ALGEBRAIC INDEPENDENCE OF ELEMENTS IN IMMEDIATE EXTENSIONS OF VALUED FIELDS

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Abstract. Refining a constructive combinatorial method due to MacLane and Schilling, we give several criteria for a valued field that guarantee that all of its maximal immediate extensions have infinite transcendence degree. If the value group of the field has countable cofinality, then these criteria give the same information for the completions of the field. The criteria have applications to the classification of valuations on rational function fields. We also apply the criteria to the question which extensions of a maximal valued field, algebraic or of finite transcendence degree, are again maximal. In the case of valued fields of infinite p-degree, we obtain the worst possible examples of nonuniqueness of maximal immediate extensions: fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree.

1. Introduction

In this paper, we denote a valued field by (K, v), its value group by vK, and its residue field by Kv. When we talk of a valued field extension $(L|K, v)$ we mean that (L, v) is a valued field, $L|K$ a field extension, and K is endowed with the restriction of v. For the basic facts about valued fields, we refer the reader to [2, 3, 9, 14, 16, 17].

One of the basic problems in valuation theory is the description of the possible extensions of a valuation from a valued field (K, v) to a given extension field L. The case of an algebraic extension $L|K$ is taken care of by ramification theory.

Another important case is given when $L|K$ is an algebraic function field. Valuations on algebraic function fields appear naturally in algebraic geometry, real algebraic geometry and the model theory of valued fields, to name only a few areas. Local uniformization, the local form of resolution of singularities, is essentially a property of valued algebraic function fields (cf. [6]). This problem, which is still open in positive characteristic, requires a detailed knowledge of all possible valuations on such function fields. The same is true for corresponding problems in the model theory of valued fields.

By means of ramification theory, the problem of describing all appearing valuations is reduced to the case of rational function fields. The case of a single variable attracted many authors; see the references in [7] for a selection from the extensive literature on this case. The case of higher transcendence degree was treated

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in [7]. What at first glance appeared to be problem easily solvable by induction, turned out to tightly connected with the intricate question whether the maximal immediate extensions of a given valued field have finite or infinite transcendence degree.

An extension $(L|K, v)$ of valued fields is called **immediate** if the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto. It was shown by W. Krull [11] that maximal immediate extensions exist for every valued field. The proof uses Zorn's Lemma in combination with an upper bound for the cardinality of valued fields with prescribed value group and residue field. Krull's deduction of this upper bound is hard to read; later, K. A. H. Gravett [4] gave a nice and simple proof.

A valued field (K, v) is called **henselian** if it satisfies Hensel's Lemma, or equivalently, if the extension of its valuation v to its algebraic closure, which we will denote by $\tilde{K}$, is unique. A **henselization** of (K, v) is an extension which is henselian and minimal in the sense that it can be embedded over K, as a valued field, in every other henselian extension field of (K, v). Therefore, a henselization of (K, v) can be found in every henselian extension field, and henselizations are unique up to valuation preserving isomorphism over K (this is why we will speak of *the* henselization of (K, v). The henselization is an immediate separable-algebraic extension.

A valued field is called **maximal** if it does not admit any proper immediate extension; clearly, maximal immediate extensions are maximal fields. I. Kaplansky ([5]) characterized the maximal field as those in which every pseudo Cauchy sequence admits a pseudo limit. From this result it follows that power series fields are maximal fields. For example, for any field k the Laurent series field $k((t))$ with the t-adic valuation is a maximal immediate extension of $k(t)$, and it is well known that $k((t))$ is of infinite transcendence degree over $k(t)$. This can be shown by a cardinality argument (and some facts about field extensions in case k is not countable). A constructive proof was given by MacLane and Schilling in Section 3 of [12]. Our main theorem is a far-reaching generalization of their result. A part of this theorem has already been applied in [7] to the problem described above.

Theorem 1.1. *Take a valued field extension $(L|K, v)$ of finite transcendence degree ≥ 0, with v nontrivial on L. Assume that one of the following four cases holds:*

valuation-transcendental case: vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;

value-algebraic case: vL/vK contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv;

residue-algebraic case: Lv contains elements of arbitrarily high degree over Kv;

separable-algebraic case: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ is infinite.

Then each maximal immediate extension of (L, v) has infinite transcendence degree over L. If the cofinality of vL is countable (which for instance is the case if vL contains an element γ such that $\gamma > vK$), then already the completion of (L, v) has infinite transcendence degree over L.

