do not follow from just being such a (normed) isomorphism.

In the last section, we have studied the structure induced by orderings on the value group and residue field of a compatible valuation. We will do the same here for an exponential f and the natural valuation v of $(K,<)$.

First of all, we see that in view of condition 1) of the foregoing lemma, f induces an isomorphism $(\mathcal{M}_v,+,0,<) \to (1+\mathcal{M}_v,\cdot,1,<)$, which we will denote by f_R. Again in view of that condition, f induces an isomorphism $(\mathcal{O}_v,+,0,<)/\mathcal{M}_v \;\to\; (\mathcal{U}_v^{>0},\cdot,1,<)/1+\mathcal{M}_v$. Composing it with the isomorphism (10.3), we obtain an induced isomorphism

$$\begin{array}{rcl}\overline{f}:(\overline{K},+,0,<) & \to & (\overline{K}^{>0},\cdot,1,<)\\ \overline{a} & \mapsto & \overline{f(a)}\end{array}$$

which is an exponential on $\overline{K}$.

Using once more our condition that $f(\mathcal{O}_v)=\mathcal{U}_v^{>0}$, we find that f induces an order isomorphism $(K,+,0,<)/\mathcal{O}_v \;\to\; (\overline{K}^{>0},\cdot,1,<)/\mathcal{U}_v^{>0}$. By (10.2), the latter group is order isomorphic to the value group $(vK,+,0,<)$. Let us write $\mathbf{G}=(G,<)$ for $(vK,+,0,<)$, and let $v_{\mathbf{G}}$ denote the natural valuation of this ordered group. So the exponential f induces an order isomorphism $(K,+,0,<)/\mathcal{O}_v \;\to\; \mathbf{G}$, which we denote by f_L. This isomorphism in turn induces an isomorphism between the ordered skeletons of these groups. The value set of $(K,+,0,<)$ is G, and its components are all isomorphic to $(\overline{K},+,0,<)$. From Corollary 2.23 we infer that the value set of $(K,+,0,<)/\mathcal{O}_v$ is $G^{<0}=\{\alpha\in G\mid \alpha<0\}$, and its components are all isomorphic to $(\overline{K},+,0,<)$. So f induces an isomorphism

$$\begin{array}{rcl}\varphi_f:(G^{<0},<) & \to & (v_{\mathbf{G}}G,<)\\ va & \mapsto & v_{\mathbf{G}}(-vf(a))=v_{\mathbf{G}}(vf(a))\end{array}$$

of ordered sets, and isomorphisms from $(\overline{K},+,0,<)$ onto every component of $\mathbf{G}$. We have proved the following result due to N. Alling (cf. [ALL4], Section 1):

Theorem 10.24 *Let $(K,<)$ be an ordered field with natural valuation v and value group $\mathbf{G}$. Let $v_{\mathbf{G}}$ be the natural valuation of $\mathbf{G}$. If K admits an exponential f, then*
a) there is an isomorphism $f_L:(K,+,0,<)/\mathcal{O}_v \;\to\; \mathbf{G}$,
b) there is an isomorphism $\varphi_f: G^{<0}\cong v_{\mathbf{G}}G$ of ordered sets,
c) all components of $\mathbf{G}$ are isomorphic to $(\overline{K},+,0,<)$,
d) there is an exponential $\overline{f}$ on $\overline{K}$,
e) there is an isomorphism $f_R:(\mathcal{M}_v,+,0,<) \;\to\; (1+\mathcal{M}_v,\cdot,1,<)$.

