Chapter 1

Ordered Sets and Ultrametric spaces

1.1 Ordered sets

Let T be a set and $<$ a binary relation on T. If for all $x \in T$,

(O$\not$R) $\quad x \not< x\,,$

(OT) $\quad x < y \,\wedge\, y < z \implies x < z\,,$

(OC) $\quad x < y \,\vee\, x = y \,\vee\, y < x\,,$

then $(T,<)$ is called an **ordered set** and $<$ is called an **ordering of** T. Axiom (O$\not$R) says that $<$ is anti-reflexive, axiom (OT) states transitivity and axiom (OC) declares that every two elements are comparable. In the presence of axiom (OC), one also speaks of a **total ordering**, but we will omit "total" since we will only consider total orders in this book. In the literature, a totally ordered set is also called a **fully ordered set**, a **linearly ordered set** or a **chain**. In contrast to this, $(T,<)$ is called a **partially ordered set** when (O$\not$R) and (OT) are satisfied.

Let $(T,<)$ be an ordered set. If $a,b \in T$, then $a \le b$ stands for $a < b \vee a = b$, as usual. If S_1, S_2 are non-empty subsets of T and $a \in T$, we will write $a < S_2$ if $a < b$ for all $b \in S_2$, and further, we will write $S_1 < S_2$ if $a < S_2$ for all $a \in S_1$. Similarly, we use the relations $>$, $\le$ and $\ge$.

If $(T', <')$ is another ordered set and f is a map from T into T', then f is said to **preserve the ordering** (or to be **order preserving**) if for all $a,b \in T$, $a < b$ implies $f(a) <' f(b)$. It then follows that f is injective. If f is also onto, then it is called an **order isomorphism**. The **order type of** $(T,<)$ is the class of all ordered sets which admit an order isomorphism onto $(T,<)$. The finite ordered sets can be represented by the number of their elements. The above map f is said to **reverse the ordering** (or to be **order reversing**) if for all $a,b \in T$, $a < b$ implies $f(b) <' f(a)$.

In the following, let S be a subset of T. We will call $(T,<)$ an **extension of** $(T,<)$ if the ordering $<$ on S is the restriction of the ordering $<$ of T. Further, S is said to be **dense in** $(T,<)$ if for all elements $\alpha, \beta \in T$ with $\alpha < \beta$, there exists $\gamma \in S$ such that $\alpha < \gamma < \beta$. T is said to be **dense** if it is dense in itself, that is, if for every two unequal elements of T there is a third element of T strictly between them. T is called **discretely ordered**, if for every $\alpha \in T$, the set $\{\beta \in T \mid \beta > \alpha\}$, if non-empty, admits a minimal element, and the set $\{\gamma \in T \mid \gamma < \alpha\}$, if non-empty, admits a maximal element. That means, if α is not the smallest element in T, then it admits an immediate predecessor $\beta < \alpha$ such that no element of T lies properly between β and α, and if α is not the maximal element in T, then it admits an immediate successor γ such that no element of T lies properly inbetween α and γ. The properties "dense" and "discretely ordered" are mutually exclusive. An element $\alpha \in T$ is called an **endpoint of** T if it is the maximal or the minimal element of T. For example, $(\mathbb{Q},<)$ is a densely ordered set without endpoints, and $(\mathbb{Z},<)$ is a discretely ordered set without endpoints (while $(\mathbb{N},<)$ has one endpoint).

Let $(T_1,<)$ and $(T_2,<)$ be two ordered sets. Their **lexicographic product**, denoted by $T_1 \amalg T_2$, is the cartesian product $T_1 \times T_2$ endowed with the ordering given by

(LEX) $$(x,y) < (x',y') \iff x < x' \vee (x = x' \wedge y < y')\ .$$

If T_2 is densely ordered without endpoints, then so is $T_1 \amalg T_2$. If T_2 is discretely ordered without endpoints, then so is $T_1 \amalg T_2$. The **sum of** $(T_1,<)$ **and** $(T_2,<)$, denoted by $T_1 + T_2$, is the disjoint union of T_1 and T_2, endowed with an extension of the orders of T_1 and T_2 such that every element of T_1 is smaller than every element of T_2. This sum can be viewed as the subset $\{(i,\alpha) \mid \alpha \in T_i \wedge i \in \{1,2\}\}$ of the lexicographic product $\{1,2\} \amalg (T_1 \cup T_2)$, endowed with the restriction of the lexicographic ordering. We will most often use the sum in the case where we have to add a **last element** to an ordered set T. Indeed, in the sum $T + \{\infty\}$ the element ∞ is bigger than every element of T. Instead of $T + \{\infty\}$ we will also write $T\infty$.

The subset S of T is called **convex in** $(T,<)$ if for every two elements $\alpha,\beta \in S$ and every $\gamma \in T$ such that $\alpha \leq \gamma \leq \beta$, it follows that $\gamma \in S$. The **convex hull of** S **in** T is the set

$$\{\gamma \in T \mid \exists \alpha,\beta \in S:\ \alpha \leq \gamma \leq \beta\}\ .$$

The subset S of T is an **initial segment of** T if for every $\alpha \in S$ and every $\gamma \in T$ with $\gamma \leq \alpha$, it follows that $\gamma \in S$. Symmetrically, S is a **final segment of** T if for every $\alpha \in S$ and every $\gamma \in T$ with $\gamma \geq \alpha$, it follows that $\gamma \in S$. Consequently, S is an initial segment of T if and only if $T \setminus S$ is a final segment of T. Note that S is an initial segment of T if and only if $S < T \setminus S$. Note also that $\emptyset < T$ and $T < \emptyset$ by definition; so $\emptyset$ is an initial segment as well as a final segment of T.

The subset S of T is called **coinitial in** T if its convex hull is an initial segment of T. Symmetrically, S is called **cofinal in** T if its convex hull is a final segment of T. Consequently, S is cofinal in T if and only if for every $\beta \in T$ there is some $\alpha \in S$ such that $\alpha \geq \beta$. The **coinitiality type** of $T \neq \emptyset$ is the minimal ordinal λ such that there is a coinitial subset S in T which admits an order reversing bijection onto λ. Similarly, the **cofinality type** of $T \neq \emptyset$ is the minimal ordinal λ such that there is a cofinal subset S in T of order type λ. Note that λ is 1 or a limit ordinal.

Example 1.1 The ordered set $(\mathbb{Z},<)$ is discretely ordered, but coinitial and cofinal in $(\mathbb{Q},<)$. Hence, the convex hull of $(\mathbb{Z},<)$ in $(\mathbb{Q},<)$ is $\mathbb{Q}$. The negative integers form an initial segment of $(\mathbb{Z},<)$, and the positive integers form a final segment. $(\mathbb{Q},<)$ is dense, and it is dense in $(\mathbb{R},<)$. The lexicographic product $\mathbb{Q}\amalg\mathbb{Z}$ is discretely ordered, while $\mathbb{Z}\amalg\mathbb{Q}$ is densely ordered. $(\mathbb{Z},<)$ is the sum $(-\mathbb{N},<) + \{0\} + (\mathbb{N},<)$. The sum $(\mathbb{Z},<) + (\mathbb{Q},<)$ is neither densely nor discretely ordered. ◇

Lemma 1.2 *Let S be a subset of the ordered set T. If S is dense in T, then for every element $\alpha \in T$, the initial segment $\{\beta \in S \mid \beta < \alpha\}$ of S is cofinal in the initial segment $\{\beta \in T \mid \beta < \alpha\}$ of T. The converse holds if T is dense.*

Proof: Assume that S is dense in T and that $\alpha \in T$. We have to show that the set $\{\beta \in S \mid \beta < \alpha\}$ is cofinal in $\{\beta \in T \mid \beta < \alpha\}$. This is equivalent to: for every $\beta \in T$, $\beta < \alpha$, there is some $\gamma \in S$ such that $\beta \leq \gamma < \alpha$. But this follows from the density of S in T.

For the converse, assume that T is dense and that $\{\beta \in S \mid \beta < \alpha\}$ is cofinal in $\{\beta \in T \mid \beta < \alpha\}$ for every $\alpha \in T$. Given $\alpha, \beta \in T$ with $\beta < \alpha$, we have to show that there exists $\gamma \in S$ such that $\beta < \gamma < \alpha$. By the density of T, there exists $\delta \in T$ such that $\beta < \delta < \alpha$, hence $\delta \in \{\beta \in T \mid \beta < \alpha\}$. Now the existence of γ follows by the cofinality of $\{\beta \in S \mid \beta < \alpha\}$ in $\{\beta \in T \mid \beta < \alpha\}$. □

1.2 Cuts

Throughout this section, let S be a non-empty subset of the ordered set T. If $S_1 \subseteq S$ and $S_2 \subseteq S$ are such that $S_1 \leq S_2$ and $S = S_1 \cup S_2$, then we will call (S_1, S_2) a **quasi-cut** in S. Then S_1 is an initial segment of S, S_2 is a final segment of S, and the intersection of S_1 and S_2 consists of at most one element. If this intersection is empty, then (S_1, S_2) will be called a **cut** in S. In this case, we will write $\Lambda^L = S_1$, $\Lambda^R = S_2$ and

$$\Lambda \;=\; (\Lambda^L, \Lambda^R)\;.$$

Since $\Lambda^R = S \setminus \Lambda^L$, the cut is uniquely determined by the initial segment Λ^L.

A cut (Λ^L, Λ^R) with $\Lambda^L \neq \emptyset$ and $\Lambda^R \neq \emptyset$ is called a **proper cut** or a **Dedekind cut**. Each (non-empty) ordered set $(S, <)$ has exactly two improper cuts: $(\emptyset, S)$ and $(S, \emptyset)$. (Our notion which includes improper cuts is sometimes called **Cuesta Dutari cut** in the literature.)

For any subset $M \subseteq S$, we let M^+ denote the cut

$$M^+ \;=\; (\{\alpha \in S \mid \exists \gamma \in M : \alpha \leq \gamma\}\,,\, \{\beta \in S \mid \beta > M\})\;.$$

That is, if $M^+ = (\Lambda^L, \Lambda^R)$ then Λ^L is the least initial segment of S which contains M, and Λ^R is the largest final segment which has empty intersection with M. If $M = \emptyset$ then $\Lambda^L = \emptyset$ and $\Lambda^R = S$, and if $M = S$, then $\Lambda^L = S$ and $\Lambda^R = \emptyset$. Symmetrically, we set

$$M^- \;=\; (\{\alpha \in S \mid \alpha < M\}\,,\, \{\beta \in S \mid \exists \gamma \in M : \beta \geq \gamma\})\;.$$

That is, if $M^- = (\Lambda^L, \Lambda^R)$ then Λ^L is the largest initial segment which has empty intersection with M, and Λ^R is the least final segment of S which contains M. If $M = \emptyset$ then $\Lambda^L = S$ and $\Lambda^R = \emptyset$, and if $M = S$, then $\Lambda^L = \emptyset$ and $\Lambda^R = S$.

If $M = \{\alpha\}$, we will write α^+ instead of $\{\alpha\}^+$ and α^- instead of $\{\alpha\}^-$. Note that

$$\alpha^+ = (\{\beta \in S \mid \beta < \alpha\}, \{\beta' \in S \mid \beta' \geq \alpha\}) \;\text{ and }\; \alpha^- = (\{\beta \in S \mid \beta \leq \alpha\}, \{\beta' \in S \mid \beta' > \alpha\})\;. \tag{1.1}$$

These two cuts are called **principal cuts**.

If $\gamma \in T$ is such that $\Lambda^L \leq \gamma \leq \Lambda^R$, then we will say that γ **realizes** (Λ^L, Λ^R) (in $(T, <)$). For $\gamma \in T$, the cut

$$(\{\alpha \in S \mid \alpha \leq \gamma\}, \{\beta \in S \mid \beta > \gamma\})$$

is called the **cut induced by** γ **in** S; this cut is realized by γ in $(T, <)$. We have:

Lemma 1.3 *If two elements γ and δ in two extensions of $(S,<)$ induce the same cut in $(S,<)$, and if this cut is not realized by any element of S, then $(S\cup\{\gamma\},<)$ and $(S\cup\{\delta\},<)$ are order isomorphic over S.*

Proof: As the cut is not realized by any element of S, we know that $\gamma,\delta\notin S$. Thus for every $\beta\in S$, $\gamma>\beta$ or $\gamma<\beta$. Since γ and δ realize the same cut (Λ^L,Λ^R) in $(S,<)$, we have $\gamma>\beta\Leftrightarrow\beta\in\Lambda^L\Leftrightarrow\delta>\beta$ and $\gamma<\beta\Leftrightarrow\beta\in\Lambda^R\Leftrightarrow\delta<\beta$. This shows that $\gamma\mapsto\delta$ induces an order preserving bijection between $(S\cup\{\gamma\},<)$ and $(S\cup\{\delta\},<)$ over S. □

Every cut in S is realized by at most two elements of S. If $\alpha<\alpha'$ are two elements of S realizing (Λ^L,Λ^R), then α is the maximal element of Λ^L and α' is the minimal element of Λ^R and there is no element of S between α and α'. Consequently, this situation can not appear if S is dense. On the other hand, every $\alpha\in S$ realizes exactly two cuts in $(S,<)$, namely α^+ and α^-. This shows that the cuts realized in $(S,<)$ are not in one to one correspondence with the elements of S.

If it is clear from the context which ordering on S we are working with, then we will often writen S instead of $(S,<)$, and we denote by $\breve{S}$ the set of cuts in S (including the improper cuts). A natural ordering can be introduced on $\breve{S}$ by setting $\Lambda_1\leq\Lambda_2$ if and only if $\Lambda_1^L\subseteq\Lambda_2^L$ (or, equivalently, $\Lambda_1^R\supseteq\Lambda_2^R$). Again, we write $\breve{S}$ for $(\breve{S},<)$.

There are two natural embeddings of S into $\breve{S}$ as ordered sets, given by $S\ni\alpha\mapsto\alpha^+\in\breve{S}$ and by $S\ni\alpha\mapsto\alpha^-\in\breve{S}$. We will work with the former, so $\alpha\in S$ will be identified with the cut whose left cut set has α as its largest element. Then we have

$$\alpha \;=\; \alpha^+ \;>\; \alpha^- \,.$$

The set S^{qc} of all quasi-cuts in S contains $\breve{S}$, and the ordering of $\breve{S}$ can be extended to S^{qc} as follows: if (A,B) is a quasicut in S with $A\cap B=\{\alpha\}$, then $\alpha^-<(A,B)<\alpha^+$. Sending α to this cut (A,B) yields a natural order preserving embedding of S in S^{qc} with image disjoint from $\breve{S}$.

1.3 Ultrametric spaces

Remark 1.4 Most of the valuation theorists who have dealt with Krull valuations, and in particular with those of rank >1, have preferred the additive notation where the value group is written additively. Not necessarily, but usually connected with this notation is the "philosophy" of taking the value of 0 to be an element bigger than all others, and to take the valuation ideal to be the set of all elements having positive value. In this philosophy, the world appears somewhat inverted: $v(a-b)$ is large if the elements a and b are near to each other; like gravitation, $v(a-b)$ measures proximity and not remoteness. Although it may be unfamiliar to many readers, we also prefer to work with this philosophy throughout the book, even when considering ultrametric spaces. So let us introduce the following mnemonic:

It is of high value to be near to each other.