Note that the cofinality of vL is equal to the cofinality of vK unless there is some γ in vL which is larger than every element of vK. In that case, because $L|K$

has finite transcendence degree, vL will have countable cofinality, no matter what the cofinality of vK is.

Note further that the condition in the residue-algebraic case always holds when $Lv|Kv$ contains an infinite separable-algebraic subextension; this is a consequence of the Theorem of the Primitive Element. There is no analogue of this theorem in abelian groups; therefore, the first condition in the value-algebraic case does not follow from the second. As an example, take q to be a prime different from $\operatorname{char} Kv$ and consider the case where vL/vK is an infinite product of $\mathbb{Z}/q\mathbb{Z}$. Under the second condition, however, the result can easily be deduced from the separable-algebraic case by passing to a henselization L^h of L and using Hensel's Lemma to show that $L^h|K$ admits the required subextension. For the details, see the proof of Theorem 1.1 in Section 3.

The key assumption in the separable-algebraic case is that the separable-algebraic subextension remains infinite when passing to the respective henselizations. We show that this condition is crucial. Take a valued field (k, v) which has a transcendental maximal immediate extension (M, v). We know that (M, v) is henselian (cf. Lemma 2.1). Take a transcendence basis $\mathcal{T}$ of $M|k$ and set $K := k(\mathcal{T})$. Then from Lemma 2.3 it follows that the henselization K^h of K inside of (M, v) is an infinite separable-algebraic subextension of $(M|K, v)$. But M is a maximal immediate extension of $L := K^h$ and $M|L$ is algebraic. Hence the assertion of Theorem 1.1 does not necessarily hold without the condition that $L_0^h|K^h$ is infinite.

An interesting special case is given when (K, v) is itself a maximal field. In this case, it is well known that if $(L|K, v)$ is a finite extension, then (L, v) is itself a maximal field. So we ask what happens if $(L|K, v)$ is infinite algebraic or transcendental of finite transcendence degree. Under which conditions could (L, v) be again a maximal field? This question will be addressed in Section 4, where all of the following theorems will be proved.

Theorem 1.2. *Take a maximal field (K, v) and an infinite algebraic extension $(L|K, v)$. Assume that $L|K$ contains an infinite separable subextension or that*

$$(vK : pvK)[Kv : Kv^p] < \infty\,, \tag{1}$$

where p is the characteristic exponent of Kv. Then every maximal immediate extension of (L, v) has infinite transcendence degree over L.

As an immediate consequence, we obtain:

Corollary 1.3. *Take a maximal field (K, v) of characteristic 0 and an algebraic extension $(L|K, v)$. Then (L, v) is maximal if and only if $L|K$ is a finite extension.*

It remains to discuss the case where $L|K$ is an infinite extension, its maximal separable subextension $K'|K$ is finite, and condition (1) fails. Since then also (K', v) is maximal, we can replace K by K' and simply concentrate on the case where $L|K$ is purely inseparable.

Note that if the maximal field K is of characteristic p, then condition (1) implies that the p-degree of K is finite, as it is equal to $(vK : pvK)[Kv : Kv^p]$. If condition (1) does not hold, then the purely inseparable extension $K^{1/p}|K$ is infinite; since $vK^{1/p} = \frac{1}{p}vK$ and $K^{1/p}v = (Kv)^{1/p}$, we then have that $vK^{1/p}/vK$ is of exponent p, every element in $K^{1/p}v \setminus Kv$ has degree p over Kv, and at least one of the two extensions is infinite. Since $(K^{1/p}, v)$ is again maximal (regardless of the p-degree

of K, see Lemma 4.1), this case shows that the assertion of Theorem 1.1 may fail even when vL/vK is an infinite torsion group or $Lv|Kv$ is an infinite algebraic extension. In fact, all possible cases can appear for infinite p-degree:

Theorem 1.4. *Take a maximal field (K,v) of characteristic $p>0$ for which condition (1) fails (which is equivalent to K having infinite p-degree). Take κ to be the maximum of $(vK:pvK)$ and $[Kv:Kv^p]$, considered as cardinals. Then:*

a) The valued field $(K^{1/p},v)$ is again maximal, although $vK^{1/p}/vK$ is an infinite torsion group or $K^{1/p}v|Kv$ is an infinite algebraic extension.