Now the question arises whether there exists a converse of this theorem. This would mean that we have to build an exponential from three different parts: an order isomorphism f_R, an exponential $\overline{f}$ on the residue field, and an order isomorphism f_L. We have to lift $\overline{f}$ and f_L up into the additive and positive multiplicative group of K. Since the additive group of a field of characteristic 0 is always divisible, there is a group complement $\mathbf{A}'\cong(\overline{K},+,0,<)$ for $\mathcal{M}_v$ in $\mathcal{O}_v$, and there is a group complement $\mathbf{A}\cong(K,+,0,<)/\mathcal{O}_v$ for $\mathcal{O}_v$ in $(K,+,0,<)$. The reader may show that we have a lexicographic decomposition

$$(K,+,0,<)\cong \mathbf{A}\amalg\mathbf{A}'\amalg\mathcal{M}_v\,. \tag{10.5}$$

To do the same for the group $(K^{>0},\cdot,1,<)$, we have to assume that it is divisible, that is, the field is **root closed for positive elements**. Note that this condition is necessary for the existence of an exponential since an isomorphism between additive and positive multiplicative group yields that the latter is divisible. Under this condition, there is a group complement $\mathbf{B}' \cong (\overline{K}^{>0},\cdot,1,<)$ for $1+\mathcal{M}_v$ in $\mathcal{U}_v^{>0}$, and there is a group complement $\mathbf{B} \cong (vK,+,0,<)$ for $\mathcal{U}_v^{>0}$ in $(K^{>0},\cdot,1,<)$. This gives a lexicographic decomposition

$$(K^{>0},\cdot,1,<) \cong \mathbf{B} \amalg \mathbf{B}' \amalg 1+\mathcal{M}_v\;. \tag{10.6}$$

Now f_R may be viewed as an exponential of the right sides of these decompositions, and we thus call it a **right exponential**. Similarly, f_L may be viewed as an exponential of the left sides of these decompositions, and we call it a **left exponential**. Putting the exponentials f_L, $\overline{f}$ and f_R together, we obtain an exponential of K which induces these partial exponentials. The problem is now to find criteria for the existence of right and left exponentials.

In [ALL4], Theorem 3.1, Alling gives a criterion for the existence of f_R. He shows that on every power series field, a right exponential from the valuation ideal onto the set of 1-units can be defined by the usual Taylor expansion for exp. Thus, an arbitrary ordered field admits a right exponential if it is embeddable into a power series field in such a way that the image of its natural valuation ideal is closed under that right power series exponential and its inverse.

We have deduced information about the skeletons of the groups from the existence of an exponential. To some extent, this procedure can be reversed. Recall that divisible ordered abelian groups are ordered $\mathbb{Q}$-vector spaces. The idea introduced by S. Kuhlmann in [KUS1] and [KUS2] is to employ the fact that in two special cases, these are determined up to isomorphism by their skeletons, namely if they are countable (cf. Theorem 3.46) or if they are maximal (cf. Theorem 3.52). (Recall that by Theorem 3.51, for divisible ordered abelian groups the properties “maximal”, “spherically complete” and “being (isomorphic to) a Hahn product” coincide.) In order to apply this idea, S. Kuhlmann computes the skeleton of $(1+\mathcal{M}_v,\cdot,1,<)$. Let w denote its natural valuation. Implicitly, we have already above determined the relation between v and w. Explicitly, it reads as follows (up to equivalence, as usual):

Theorem 10.25 *Let v be the natural valuation of $(K,<)$. Then*

$$w(1+a) = va \quad \textit{for all } a \in \mathcal{M}_v \tag{10.7}$$

is the natural valuation on $(1+\mathcal{M}_v,\cdot,1,<)$. Hence, its value set is $w(1+\mathcal{M}_v) = v\mathcal{M}_v = (vK)^{>0}$. The components are all isomorphic to $(\overline{K},+,0,<)$. Consequently, $(\mathcal{M}_v,+,0,<)$ and $(1+\mathcal{M}_v,\cdot,1,<)$ have the same skeleton.