Let Γ be an ordered set. By $\Gamma\infty$ we will denote the ordered set that we obtain from Γ by adding a new element ∞ and extending the ordering of Γ such that ∞ becomes the last element of $\Gamma\infty$. Written as a sum of two ordered sets, $\Gamma\infty=\Gamma+\{\infty\}$.

A set X together with a map $u:\;X\times X\longrightarrow\Gamma\infty$ is called an **ultrametric space** and u is called an **ultrametric** if the following axioms hold, for all $x,y,z\in X$:

(UM 0) $\quad u(x,y)=\infty \iff x=y\,,$

(UMT) $\quad u(x,y)\geq \min\{u(x,z),u(z,y)\}\,,$

(UMS) $\quad u(x,y)=u(y,x)\,.$

Axiom (UMT) is called the **ultrametric triangle law**, and (UMS) expresses the symmetry of the ultrametric. We will call $uX := \{u(x,y)\mid x,y\in X\,\wedge\, x\neq y\}\subseteq\Gamma$ the **value set of** (X,u). Note that we exclude ∞ from uX. We call $u(x,y)$ the **(ultrametric) distance between** x **and** y.

Lemma 1.5 *Axioms (UMT) and (UMS) imply*

(UM$\neq$) $\quad u(x,z)\neq u(z,y)\Longrightarrow u(x,y)=\min\{u(x,z),u(z,y)\}\,,$

which in view of (UMT) can also be written as

(UM=) $\quad u(x,y)>\min\{u(x,z),u(z,y)\}\Longrightarrow u(x,z)=u(z,y)\,.$

Proof: Assume $u(x,z)<u(z,y)$ so that $u(x,z)=\min\{u(x,z),u(z,y)\}$; we have to show that $u(x,y)=u(x,z)$. If this were not the case, then $u(x,y)>u(x,z)$, by virtue of (UMT). Applying (UMT) again, we find $u(x,z)\geq\min\{u(x,y),u(y,z)\}$. By (UMS), $u(y,z)=u(z,y)$, and we can now deduce $u(x,z)\geq\min\{u(x,y),u(z,y)\}>u(x,z)$, a contradiction. □

By induction, axiom (UMT) yields

$$\forall x_1,\ldots,x_n\in X:\; u(x_1,x_n)\geq\min_{1\leq i\leq n-1}u(x_i,x_{i+1})\,. \tag{1.2}$$

An ultrametric space (X,u) is called **homogeneous** if for every $x\in X$ and every $\alpha\in uX$ there is $x'\in X$ such that $u(x,x')=\alpha$.

The main examples of ultrametric spaces will be provided by valued abelian groups and valued fields. All ultrametric spaces induced by valuations of abelian groups or fields are homogeneous.

1.4 Balls

Let (X,u) be an ultrametric space. A subset $B\subseteq X$ will be called a **ball (in** X**)** if

(BALL) $\quad \forall y,z\in B\;\forall w\in X:\; u(y,w)\geq u(y,z)\Rightarrow w\in B\,.$

Balls are the ultrametric analogue of convex sets. We are giving an alternative representation of balls. For every $x\in X$ and every final segment S of $uX\infty$, we define

$$B_S(x):=\{y\in X\mid u(x,y)\in S\}\,.$$

We note that $S\subseteq S'$ implies that $B_S(x)\subseteq B_{S'}(x)$. The sets $B_S(x)$ are precisely all balls in X:

Lemma 1.6 *Every set $B_S(x)$ is a ball in X. Conversely, if B is a ball in X and S is the least final segment containing the elements $u(y,z)$ for all $y,z \in B$, then for every $x \in B$,*

$$B = B_S(x) .$$

In particular, $B_S(x) = B_S(y)$ for every $y \in B_S(x)$.

Proof: Assume that $y,z \in B_S(x)$, that is, $u(y,x) = u(x,y) \in S$ and $u(x,z) \in S$. Suppose in addition that $w \in X$ with $u(y,w) \geq u(y,z)$. Then $u(x,w) \geq \min\{u(x,y),u(y,w)\} \geq \min\{u(x,y),u(y,z)\} \geq \min\{u(x,y),u(y,x),u(x,z)\} \in S$. Since S is a final segment of $uX\infty$, it follows that $u(x,w) \in S$. Hence, $w \in B_S(x)$. We have proved that $B_S(x)$ is a ball.

For the converse, assume that B is a ball, and let S be as in the assertion. Further, let x be any element in B. If $y \in B$, then $u(x,y) \in S$ and thus, $y \in B_S(x)$. Conversely, if $y \in B_S(x)$, then $u(x,y) \in S$. So by definition of S, there is some $z \in B$ such that $u(x,z) \leq u(x,y)$. Since B is a ball, it follows that $y \in B$. We have proved that $B = B_S(x)$. □

The last assertion of the lemma is sometimes described in the following words:

Every element of a ball in an ultrametric space is a center of the ball.

Note that also X, singletons and the empty set are balls in X. Indeed, for every $x \in X$,

$$X = B_{uX\infty}(x) , \quad \{x\} = B_{\{\infty\}}(x) , \quad \emptyset = B_{\emptyset}(x) .$$

For every $\alpha \in uX\infty$, we define the following **balls of radius α around** x:

$$B_\alpha(x) := \{y \in X \mid u(x,y) \geq \alpha\} , \quad B^\circ_\alpha(x) := \{y \in X \mid u(x,y) > \alpha\} .$$

Hence, $B_\alpha(x) = B_S(x)$ for the final segment $S = \{\gamma \in uX\infty \mid \gamma \geq \alpha\}$, and $B^\circ_\alpha(x) = B_S(x)$ for the final segment $S = \{\gamma \in uX\infty \mid \gamma > \alpha\}$. We will also write $B_\alpha(X,x)$, $B^\circ_\alpha(X,x)$ to indicate the space in which we are working. Note that $B_\infty(x) = \{x\}$ and $B^\circ_\infty(x) = \emptyset$. For $x,y \in X$, we define:

$$B(x,y) \;:=\; B_{u(x,y)}(x) \;=\; B_{u(x,y)}(y) ,$$

which is the smallest ball containing both x and y.

In the literature, a ball of the form $B^\circ_\alpha(x)$ is sometimes called "open ball", and a ball of the form $B_\alpha(x)$ "closed ball". But as we will see, both are closed and open in the topology induced by the ultrametric. Therefore, we will speak of **o-ball** and **c-ball** instead.

We will work with the following sets of balls:

$$\mathbf{B}_\alpha(X) := \{B_\alpha(x) \mid x \in X\} , \qquad \mathbf{B}^\circ_\alpha(X) := \{B^\circ_\alpha(x) \mid x \in X\}$$
$$\mathbf{B}(X) := \bigcup_{\alpha \in uX\infty} \mathbf{B}_\alpha(X) , \qquad \mathbf{B}^\circ(X) := \bigcup_{\alpha \in uX\infty} \mathbf{B}^\circ_\alpha(X) .$$

Hence $\mathbf{B}(X)$ is the set of all c-balls and $\mathbf{B}^\circ(X)$ the set of all o-balls in X.

Lemma 1.7 *Every two balls with non-empty intersection are comparable by inclusion: If B, B' are balls in X with $B \cap B' \neq \emptyset$, then $B \subseteq B'$ or $B' \subseteq B$. In particular, for all $\alpha, \beta \in uX$ and all $x, y \in X$,*

$$\alpha \geq \beta \wedge B_\alpha(x) \cap B_\beta(y) \neq \emptyset \implies B_\alpha(x) \subseteq B_\beta(y) , \tag{1.3}$$

and the same holds for the balls B°.

Proof: Suppose that $z \in B \cap B'$. Then by the last lemma, there are final segments S and S' of $uX\infty$ such that $B = B_S(z)$ and $B' = B_{S'}(z)$. Since S and S' are final segments, we must have $S \subseteq S'$ or $S' \subseteq S$. Hence, $B \subseteq B'$ or $B' \subseteq B$. For the second assertion, we just have to note that $\alpha \geq \beta$ implies that $\{\gamma \in uX\infty \mid \gamma \geq \alpha\} \subseteq \{\gamma \in uX\infty \mid \gamma \geq \beta\}$. □

Lemma 1.8 *Let B be a ball in X and $y \in X \setminus B$. Then for every $x \in B$, $B \cap B^\circ_\alpha(y) = \emptyset$ for $\alpha = u(x, y)$.*

Proof: Suppose that $z \in B^\circ_\alpha(y)$. Then $u(y, z) > \alpha = u(x, y)$, and thus $u(x, z) = \min\{u(x, y), u(y, z)\} = u(x, y)$. If z were an element of B, then it would follow from (BALL) that $y \in B$, a contradiction. □

Exercise 1.1 *Prove that for all $x, y \in X$ and $\alpha, \beta \in uX\infty$,*

$$\begin{aligned} \alpha \geq \beta \wedge B^\circ_\alpha(x) \cap B_\beta(y) \neq \emptyset &\implies B^\circ_\alpha(x) \subseteq B_\beta(y) \\ \alpha > \beta \wedge B_\alpha(x) \cap B^\circ_\beta(y) \neq \emptyset &\implies B_\alpha(x) \subseteq B^\circ_\beta(y) . \end{aligned}$$

Exercise 1.2 *Assume that (X, u) is homogenous. Prove that every c-ball is of the form B(x,x') for suitable $x, x' \in X$. Further, prove that if $x \in X$ $\alpha, \beta \in uX$ with $\alpha < \beta$, then $B_\beta(x) \subsetneq B_\alpha(x)$. Construct an example of a non-homogenous ultrametric space in which this does not hold.*

1.5 The topology induced by an ultrametric

For all $x, y \in X$ and $\alpha, \beta \in uX\infty$ we have that $B^\circ_\alpha(x) \cap B^\circ_\beta(y)$ is equal to either $\emptyset = B^\circ_\infty(x)$, or $B^\circ_\gamma(z)$ for any z in the intersection and $\gamma = \max\{\alpha, \beta\}$. Consequently, the balls $B^\circ_\alpha(x)$, $\alpha \in uX\infty$, $x \in X$, form a basis for a topology on X, canonically induced by the ultrametric u. One might be tempted to think of the $B_\alpha(x)$ as "closed" balls in contrast to the open balls $B^\circ_\alpha(x)$, but this does not reflect the reality:

Lemma 1.9 *In the topology induced by the ultrametric u on X, the following holds:*
a) all balls in X are closed,
b) all balls $B_S(x)$ in X with $S \neq \{\infty\}$ are open,
c) X is Hausdorff and totally disconnected.

Proof: The balls $\emptyset$ and X are closed. Now take a non-empty ball $B \neq X$, $x \in B$ and $y \notin B$. Then the ball $B_\alpha(y)$ with $\alpha = u(x, y)$ is an open neighborhood of y, and Lemma 1.8 shows that it does not intersect with B. This proves a).

Now take $B = B_S(x)$ with $S \neq \{\infty\}$. If $S = \emptyset$, then $B = \emptyset$ is open. So assume that S contains at least one element α of uX. Then for every $y \in B$ the open neighborhood $B^\circ_\alpha(y)$ is contained in B. This proves b).

Given $x, y \in X$ with $x \neq y$, we set $\alpha = u(x,y) < \infty$. Thus, $x \in B^\circ_\alpha(x)$ and $y \in B^\circ_\alpha(y)$. On the other hand, we see that $x \notin B^\circ_\alpha(y)$ and $y \notin B^\circ_\alpha(x)$. Hence by Lemma 1.7, $B^\circ_\alpha(x) \cap B^\circ_\alpha(y) = \emptyset$. This proves that the topology is Hausdorff. Since the two balls $B^\circ_\alpha(x)$ and $B^\circ_\alpha(y)$ are open and closed, the connected component of x is just $\{x\}$. This proves c). $\square$

Note that the topology is discrete if and only if all balls $B_S(x)$ with $S = \{\infty\}$ are open (because these balls are precisely the singletons in X). This is the case e.g. if uX has a greatest element γ, since then $\{x\} = B^\circ_\gamma(x)$.

Let (X, u) and (X', u') be ultrametric spaces and $f : X \to X'$ an arbitrary map. For the topologies induced on X and X' by the ultrametrics, f is **continuous** if and only if for every $x \in X$ and every $\beta \in u'X'$ there is $\alpha \in uX$ such that $f(B_\alpha(x)) \subseteq B_\beta(f(x))$.

1.6 Nests of balls and spherical completeness

The set of all balls in X is partially ordered by inclusion. Lemma 1.7 states that every two balls with a non-empty intersection are comparable. The same can be deduced for the set $\mathbf{B}(X) \cup \mathbf{B}^\circ(X)$. Thus, a set of non-empty balls will be totally ordered as soon as every two balls in the set have a non-empty intersection; in this case, it is called a **nest of arbitrary balls**. In accordance with the literature, it will be called a **nest of balls** if it consists only of c-balls. It will be called a **nest of co-balls** if it consists only of c-balls and o-balls.

The ultrametric space (X, u) is called **spherically complete** if for every nest $\mathbf{B} \subseteq \mathbf{B}(X)$ of balls, their intersection $\bigcap \mathbf{B} = \bigcap_{B \in \mathbf{B}} B$ is non-empty. In the literature, the names **spherically compact** and **ultracomplete** are also used for the same property.

A nest $\{B_\alpha(x_\alpha) \in \mathbf{B}(X) \mid \alpha \in S\}$ of balls will be called a **completion nest** if S is a cofinal subset of $uX\infty$. Ultracompleteness is stronger than the following property: (X, u) is called **complete** if for every completion nest $\mathbf{B} \subseteq \mathbf{B}(X)$, the intersection $\bigcap \mathbf{B}$ is non-empty.

Given a nest of balls $\mathbf{B} \subseteq \mathbf{B}(X)$, it can be written as $\{B_\alpha(x_\alpha) \mid \alpha \in S\}$ with S a subset of $uX\infty$ and x_α elements of X. We denote by $\Lambda^L(\mathbf{B})$ the minimal initial segment of $uX\infty$ containing S and call it the **support segment of B**. If $S = \Lambda^L(\mathbf{B})$, then we call $\mathbf{B}$ a **full nest of balls**. If S is not an initial segment of uX, that is, if $S \neq \Lambda^L(\mathbf{B})$, then we can **fill up B** to a full nest $\mathbf{B}' = \{B_\alpha(x_\alpha) \mid \alpha \in \Lambda^L(\mathbf{B})\}$ of balls such that $\mathbf{B} \subseteq \mathbf{B}'$ and $\mathbf{B}'$ has the same intersection as $\mathbf{B}$. Indeed, if $\alpha \in \Lambda^L(\mathbf{B}) \setminus S$ then we can pick $\beta \in S$ such that $\beta > \alpha$ and we can set $x_\alpha = x_\beta$. Note that $B_\alpha(x_\alpha)$ does not depend on the actual choice of β: If we take $\gamma \in S$ such that $\gamma > \alpha$, then the balls $B_\beta(x_\beta)$ and $B_\gamma(x_\gamma)$ have a non-empty intersection since $\mathbf{B}$ is a nest of balls. Consequently, $u(x_\beta, x_\gamma) \geq \min\{\beta, \gamma\} > \alpha$, showing that $B_\alpha(x_\beta) = B_\alpha(x_\gamma)$. Since $\mathbf{B} \subseteq \mathbf{B}'$ and every $B_\alpha(x_\alpha) \in \mathbf{B}'$ contains some $B_\beta(x_\beta) \in \mathbf{B}$, we have $\bigcap \mathbf{B}' = \bigcap \mathbf{B}$.