b) For every $n\in\mathbb{N}$ and every infinite cardinal $\lambda\leq\kappa$, there are subextensions $(L_n|K,v)$ and $(L_\lambda|K,v)$ of $(K^{1/p}|K,v)$ such that $(K^{1/p}|L_\lambda,v)$ is an immediate algebraic extension of degree λ and $(K^{1/p}|L_n,v)$ is an immediate algebraic extension of degree p^n.

c) There is a purely inseparable extension $(L|K,v)$ with

- *$vL=\frac{1}{p}vK$ and $Lv=Kv$ if $(vK:pvK)=\infty$,*
- *$vL=vK$ and $Lv=(Kv)^{1/p}$ if $[Kv:Kv^p]=\infty$,*

such that every maximal immediate extension of (L,v) has transcendence degree at least κ. In both cases, L can also be taken to simultaneously satisfy $vL=\frac{1}{p}vK$ and $Lv=(Kv)^{1/p}$.

If the cofinality of vK is countable, then in b), $K^{1/p}$ can be replaced by the completion of L_λ or L_n, respectively, and in c), "maximal immediate extension" can be replaced by "completion".

Case b) of this theorem is a generalization of Nagata's example ([13, Appendix, Example (E3.1), pp. 206-207]). Similar to that example, the valued fields in b) are nonmaximal fields admitting an algebraic maximal immediate extension. We note that the field L of part c) is *not* contained in $K^{1/p}$.

It was shown by Kaplansky that if a valued field satisfies "hypothesis A", then its maximal immediate extensions are unique up to isomorphism ([5, Theorem 5]; see also [10]). Kaplansky also gives an example for a valued field for which uniqueness fails ([5, Section 5]). The question whether uniqueness always fails when hypothesis A is violated is open. Different partial answers were given in [10] and in [15]. To the best knowledge of the authors, the next theorem presents, for the first time in the literature, the worst case of nonuniqueness:

Theorem 1.5. *Take a maximal field (K,v) of characteristic $p>0$ satisfying one of the following conditions:*

i) vK/pvK is infinite and vK admits a set of representatives of the cosets modulo pvK which contains an infinite bounded subset, or

ii) the residue field extension $Kv|(Kv)^p$ is infinite and the value group vK is not discrete.

Then there is an infinite purely inseparable extension (L,v) of (K,v) which admits $(K^{1/p},v)$ as an algebraic maximal immediate extension, but also admits a maximal immediate extension of infinite transcendence degree.

Let us mention that the condition of i) always holds when vK/pvK is infinite and vK is of finite rank, or $\Gamma/p\Gamma$ is infinite for some archimedean component Γ of vK. For example, if $\mathbb{F}_p$ denotes the field with p elements and G is an ordered subgroup

of the reals of the form $\bigoplus_{i\in\mathbb{N}} r_i\mathbb{Z}$, then the power series field $\mathbb{F}_p((G))$ satisfies the condition of i). If t_i , $i \in \mathbb{N}$, are algebraically independent over $\mathbb{F}_p$, then the power series field $\mathbb{F}_p(t_i \mid i \in \mathbb{N})((\mathbb{Q}))$ with residue field $\mathbb{F}_p(t_i \mid i \in \mathbb{N})$ satisfies the condition of ii).

Finally, let us discuss the case of transcendental extensions (L, v) of a maximal field (K, v). In view of the valuation-transcendental case of Theorem 1.1, it remains to consider the **valuation-algebraic case** where vL/vK is a torsion group and $Lv|Kv$ is algebraic. Here is a partial answer to our above question:

Theorem 1.6. *Take a maximal field (K, v) and a transcendental extension (L, v) of (K, v) of finite transcendence degree. Assume that $Lv|Kv$ is separable-algebraic and vL/vK is a torsion group such that the characteristic of Kv does not divide the orders of its elements. Then $Lv|Kv$ or vL/vK is infinite and every maximal immediate extension of (L, v) has infinite transcendence degree over L.*

We do not know an answer in the case where the conditions on the value group and residue field extensions fail.

2. Preliminaries

By va we denote the value of an element $a \in K$, and by av its residue. Given any subset S of K, we define

$$vS \;=\; \{va \mid 0 \neq a \in S\} \;\text{ and }\; Sv \;=\; \{av \mid a \in S, va \geq 0\}\,.$$

The valuation ring of (K, v) will be denoted by $\mathcal{O}_K$.