Proof: Suppose that $a,b \in \mathcal{M}_v$ with $a,b>0$. Assume that $va \geq vb$. Then there is $n \in \mathbb{N}$ such that $nb \geq a$. This yields that $(1+b)^n \geq 1+nb \geq 1+a$, hence $w(1+a) \geq w(1+b)$. Conversely, assume that $w(1+a) \geq w(1+b)$ and choose $n \in \mathbb{N}$ such that $(1+b)^n \geq 1+a$. Hence $(1+b)^n - 1 \geq a$, showing that $va \geq v((1+b)^n-1)$. Since $vb>0$, we have $v((1+b)^n-1) = v(nb+\ldots+b^n) = vnb = vb$, which proves that $va \geq vb$. We have shown that $va \geq vb \Leftrightarrow w(1+a) \geq w(1+b)$; consequently, $va = w(1+a)$ for all positive $a \in \mathcal{M}_v$ (up to equivalence of valuations). To deduce that this also holds for negative $a \in \mathcal{M}_v$, it suffices to show that $w(1+b) = w(1-b)$ for every positive $b \in \mathcal{M}_v$, or in other words, that $1+b$ is archimedean equivalent to $1-b$ in the multiplicative group $1+\mathcal{M}_v$. Since b is positive, we have that $(1-b)^{-1} > 1+b$. On the other hand, $(1+b)^2(1-b) = 1+b-b^2-b^3$ which is > 1, since $vb < 2vb = v(b^2+b^3)$ yields that $b > b^2+b^3$. This proves that $(1+b)^2 > (1-b)^{-1}$. Hence indeed, $1+b$ is archimedean equivalent to $1-b$. Now it also follows that $w(1+\mathcal{M}_v) = v\mathcal{M}_v = (vK)^{>0}$.

Let $\alpha \in w(1+\mathcal{M}_v) = (vK)^{>0}$. From what we have proved, we see that $\{c \in 1+\mathcal{M}_v \mid wc \geq \alpha\} = 1+\mathcal{O}_v^\alpha$ and $\{c \in 1+\mathcal{M}_v \mid wc > \alpha\} = 1+\mathcal{M}_v^\alpha$. Hence, the α-component of $(1+\mathcal{M}_v,\cdot,1,<)$ is $(1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha)$. Define a map

$$\mathcal{O}_v^\alpha \ni a \mapsto (1+a)(1+\mathcal{M}_v^\alpha) \in (1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha)\,.$$

Let $va = vb = \alpha$. Since $\alpha > 0$, we have that $vab > \alpha$ and consequently, $(1+a+b+ab)/(1+a+b) \in 1+\mathcal{M}_v^\alpha$. This yields that $(1+a+b)(1+\mathcal{M}_v^\alpha) = (1+a+b+ab)(1+\mathcal{M}_v^\alpha) = (1+a)(1+b)(1+\mathcal{M}_v^\alpha)$. Hence, our map is a group homomorphism. Since $a \mapsto a+1$ preserves the order and since the order on $(1+\mathcal{O}_v^\alpha)/(1+\mathcal{M}_v^\alpha)$ is the order induced by $(1+\mathcal{O}_v^\alpha,\cdot,1,<)$, our map preserves $\leq$. Its kernel is $\mathcal{M}_v^\alpha$. Now Corollary 2.22 shows that it induces an order isomorphism from the component $(\mathcal{O}_v^\alpha,+,0,<)/\mathcal{M}_v^\alpha$ (which is isomorphic to $(\overline{K},+,0,<)$) onto the component $(1+\mathcal{O}_v^\alpha,\cdot,1,<)/(1+\mathcal{M}_v^\alpha)$. □

Since every exponential f (and every right exponential f_R) induces an isomorphism of the skeletons of $\mathcal{M}_v$ and $1+\mathcal{M}_v$, the equality of the value sets shows that it induces an automorphism ψ_f of $(vK)^{>0}$.