This construction is doing nothing else than adding to $\mathbf{B}$ all those c-balls in X that lie between two c-balls in $\mathbf{B}$ (that is, contain one and are contained in the other). In the same way, we can fill up any nest $\mathbf{B} \subseteq \mathbf{B}(X) \cup \mathbf{B}^\circ(X)$ with c-balls and o-balls, and any nest of general balls with general balls. The nests obtained in this way are uniquely determined. (We leave the details as an exercise to the reader.)

Given a nest $\mathbf{B}$ of arbitrary balls, we can first fill it up and then extract the nest $\mathbf{B}''$ of all c-balls which the filled-up nest contains. If $\mathbf{B}$ contains no smallest ball, then $\bigcap \mathbf{B}'' = \bigcap \mathbf{B}$. The containment "$\subseteq$" is clear. The converse follows from the fact that if B_1, B_2 are arbitrary balls such that $B_1 \subsetneq B_2$, then there is a c-ball B such that $B_1 \subseteq B \subseteq B_2$. Indeed, by Lemma 1.6 we can write $B_i = B_{S_i}(x_i)$ for $i = 1, 2$ with final segments $S_1 \subsetneq S_2$; taking any $\alpha \in S_2 \setminus S_1$, the ball $B = B_\alpha(x_1)$ is what we need.

Since every nest of balls containing a smallest ball has a non-empty intersection, our above construction proves:

Lemma 1.10 *The ultrametric space (X, u) is spherically complete if and only if every nest of arbitrary balls has a non-empty intersection.*

A finite nest of balls always contains a smallest ball. Hence, every ultrametric space with a finite value set is spherically complete. More generally, if the value set uX is a reversed well-ordering, then every ascending chain of values in uX is finite, which yields that all nests of balls contain a minimal ball, and again, (X, u) is spherically complete. Similarly, if uX contains a maximal element $\alpha_{\max}$, then (X, u) is complete since every cofinal subset in uX must contain $\alpha_{\max}$.

1.7 Products of ultrametric spaces

Let (X_i, u_i), $i \in I$, be ultrametric spaces whose value sets u_iX_i are all contained in a common ordered set, and assume that I is finite or $\bigcup_{i\in I} u_iX_i$ is wellordered. Then their **product** will be the cartesian product $\prod_{i\in I} X_i$ equipped with the ultrametric $\prod_{i\in I} u_i$: $\prod_{i\in I} X_i \times \prod_{i\in I} X_i \to (\bigcup_{i\in I} u_iX_i)\infty$ defined by

$$(\prod_{i\in I} u_i)\,((x_i)_{i\in I}, (y_i)_{i\in I}) := \min_{i\in I} u_i(x_i, y_i)\ .$$

We leave it to the reader to verify that this map satisfies (UM 0), (UMT) and (UMS). Note that indeed every element of $\bigcup_{i\in I} u_iX_i$ appears as the distance of two suitably chosen elements of $\prod_{i\in I} X_i$. In case of two ultrametric spaces, we write $(X_1 \times X_2, u_1 \times u_2)$. Examples for products of ultrametric spaces are provided by Lemma 3.11 below.

Lemma 1.11 *If the product $(\prod_{i\in I} X_i\,, \prod_{i\in I} u_i)$ of ultrametric spaces is spherically complete, then the same holds for all (X_i, u_i), $i \in I$. If for $1 \le i \le n$, the ultrametric spaces (X_i, u_i) are spherically complete (resp. complete), then the same holds for the finite product $(\prod_{i=1}^n X_i\,, \prod_{i=1}^n u_i)$.*

Proof: Assume that the product $(X, u) := (\prod_{i\in I} X_i\,, \prod_{i\in I} u_i)$ is spherically complete and that $\mathbf{B} \subseteq \mathbf{B}(X)$ is a nest of balls in (X_j, u_j) for some fixed $j \in I$. Let us write $\mathbf{B} = \{B_\alpha(X_j, x_\alpha) \mid \alpha \in S\}$ with S a subset of u_jX_j and x_α elements in X_j. Pick some elements $x_i \in X_i$, $i \in I$, and set $x_{j,\alpha} = x_j$ and $x_{i,\alpha} = x_i$ for $i \neq j$. Define $\underline{x}_\alpha := (x_{i,\alpha})_{i\in I} \in \prod_{i\in I} X_i$ for every $\alpha \in S$. Then $\{B_\alpha(X, \underline{x}_\alpha) \mid \alpha \in S\}$ is a nest of balls in the product. By assumption, its intersection contains an element $\underline{y}$. In view of the definition of $\prod_{i\in I} u_i$ it follows that its j-th coordinate $y_j \in X_j$ lies in the intersection of $\mathbf{B}$.

For the converse, assume that the finitely many ultrametric spaces (X_i, u_i), $1 \leq i \leq n$, are spherically complete. Let $\mathbf{B} = \{B_\alpha(X, \underline{x}_\alpha) \mid \alpha \in S\}$ be a nest of balls in the product $(X, u) := (\prod_{i=1}^n X_i\,, \prod_{i=1}^n u_i)$, with $S \subseteq \bigcup_{i=1}^n u_i X_i$. We have to show that the intersection of $\mathbf{B}$ is non-empty, so we can assume w.l.o.g. that S has no maximal element. We write $\underline{x}_\alpha = (x_{i,\alpha})_{1\leq i\leq n}$. Although $\alpha \in S$ may not lie in $u_j X_j$, the notation $B_\alpha(X_j, x_{j,\alpha})$ still makes sense since by assumption, $\bigcup_{i=1}^n u_i X_i$ is an ordered set containing all ordered sets $u_i X_i$. By definition, $\prod_{i=1}^n u_i(\underline{x}, \underline{y}) \geq \alpha$ implies that $u_i(x_i, y_i) \geq \alpha$ for all i. This yields that every two sets in $\{B_\alpha(X_j, x_{j,\alpha}) \mid \alpha \in S\}$ have non-empty intersection since the same is true for the sets in $\mathbf{B}$. However, $B_\alpha(X_j, x_{j,\alpha})$ is not a ball in the original sense. But we can extract a single element or a nest of balls $\mathbf{B}_j$ in (X_j, u_j) from $\{B_\alpha(X_j, x_{j,\alpha}) \mid \alpha \in S\}$ as follows. If $u_j X_j \cap S$ is not cofinal in S then choose $\alpha \in S$ such that $\alpha > u_j X_j \cap S$ and define $y_j := x_{j,\alpha}$. It will follow that for every $\beta \in S$ with $\beta \geq \alpha$ we have $u_j(y_j, x_{j,\beta}) > S$. Now assume that $u_j X_j \cap S$ is cofinal in S. By virtue of our filling up procedure for nests of balls, we can assume w.l.o.g. that S is an initial segment of $\bigcup_{i=1}^n u_i X_i$. So $u_j X_j \cap S$ will be an initial segment of $u_j X_j$ which is cofinal in S, and $\mathbf{B}_j := \{B_\alpha(X_j, x_{j,\alpha}) \mid \alpha \in u_j X_j \cap S\}$ is a set of balls which has the same intersection as $\{B_\alpha(X_j, x_{j,\alpha}) \mid \alpha \in S\}$. Since every two elements in the latter have non-empty intersection, it follows that this also holds for $\mathbf{B}_j$, which is consequently a nest of balls in (X_j, u_j). By assumption on (X_j, u_j), we find that $\bigcap \mathbf{B}_j$ contains an element y_j. By construction of the elements y_j for $1 \leq j \leq n$, it follows that the element $\underline{y} := (y_1, \ldots, y_n)$ lies in the intersection of $\mathbf{B}$.

Now assume that the finitely many ultrametric spaces (X_i, u_i), $1 \leq i \leq n$, are complete; we have to show that their product is complete. We proceed as before with the same $\mathbf{B}$, but assuming that S is cofinal in $\bigcup_{i=1}^n u_i X_i$. Assuming again that S is an initial segment, we find that $S = \bigcup_{i=1}^n u_i X_i$ and consequently, $u_j X_j \cap S = u_j X_j$ for every j. So for the construction of the elements y_j it suffices that every (X_i, u_i) be complete. □

For an application of this lemma, see the proof of Theorem 11.27.

Exercise 1.3 *Show that the first assertion of Lemma 1.11 does not hold for "complete" in the place of "spherically complete", and that its second assertion does not hold for the product of infinitely many ultrametric spaces.*

1.8 The Ultrametric Fixed Point Theorem

We shall now prove an Ultrametric Fixed Point Theorem which is due to S. Prieß-Crampe (cf. [PC2]). A generalization to ultrametric spaces whose value sets are not totally ordered, was given by S. Prieß-Crampe and P. Ribenboim (cf. [PC–RIB1,2]). In the latter paper, it was remarked that the Fixed Point Theorem holds for a slightly more general class of maps than that which are commonly called **contractive**. This generalization, however, will be significant for some of our applications, so we will make it available by the following definition. A map $\Xi : X \to X$ will be called **self-contractive** if it satisfies

(SC1) $\quad x \neq \Xi x \;\Rightarrow\; u(\Xi x, \Xi(\Xi x)) > u(x, \Xi x)\,,$

(SC2) $\quad u(\Xi x, \Xi y) \geq u(x, y)$

for all $x, y \in X$. The map Ξ is called **contractive** if it satisfies $u(\Xi x, \Xi y) > u(x, y)$ for all $x, y \in X$, $x \neq y$. Hence, every contractive map is self-contractive. An element x is called a fixed point of Ξ if $\Xi x = x$. Note that a contractive map Ξ has at most one fixed point; indeed, if $x \neq y$ were fixed points of Ξ, then $u(x,y) = u(\Xi x, \Xi y) > u(x,y)$, a contradiction.

Theorem 1.12 (Ultrametric Fixed Point Theorem)
The ultrametric space (X,u) is spherically complete if and only if every self-contractive map $\Xi : X \to X$ admits a fixed point.

Proof: Assume first that (X,u) is spherically complete and that $\Xi : X \to X$ is a self-contractive map. Let $\alpha_x := u(x, \Xi x)$ and $B_x := B_{\alpha_x}(x)$. The set $S = \{B_x \mid x \in X\}$ is partially ordered by inclusion. Let T be a maximal totally ordered subset of S. Note that no ball is empty, and thus, the intersection of any two balls in T is non-empty. Since (X,u) is spherically complete, there is some $z \in \bigcap_{B_x \in T} B_x$. Pick an arbitrary $B_x \in T$; we wish to show that $B_z \subseteq B_x$. By (SC2) we know that $u(\Xi z, \Xi x) \geq u(z,x)$. Since $z \in B_x$, we have $u(z,x) \geq u(x, \Xi x)$. Altogether, $u(z, \Xi z) \geq \min\{u(z,x), u(x,\Xi x), u(\Xi x, \Xi z)\} = u(x, \Xi x)$. By (1.3) it now follows that $B_z \subseteq B_x$. We find that B_z must be the smallest element of T. We claim that z is a fixed point of Ξ. Otherwise, $u(\Xi z, \Xi(\Xi z)) > u(z, \Xi z)$ by assumption on Ξ. This yields that $z \notin B_{\Xi z}$ and in view of (1.3), $B_{\Xi z} \subsetneq B_z$. But this contradicts the maximality of T.

For the converse, suppose that $\mathbf{B} \subseteq \mathbf{B}(X)$ is a nest of balls. Then (1.3) yields that inclusion imposes a total ordering on $\mathbf{B}$. Let $B_\nu = B_{\alpha_\nu}(x_\nu)$, $\nu < \lambda$, be a coinitial descending sequence in $\mathbf{B}$, where λ is an ordinal. Let us assume that $\bigcap_{B \in \mathbf{B}} B = \emptyset$. Then also $\bigcap_{\nu<\lambda} B_\nu = \emptyset$ since the sequence B_ν is coinitial in $\mathbf{B}$. Hence, for every $x \in X$, there is a minimal ordinal ν such that $x \notin B_\nu$, and we set $\Xi x := x_\nu$. Since $x_\nu \in B_\nu$ and $x \notin B_\nu$, we have $x \neq \Xi x$, showing that Ξ admits no fixed point. Let us show that Ξ is contractive. Let $x, y \in X$ such that $x \neq y$ and $\Xi x = x_\nu$, $\Xi y = x_\mu$. If $x_\nu = x_\mu$ then $u(\Xi x, \Xi y) = \infty > u(x,y)$. Otherwise, assume w.l.o.g. that $\mu < \nu$, that is, $B_\nu \subseteq B_\mu$. Then by our definition of Ξ, $x \in B_\mu$ but $y \notin B_\mu$. It follows that $u(x,y) < \alpha_\mu$ and further, $u(\Xi x, \Xi y) = u(x_\mu, x_\nu) \geq \alpha_\mu > u(x,y)$, as required. We have shown the existence of a contractive map $\Xi : X \to X$ without a fixed point. □

The first direction of the above proof is taken from the paper [PC–RIB2]; it also works for ultrametric spaces with partially ordered value sets. The proof shows that in the assertion of our theorem, we could replace "self-contractive" by "contractive". Strengthening the properties of the map Ξ, we can weaken the assumptions on the ultrametric space:

Lemma 1.13 *Let (X,u) be a complete ultrametric space. Assume that the self-contractive map $\Xi : X \to X$ satisfies in addition that for every $x \in X$ and $\alpha \in uX$ there is some $n \in \mathbb{N}$ such that $u(\Xi^{n+1}x, \Xi^n x) > \alpha$. Then Ξ admits a fixed point.*

Proof: By a modification of the first part of the proof of the preceding theorem. Choose an arbitrary $x \in X$. By (SC1) and (1.3), the balls $B_{\Xi^n x}$ form a totally ordered subset T' of S. Let T be a maximal totally ordered subset of S containing T'. Then by our additional assumption on Ξ, T contains non-empty balls of arbitrarily high radius. Now the existence of z is guaranteed by our assumption that (X,u) be complete. □

1.9 Extensions of ultrametric spaces

Let (X, u) and (X', u') be ultrametric spaces. An injective map $\iota : X \to X'$ is called an **embedding of** (X, u) **in** (X', u') if it satisfies

$$u(x, y) < u(z, t) \iff u'(\iota x, \iota y) < u'(\iota z, \iota t) \tag{1.4}$$

for all $x, y, z, t \in X$. If in addition ι is onto, then it is called an **isomorphism of** (X, u) **onto** (X', u'). Since we have assumed $u : X \times X \to uX$ to be onto for every ultrametric space (X, u), it follows that an embedding ι of (X, u) in (X', u') induces an order preserving injective map $\rho : uX \to u'X'$ such that $u'(\iota x, \iota y) = \rho u(x, y)$ for all $x, y \in X$. If ι is an isomorphism, then ρ is also an order isomorphism.

Take two ultrametric spaces (X, u) and (Y, u'). We will call (Y, u') an **extension of** (X, u) and write $(X, u) \subseteq (Y, u')$ if $X \subseteq Y$ and u' restricted to $X \times X$ coincides with u. Then we will also just write $(X, u) \subseteq (Y, u)$ for extensions of ultrametric spaces. For $x \in X$ we denote by $B_\alpha(X, x)$ the ball $B_\alpha(x)$ in X and by $B_\alpha(Y, x)$ the ball $B_\alpha(x)$ in Y. Similarly for the balls B_α° and B_S .