2.1. **The fundamental inequality.** Every finite extension $(L|K, v)$ of valued fields satisfies the **fundamental inequality** (cf. (17.5) of [2] or Theorem 19 on p. 55 of [17]):

$$[L : K] \;\geq\; (vL : vK)[Lv : Kv]\,. \tag{2}$$

The nature of the "missing factor" on the right hand side is determined by the **Lemma of Ostrowski** which says that whenever the extension of v from K to L is unique, then

$$[L : K] \;=\; p^\nu \cdot (vL : vK) \cdot [Lv : Kv] \quad \text{with } \nu \geq 0\,, \tag{3}$$

where p is the **characteristic exponent** of Lv, that is, $p = \operatorname{char} Lv$ if this is positive, and $p = 1$ otherwise. For the proof, see [14, Theoreme 2, p. 236]) or [17, Corollary to Theorem 25, p. 78]).

The factor $\mathrm{d} = \mathrm{d}(L|K, v) = p^\nu$ is called the **defect** of the extension $(L|K, v)$. If $\mathrm{d} = 1$, then we call $(L|K, v)$ a **defectless extension**. Note that $(L|K, v)$ is always defectless if $\operatorname{char} Kv = 0$.

We call a henselian field (K, v) a **defectless field** if equality holds in the fundamental inequality (2) for every finite extension L of K.

Theorem 2.1. *Every maximal field is henselian and a defectless field.*

Proof. The henselization of a valued field is an immediate extension. Therefore, a maximal field is equal to its henselization and thus henselian. For a proof that maximal fields are defectless fields, see [16, Theorem 31.21]. □

2.2. **Some facts about henselian fields and henselizations.** Let (K,v) be any valued field. If $a \in \tilde{K} \setminus K$ is not purely inseparable over K, we choose some extension of v from K to $\tilde{K}$ and define

$$\operatorname{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \operatorname{Gal}(\tilde{K}|K) \;\text{ and }\; \tau a \neq \sigma a\} \;\in\; v\tilde{K}$$

and call it the **Krasner constant of a over K**. Since all extensions of v from K to $\tilde{K}$ are conjugate, this does not depend on the choice of the particular extension of v. For the same reason, over a henselian field (K,v) our Krasner constant $\operatorname{kras}(a,K)$ is equal to

$$\max\{v(a - \sigma a) \mid \sigma \in \operatorname{Gal}(\tilde{K}|K) \;\text{ and }\; a \neq \sigma a\}\,.$$

Lemma 2.2. *Take an extension $(K(a)|K,v)$ of henselian fields, where a is an element in the separable-algebraic closure of K with $va \geq 0$. Then*

$$va \;\leq\; \operatorname{kras}(a,K)\,, \tag{4}$$

and for every polynomial $f = d_m X^m + \ldots + d_0 \in K[X]$ of degree $m < [K(a):K]$,

$$vf(a) \;\leq\; vd_m \,+\, m\operatorname{kras}(a,K)\,. \tag{5}$$

Proof. Since (K,v) is henselian, $v\sigma a = a$ and therefore $v(a-\sigma a) \geq va$ for all σ. This yields inequality (4).

Take any element b in the separable-algebraic closure of K with $[K(b):K] < [K(a):K]$. Then $v(a-b) \leq \operatorname{kras}(a,K)$ since otherwise, Krasner's Lemma would yield that $a \in K(b)$ and $[K(b):K] \geq [K(a):K]$. If we write $f(X) = d_m \prod_{i=1}^m (X - b_i)$, then $[K(b_i):K] \leq \deg(f) < [K(a):K]$. Hence,

$$vf(a) \;=\; vd_m \,+\, \sum_{i=1}^m v(a-b_i) \;\leq\; vd_m \,+\, m\operatorname{kras}(a,K)\,.$$

This proves inequality (5). □

Lemma 2.3. *Take a nontrivially valued field $(k(\mathcal{T}),v)$, where $\mathcal{T}$ is a nonempty set of elements algebraically independent over k. Then the henselization of $(k(\mathcal{T}),v)$ inside of any henselian valued extension field is an infinite extension of $k(\mathcal{T})$.*