From (10.7), we deduce

$$w\left(\frac{1+a}{1+b}\right) = v\left(\frac{1+a}{1+b}-1\right) = v\left(\frac{a-b}{1+b}\right) = v(a-b)$$

for all $a,b \in \mathcal{M}_v$. This shows that

$$\mathcal{M}_v \ni a \mapsto 1+a \in 1+\mathcal{M}_v$$

is an isomorphism of ultrametric spaces. In view of Lemma **??** and Theorem 3.6, we obtain:

Corollary 10.26 *If K is spherically complete with respect to its natural valuation, then the groups $(\mathcal{M}_v,+,0,<)$ and $(1+\mathcal{M}_v,\cdot,1,<)$ are spherically complete (and hence maximal) with respect to their natural valuations.*

We have mentioned in the last section that the group $1+\mathcal{M}_v$ of 1-units (with respect to the natural valuation) is divisible. Hence, it is a $\mathbb{Q}$-vector space like the additive group $\mathcal{M}_v$. Hence, we may use Theorem 3.46 and Theorem 3.52 to obtain:

Corollary 10.27 *Suppose that $(K,<)$ is countable or spherically complete with respect to its natural valuation. Then $(K,<)$ admits a right exponential.*

The conditions for the skeletons of $\mathbf{A} \cong (\overline{K},+,0,<)/\mathcal{O}_v$ and $\mathbf{B} \cong \mathbf{G}$ to be isomorphic are given by b) and c) of Theorem 10.24. Hence, we have: *If $(K,<)$ is countable or spherically complete with respect to its natural valuation, then conditions b), c) and d) of Theorem 10.24 imply the existence of an exponential on K.* We shall improve this result by determining all possible exponentials (in the spirit of Theorem 10.5). But this is quite hopeless if we allow the exponentials to be too messy on the components. So we will confine ourselves to **$\overline{K}$-linear exponentials**. To explain what we mean by this notion, we show that we are actually working with $\overline{K}$-vector spaces. Observe that the components of $(\mathcal{M}_v,+,0,<)$, $(1+\mathcal{M}_v,\cdot,1,<)$, and $(\overline{K},+,0,<)/\mathcal{O}_v$ are all $\overline{K}$, and the same is true for $\mathbf{G}$ if it satisfies condition c). We shall use the following fact:

Lemma 10.28 *Let k be a field and $\mathbf{G}$ a countable or spherically complete divisible ordered (resp. valued) group whose components are k-vector spaces. Then $\mathbf{G}$ is a k-vector space (with value preserving scalar multiplication).*

Proof: If $\mathbf{G}$ is countable, then by Theorem 3.46 it is a Hahn sum over its skeleton. If $\mathbf{G}$ is spherically complete, then by Theorem 3.51 it is a Hahn product over its skeleton. In both cases, k-scalar multiplication can be defined componentwise since the components are k-vector spaces. Multiplication by a nonzero scalar does not change the minimum of the support and therefore, it preserves the value. □

It now remains to consider the component $(\mathcal{U}_v^{>0},\cdot,1,<)/1+\mathcal{M}_v \cong (\overline{K}^{>0},\cdot,1,<)$. If $\overline{f}$ is an exponential on $\overline{K}$, then it is isomorphic to $(\overline{K},+,0,<)$ and through this isomorphism, it inherits the structure of a $\overline{K}$-vector space, with respect to which $\overline{f}$ is $\overline{K}$-linear. With this in mind, we will consider f to be $\overline{K}$-linear if and only if f_L and f_R are $\overline{K}$-linear. If $\mathbf{G}$ satisfies condition c), then every $\overline{K}$-linear f_L is uniquely determined by the order isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$. Similarly, every $\overline{K}$-linear f_R is uniquely determined by the order automorphism ψ_f of $G^{>0}$. We leave it to the reader to prove the following:

Lemma 10.29 *Let k be a field and $(V,<)$ and $(V',<)$ two ordered Hahn sums or Hahn products whose components are all equal to $(k,+,0,<)$ (hence, we view them as k-vector spaces). Then for every isomorphism φ from the value set of V onto the value set of V' there is a unique k-linear isomorphism form $(V,<)$ onto $(V',<)$ which induces φ.*

Fixing the isomorphism of the considered vector spaces onto the respective Hahn sums (or Hahn products), we now obtain the desired description of $\overline{K}$-linear exponentials:

Theorem 10.30 *Suppose that $(K,<)$ is root closed for positive elements and countable. Assume further that all components of its value group $\mathbf{G}$ are isomorphic to $(\overline{K},+,0,<)$. Then there is a bijection*

$$f \mapsto (\varphi_f, \overline{f}, \psi_f)$$

from the set of all $\overline{K}$-linear exponentials of $(K,<)$ onto the product of the set of all order isomorphisms $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$, the set of all exponentials on $\overline{K}$ and the set of all order automorphisms ψ_f of $G^{>0}$.