The easy proof of the following lemma is left to the reader as an exercise:

Lemma 1.14 *Take an extension $(X, u) \subseteq (Y, u)$ of ultrametric spaces and a ball B in (Y, u) such that $B \cap X \neq \emptyset$. Then $B \cap X$ is a ball in (X, u). If B is a c-ball with radius α, then so is $B \cap X$. If B is an o-ball with radius α, then so is $B \cap X$.*

Take an extension $(X, u) \subseteq (Y, u)$ of ultrametric spaces. Then (X, u) is said to have the **optimal approximation property in** (Y, u) if for every $y \in Y$ the set

$$u(y, X) := \{u(y, x) \mid x \in X\} \subseteq uY\infty \tag{1.5}$$

admits a maximum (i.e., for every element in Y there is a nearest element in X; however, it may not be uniquely determined). Note that if $y \in X$, then $\max\{u(y, x)\} = \infty$. Further, (X, u) is said to be **spherically closed in** (Y, u), if every nest of balls $\{B_\alpha(X, x_\alpha) \mid \alpha \in S\}$, $S \subseteq uX$, in (X, u) has a non-empty intersection whenever the associated nest of balls $\{B_\alpha(Y, x_\alpha) \mid \alpha \in S\}$ in (Y, u) has a non-empty intersection.

Lemma 1.15 *(X, u) is spherically closed in (Y, u) if and only if it has the optimal approximation property in (Y, u).*

Proof: Suppose that (X, u) is spherically closed in (Y, u); we wish to show that (X, u) has the optimal approximation property in (Y, u). Let $y \in Y$. Define $B_\alpha := \{x \in X \mid u(x, y) \geq \alpha\}$ for all $\alpha \in uX\infty$. Consider all such balls which are non-empty. They form a nest $\{B_\alpha(X, x_\alpha) \mid \alpha \in S\}$ of balls, since by definition $B_\alpha \subseteq B_\beta$ for $\beta \leq \alpha$. By construction, the corresponding nest $\{B_\alpha(Y, x_\alpha) \mid \alpha \in S\}$ in (Y, u) contains y in its intersection. Hence by hypothesis, the intersection of $\{B_\alpha(X, x_\alpha) \mid \alpha \in S\}$ is non-empty. Let x be an element of this intersection. If $u(y, x)$ is the maximum of $u(y, X)$ then we are done. If there exists some $x' \in X$ such that $u(y, x) < u(y, x')$, then we wish to show that $u(y, x')$ is the maximum of $u(y, X)$. Suppose there is $x'' \in X$ such that $u(y, x') < u(y, x'')$. Then $u(y, x) < u(y, x') = u(x', x'')$ by (UM=), showing that $x \notin B_\alpha$ for $\alpha = u(x', x'') \in uX$. This contradicts our choice of x.

Now suppose that (X,u) has the optimal approximation property in (Y,u). Take a nest $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ of balls in (X,u) such that $\{B_\alpha(Y,x_\alpha) \mid \alpha \in S\}$ in (Y,u) admits an element $y \in Y$ in its intersection. Take $x \in X$ such that $u(y,x)$ is maximal. Then $u(y,x) \geq u(y,x_\alpha)$ for all $\alpha \in S$. This shows that x lies in all $B_\alpha(X,x_\alpha)$ and thus in their intersection. We have proved that (X,u) is spherically closed in (Y,u). □

An extension $(X,u) \subseteq (Y,u)$ is said to be an **immediate** if for all $y \in Y$ and all $x \in X \setminus \{y\}$ there is some $x' \in X$ such that $u(y,x') > u(y,x)$. This means that for no $y \in Y \setminus X$ there is a best approximation by elements from X; however, $u(y,X) \cap uX$ may be bounded from above by some element in uX. A stronger property is the following: (X,u) **is dense in** (Y,u) if for all $y \in Y$ and all $\alpha \in uY$ there is some $x \in X$ such that $u(y,x) > \alpha$. This means that every element in Y can be approximated arbitrarily well by elements from X.

Lemma 1.16 *Let $(X,u) \subseteq (Y,u)$ be an extension of ultrametric spaces. If (X,u) is dense in (Y,u), then $(X,u) \subseteq (Y,u)$ is immediate and $uX = uY$.*

Proof: Suppose that (X,u) is dense in (Y,u). Then for every $y \in Y$ and $x \in X$, $\alpha := u(y,x) \in uY$ and hence there is some $x' \in X$ such that $u(y,x') > u(y,x)$. This proves that $(X,u) \subseteq (Y,u)$ is immediate. Now let $y, y' \in Y$ and choose $x, x' \in X$ such that $u(y,x) > u(y,y')$ and $u(y',x') > u(y,y')$. It follows that $u(x,x') = \min\{u(y,x), u(y,y'), u(y',x')\} = u(y,y')$. This proves that $uY = uX$. □

The properties "(X,u) has the optimal approximation property in (Y,u)" (or, equivalently, "(X,u) is spherically closed in (Y,u)") and "$(X,u) \subseteq (Y,u)$ is an immediate extension" are mutually exclusive. All three properties discussed above are transitive:

Lemma 1.17 (Transitivity) *Let $(X,u) \subseteq (Y,u) \subseteq (Z,u)$ be ultrametric spaces.*
a) If (X,u) is spherically closed in (Y,u) and (Y,u) is spherically closed in (Z,u), then (X,u) is spherically closed in (Z,u).
b) If $(X,u) \subseteq (Y,u)$ and $(Y,u) \subseteq (Z,u)$ are immediate, then so is $(X,u) \subseteq (Z,u)$.
c) (X,u) is dense in (Y,u) and (Y,u) is dense in (Z,u), if and only if (X,u) is dense in (Z,u).

Proof: a): Take a nest $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$, $S \subseteq uX$, in (X,u) such that the associated nest $\{B_\alpha(Z,x_\alpha) \mid \alpha \in S\}$ in (Z,u) has a non-empty intersection. Since (Y,u) is spherically closed in (Z,u), the nest $\{B_\alpha(Y,x_\alpha) \mid \alpha \in S\}$ has a non-empty intersection. Since (X,u) is spherically closed in (Y,u), it follows that the nest $\{B_\alpha(X,x_\alpha) \mid \alpha \in S\}$ has a non-empty intersection.

b): Take $z \in Z$ and $x \in X \setminus \{z\}$. We show the existence of $x' \in X$ satisfying $u(z,x') > u(z,x)$. By assumption, there is $y' \in Y$ such that $u(z,y') > u(z,x)$. Hence, $u(x,y') = \min\{u(z,y'), u(z,x)\} = u(z,x)$. Again by assumption, there is $x' \in X$ such that $u(y',x') > u(y',x) = u(z,x)$. It follows that $u(z,x') \geq \min\{u(z,y'), u(y',x')\} > u(z,x)$, as desired.

c): Take $z \in Z$ and $\alpha \in uZ$. We show the existence of $x \in X$ satisfying $u(z,x) > \alpha$. By assumption, there is $y \in Y$ such that $u(z,y) > \alpha$. By Lemma 1.16, $uZ = uY$ and hence, $\alpha \in uY$. Again by assumption, there is $x \in X$ such that $u(y,x) > \alpha$. It follows that $u(z,x) \geq \min\{u(z,y),u(y,x)\} > \alpha$, as desired. The converse follows directly from the definition. □

As to the converses of a) and b), it follows from the definitions that if (X,u) is spherically closed in (Z,u), then (X,u) is spherically closed in (Y,u), and that if $(X,u) \subseteq (Z,u)$ is immediate, then so is $(X,u) \subseteq (Y,u)$. But the respective properties do not necessarily follow for $(Y,u) \subseteq (Z,u)$ (cf. Exercise **??**).

Lemma 1.18 *If $(X,u) \subseteq (Y,u)$ is immediate and (Y,u) is homogeneous, then $uY = uX$.*

Proof: For $y \neq y' \in Y$ we have to show that $u(y,y') \in uX$. Take $x \in X$. Since (Y,u) is assumed to be homogeneous, there is $y'' \in Y$ such that $u(x,y'') = u(y,y')$. Since $(X,u) \subseteq (Y,u)$ is immediate, there is $x' \in X$ such that $u(y'',x') > u(x,y'')$. Consequently, $u(y,y') = u(x,y'') = \min\{u(x,y''),u(y'',x')\} = u(x,x') \in uX$. □

An ultrametric space will be called **maximal** if it does not admit proper immediate extensions.

Lemma 1.19 *If (X,u) is spherically complete then it is spherically closed in every ultrametric space extension, and it is maximal.*

Proof: If (X,u) is spherically complete then every nest of balls has a non-empty intersection and thus, (X,u) is spherically closed in every ultrametric space extension. Now the second assertion follows from Lemma 1.15 and the fact that "optimal approximation property" and "immediate extension" are mutually exclusive. □

Suppose that (X,u) is not spherically complete. Does there exist an extension in which (X,u) is not spherically closed? Let $\mathbf{B} \subseteq \mathbf{B}(X)$ be a nest of balls in (X,u) having an empty intersection. Take $y \notin X$. We define an extension of u to $Y := X \cup \{y\}$ as follows. First, we set $u(y,y) = \infty$. Given $x \in X$, there is some non-empty ball in $\mathbf{B}$ which does not contain x. Choose some x' in that ball and set $u(x,y) = u(y,x) := u(x',x)$. This is welldefined: suppose x'' is in some other ball which does not contain x; we know that one of the balls is contained in the other, so x' and x'' are contained in a common ball which does not contain x, that is, $u(x'',x') > u(x,x')$, which yields $u(x,x') = u(x,x'')$. By our definition, for $x \neq y$ we have $u(y,x) \neq \infty$; this shows that (Y,u) satisfies (UM 0). It also satisfies symmetry by definition. It remains to show (UMT). The only non-trivial case is where precisely one of the appearing elements is our new element y. Note that it suffices to show $u(x,z) \geq \min\{u(x,y),u(y,z)\}$ since this implies that $u(x,y) \geq \min\{u(x,z),u(z,y)\}$. So take x' and z' such that by our definition, $u(y,x) = u(x',x)$ and $u(y,z) = u(z',z)$. Then there is a ball containing x' and z' such that x or z does not lie in this ball, that is, $u(x',z') > u(x',x)$ or $u(x',z') > u(z',z)$. In both cases, we obtain $u(x,z) \geq \min\{u(x,x'),u(x',z'),u(z',z)\} = \min\{u(x,x'),u(z',z)\} = \min\{u(x,y),u(y,z)\}$.

We have now proved that $(X,u) \subseteq (Y,u)$ is an extension of ultrametric spaces. Let us show that $u(y,X)$ does not admit a maximum. Take $x \in X$ and x' in a ball B' so that $u(y,x) = u(x',x)$ by our definition. Now let x'' be an element in a ball B'' not containing x' so that we have $u(y,x') = u(x'',x')$. Since one of the balls contains the other and since

$x' \notin B''$, we conclude that $x'' \in B'$. Since $x \notin B'$, we have $u(x', x) < u(x'', x')$ showing that $u(y, x) < u(y, x')$. Since $x \in X$ was arbitrary, this proves our claim. Since $Y \setminus X$ consists only of the element y, we have even proved that $(X, u) \subseteq (Y, u)$ is an immediate extension. By our construction and the foregoing lemma, we have now proved:

Theorem 1.20 *An ultrametric space is spherically complete if and only if it is spherically closed in every ultrametric space extension, and this is the case if and only if it is maximal.*

A closed subspace of a spherically complete ultrametric space is not necessarily again spherically complete (therefore, we have introduced the stronger notion "spherically closed"). Indeed, take a spherically complete ultrametric space (X, u) with a value set that is not discretely ordered, and such that for every $x \in X$ and $\alpha \in uX$ there is $y \in X$ with $u(x, y) = \alpha$ (such ultrametric spaces appear for example as the underlying ultrametric spaces of power series fields with their canonical valuations). Choose some $x \in X$ and $\alpha \in uX$ without immediate predecessor. As $B_\alpha(x)$ is open, $X' = X \setminus B_\alpha(x)$ is a closed subspace of X. It is not spherically complete, as the intersection of the non-empty balls $B_\beta(x) \setminus B_\alpha(x)$ in X' for $\beta < \alpha$ is empty.

The easy proof of the following lemma is left to the reader as an exercise:

Lemma 1.21 *Every ball in an ultrametric space is a spherically closed subspace. Hence if the ultrametric space is spherically complete, then all of its balls are spherically complete subspaces.*

Exercise 1.4 *Show that every extension of a spherically complete ultrametric space by finitely many points is spherically complete again.*

Exercise 1.5 *Construct an example of an immediate extension $(X, u) \subset (Y, u)$ of ultrametric spaces such that $uY \neq uX$.*

1.10 The ultrametric main theorem

Let (Y, u) and (Y', u') be non-empty ultrametric spaces and $f : Y \to Y'$ a map. For $y \in Y$, we will write fy instead of $f(y)$. An element $z' \in Y'$ is called **attractor for** f if for every $y \in Y$ such that $z' \neq fy$, there is an element $z \in Y$ which satisfies:

(AT1) $u'(fz, z') > u'(fy, z')$,
(AT2) $f(B(y, z)) \subseteq B(fy, z')$.

Condition (AT1) says that the approximation fy of z' from within the image of f can be improved, and condition (AT2) says that this can be done in a somewhat continuous way.

Theorem 1.22 (Attractor Theorem) *Assume that $z' \in Y'$ is an attractor for $f : Y \to Y'$ and that (Y, u) is spherically complete. Then $z' \in f(Y)$.*

For the proof of Theorem 1.22, we show the following more precise statement:

Lemma 1.23 *Assume that $z' \in Y'$ is an attractor for $f : Y \to Y'$ and that (Y, u) is spherically complete. Then for every $y \in Y$ there is $z_0 \in Y$ such that $fz_0 = z'$ and $f(B(y, z_0)) \subseteq B(fy, z')$.*

Proof: If $z' = fy$ then we set $z_0 = y$ and there is nothing to show. So assume that $z' \neq fy$. Then by assumption on z' there is $z \in Y$ such that (AT1) and (AT2) hold. Take elements $y_i, z_i \in B(y,z)$, $i \in I$, such that the balls $B(y_i,z_i)$ form a nest inside of $B(y,z)$, maximal with the following properties, for all i:

i) $z' = fy_i = fz_i$ or $u'(z',fz_i) > u'(z',fy_i)$,
ii) $f(B(y_i,z_i)) \subseteq B(fy_i,z')$,
iii) for all $j \in I$, $u(y_i,z_i) < u(y_j,z_j)$ implies that $u'(fy_i,z') < u'(fy_j,z')$.

Non-empty nests with these properties exist. Indeed, the singleton $\{B(y,z)\}$ is such a nest. Maximal nests with these properties exist by Zorn's Lemma. Take one such maximal nest. As soon as we find $z_0 \in B(y,z)$ such that $z' = fz_0$ we are done because $f(B(y,z_0)) \subseteq f(B(y,z)) \subseteq B(fy,z')$.