Proof. Set $F := k(\mathcal{T})$ and take a henselization F^h of F inside of some henselian valued extension field. Pick an arbitrary $t \in \mathcal{T}$. Without loss of generality we can assume that $vt > 0$. By Hensel's Lemma, F^h contains a root ϑ_1 of the polynomial $X^2 - X - t$ such that $v\vartheta_1 > 0$. We proceed by induction. Once we have constructed ϑ_i with $v\vartheta_i > 0$ for some $i \in \mathbb{N}$, we again use Hensel's Lemma to obtain a root $\vartheta_{i+1} \in F^h$ of the polynomial $X^2 - X - \vartheta_i$ with $v\vartheta_{i+1} > 0$.

It now suffices to show that the extension $F(\vartheta_i \,|\, i \in \mathbb{N})|F$ is infinite. To this end, we consider the t^{-1}-adic valuation w on $F = k(\mathcal{T} \setminus \{t\})(t^{-1})$ which is trivial on $k(\mathcal{T} \setminus \{t\})$. We note that $wF = \mathbb{Z}$. Since $wt < 0$, we obtain that $w\vartheta_1 = \frac{1}{2}wt$ and by induction, $w\vartheta_i = \frac{1}{2^i}wt$. Therefore, the 2-divisible hull of $\mathbb{Z}$ is contained in $wF(\vartheta_i \,|\, i \in \mathbb{N})$. In view of the fundamental inequality (2), this shows that $F(\vartheta_i \,|\, i \in \mathbb{N})|F$ cannot be a finite extension. □

Lemma 2.4. *Assume (L,v) to be henselian and K to be relatively separable-algebraically closed in L. Then Kv is relatively separable-algebraically closed in Lv. If in addition $Lv|Kv$ is algebraic, then the torsion subgroup of vL/vK is a p-group, where p is the characteristic exponent of Kv.*

Proof. Take $\zeta \in Lv$ separable-algebraic over Kv. Choose a monic polynomial $g(X) \in K[X]$ whose reduction $gv(X) \in Kv[X]$ modulo v is the minimal polynomial of ζ over Kv. Then ζ is a simple root of gv. Hence by Hensel's Lemma, there is a root $a \in L$ of g whose residue is ζ. As all roots of gv are distinct, we can lift them all to distinct roots of g. Thus, a is separable-algebraic over K. From the assumption of the lemma, it follows that $a \in K$, showing that $\zeta \in Kv$. This proves that Kv is relatively separable-algebraically closed in Lv.

Now assume in addition that $Lv|Kv$ is algebraic. Then Kv is relatively separable-algebraically closed in Lv, by what we have proved already. Take $\alpha \in vL$ and $n \in \mathbb{N}$ not divisible by p such that $n\alpha \in vK$. Choose $a \in L$ and $b \in K$ such that $va = \alpha$ and $vb = n\alpha$. Then $v(a^n/b) = 0$. Since $Lv|Kv$ is a purely inseparable extension, there exists $m \in \mathbb{N}$ such that $((a^n/b)v)^{p^m} \in Kv$. We choose $c \in K$ satisfying $vc = 0$ and $cv = ((a^n/b)v)^{p^m}$, to obtain that $(a^{np^m}/cb^{p^m})v = 1$. So the reduction of the polynomial $X^n - a^{np^m}/cb^{p^m}$ modulo v is $X^n - 1$. Since n is not divisible by p, 1 is a simple root of this polynomial. Hence by Hensel's Lemma, there is a simple root $d \in L$ of the polynomial $X^n - a^{np^m}/cb^{p^m}$ with $dv = 1$, whence $vd = 0$. Consequently, a^{p^m}/d is a simple root of the polynomial $X^n - cb^{p^m}$ and thus is separable algebraic over K. Since K was assumed to be relatively separable-algebraically closed in L, we find that $a^{p^m}/d \in K$. As n is not divisible by p, there are $k, l \in \mathbb{Z}$ such that $1 = kn + lp^m$. This yields:

$$\alpha \;=\; kn\alpha + lp^m\alpha \;=\; kn\alpha + l(p^m va - vd) \;=\; k(n\alpha) + lv\left(\frac{a^{p^m}}{d}\right) \in vK.$$

□

2.3. **Valuation independence.** For the easy proof of the following lemma, see [1, chapter VI, §10.3, Theorem 1] .