The same holds if $(K,<)$ and $\mathbf{G}$ are spherically complete with respect to their natural valuations.

Observe that our theorem does not assert the existence of the isomorphism φ_f (while ψ_f can always be taken to be the identity). And in fact, not every value group admits such an isomorphism. In the countable case, the value group of an exponential field K is of the form

$$\coprod_{\mathbb{Q}} (\overline{K},+,0,<)\ .$$

Indeed, the value group $\mathbf{G}$ of a root closed field must be divisible. If it is countable, it is thus isomorphic to a Hahn sum, by virtue of Theorem 3.46. From divisibility, it also follows that its ordering is dense and without endpoints and that the same is true for $G^{<0}$. If G and hence also $G^{<0}$ are countable, then $G^{<0}$ is a countable dense linearly ordered set without endpoints and as such, it is isomorphic to the ordered set $\mathbb{Q}$ (cf. [ROS]). Hence, a non-trivial countable divisible ordered abelian group $\mathbf{G}$ admits an isomorphism $G^{<0} \cong v_{\mathbf{G}} G$ if and only if it is (isomorphic to) a Hahn sum with value set $\mathbb{Q}$. We see that there exist countable groups satisfying conditions b) and c), so our theorem indeed yields the existence of countable exponential fields as soon as we are able to construct countable valued fields with given value group and residue field. This will be done in section 6.5. As to the problem of the existence of exponential power series fields, see [KU–KUS2].

Let us now consider the valuation theoretical interpretation of growth axioms for an exponential of a nonarchimedean ordered field. As an example, we choose an axiom scheme which plays a distinguished role in the model theory of exponential fields, according to a theorem of Ressayre ([RES]; cf. also Theorem 1.1 of [VDD–MAC–MAR2]). It describes the growth of the exponential f in comparison with polynomials:

(GRE) $\qquad x > n^2 \implies f(x) > x^n \qquad (n \in \mathbb{N})$.

Because of the condition "$x > n^2$", this axiom scheme is void for infinitesimals. That is, it gives information only in the case of $vx \leq 0$. It holds in the case $vx = 0$ if f coincides with exp on a maximal archimedean subfield of K (viewed as a subfield of $\mathbb{R}$). (Recall that exp itself satisfies (GRE) and even $x \geq n^2 \Rightarrow f(x) > x^n$.) Indeed, assume that $vx = 0$; then $x = r + \varepsilon$ with $r \in \mathbb{R}$ and $v\varepsilon > 0$, and if $x > n^2$, then $r \geq n^2$. So we have $f(r) = \exp(r) > r^n$. But $f(r)$ is the residue of $f(x)$ and r^n is the residue of x^n modulo the natural valuation v since ε is infinitesimal. Hence, $f(r) > r^n$ implies that $f(x) > x^n$.

Now we have to consider the case of $vx < 0$. In this case, "$x > n^2$" holds for all $n \in \mathbb{N}$ if only x is positive. Restricted to $K \setminus \mathcal{O}_{\mathbf{K}}$, axiom scheme (GRE) is thus equivalent to the assertion

$$vx < 0 \wedge x > 0 \implies \forall n \in \mathbb{N}: \ f(x) > x^n \ . \tag{10.8}$$

But "$\forall n \in \mathbb{N} : f(x) > x^n$" means that $wf(x) < wx$. Observe that the condition "$vx < 0 \wedge x > 0$" implies that $x, f(x) \in K^{>0} \setminus \mathcal{U}_v^{>0}$. Through the isomorphism (10.2) which is induced by the map $a \mapsto -va$, we see that the natural valuation w is given on $K^{>0} \setminus \mathcal{U}_v^{>0}$ by

$$wa = v_{\mathbf{G}}(va) \quad \text{for all } a \in K^{>0} \setminus \mathcal{U}_v^{>0}$$