Assume first that this nest has a minimal ball, say, $B(y_0,z_0)$. If $z' = fz_0$ then we are done. So assume that $z' \neq fz_0$, and set $\tilde{y} := z_0$. Then by assumption on z', we can find $\tilde{z} \in Y$ such that

$$u'(f\tilde{z},z') > u'(f\tilde{y},z') \quad \text{and} \quad f(B(\tilde{y},\tilde{z})) \subseteq B(f\tilde{y},z') .$$

We have that

$$u'(f\tilde{y},z') = u'(fz_0,z') > u'(fy_0,z') = u'(f\tilde{y},fy_0) , \tag{1.6}$$

where the last equality follows from the ultrametric triangle law. So we know that $fy_0 \notin B(f\tilde{y},z')$ and thus, $y_0 \notin B(\tilde{y},\tilde{z})$. This shows that $u(\tilde{y},\tilde{z}) > u(\tilde{y},y_0) = u(z_0,y_0)$, and since $\tilde{y} = z_0 \in B(z_0,y_0)$, it follows that $B(\tilde{z},\tilde{y}) \subsetneq B(z_0,y_0)$. So we can enlarge our nest of balls by adding $B(\tilde{z},\tilde{y})$, and conditions i) and ii) hold for the new nest. From iii) we see that $u'(fy_0,z')$ is maximal among the $u'(fy_i,z')$, $i \in I$; so (1.6) shows that also iii) holds for the new nest. But this contradicts the maximality of the chosen nest.

Now assume that the nest contains no smallest ball. Since (Y,u) is spherically complete by assumption, there is some $z_0 \in \bigcap_{i\in I} B(y_i,z_i)$. Suppose that $fz_0 \neq z'$. Then we set $\tilde{y} := z_0$. For all i, we have $\tilde{y} \in B(y_i,z_i)$ and $f\tilde{y} \in f(B(y_i,z_i)) \subseteq B(fy_i,z')$, showing that $u'(f\tilde{y},z') \geq u'(fy_i,z')$. We choose $\tilde{z}$ as before. We have $f(B(\tilde{y},\tilde{z})) \subseteq B(f\tilde{y},z') \subseteq B(fy_i,z')$ for all i. On the other hand, since the nest contains no smallest ball, the set $\{u(y_i,z_i) \mid i \in I\}$ has no maximal element. So iii) implies that also the set $\{u'(fy_i,z') \mid i \in I\}$ has no maximal element. Consequently, for all $i \in I$ there is $j \in I$ such that $u'(f\tilde{y},z') \geq u'(fy_j,z') > u'(fy_i,z')$. Consequently, $fy_i \notin B(f\tilde{y},z')$, which yields that $y_i \notin B(\tilde{y},\tilde{z})$. Therefore, $B(\tilde{y},\tilde{z}) \subsetneq B(y_i,z_i)$ and $u(\tilde{y},\tilde{z}) > u(y_i,z_i)$ for all i. So we can enlarge our nest of balls by adding $B(\tilde{y},\tilde{z})$, and conditions i), ii) and iii) hold for the new nest. This again contradicts the maximality of the chosen nest. Hence, $fz_0 = z'$ and we are done. □

The map f will be called **immediate** if every $z' \in Y'$ is an attractor for f. Hence by the Attractor Theorem, every immediate map from a spherically complete space is surjective. But we will show more, and for that we need the next corollary and another lemma.

Corollary 1.24 *Assume that $f : Y \to Y'$ is immediate and that (Y,u) is spherically complete. Then the following holds:*

(BB) *for every $y \in Y$ and every ball B' in Y' around fy, there is a ball B in Y around y such that $f(B) = B'$.*

Proof: Assume that $y \in Y$ and that B' is any ball in Y' which contains fy. Then we can write

$$B' = \bigcup_{z' \in B'} B(z', fy) .$$

According to the foregoing lemma, for every z' there is $z_0 \in Y$ such that $z' \in f(B(y, z_0)) \subseteq B(fy, z') \subseteq B'$. Take B to be the union over all such balls $B(y, z_0)$ when z' runs through all elements of B'. Then B is a ball around y satisfying $f(B) = B'$. □

Lemma 1.25 *Assume that $f : Y \to Y'$ is a map which satisfies (BB), and that (Y, u) is spherically complete. Then f is surjective, and (Y', u') is spherically complete.*

Proof: Taking $B' = Y'$, we obtain the surjectivity of f.

Now we take any nest of balls $\{B'_j \mid j \in J\}$ in Y'. We have to show that this nest has a non-empty intersection. We claim that in Y there exists a nest of balls B_i, $i \in I$, maximal with the property that

$$I \subseteq J, \text{ and for all } i \in I,\ f(B_i) = B'_i . \tag{1.7}$$

To show this, we first take any $j \in J$ and choose some $y_j \in Y$ such that $fy_j \in B'_j$, making use of the surjectivity of f. As f satisfies (BB), we can choose a ball B_j in Y around y_j and such that $f(B_j) = B'_j$. So the nest $\{B_j\}$ has property (1.7). Hence, a maximal nest $\{B_i \mid i \in I\}$ with property (1.7) exists by Zorn's Lemma.

We wish to show that the balls B'_i, $i \in I$, are coinitial in the nest B'_j, $j \in J$, that is, for every ball B'_j there is some $i \in I$ such that $B'_i \subseteq B'_j$. Once we have shown this we are done: as Y is spherically complete, there is some $y \in \bigcap_{i \in I} B_i$, and

$$fy \in \bigcap_{i \in I} f(B_i) = \bigcap_{i \in I} B'_i = \bigcap_{j \in J} B'_j$$

shows that $\bigcap_{j \in J} B'_j$ is non-empty.

Suppose the balls B'_i, $i \in I$, are not coinitial in the nest B'_j, $j \in J$. Then there is some $j \in J$ such that $B'_j \subsetneqq B'_i$ for all $i \in I$. Since Y is spherically complete, there is some $y \in \bigcap_{i \in I} B_i$. We have that $fy \in \bigcap_{i \in I} B'_i =: B'$, and also that $B'_j \subseteq B'$. By assumption, there is a ball B around y such that $f(B) = B'$. If B' happens to be the smallest ball among the B'_i, say, $B' = B'_{i_0}$ with $i_0 \in I$, then we just take $B = B_{i_0}$. If $B' \subsetneqq B'_i$, then it follows that $B \subsetneqq B_i$. Hence in all cases, $B \subseteq B_i$ for all i. Since $B'_j \subseteq B'$, we can choose $\tilde{y} \in B$ such that $f\tilde{y} \in B'_j$. By assumption, there is a ball B_j around $\tilde{y}$ such that $f(B_j) = B'_j$. Since $\tilde{y} \in B_i$ for all $i \in I$, we know that B_i, $i \in I \cup \{j\}$ is a nest of balls. By construction, it has property (1.7). Since $j \notin I$, this contradicts our maximality assumption on I. This proves that the balls B'_i, $i \in I$, must be coinitial in the nest B'_j, $j \in J$. □

From the Attractor Theorem, Corollary 1.24 and the last lemma, we obtain:

Theorem 1.26 (Ultrametric Main Theorem) *Assume that $f : Y \to Y'$ is immediate and that (Y, u) is spherically complete. Then f is surjective and (Y', u') is spherically complete. Moreover, for every $y \in Y$ and every ball B' in Y' containing fy, there is a ball B in Y containing y and such that $f(B) = B'$.*

Compared to the Ultrametric Fixed Point Theorem, the Attractor Theorem and the Ultrametric Main Theorem have the advantage that they can be applied to situations where a natural contracting map is not readily at hand.

If f is just the embedding of an ultrametric subspace Y in an ultrametric space Y', then (AT2) will automatically hold. Hence, Theorem 1.26 reproves the following assertion, which was already shown in Lemma 1.19: *If* (Y', u) *is an immediate extension of* (Y, u) *and* (Y, u) *is spherically complete, then* $Y = Y'$.

Exercise 1.6 *Construct a map of ultrametric spaces which is not immediate, but continuous with respect to the topology given by the basic open sets* $B_\alpha(x)$, $\infty \neq \alpha$, $x \in X$. *(Hint: Take a suitable space* (X, u) *with value set* $uX = \omega + 1$. *Let* $\gamma = \{\alpha \mid \alpha < \omega\}$, *the set of finite ordinals, and consider the space* $(X, u)/\sim_\gamma$. *Construct some map* $\overline{f}$ *on this space which is not spherically continuous, and lift it to a continuous map* f *on* (X, u).*)*

1.11 Approximation types

When considering extensions of ultrametric spaces, the question arises: when are two extensions of the same ultrametric space (X, u) isomorphic over X? The simplest answer to this question is found in the case where the extensions contain just one new element each, say, y and y' respectively. In this case, an isomorphism over X can only send y to y'. The answer is as simple as the question: it is necessary and sufficient that $u(y, x) = u(y', x)$ for all $x \in X$. If $u(X \cup \{y\}) = uX = u(X \cup \{y'\})$, our answer is immediately seen to be correct since it contains all information that has to be checked for an isomorphism of the ultrametric spaces $(X \cup \{y\}, u)$ and $(X \cup \{y'\}, u)$. If $u(X \cup \{y\}) \neq uX$, then we also have to construct the isomorphism $\rho : u(X \cup \{y\}) \to u(X \cup \{y'\})$ over uX. But $u(X \cup \{y\}) \setminus uX$ can only contain one element $u(y, x_0)$ (cf. Exercise **??**), and if we understand the assertion "$u(y, x_0) = u(y', x_0)$" to say also that $u(y', x_0) \notin uX$, and that both $u(y, x_0)$ and $u(y', x_0)$ induce the same cut in uX, then Lemma 1.3 gives the required isomorphism of the value sets, and we are done.

It is less easy to determine how the information about the values can be encoded in the ultrametric space (X, u) itself, without using the symbols y and y'. We are looking for a structure induced by y in (X, u) which, if equal to that induced by y', gives us the isomorphism we have asked for. For instance, for ordered sets the adequate structure is the cut induced by an element of an arbitrary extension, as Lemma 1.3 shows. A second role of such intrinsic structures is that they may tell us what sort of extensions the ultrametric would admit. For example, if there are nests of balls with empty intersection, then the space admits proper immediate extensions, and vice versa, according to Theorem 1.20. But nests of balls in (X, u) will in general not suffice to describe extensions which are not immediate:

Example 1.27 Let $(X \cup \{y\}, u)$ be such that there is some $x_0 \in X$ with $u(y, x_0) = \max_{x \in X} u(y, x)$ and $\alpha := u(y, x_0) \in uX$. On the other hand, let $(X \cup \{y'\}, u)$ be such that $u(y', x_0) = \max_{x \in X} u(y', x)$ with $u(y', x_0) > \alpha$, but $u(y', x_0) < \beta$ for every $\beta \in uX$, $\beta > \alpha$. Then for all $\beta \in uX$, the balls $B_\beta(X \cup \{y\}, y) \cap X$ and $B_\beta(X \cup \{y'\}, y') \cap X$ are equal. Indeed, for $\beta > \alpha$, they are both empty, and for $\beta \leq \alpha$, they are both equal to $B_\beta(X, x_0)$. On the other hand, $B^\circ_\alpha(X \cup \{y\}, y) \cap X = \emptyset$ whereas $B^\circ_\alpha(X \cup \{y'\}, y') \cap X = B^\circ_\alpha(X, x_0)$.

This example shows that the required structure should also pay attention to the o-balls. Further, the value $u(y', x_0) \notin uX$ induces in uX the cut whose initial segment is

$\{\beta \mid \beta \leq \alpha\}$, which is the set of radii β for which the balls $B_\beta(X \cup \{y'\}, y') \cap X$ are non-empty. $\diamondsuit$

We will now introduce approximation types, which constitute the required structure for dealing with immediate extensions of ultrametric spaces. Approximation types will play an important role in our investigation of immediate extensions of valued fields, groups and modules. In the following, let (X, u) be an ultrametric space with value set uX. Recall that we always exclude ∞ from uX, and that we write $uX\infty$ for $uX \cup \{\infty\}$.

An **approximation type over** (X, u) is a full nest of balls in (X, u), that is, a nest $\mathbf{A} = \{B_\alpha(x_\alpha) \mid \alpha \in S\}$ with S an initial segment of $uX\infty$ and x_α elements of X; hence, $S = \Lambda^L(\mathbf{A})$. We write $\mathbf{A}_\alpha = B_\alpha(x_\alpha)$ for $\alpha \in \Lambda^L(\mathbf{A})$, and $\mathbf{A}_\alpha = \emptyset$ otherwise. An approximation type will be called **immediate** if its intersection is empty.

Every nest of balls determines uniquely an approximation type which is obtained by filling up the nest as described in Section 1.6. If the nest has empty intersection, the corresponding approximation type will be immediate.

Take any extension $(X, u) \subseteq (Y, u)$, and $y \in Y$. For all $\alpha \in uX\infty$, we set

$$\text{at}\,(y, X)_\alpha \;:=\; \{x \in X \mid u(y, x) \geq \alpha\} \;=\; B_\alpha(Y, y) \cap X\;, \tag{1.8}$$

$$\text{at}\,(y, X)^\circ_\alpha \;:=\; \{x \in X \mid u(y, x) > \alpha\} \;=\; B^\circ_\alpha(Y, y) \cap X\;. \tag{1.9}$$

By Lemma 1.14, $\text{at}\,(y, X)_\alpha$ is empty or a c-ball of radius α, and $\text{at}\,(y, X)^\circ_\alpha$ is empty or an o-ball of radius α. If $\text{at}\,(y, X)_\alpha \neq \emptyset$ and $\beta < \alpha$, then also $\text{at}\,(y, X)_\beta \neq \emptyset$. This shows that

$$\Lambda^L(y, X) \;:=\; \{\alpha \in uX\infty \mid \text{at}\,(y, X)_\alpha \neq \emptyset\} \tag{1.10}$$

is an initial segment of uX and therefore,

$$\text{at}\,(y, X) \;:=\; \{\text{at}\,(y, X)_\alpha \mid \alpha \in \Lambda^L(y, X)\} \tag{1.11}$$

is an approximation type over (X, u). We call it the **approximation type of** y **over** (X, u). If we have to make clear which ultrametric we refer to, we will write "$\text{at}_{\,u}$" (resp. "$\text{at}_{\,v}$" if we are dealing with the ultrametric induced by a valuation v).

Let us compare $\Lambda^L(y, X)$ with $u(y, X)$:

Lemma 1.28 *We have that $u(y, X) \cap uX\infty \subseteq \Lambda^L(y, X)$, and if $u(y, X) \not\subseteq uX\infty$, then $u(y, X) \setminus uX = \{\gamma\}$ with $\gamma > \Lambda^L(y, X)$. If in addition (X, u) is homogeneous, then $u(y, X) \cap uX = \Lambda^L(y, X)$, hence if $u(y, X) \subseteq uX\infty$, then $u(y, X) = \Lambda^L(y, X)$, and if $u(y, X) \not\subseteq uX\infty$, then $u(y, X) = \Lambda^L(y, X) \cup \{\gamma\}$ with $\Lambda^L(y, X) < \gamma \notin uX$.*

Proof: Take $x \in X$ such that $u(y, x) \in uX$. Then $x \in \text{at}\,(y, X)_{u(y,x)}$ and thus $u(y, x) \in \Lambda^L(y, X)$. On the other hand, if $x \in X$ such that $\gamma := u(y, x) \notin uX$, then for each $x' \in X$, $u(y, x') = \min\{u(y, x), u(x, x')\}$ since $u(y, x) \neq u(x, x')$. Hence if $u(y, x') \in uX\infty$, then $u(y, x') < \gamma$, and if $u(y, x') \notin uX$, then $u(y, x') = \gamma$.