Lemma 2.5. *Let $(L|K, v)$ be an extension of valued fields. Take elements $x_i, y_j \in L$, $i \in I$, $j \in J$, such that the values vx_i , $i \in I$, are rationally independent over vK, and the residues y_jv, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.*

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k vc_k \;+\; \sum_{i\in I} \mu_{k,i} vx_i \;. \tag{6}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z}vx_i \\ K(x_i, y_j \mid i \in I, j \in J)v &= Kv\,(y_jv \mid j \in J)\;. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues y_jv.

Conversely, if (K,v) is any valued field and we assign to the vx_i any values in an ordered group extension of vK which are rationally independent, then (6) defines a valuation on L, and the residues y_jv, $j \in J$, are algebraically independent over Kv.

Corollary 2.6. *Let $(L|K,v)$ be an extension of valued fields. Then*

$$\operatorname{trdeg} L|K \;\geq\; \operatorname{trdeg} Lv|Kv \;+\; \operatorname{rr}(vL/vK)\;. \tag{7}$$

If in addition $L|K$ is a function field and if equality holds in (7), then the extensions $vL|vK$ and $Lv|Kv$ are finitely generated.

Proof. Choose elements $x_1,\ldots,x_\rho,y_1,\ldots,y_\tau \in L$ such that the values $vx_1,\ldots,vx_\rho$ are rationally independent over vK and the residues $y_1v,\ldots,y_\tau v$ are algebraically independent over Kv. Then by the foregoing lemma, $\rho+\tau \leq \operatorname{trdeg} L|K$. This proves that $\operatorname{trdeg} Lv|Kv$ and the rational rank of vL/vK are finite. Therefore, we may choose the elements x_i, y_j such that $\tau = \operatorname{trdeg} Lv|Kv$ and $\rho = \dim_{\mathbb{Q}} \mathbb{Q}\otimes(vL/vK)$ to obtain inequality (7).

Set $L_0 := K(x_1,\ldots,x_\rho,y_1,\ldots,y_\tau)$ and assume that equality holds in (7). This means that the extension $L|L_0$ is algebraic. Since $L|K$ is finitely generated, it follows that this extension is finite. By the fundamental inequality (2), this yields that $vL|vL_0$ and $Lv|L_0v$ are finite extensions. Since already $vL_0|vK$ and $L_0v|Kv$ are finitely generated by the foregoing lemma, it follows that also $vL|vK$ and $Lv|Kv$ are finitely generated. □

The algebraic analogue to the transcendental case discussed in Lemma 2.5 is the following lemma (see [17] for a proof):

Lemma 2.7. *Let $(L|K,v)$ be an extension of valued fields. Suppose that $\eta_1,\ldots,\eta_k \in L$ such that $v\eta_1,\ldots,v\eta_k \in vL$ belong to distinct cosets modulo vK. Further, assume that $\vartheta_1,\ldots,\vartheta_\ell \in \mathcal{O}_L$ such that $\vartheta_1v,\ldots,\vartheta_\ell v$ are Kv-linearly independent. Then the elements $\eta_i\vartheta_j$, $1\leq i\leq k$, $1\leq j\leq \ell$, are K-linearly independent, and for every choice of elements $c_{ij}\in K$, we have that*

$$v\sum_{i,j} c_{ij}\eta_i\vartheta_j \;=\; \min_{i,j} vc_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}(vc_{ij}\;+\;v\eta_i)\;.$$

If the elements $\eta_i\vartheta_j$ form a K-basis of L, then

$$vL \;=\; vK + \bigoplus_{1\leq i\leq k} \mathbb{Z}v\eta_i \qquad \text{and} \qquad Lv \;=\; Kv(\vartheta_jv \mid 1\leq j\leq \ell)\;.$$

For any element x in a field extension of K and every nonnegative integer n, we set

$$K[x]_n \;:=\; K + Kx + \ldots + Kx^n\;.$$

Since $\dim_K K[x]_n \leq n+1$, we obtain the following corollary from Lemma 2.7:

Corollary 2.8. *Take a valued field extension $(K(x)|K,v)$. Then for every $n\geq 0$,*

a) the elements of $vK[x]_n$ lie in at most $n+1$ many distinct cosets modulo vK,

b) the Kv-vector space $K[x]_nv$ is of dimension at most $n+1$.