(note that $v_{\mathbf{G}}(-va) = v_{\mathbf{G}}(va)$). Hence, $wf(x) < wx$ is equivalent to $v_{\mathbf{G}}(vf(x)) < v_{\mathbf{G}}(vx)$. In view of $vx < 0$ and the definition of φ_f, this in turn is equivalent to $\varphi_f(vx) < v_{\mathbf{G}}(vx)$. Therefore, assertion (10.8) is equivalent to

$$\varphi_f\, g < v_{\mathbf{G}}\, g \quad \text{for all } g \in G^{<0} \ . \tag{10.9}$$

For the conclusion of this section, let us sketch how to obtain a map induced on the natural value group by the exponential f. One way to define this map is to compose the natural valuation $v_{\mathbf{G}}$ with the inverse of the isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}} G$. This gives a map

$$\varphi_f^{-1} \circ v_{\mathbf{G}} : \ G \setminus \{0\} \ \to \ G^{<0} \ . \tag{10.10}$$

Using the definition of φ_f, we find that this map sends $vf(a)$ to va, provided that $va < 0$. Setting $b = f^{-1}(a)$, we see that the above map is in fact induced by the **logarithm** f^{-1}:

$$\begin{array}{rcl} G\setminus\{0\} & \to & G^{<0} \\ vb & \mapsto & vf^{-1}(b) \end{array}$$

Since the natural valuation $v_{\mathbf{G}}$ sends archimedean equivalent elements to the same value, it is seen immediately from (10.10) that the above map does the same. In other words, the map is constant on every archimedean class (which consists of two convex sets, one in $G^{<0}$ and one in $G^{>0}$). Nevertheless, its image is all of $G^{<0}$ since both $v_{\mathbf{G}}$ and φ_f^{-1} are surjective. This shows that ordered groups admitting such maps resp. an isomorphism $\varphi_f : G^{<0} \cong v_{\mathbf{G}}G$ must be "quite big". For the study of their properties it turned out to be convenient to change these maps a little bit. We define

$$\chi_f\, g := \varphi_f^{-1}(v_{\mathbf{G}}\, g) \quad \text{for all } g \in G^{<0}$$

and extend χ_f to all of G by symmetry, setting $\chi_f\, 0 = 0$. in this way, we obtain a map $\chi_f : G \to G$ which satisfies the following axioms:

(C0) $\chi_f\, x = 0 \Leftrightarrow x = 0\,,$

(C−) $\chi_f(-x) = -\chi_f\, x\,,$

(CA) $v_{\mathbf{G}}\, x = v_{\mathbf{G}}\, y \wedge \operatorname{sign}(x) = \operatorname{sign}(y) \;\Rightarrow\; \chi_f\, x = \chi_f\, y\,,$

(C≤) χ_f preserves $\leq$,

(CS) χ_f is surjective.

The axioms (C0) and (C−) are direct consequences of our definition. Axiom (CA) follows from the fact that (10.10) is constant on archimedean classes, and the extension by symmetry. Axiom (C≤) is seen as follows. On $G^{<0}$, the natural valuation $v_{\mathbf{G}}$ preserves $\leq$ (cf. (2.7)), so its composition with the order preserving map φ_f^{-1} also preserves $\leq$. On $G^{>0}$, $v_{\mathbf{G}}$ reverses $\leq$, but χ_f again preserves $\leq$ since there it is equal to $-\varphi_f^{-1}\circ v_{\mathbf{G}}$ by virtue of our symmetric extension. Finally, axiom (CS) is a consequence of symmetry and the fact that the image of $\varphi_f^{-1}\circ v_{\mathbf{G}}$ is all of $G^{<0}$. The advantage of the map χ_f in comparison with $\varphi_f^{-1}\circ v_{\mathbf{G}}$ is that it is a map from G to G and that it preserves $\leq$ on all of G. Observe that all axioms except (CS) are universal (cf. Section 20.1). Since axiom (CA) says that the map contracts archimedean classes to just two points, we call a map χ_f a **precontraction** if it satisfies the universal axioms (C0), (C−), (CA) and (C≤), and we call it a **contraction** if it satisfies in addition axiom (CS).