Now assume that (X, u) is homogeneous. It suffices to show that $\Lambda^L(y, X) \subseteq u(y, X)$. Take $\alpha \in \Lambda^L(y, X)$ and $x \in X$ such that $u(y, x) \geq \alpha$. If $\alpha = u(y, x) \in u(y, X)$, then we are done. Suppose that $\alpha < u(y, x)$. Since (Y, u) is homogeneous, there is $x' \in X$ such that $u(x, x') = \alpha$ and thus, $\alpha = u(x, x') = \min\{u(y, x), u(x, x')\} = u(y, x') \in u(y, X)$. $\square$

The proof of the following lemma is straightforward:

Lemma 1.29 *Let $(X,u) \subseteq (Y,u) \subseteq (Z,u)$ be extensions of ultrametric spaces and $z \in Z$. Then $\Lambda^L(z,X) \subseteq \Lambda^L(z,Y)$ and $u(z,X) \subseteq \Lambda^L(z,Y)$.*

Which elements in an ultrametric space extension of (X,u) have equal approximation types over (X,u)? The next lemma answers this question:

Lemma 1.30 *Let $(X,u) \subseteq (Y,u)$ be an extension of ultrametric spaces and $y,y' \in Y$.*

a) For every $\alpha \in \Lambda^L(y,X)$, $\mathrm{at}\,(y,X)_\alpha = \mathrm{at}\,(y',X)_\alpha$ holds if and only if $u(y,y') \geq \alpha$.

b) Further,

$$\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X) \iff u(y,y') \geq \Lambda^L(y,X) = \Lambda^L(y',X)\ .$$

Proof: a): Let $\alpha \in uX\infty$. If $u(y,y') \geq \alpha$, then $B_\alpha(Y,y) = B_\alpha(Y,y')$, which yields that $\mathrm{at}\,(y,X)_\alpha = B_\alpha(Y,y) \cap X = B_\alpha(Y,y') \cap X = \mathrm{at}\,(y',X)_\alpha$. If $u(y,y') < \alpha$, then $B_\alpha(Y,y) \cap B_\alpha(Y,y') = \emptyset$, whence $\mathrm{at}\,(y,X)_\alpha \cap \mathrm{at}\,(y',X)_\alpha = \emptyset$; for $\alpha \in \Lambda^L(y,X)$, this yields that $\mathrm{at}\,(y,X)_\alpha \neq \mathrm{at}\,(y',X)_\alpha$ since then $\mathrm{at}\,(y,X)_\alpha$ is non-empty.

b): If $\Lambda^L(y,X) \neq \Lambda^L(y',X)$, then $\mathrm{at}\,(y,X) \neq \mathrm{at}\,(y',X)$. If $u(y,y') \geq \Lambda^L(y,X)$ does not hold, then there is some $\alpha \in \Lambda^L(y,X)$ such that $\alpha > u(y,y')$. By part a), it follows that $\mathrm{at}\,(y,X)_\alpha \neq \mathrm{at}\,(y',X)_\alpha$. This proves implication "$\Rightarrow$".

If $u(y,y') \geq \Lambda^L(y,X)$ holds, then $u(y,y') \geq \alpha$ for all $\alpha \in \Lambda^L(y,X)$. Again by part a), it follows that $\mathrm{at}\,(y,X)_\alpha = \mathrm{at}\,(y',X)_\alpha$ for all $\alpha \in \Lambda^L(y,X) = \Lambda^L(y',X)$, that is, $\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X)$. □

If $\mathbf{A}$ is an approximation type over (X,u) and there exists an element $y \in Y$ such that $\mathbf{A} = \mathrm{at}\,(y,X)$, then we say that y **realizes A** (in (Y,u)). If $\mathbf{A}$ is realized by some $x \in X$, then $\mathbf{A}$ will be called **trivial**. We leave the easy proof of the following lemma to the reader as an exercise.

Lemma 1.31 *Let $\mathbf{A}$ be an approximation type over (X,u) and $(X,u) \subseteq (Y,u)$ an extension of ultrametric spaces.*

a) The approximation type $\mathbf{A}$ is trivial if and only if $\mathbf{A}_\infty \neq \emptyset$. In this case, the element x realizing $\mathbf{A}$ in X is the only element that realizes $\mathbf{A}$ in Y.

b) The element $y \in Y$ realizes $\mathbf{A}$ if and only if

1) for all $\alpha \in \Lambda^L(\mathbf{A})$, $u(y,x) \geq \alpha$ for some $x \in \mathbf{A}_\alpha$,

2) for all $\alpha > \Lambda^L(\mathbf{A})$, $u(y,x) < \alpha$ for all $x \in X$.

Note that since $\mathbf{A}_\alpha$ is a c-ball of radius α, "$u(y,x) \geq \alpha$ for some $x \in \mathbf{A}_\alpha$" implies "$u(y,x) \geq \alpha \Leftrightarrow x \in \mathbf{A}_\alpha$".

1.12 Immediate approximation types

In this section we will consider the important special case of immediate approximation types. We note that immediate approximation types $\mathbf{A}$ contain no smallest c-balls as their intersection is empty, hence $\Lambda^L(\mathbf{A})$ has no largest element, and $\Lambda^L(\mathbf{A}) \subseteq uX$. The following is a corollary to Lemma 1.31:

Corollary 1.32 *Let* $\mathbf{A}$ *be an immediate approximation type over* (X,u) *and* $(X,u) \subseteq (Y,u)$ *an extension of ultrametric spaces. Then* $y \in Y$ *realizes* $\mathbf{A}$ *if and only if for all* $\alpha \in \Lambda^L(\mathbf{A})$*, we have that* $u(y,x) \geq \alpha$ *for all* $x \in \mathbf{A}_\alpha$*.*

Proof: We have to show that for an immediate approximation type $\mathbf{A}$ over (X,u), condition 1) of Lemma 1.31 implies condition 2). Take $\alpha > \Lambda^L(\mathbf{A})$ and $x \in X$. Then by the definiton of "immediate" there is some $\beta \in \Lambda^L(\mathbf{A})$ such that $x \notin \mathbf{A}_\beta$. Hence by 1) and the remark following Lemma 1.31, $u(y,x) < \beta < \alpha$. □

As a corollary to Lemma 1.30, we have:

Corollary 1.33 *Let* $(X,u) \subseteq (Y,u)$ *be an extension of ultrametric spaces and* $y, y' \in Y$*. If* $\mathrm{at}\,(y,X)$ *is immediate, then*

$$\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X) \iff u(y,y') > \Lambda^L(y,X) \iff u(y,y') \geq \Lambda^L(y,X)\,.$$

Proof: Since $\Lambda^L(y,X) = \Lambda^L(\mathrm{at}\,(y,X))$ has no largest element, $u(y,y') > \Lambda^L(y,X)$ if and only if $u(y,y') \geq \Lambda^L(y,X)$. It remains to show that $u(y,y') > \Lambda^L(y,X)$ implies that $\Lambda^L(y,X) = \Lambda^L(y',X)$. Take $\alpha \in \Lambda^L(y,X)$ and $x \in X$ such that $u(y,x) \geq \alpha$. With $u(y,y') > \alpha$ it follows that $u(y',x) \geq \alpha$, hence $\alpha \in \Lambda^L(y',X)$. Now take $\alpha \in \Lambda^L(y',X)$ and $x \in X$ such that $u(y',x) \geq \alpha$, and suppose that $\alpha \notin \Lambda^L(y,X)$. Then $\alpha > \Lambda^L(y,X)$, and with $u(y,y') > \Lambda^L(y,X)$ we find that $u(y,x) > \Lambda^L(y,X)$. Hence for every $\beta \in \Lambda^L(y,X)$, $u(y,x) > \beta$ and thus, $x \in \mathrm{at}\,(y,X)_\beta$. This shows that the intersection of all balls in $\mathrm{at}\,(y,X)$ is non-empty, contradicting our assumption that it is an immediate approximation type. Therefore, $\alpha \in \Lambda^L(y,X)$. □

The following corollary deals with the behaviour of immediate approximation types under ultrametric space extensions.

Corollary 1.34 *Let* $(X,u) \subseteq (Y,u) \subseteq (Z,u)$ *be extensions of ultrametric spaces and* $z \in Z$ *with non-trivial immediate approximation type* $\mathrm{at}\,(z,X)$*. If* $\mathrm{at}\,(z,Y)$ *is not immediate, then there exists* $y \in Y$ *such that* $\mathrm{at}\,(z,X) = \mathrm{at}\,(y,X)$*.*

Proof: If $\mathrm{at}\,(z,Y)$ is not immediate, then by definition there is some $y \in Y$ contained in the intersection of all balls in $\mathrm{at}\,(z,Y)$. Hence, $u(z,y) \geq \alpha$ for all $\alpha \in \Lambda^L(z,Y)$. In view of Lemma 1.29, this also holds for all $\alpha \in \Lambda^L(z,X)$, that is, $u(z,y) \geq \Lambda^L(z,X)$. From Corollary 1.33 we can now infer that $\mathrm{at}\,(y',X) = \mathrm{at}\,(y,X)$. □

Corollary 1.35 *Let* $(X,u) \subseteq (Y,u)$ *be an extension of ultrametric spaces and* $y, y' \in Y$*. If* $\mathrm{at}\,(y,X)$ *is immediate, then there exists* $\alpha \in \Lambda^L(y,X)$ *such that* $u(y',y) \geq u(y',x)$ *for all* $x \in \mathrm{at}\,(y,X)_\alpha$*.*

Proof: If $u(y,y') \geq \Lambda^L(y,X)$ then $\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X)$ by Corollary 1.33. Then it follows that $u(y,y') \geq u(y,x) = u(y',x)$ for all $x \in X$.

Now assume that $u(y,y') \geq \Lambda^L(y,X)$ does not hold. Then there is some $\alpha \in \Lambda^L(y,X)$ such that $u(y',y) < \alpha$. Hence, $u(y',x) = \min\{u(y',y), u(y,x)\} = u(y',y)$ for all $x \in \mathrm{at}\,(y,X)_\alpha$. □

The following lemma shows the motivation for the name "immediate approximation type":

Lemma 1.36 *Let $(X,u) \subseteq (Y,u)$ be an extension of ultrametric spaces and $y \in Y \setminus X$. Then* $\mathrm{at}\,(y,X)$ *is immediate if and only if for every $x \in X$ there is some $x' \in X$ such that $u(y,x') > u(y,x)$. The extension $(X,u) \subseteq (Y,u)$ is immediate if and only if for every $y \in Y \setminus X$, the approximation type* $\mathrm{at}\,(y,X)$ *is immediate.*

Proof: Suppose that $\mathrm{at}\,(y,X)$ is immediate and that x is an arbitrary element of X. By definition of "immediate", there is some $\alpha \in \Lambda^L(y,X)$ such that $x \notin \mathrm{at}\,(y,X)_\alpha$. Choose some $x' \in \mathrm{at}\,(y,X)_\alpha$. By Lemma 1.31, we obtain that $u(y,x) < \alpha \leq u(y,x')$.

Now let $y \in Y \setminus X$ and suppose that for arbitrary $x \in X$, there is $x' \in X$ such that $u(y,x') > u(y,x)$. Then there is also some $x'' \in X$ such that $u(y,x'') > u(y,x')$. By (UM$\neq$) we obtain that $u(x',x) = u(y,x) < u(y,x') = u(x'',x')$. Hence by Lemma 1.31, $u(x'',x') \in \Lambda^L(y,X)$ and $x \notin \mathrm{at}\,(y,X)_{u(x'',x')}$. As $x \in X$ was arbitrary, this shows that $\mathrm{at}\,(y,X)$ is immediate.

The second assertion follows from the first and the definition of "immediate extension". □

We will now come back to the question that has led to the introduction of approximation types. We will show how immediate approximation types can serve to establish isomorphisms of immediate extensions of ultrametric spaces. In the course of this book, this will gain even more importance when isomorphisms of richer ultrametric structures like valued groups and fields can be traced back to approximation types.

Lemma 1.37 *Let (Y,u) and (Y',u') be extensions of the ultrametric space (X,u). Further, let $y \in Y$ and $y' \in Y'$. If* $\mathrm{at}\,(y,X)$ *is immediate, then there is an isomorphism $(X \cup \{y\},u) \to (X \cup \{y'\},u')$ over (X,u) if and only if* $\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X)$.

Proof: If there is such an isomorphism, then it has to send y to y', and $u(y,x) = u'(y',x)$ for all $x \in X$. This implies that $\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X)$.

For the converse, assume that $\mathrm{at}\,(y,X)$ is immediate and that $\mathrm{at}\,(y,X) = \mathrm{at}\,(y',X)$. Let x be an arbitrary element of X. As the intersection of the balls in $\mathrm{at}\,(y,X)$ is empty, there is some $\alpha \in \Lambda^L(y,X)$ such that $x \notin \mathrm{at}\,(y,X)_\alpha$ and hence also $x \notin \mathrm{at}\,(y',X)_\alpha$. Choose some $x' \in \mathrm{at}\,(y,X)_\alpha = \mathrm{at}\,(y',X)_\alpha$. We find that $u(y,x) < \alpha \leq u(y,x')$, hence $u(y,x) = \min\{u(y,x), u(y,x')\} = u(x,x')$. The same holds with y' in the place of y. Therefore, $u(y,x) = u(x,x') = u(y',x)$. This shows that $y \mapsto y'$ induces an isomorphism $(X \cup \{y\},u) \to (X \cup \{y'\},u')$ over (X,u). □

To conclude this section, we will show the connection between immediate approximation types and the notion "spherically complete" and "complete".

Lemma 1.38 *The ultrametric space (X,u) is spherically complete if and only if there are no immediate approximation types over (X,u).*

Proof: An immediate approximation type is a nest of balls with empty intersection, hence if there is one over (X,u), then (X,u) is not spherically complete. Conversely, if (X,u) is not spherically complete, then it admits a nest of balls with empty intersection; filling this ball up as described in Section 1.6 will yield an immediate approximation type over (X,u). □

What happens to immediate approximation types over (X, u) in a spherically complete extension of (X, u)? To answer this question, we need:

Lemma 1.39 *Let* $\mathbf{A}$ *be an immediate approximation type over* (X, u) *and* (Y, u) *an extension of* (X, u)*. If* $y \in Y$ *is such that* $y \in B_\alpha(Y, x_\alpha)$ *for* $x_\alpha \in \mathbf{A}_\alpha$ *and all* $\alpha \in \Lambda^L(\mathbf{A})$*, then* $\mathbf{A} = \mathrm{at}\,(y, X)$*.*

Proof: The assumption yields that for all $\alpha \in \Lambda^L(\mathbf{A})$, we have $B_\alpha(Y, x_\alpha) = B_\alpha(Y, y)$, whence $\mathbf{A}_\alpha = B_\alpha(Y, x_\alpha) \cap X = B_\alpha(Y, y) \cap X$ for all $\alpha \in \Lambda^L(\mathbf{A})$. Since already these balls determine the immediate approximation type $\mathbf{A}$ (cf. Corollary **??**), we have proved that $\mathbf{A} = \mathrm{at}\,(y, X)$. □

Lemma 1.40 *If the extension* (Y, u) *of* (X, u) *is spherically complete, then every immediate approximation type over* (X, u) *is realized by some element of* (Y, u)*.*

Proof: Let $\mathbf{A}$ be an immediate approximation type over (X, u). For every $\alpha \in \Lambda^L(\mathbf{A})$, let $x_\alpha \in \mathbf{A}_\alpha$. Then $B_\alpha(Y, x_\alpha)$ is a nest of balls in (Y, u), which by assumption must have a non-empty intersection. If $y \in Y$ lies in this intersetion, then it satisfies the assumption of the foregoing lemma, which shows that $\mathbf{A} = \mathrm{at}\,(y, X)$. □

1.13 Completions and completion types

Let us return to the notion of complete ultrametric spaces. If a given space (X, u) is not complete, the question arises whether there is an extension which is complete and which is as small as possible. We will call (Y, u) a **completion of** (X, u) if (Y, u) is complete and (X, u) is dense in (Y, u). We will now show that completions of (X, u) exist and are unique up to isomorphism over X. In view of the latter, we will talk of "the" completion of (X, u) and denote it by $(X, u)^c$ or (X^c, u).