2.4. **Immediate extensions.** We will assume some familiarity with the basic properties of pseudo Cauchy sequences; we refer the reader to Kaplansky's paper "Maximal fields with valuations" ([5]). In particular, we will use the following two main theorems:

Theorem 2.9. (Theorem 2 of [5])
For every pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) of transcendental type there exists an immediate transcendental extension $(K(x),v)$ such that x is a pseudo limit of $(a_\nu)_{\nu<\lambda}$. If $(K(y),v)$ is another valued extension field of (K,v) such that y is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, then y is also transcendental over K and the isomorphism between $K(x)$ and $K(y)$ over K sending x to y is valuation preserving.

Theorem 2.10. (Theorem 3 of [5])
Take a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) of algebraic type. Choose a polynomial $f(X)\in K[X]$ of minimal degree whose value is not fixed by $(a_\nu)_{\nu<\lambda}$, and a root z of f. Then there exists an extension of v from K to $K(z)$ such that $(K(z)|K,v)$ is an immediate extension and z is a pseudo limit of $(a_\nu)_{\nu<\lambda}$.

If $(K(z'),v)$ is another valued extension field of (K,v) such that z' is also a root of f and a pseudo limit of $(a_\nu)_{\nu<\lambda}$, then the field isomorphism between $K(a)$ and $K(b)$ over K sending a to b will preserve the valuation.

We will need a few more results that are not in Kaplansky's paper.

Lemma 2.11. *Take an algebraic algebraic field extension $(K(a)|K,v)$, where a is a pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) without a pseudo limit in K. Then $(a_\nu)_{\nu<\lambda}$ does not fix the value of the minimal polynomial of a over K.*

Proof. We denote the minimal polynomial of a over K by $f(X)=\prod_{i=1}^n(X-\sigma_i a)$ with $\sigma_i\in \mathrm{Gal}\,(\tilde{K}|K)$. Since a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, the values $v(a_\nu-a)$ are ultimately increasing. If $v(a-\sigma_i a)>v(a_\nu-a)$ for all $\nu<\lambda$, then also the values $v(a_\nu-\sigma_i a)=\min\{v(a_\nu-a),v(a-\sigma_i a)\}=v(a_\nu-a)$ are ultimately increasing. If on the other hand, $v(a-\sigma_i a)\le v(a_{\nu_0}-a)$ for some $\nu_0<\lambda$, then for $\nu_0<\nu<\lambda$, the value $v(a_\nu-\sigma_i a)=\min\{v(a_\nu-a),v(a-\sigma_i a)\}=v(a-\sigma_i a)$ is fixed. We conclude that the values $vf(a_\nu)=\sum_{i=1}^n v(a_\nu-\sigma_i a)$ are ultimately increasing. □

Lemma 2.12. *Take a henselian field (K,v) of positive characteristic p and a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) without a pseudo limit in K. If $(K(a)|K,v)$ is a valued field extension of degree p such that a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, then $(K(a)|K,v)$ is immediate.*

Proof. By the previous lemma, $(a_\nu)_{\nu<\lambda}$ does not fix the value of the minimal polynomial f of a over K. On the other hand, we will show that $(a_\nu)_{\nu<\lambda}$ fixes the value of every polynomial of degree less than $\deg f=p$. We take $g\in K[X]$ to be a polynomial of smallest degree such that $(a_\nu)_{\nu<\lambda}$ does not fix the value of g. Since $(a_\nu)_{\nu<\lambda}$ admits no pseudo limit in (K,v), the polynomial g is of degree at least 2. Take a root b of g. By Theorem 2.10, there is an extension of the valuation v from K to $K(b)$ such that $(K(b)|K,v)$ is immediate. Since $[K(b):K]\ge 2$ and (K,v) is henselian, the Lemma of Ostrowski implies that $[K(b):K]\ge p$. This shows that f is a polynomial of smallest degree whose value is not fixed by $(a_\nu)_{\nu<\lambda}$. Hence again by Theorem 2.10, there is an extension of the valuation v from K to $K(a)$ such that $(K(a)|K,v)$ is immediate. Since (K,v) is henselian, this extension coincides with the given valuation on $K(a)$ and we have thus proved that the extension $(K(a)|K,v)$ is immediate. □

The following result is Proposition 4.3 of [8]:

Proposition 2.13. *Take a valued field (F,v) of positive characteristic p. Assume that F admits an immediate purely inseparable extension $F(\eta)$ of degree p such that the element η does not lie in the completion of (F,v). Then for each element $b\in F^\times$ such that*

$$(p-1)vb+v\eta \;>\; pv(\eta-c) \tag{8}$$

holds for every $c\in F$, any root ϑ of the polynomial

$$X^p-X-\left(\frac{\eta}{b}\right)^p$$

generates an immediate Galois extension $(F(\vartheta)|F,v)$ of degree p with a unique extension of the valuation v from F to $F(\vartheta)$.