Furthermore, assertion (10.9) is equivalent to

$$g < \chi_f\, g \quad \text{for all } g \in G^{<0}\,.$$

In view of the symmetry axiom (C−), this in turn is equivalent to the axiom

(CP) $x \neq 0 \;\Rightarrow\; |\chi_f\, x| < |x|\,.$

This axiom expresses the assertion that χ_f sends the elements towards the middle point 0 of the group. Therefore, we call a precontraction **centripetal** if it satisfies (CP). For a precontraction, (CP) is equivalent to the seemingly stronger assertion

$$v_{\mathbf{G}}(\chi_f\, g) > v_{\mathbf{G}}\, g \quad \text{for all } g \in G\setminus\{0\}\,. \tag{10.11}$$

Indeed, (CP) implies $v_{\mathbf{G}}\chi_f\, g \geq v_{\mathbf{G}}\, g$. But equality can not hold for $g \neq 0$ since otherwise we would obtain $\chi_f\chi_f\, g = \chi_f\, g$ by virtue of (CA), in contradiction to (CP). We have proved:

Lemma 10.31 *Every exponential f on $(K,<)$ induces a contraction χ_f on its natural value group. Further, f satisfies (10.8) if and only if χ_f is centripetal.*

Ordered abelian groups with contractions and precontractions are studied in detail in the papers [KU4] and [KU5], making intense use of the natural valuation of such groups.

Remark 10.32 In Ressayre's proof of his main theorem in [RES], a crucial tool is the **integer part** (also called **integral part**) of an ordered field $(K,<)$, that is, a subring R of K which satisfies

(IP1) $(R,<)$ is discretely ordered (i.e., $\forall x \in R : x \leq 0 \vee x \geq 1$),

(IP2) $\forall x \in K\, \exists y \in R :\ |x-y| < 1$.

M. H. Mourgues and J.-P. Ressayre [MOU–RES] showed that every real closed field (cf. Section 10.2) has an integer part; see also S. Boughattas [BOUG]. If f is an exponential on $(K,<)$, then an integer part will be called **exponential integer part** if it is closed under f. Ressayre has shown that every real closed field has an exponential integer part, provided that the exponential satisfies $f(1) = 2$ (such an exponential is commonly denoted by 2^x). The reader may show the following:

Let $(K,<)$ be an ordered field with natural valuation v. If the group complement $\mathbf{A}$ of $\mathcal{O}_v$ in $(K,+,0)$ is closed under multiplication, then the subring R of K generated by $\mathbf{A}$ and $\mathbb{Z}$ is an integer part of $(K,<)$. If in addition, f is an exponential on $(K,<)$ satisfying $f(1) = 2$ and if $\mathbf{A}$ is closed under f, then R is an exponential integer part of $(K,<,f)$.

Exercise 10.5 *For an exponential f satisfying (GRE), deduce (10.11) directly from the definition $\chi_f(vb) = vf^{-1}(b)$.*

10.4 Rolle fields

In this section, we shall give a theorem on Rolle fields (cf. P. Ribenboim [RIB29]). It follows the spirit of Section 10.2. See also F. Delon [DEL6] for further information on Rolle fields and Rolle rings. It should be noted that the following results admit a straitforward generalization to Rolle rings.