Let us call an approximation type $\mathbf{A}$ over (X, u) a **completion type** if it is immediate and $\Lambda^L(\mathbf{A}) = uX$.

Lemma 1.41 *The ultrametric space* (X, u) *is complete if and only if there are no completion types over* (X, u)*.*

Proof: A completion type over (X, u) is a nest $\mathbf{A}$ of balls with empty intersection and $\Lambda^L(\mathbf{A})$, hence if there is one, then (X, u) is not complete. Conversely, if (X, u) is not complete, then it admits a completion nest with empty intersection; filling this ball up as described in Section 1.6 will yield an completion type over (X, u). □

Now take an ultrametric space (X, u) which is not complete. Then there exists a completion type over (X, u). We take X^c to be the set of all completion types and all trivial approximation types over (X, u). If $x \neq x' \in X$, then $\mathrm{at}\,(x, X) \neq \mathrm{at}\,(x', X)$. We can thus view X as a subset of X^c. We define an extension of the ultrametric u from X to

X^c as follows. Let $\mathbf{A}, \mathbf{A}' \in X^c$. If $\mathbf{A} = \mathbf{A}'$ then $u(\mathbf{A}, \mathbf{A}') := \infty$. If $\mathbf{A} \neq \mathbf{A}'$ then $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$ for some $\alpha \in uX$, and we set

$$u(\mathbf{A}, \mathbf{A}') = u(x_\alpha, x'_\alpha) \quad \text{for arbitrary } x_\alpha \in \mathbf{A}_\alpha\,,\ x'_\alpha \in \mathbf{A}'_\alpha\,. \tag{1.12}$$

Then $u(x_\alpha, x'_\alpha) < \alpha$ since otherwise, the balls would coincide by virtue of Lemma 1.7. Using this fact, the reader may show that our definition does not depend on the choice of α and the elements x_α, x'_α of the respective balls, and that the map so defined coincides with u on X. It evidently satisfies (UM 0) and (UMS). For the proof of (UMT), we note the following: if we have $u(\mathbf{A}, \mathbf{A}') = u(x_\alpha, x'_\alpha)$ by our above definition, then also $u(\mathbf{A}, \mathbf{A}') = u(x_\gamma, x'_\gamma)$ for every $\gamma \geq \alpha$ and $x_\gamma \in \mathbf{A}_\gamma$, $x'_\gamma \in \mathbf{A}'_\gamma$. So if we have to compare the u-distance between three completion types, then we can choose γ large enough as to represent these u-distances as u-distances between three elements $x_\gamma, x'_\gamma, x''_\gamma \in X$, and the ultrametric triangle law on X^c will follow from that on X.

As a direct consequence of our definition, we have $uX^c = uX$. Furthermore, let us observe the following: if $\mathbf{A}_\alpha = B_\alpha(x_\alpha)$ for all $\alpha \in uX$, then for arbitrary $\beta \in uX$, we can choose $\mathbf{A}' = \mathrm{at}\,(x_\beta, X)$ to obtain $u(\mathbf{A}, \mathbf{A}') > \beta$. Indeed, for $\alpha \leq \beta$, we have $\mathbf{A}_\alpha = B_\alpha(x_\beta) = \mathbf{A}'_\alpha$. Consequently, if α is chosen such that $\mathbf{A}_\alpha \neq \mathbf{A}'_\alpha$, then $\alpha > \beta$. For $x_\alpha \in \mathbf{A}_\alpha$, this yields that $x_\alpha \in \mathbf{A}_\beta(x_\beta)$, that is, $u(\mathbf{A}, \mathbf{A}') = u(x_\alpha, x_\beta) \geq \beta$. If equality holds, then we replace x_β by x_α and set $\mathbf{A}' = \mathrm{at}\,(x_\alpha, X)$. Then we obtain $u(\mathbf{A}, \mathbf{A}') > \beta$. We have proved:

$$\forall y \in X^c\, \forall \beta \in uX^c\, \exists x \in X:\ u(y, x) > \beta\,, \tag{1.13}$$

that is, (X, u) is dense in (Y, u).

We have not yet shown that (X^c, u) is complete. So let $\{B_\alpha(X^c, y_\alpha) \mid \alpha \in S\}$ be a nest of balls in (X^c, u), with S a cofinal subset of $uX = uX^c$. By (1.13), for every $\alpha \in S$ there is some $x_\alpha \in X$ such that $u(x_\alpha, y_\alpha) > \alpha$, whence $B_\alpha(X^c, y_\alpha) = B_\alpha(X^c, x_\alpha)$. So $B_\alpha(X^c, y_\alpha) \cap X = B_\alpha(X, x_\alpha) \neq \emptyset$ for every $\alpha \in uX$. It follows that $\{B_\alpha(X, x_\alpha) \mid \alpha \in S\}$ is a nest of balls in (X, u). Its associated immediate approximation type $\mathbf{A}$ is a completion type in (X, u) and hence an element of X^c; it is uniquely determined since by assumption, uX admits no maximal element. By construction, $\mathbf{A}$ satisfies $u(y_\alpha, \mathbf{A}) \geq \min\{u(y_\alpha, x_\alpha), u(x_\alpha, \mathbf{A})\} \geq \alpha$ for every α. That is, $\mathbf{A}$ is an element of $\bigcap\{B_\alpha(X^c, y_\alpha) \mid \alpha \in uX^c\}$. This proves that (X^c, u) is complete.

We have shown the existence of completions of (X, u). We will now work towards showing their uniqueness, up to isomorphism over X. The following lemma gives a useful characterization of density:

Lemma 1.42 *Let $(X, u) \subseteq (Y, u)$ be an extension of ultrametric spaces. Then (X, u) is dense in (Y, u) if and only if uX is cofinal in uY and for every $y \in Y \setminus X$, $\mathrm{at}\,(y, X)$ is a completion type. In particular, if (X, u) is complete and dense in (Y, u), then $X = Y$.*

Proof: Suppose that uX is cofinal in uY and that for every $y \in Y$, $\mathrm{at}\,(y, X)$ is a completion type. Then for every $\alpha \in uY$ there is some $\beta \in uX$ such that $\alpha \leq \beta$. On the other hand, for every $\beta \in uX$ the ball $\mathrm{at}\,(y, X)_\beta$ is non-empty. As the intersection of all balls in $\mathrm{at}\,(y, X)$ is empty, there must be some $\gamma \in uX$ such that $\gamma > \beta$ and $\mathrm{at}\,(y, X)_\gamma$ is non-empty. That is, there is some $x \in X$ such that $u(y, x) \geq \gamma > \beta \geq \alpha$. This proves that (X, u) is dense in (Y, u).

For the converse, suppose that (X, u) is dense in (Y, u). We have already shown in Lemma 1.16 that this implies that $(X, u) \subseteq (Y, u)$ is immediate and $uX = uY$. From

Lemma 1.36 we infer that at (y, X) is immediate for every $y \in Y$. Take any $\alpha \in uX$. Choose $x' \in X$ such that $u(y, x') > \alpha$. Then either $y = x' \in X$, or $u(y, x') \in uY$. In the latter case, choose $x \in X$ such that $u(y, x) > u(y, x')$. Then by (UM=), $u(y, x') = u(x', x) \in uX$, showing that $\alpha < u(x', x) \in \Lambda^L(y, X)$. We have proved that $\Lambda^L(y, X)$ is cofinal in uX, hence at (y, X) is a completion type.

Assume that (X, u) is dense in (Y, u) and that (X, u) is complete. Then by Lemma 1.41, there are no completion types over (X, u). Hence by what we have already shown, there is no $y \in Y \setminus X$. □

How many different elements can realize a completion type?

Lemma 1.43 *Let $(X, u) \subseteq (Y, u)$ be an extension of ultrametric spaces such that uX is cofinal in uY. Then every completion type over (X, u) is realized by at most one element of (Y, u).*

Proof: Let $\mathbf{A}$ be a completion type over (X, u) which is realized by the elements $y, y' \in Y$. That is, at $(y, X) = \mathbf{A} =$ at (y', X). Since every completion type is an immediate approximation type, it follows from Lemma 1.33 that $u(y, y') > \Lambda^L(y, X) = uX$. Since uX is cofinal in uY by assumption, it follows that $u(y, y') = \infty$, that is, $y = y'$. □

Using this lemma in conjunction with Lemma 1.40, we obtain:

Lemma 1.44 *If $(X, u) \subseteq (Y, u)$ with uX cofinal in uY, and if (Y, u) is complete, then every completion type over (X, u) is realized by a unique element of (Y, u).*

Lemma 1.45 *If (X, u) is dense in (Y, u), then (Y, u) admits a unique embedding in (X^c, u) over X. If in addition (Y, u) is complete, then the embedding is onto.*

Proof: Assume that (X, u) is dense in (Y, u). By the standard argument using Zorn's Lemma, there is a maximal subspace $(X_1, u) \subseteq (Y, u)$ containing (X, u) which admits a unique embedding in $(X, u)^c$ over (X, u). We can identify (X_1, u) with its image in (X^c, u). Since (X, u) is dense in (X^c, u), we then know from Lemma 1.17 that (X_1, u) is dense in (X^c, u). If $(X_1, u) \neq (Y, u)$, then there is some $y \in Y \setminus X_1$. By Lemma 1.42, at (y, X_1) is a completion type over (X_1, u). By the foregoing lemma (which we can apply since we know from Lemma 1.42 that $uX_1 = uX^c$), it is realized by a unique element $y' \in X^c$. Now Lemma 1.37 shows that there is a unique embedding of $(X_1 \cup \{y\}, u)$ in (X^c, u) over X_1, contrary to our maximality assumption for (X_1, u). Hence $X_1 = Y$, showing that (Y, u) can be embedded in (X^c, u) over (X, u). If in addition (Y, u) is complete, then the same holds for its image in (X^c, u), and since this is dense in (X^c, u) by , it follows from Lemma 1.42 that the embedding is onto X^c. □

Now the proof of the following theorem is easy, and we leave it to the reader:

Theorem 1.46 *Every ultrametric space (X, u) admits a completion. Between every two completions of (X, u) there is a unique isomorphism over X. If (Z, u) is any completion of (X, u) and (X, u) is dense in (Y, u), then (Y, u) admits a unique embedding in (Z, u) over X.*

1.14 Pseudo Cauchy sequences

Pseudo Cauchy sequences have played an important role in the development of valuation theory. We will thus introduce them here, although we prefer to work with approximation types since they express more information. However, there are some situations where it is more convenient to work with pseudo Cauchy sequences. For instance, certain maps may send pseudo Cauchy sequences into pseudo Cauchy sequences while it is not easy to describe how the approximation types are transformed. Another example is the task of constructing immediate extensions with certain properties. In this case, pseudo Cauchy sequences can represent a natural approach.

In what follows, let (X,u) always be an ultrametric space. Take a sequence $(x_\nu)_{\nu<\lambda}$ of elements in X, indexed by ordinals $\nu<\lambda$ where λ is a limit ordinal. It is called a **pseudo Cauchy sequence** if

(PCS) $\quad u(x_\rho,x_\sigma) < u(x_\sigma,x_\tau)$ whenever $\rho<\sigma<\tau<\lambda$.

More generally, the sequence is **ultimately a pseudo Cauchy sequence** if there is some $\nu_0<\lambda$ such that the condition in (PCS) holds whenever $\nu_0\leq\rho<\sigma<\tau<\lambda$. A sequence that is ultimately a pseudo Cauchy sequence can be made into a pseudo Cauchy sequence by deleting sufficiently many initial members. Hence the results we will prove for pseudo Cauchy sequences hold in a corresponding form also for such sequences.

Let us state some properties of pseudo Cauchy sequences.

Lemma 1.47 *Let $(x_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence in (X,u). Then*

$$u(x_\mu,x_\nu)=u(x_\mu,x_{\mu+1}) \ \ \textit{whenever } \mu<\nu<\lambda \tag{1.14}$$

and

$$u(x_\mu,x_\nu)<u(x_{\mu'},x_{\nu'}) \ \ \textit{whenever } \mu<\nu<\lambda \textit{ and } \nu<\mu'<\nu'<\lambda\,. \tag{1.15}$$

Further, if $y\in X$, then either

$$u(y,x_\mu)<u(y,x_\nu) \ \ \textit{whenever } \mu<\nu<\lambda\,, \tag{1.16}$$

or there is $\mu_0<\lambda$ such that

$$u(y,x_\nu)=u(y,x_{\mu_0}) \ \textit{whenever } \mu_0\leq\nu<\lambda\,.$$

Property (1.16) is equivalent to

$$u(y,x_\nu)=u(x_\nu,x_{\nu+1}) \ \ \textit{for all } \nu<\lambda\,. \tag{1.17}$$

Proof: Assume that $(x_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence and $\nu<\mu<\lambda$. By definition, $u(x_\mu,x_{\mu+1})<u(x_{\mu+1},x_\nu)$. Hence, $u(x_\mu,x_\nu)=\min\{u(x_\mu,x_{\mu+1}),u(x_{\mu+1},x_\nu)\}=u(x_\mu,x_{\mu+1})$ by (UM$\neq$). Having proved this, we take $\mu<\nu<\lambda$ and $\mu<\mu'<\nu'<\lambda$, hence $\mu+1<\nu'$, and we compute $u(x_{\mu'},x_{\nu'})\geq\min\{u(x_{\mu'},x_{\mu+1}),u(x_{\mu+1},x_{\nu'})\}=u(x_{\mu+1},x_{\nu'})=u(x_{\mu+1},x_{\mu+2})>u(x_\mu,x_{\mu+1})=u(x_\mu,x_\nu)$.

Now take $y\in X$ and suppose that (1.16) does not hold. Then there are μ_0,μ' such that $\mu'<\mu_0<\lambda$ and $u(y,x_{\mu_0})\leq u(y,x_{\mu'})$. Assume $\mu_0<\nu<\lambda$ and $u(y,x_\nu)\neq u(y,x_{\mu_0})$. Then $u(x_{\mu'},x_{\mu_0})\geq\min\{u(y,x_{\mu'}),u(y,x_{\mu_0})\}=u(y,x_{\mu_0})\geq\min\{u(y,x_{\mu_0}),u(y,x_\nu)\}=u(x_{\mu_0},x_\nu)$, where the last equality holds by (UM$\neq$). But this is a contradiction to assertion (1.15), which we have already proved above.

Suppose that y satisfies (1.16). Then $u(y, x_\nu) < u(y, x_{\nu+1})$ and thus, $u(x_\nu, x_{\nu+1}) = \min\{u(y, x_\nu), u(y, x_{\nu+1})\} = u(y, x_\nu)$ for all $\nu < \lambda$. Hence, y satisfies (1.17). Conversely, (1.16) follows from (1.17) and assertion (1.15). □

In the following, take $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ to be any pseudo Cauchy sequence. We will say that an assertion about its members x_ν holds **ultimately** if there is some μ_0, $\mu_0 < \lambda$, so that it holds for all x_ν with $\mu_0 \leq \nu < \lambda$. The foregoing lemma says that for every $y \in X$, the sequence $(u(y, x_\nu))_{\nu<\lambda}$ is either strictly increasing or ultimately constant.

For $\nu < \lambda$, we set

$$\gamma_\nu := u(x_\nu, x_{\nu+1}) \; .$$

Then (1.15) tells us that $(\gamma_\nu)_{\nu<\lambda}$ is a strictly increasing sequence in uX. Hence if uX admits no infinite strictly increasing sequences, then (X, u) admits no pseudo Cauchy sequences. By (1.14), we have that $\gamma_\mu = u(x_\mu, x_\nu)$ whenever $\mu < \nu < \lambda$.

Note that if (Y, u) is an extension of (X, u) and $\mathbf{S}$ is a pseudo Cauchy sequence in (X, u), then it is also a pseudo Cauchy sequence in (Y, u). An element $y \in Y$ is called a **pseudo limit** (or just **limit**) **of S** if it satisfies (1.16), or equivalently, (1.17). Since $u(y, x_{\nu+1}) \geq \gamma_{\nu+1} > \gamma_\nu$ implies that $u(y, x_\nu) = \min\{\gamma_\nu, u(y, x_{\nu+1})\} = \gamma_\nu$, both conditions are equivalent to

$$u(y, x_\nu) \geq \gamma_\nu \;\text{ for all } \nu < \lambda \; .$$

Assume that $y \in X$ is not a limit of $\mathbf{S}$. Then by the foregoing lemma, there is μ_0 such that $\mu_0 < \lambda$ and $u(y, x_\nu) = u(y, x_{\mu_0})$ whenever $\mu_0 \leq \nu < \lambda$. It follows that $\gamma_\nu > \gamma_{\mu_0} = u(x_{\mu_0}, x_{\mu_0+1}) \geq \min\{u(y, x_{\mu_0}), u(y, x_{\mu_0+1})\} = u(y, x_{\mu_0}) = u(y, x_\nu)$ whenever $\mu_0 < \nu < \lambda$. Conversely, if there is ν such that $\nu < \lambda$ and $\gamma_\nu > u(y, x_\nu)$, then in view of (1.17), y can not be a limit of $\mathbf{S}$. We have proved:

Lemma 1.48 *For an element $y \in X$ and a pseudo Cauchy sequence $\mathbf{S} = (x_\nu)_{\nu<\lambda}$, the following assertions are equivalent:*

1) $y \in X$ is not a limit of $\mathbf{S}$,

2) $u(y, x_\nu)$ is ultimately constant,

3) $u(y, x_\nu) < \gamma_\nu$ holds ultimately,

4) $u(y, x_\nu) < \gamma_\nu$ holds for some $\nu < \lambda$.

A pseudo Cauchy sequence may admit more than one limit. The following lemma describes the set of all limits.

Lemma 1.49 *Let $y \in X$ be a limit of $\mathbf{S}$. Then $z \in X$ is also a limit of $\mathbf{S}$ if and only if*

$$u(y, z) > \gamma_\nu \;\text{ for all } \nu < \lambda \; .$$

Proof: Since y is a limit of $\mathbf{S}$, we have $u(y, x_\nu) = \gamma_\nu$ for $\nu < \lambda$. If $u(y, z) > \gamma_\nu$, then $u(z, x_\nu) = \min\{u(y, z), u(y, x_\nu)\} = \min\{u(y, z), \gamma_\nu\} = \gamma_\nu$ for $\nu < \lambda$, showing that also z is a limit of $\mathbf{S}$. Conversely, suppose that z is a limit of $\mathbf{S}$, that is, $u(z, x_\nu) = \gamma_\nu$ for $\nu < \lambda$. Then $u(y, z) \geq \min\{u(y, x_{\nu+1}), u(x_{\nu+1}, z)\} \geq \gamma_{\nu+1} > \gamma_\nu$ for $\nu < \lambda$. □

The least initial segment of uX containing all γ_ν for $\nu < \lambda$ will be called the **support segment of S** and denoted by $\Lambda^L(\mathbf{S})$. Note that $\Lambda^L(\mathbf{S})$ does not have a largest element.

Now the assertion of the foregoing lemma can be expressed as follows. If y is a limit of $\mathbf{S}$, then z is a limit of $\mathbf{S}$ if and only if

$$u(y,z) \;\geq\; \Lambda^L(\mathbf{S}) \;,$$

or equivalently,

$$u(y,z) \;>\; \Lambda^L(\mathbf{S}) \;.$$

If $\Lambda^L(\mathbf{S}) = uX$, then this means that $u(y,z) = \infty$, that is, $y = z$. Consequently,

Lemma 1.50 *Assume that* $\mathbf{S}$ *has a limit in* (X,u)*. If* $\Lambda^L(\mathbf{S}) = uX$*, then the limit is unique. The converse holds if* (X,u) *is homogeneous.*

If $\Lambda^L(\mathbf{S}) = uX$, then the pseudo Cauchy sequence $\mathbf{S}$ is called a **Cauchy sequence**. Hence by the foregoing lemma, *every Cauchy sequence admits at most one limit.* The non-uniqueness of limits in the case of $\Lambda^L(\mathbf{S}) \neq uX$ is the reason for the name *pseudo* Cauchy sequence.

We shall now describe the relation between pseudo Cauchy sequences, nests of balls and approximation types. Let $\mathbf{B} \subseteq \mathbf{B}(X)$ be a nest of balls in (X,u) without a smallest ball. We want to associate a pseudo Cauchy sequence $\mathbf{S_B}$ to $\mathbf{B}$. Choose some ball $B_0 \in \mathbf{B}$; as this is not the smallest ball in $\mathbf{B}$, we can choose some $x_0 \in B_0 \setminus \bigcap \mathbf{B}$. Suppose that $\mu < \lambda$ and that for all $\nu < \mu$ we have already chosen $B_\nu \in \mathbf{B}$ and $x_\nu \in B_\nu \setminus \bigcap \mathbf{B}$ such that $x_{\nu'} \notin B_\nu$ for $\nu' < \nu$.

First consider the case of $\mu = \mu' + 1$ a successor ordinal. Since $\bigcap \mathbf{B} = \emptyset$, there is a ball $B_\mu \in \mathbf{B}$ which does not contain $x_{\mu'}$. Since the balls B_ν are linearly ordered by inclusion, this together with our induction hypothesis yields that $x_{\nu'} \notin B_\mu$ for $\nu' < \mu$.

Now consider the case of μ a limit ordinal. If the balls B_ν are coinitial in $\mathbf{B}$ (with respect to inclusion), then we set $\lambda := \mu$, and our construction is finished. Otherwise, $\mathbf{B}$ contains a ball B_μ which is properly contained in B_ν for every $\nu < \mu$. For every $\nu < \mu$, $x_\nu \notin B_{\nu+1}$ by induction hypothesis, and hence $x_\nu \notin B_\mu$.

In both cases, as B_μ is not the smallest ball in $\mathbf{B}$ we can choose some $x_\mu \in B_\mu \setminus \bigcap \mathbf{B}$. Our induction will stop at some ordinal μ because all of them are bounded by the coinitiality type of the set $\mathbf{B}$, ordered by inclusion. This ordinal μ must be a limit ordinal because $\mathbf{B}$ contains no smallest ball.

The sequence $\mathbf{S_B} := (x_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence. Indeed, if $\rho < \sigma < \tau < \lambda$, then $x_\rho \notin B_\sigma \ni x_\tau$ by construction, and hence $u(x_\rho, x_\sigma) < u(x_\sigma, x_\tau)$.

We claim that $\Lambda^L(\mathbf{B}) = \Lambda^L(\mathbf{S_B})$. Take $\alpha \in \Lambda^L(\mathbf{B})$. Then there is a ball $B_\beta(z) \in \mathbf{B}$ such that $\alpha \leq \beta$. Since this is not the smallest ball in $\mathbf{B}$, there is a ball B_ν in our above construction which is prperly contained in $B_\beta(z)$. Then $\alpha \leq \beta < u(x_\nu, x_{\nu+1}) \in \Lambda^L(\mathbf{S_B})$, so $\alpha \in \Lambda^L(\mathbf{S_B})$. For the converse, take $\gamma \in \Lambda^L(\mathbf{S_B})$. So there is some $\nu < \lambda$ such that $\gamma \leq u(x_\nu, x_{\nu+1})$. Since $x_\nu \notin B_{\nu+1}$ and $B_{\nu+1}$ is contained in some $B_\beta(z) \in \mathbf{B}$ by our construction, we have that $\gamma \leq u(x_\nu, x_{\nu+1}) < \beta$ and hence $\gamma \in \Lambda^L(\mathbf{B})$.

Let (Y,u) be an extension of (X,u). Suppose that $y \in Y$ is a limit of $\mathbf{S_B}$. Then for all $\nu < \lambda$, $u(y, x_\nu) = u(x_\nu, x_{\nu+1})$. Take any $B_\beta(z) \in \mathbf{B}$. Since the sequence $(B_\nu)_{\nu<\lambda}$ is coinitial in $\mathbf{B}$, there is some ν such that $B_\nu \subseteq B_\beta(z)$. As $x_\nu, x_{\nu+1} \in B_\nu \subseteq B_\beta(Y,z)$, we obtain that $y \in B_\beta(Y,z)$. Hence $y \in \bigcap\{B_\beta(Y,z) \mid B_\beta(z) \in \mathbf{B}\}$. For the converse, assume that the latter holds, and take $\nu < \lambda$. Write $B_{\nu+1} = B_\beta(z)$. Then $y, x_{\nu+1} \in B_\beta(Y,z) \not\ni x_\nu$, whence $u(y, x_{\nu+1}) > u(x_\nu, x_{\nu+1})$. This yields that $u(y, x_\nu) = \min\{u(y, x_{\nu+1}), u(x_{\nu+1}, x_\nu)\} = u(x_\nu, x_{\nu+1})$ for all $\nu < \lambda$, showing that y is a limit of $\mathbf{S_B}$.

Now assume that $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence. As before, we let $\gamma_\nu = u(x_\nu, x_{\nu+1})$. Then $\mathbf{B_S} := \{B_{\gamma_\nu}(x_\nu) \mid \nu_0 < \nu < \lambda\}$ is a nest of balls because every ball $B_{\gamma_\nu}(x_\nu)$ contains all balls $B_{\gamma_\mu}(x_\mu)$ for $\nu < \mu < \lambda$. If $\mathbf{S}$ has no limit in X, then $\bigcap \mathbf{B} = \emptyset$; indeed, an element $x \in \bigcap \mathbf{B}$ would satisfy $u(x_\nu, x) \geq \gamma_\nu$ for $\nu_0 < \nu < \lambda$ and would thus be a limit of $\mathbf{S}$. Our above construction applied to $\mathbf{B_S}$ will (essentially) give back $\mathbf{S}$. We have now proved:

Lemma 1.51 *Let (X, u) be an ultrametric space. Then for every nest of balls $\mathbf{B} \subseteq \mathbf{B}(X)$ in (X, u) without a smallest ball there is a pseudo Cauchy sequence $\mathbf{S_B}$ such that $\Lambda^L(\mathbf{B}) = \Lambda^L(\mathbf{S_B})$ and that for every extension $(X, u) \subseteq (Y, u)$ of ultrametric spaces, an element $y \in Y$ lies in $\bigcap\{B_\alpha(Y, b_\alpha) \mid B_\alpha(x_\alpha) \in \mathbf{B}\}$ if and only if it is a limit of $\mathbf{S_B}$. In particular, $\bigcap \mathbf{B} = \emptyset$ if and only if $\mathbf{S_B}$ does not have a limit in X. Further, $\mathbf{B}$ is a completion nest if and only if $\mathbf{S_B}$ is a Cauchy sequence.*

Conversely, for every pseudo Cauchy sequence $\mathbf{S}$ there exists a nest $\mathbf{B_S}$ of balls in (X, u) with $\bigcap \mathbf{B_S} = \emptyset$, such that the relation between $\mathbf{S}$ and $\mathbf{B_S}$ is the same as between $\mathbf{S_B}$ and $\mathbf{B}$ that we have described above.

From this lemma together with Lemma **??**, we obtain an analogous lemma which describes the relation between approximation types and pseudo Cauchy sequences. The proof is an easy exercise which we leave to the reader.

Lemma 1.52 *For every immediate approximation type $\mathbf{A}$ in (X, u), there is a pseudo Cauchy sequence $\mathbf{S_A}$ in (X, u) without a limit in X such that $\Lambda^L(\mathbf{A}) = \Lambda^L(\mathbf{S_A})$ and that for every extension $(X, u) \subseteq (Y, u)$ of ultrametric spaces, an element $y \in Y$ realizes $\mathbf{A}$ if and only if it is a limit of $\mathbf{S_A}$. Further, $\mathbf{A}$ is a completion type if and only if $\mathbf{S_A}$ is a Cauchy sequence.*

Conversely, for every pseudo Cauchy sequence $\mathbf{S}$ in (X, u) without a limit in X there exists a unique immediate approximation type $\mathbf{A_S}$ in (X, u) such that the relation between $\mathbf{S}$ and $\mathbf{A_S}$ is the same as between $\mathbf{S_A}$ and $\mathbf{A}$ that we have described above.

We obtain the following characterization of spherically complete ultrametric spaces which is analogous to Lemma 1.38:

Corollary 1.53 *The ultrametric space (X, u) is spherically complete if and only if every pseudo Cauchy sequence in (X, u) has a limit in X. Further, (X, u) is complete if and only if every Cauchy sequence in (X, u) has a limit in X.*

Exercise 1.7 *Let $\mathbf{S} = (x_\nu)_{\nu<\lambda}$ be a pseudo Cauchy sequence. Show that y is a limit of $\mathbf{S}$ if and only if $u(y, x_\nu) < u(y, x_{\nu+1})$ for all $\nu < \lambda$.*

Exercise 1.8 *Assume that every pseudo Cauchy sequence in (X, u) admits a limit in some extension (Y, u) of (X, u) (which is true; cf. Lemma 20.85). Prove: if $\mathbf{B}$ is a nest of balls and $\mathbf{S}$ is a pseudo Cauchy sequence such that for every extension (Y, u) of (X, u), the intersection $\bigcap\{B_\alpha(Y, x_\alpha) \mid B_\alpha(x_\alpha) \in \mathbf{B}\}$ is precisely the set of all limits of $\mathbf{S}$ in Y, then it follows that $\Lambda^L(\mathbf{B}) = \Lambda^L(\mathbf{S})$.*