An ordered field $(K,<)$ is called a **Rolle field** if it satisfies the following **Rolle property**:

for every polynomial $f \in K[X]$ and every pair $a < b$ of elements of K such that $f(a) = f(b) = 0$, there exists an element $c \in K$ satisfying $a < c < b$ and $f'(c) = 0$

where f' denotes the formal derivative of f. The following lemma is an easy consequence of this definition:

Lemma 10.33 *Assume $(L,<)$ to be a Rolle field and $(K,<)$ is a subfield, relatively algebraically closed in L (the ordering being the restriction of the ordering of L). Then (K,v) is also a Rolle field.*

Proof: Given a polynomial $f \in K[X]$ and elements $a < b$ of K such that $f(a) = f(b) = 0$. Since (L,v) is a Rolle field, there exists an element $c \in L$ satisfying $a < c < b$ and $f'(c) = 0$. But like f, its derivative f' is also a polynomial with coefficients in K; since K is relatively algebraically closed in L, it follows that $c \in K$. $\square$

Theorem 10.34 *Assume that P is a henselian place of the Rolle field $(L,<)$. Then LP admits an ordering with respect to which it is a Rolle field.*

Proof: Since L admits an ordering, it has characteristic 0. Consider the relative algebraic closure k of $\mathbb{Q}$ in L. Since k admits an ordering (the restriction of the ordering of L), it cannot have a non-trivial henselian p-adic valuation. Thus P is trivial on k which yields that the characteristic of LP is 0.

Since LP has characteristic 0 and (L,P) is henselian, there exists an embedding ι of LP into L such that $\forall x \in LP : (\iota x)P = x$ (i.e. ιLP is a field of representatives for the residue field LP). If ιLP would admit a proper algebraic extension L' within L then $L'P$ would be a proper extension of LP within LP which is absurd. Hence ιLP is relatively algebraically closed in L. Equipped with the restriction of the ordering of L it is thus a Rolle field by virtue of the above lemma. The same holds for LP with respect to the ordering induced by the isomorphism ι. □

In the following we will discuss the converse of Theorem 10.34. We need the characterization for Rolle fields which was given by R. Brown, T. C. Craven and M. J. Pelling in [BRW–CR–PE]. An abelian group is called **odd-divisible** if every element is divisible by every odd number.

Theorem 10.35 $(K,<)$ *is a Rolle field if and only if it admits a henselian valuation v whose residue field Kv is real closed and whose value group vK is odd-divisible.*

Note that it is not necessary to require that the valuation v be the natural valuation of $(K,<)$ (i.e. that the valuation ring be convex). Indeed, Brown, Craven and Pelling have also shown in [BRW–CR–PE] that if $(K,<)$ is a Rolle field, then also $(K,<')$ for every other ordering $<'$ of K.

The following is a generalization of one direction of the preceding theorem and gives a converse of Theorem 10.34.

Theorem 10.36 *Assume that the field L admits a henselian valuation v whose value group vL is odd-divisible. If Lv admits an ordering such that $(Lv,<)$ is a Rolle field, then L admits an ordering (inducing $<$ via v) and is a Rolle field (with respect to every ordering).*

Proof: Assume the hypothesis of the theorem. Then by virtue of the Theorem of Brown, Craven and Pelling, the Rolle field $(Lv,<)$ admits a valuation w whose residue field $(Lv)w$ is real closed and whose value group $w(Lv)$ is odd-divisible. Consider the composite $v \circ w$ on L. Its residue field $L(v \circ w) = (Lv)w$ is real closed. Its value group $v \circ wL$ is odd-divisible; this is seen as follows. $v \circ wL$ has a convex subgroup Γ isomorphic to $w(Lv)$ and hence odd-divisible. The quotiont of $v \circ wL$ by Γ is isomorphic to vL and hence also odd-divisible by hypothesis. It follows that the group $v \circ wL$ itself is odd-divisible. Again using the Theorem of Brown, Craven and Pelling, we conclude that L is a Rolle field for every ordering. Let us show that there exists at least one ordering. As L admits a place $P = P_v$ onto the ordered field $(Lv,<)$, it admits an ordering $<'$ which induces $<$ on Lv via this place, that is, the positive cone of $<$ on Lv is the image of the positive cone of $<'$ under the place P. □