2.5. **Characteristic blind Taylor expansion.** We need a Taylor expansion that works in all characteristics. For polynomials $f\in K[X]$, we define the **i-th formal derivative of f** as

$$f_i(X) \;:=\; \sum_{j=i}^{n}\binom{j}{i}c_jX^{j-i} \;=\; \sum_{j=0}^{n-i}\binom{j+i}{i}c_{j+i}X^j\ . \tag{9}$$

Then regardless of the characteristic of K, we have the **Taylor expansion** of f at c in the following form:

$$f(X) \;=\; \sum_{i=0}^{n} f_i(c)(X-c)^i\ . \tag{10}$$

3. Algebraic independence of elements in maximal immediate extensions

This section is devoted to the proof of Theorem 1.1. Our first goal is a basic independence lemma.

Take $i\in\mathbb{N}$, any field K and a polynomial $f\in K[X_1,\ldots,X_i]$. With respect to the lexicographic order on $\mathbb{Z}^i$, let $(\mu_1,\ldots,\mu_i)$ be maximal with the property that the coefficient of $X_1^{\mu_1}\cdots X_i^{\mu_i}$ in f is nonzero. Then define c_f to be this coefficient and call $(\mu_1,\ldots,\mu_i)$ the **crucial exponent** of f.

For our basic independence lemma, we consider the following situation. We choose a function

$$\varphi:\ \mathbb{N}\times\mathbb{N}\ \longrightarrow\ \mathbb{N}$$

such that

$$\varphi(k,\ell)>\max\{k,\ell\}\quad\text{and}\quad\varphi(k+1,\ell)>\varphi(k,\ell)\quad\text{for all } k,\ell\in\mathbb{N},$$

and for each $i\in\mathbb{N}$ a strictly increasing sequence $(E_i(k))_{k\in\mathbb{N}}$ of integers ≥ 2 such that for all $k\geq 1$ and $i\geq 2$,

$$\left.\begin{array}{rcl} E_1(k+1) & \geq & \varphi(k,E_1(k))+1,\\ E_i(k+1) & \geq & E_{i-1}(\varphi(k,E_i(k))+1)\ .\end{array}\right\} \tag{11}$$

Then for $i,k\in\mathbb{N}$,

$$E_i(k)\ >\ k\quad\text{and}\quad E_i(k+1)\ \geq\ \varphi(k,E_i(k))+1\ >\ E_i(k)+1\ . \tag{12}$$

Further, we take an extension $(L|K,v)$ of valued fields, elements

$$a_j\in L\quad\text{and}\quad\alpha_j\in vL\quad\text{for all } j\in\mathbb{N},$$

and K-subspaces

$$S_j \subseteq L\,, \quad j \in \mathbb{N}\,.$$

We assume that for all $i, k, \ell \in \mathbb{N}$, the following conditions are satisfied:

(A1) $0 \leq va_k \leq \alpha_k < va_{k+1}$ and $k\alpha_{E_i(k)} \leq \alpha_{\varphi(k,E_i(k))}$,

(A2) $a_1, \ldots, a_k \in S_k$ and $S_k \subseteq S_{k+1}$,

(A3) if $d_0, \ldots, d_k \in S_k$ and $u \in S_\ell$, then

$$d_0 + d_1 u + \ldots + d_k u^k \in S_{\varphi(k,\ell)}\,,$$

(A4) if $m \leq k$ and $d_0, \ldots, d_m \in S_k$, then

$$v(d_0 + d_1 a_{k+1} + \ldots + d_m a_{k+1}^m) \leq vd_m + m\alpha_{k+1}\,.$$

Now we choose any maximal immediate extension (M, v) of (L, v). For each i, we take an arbitrary pseudo limit $y_i \in M$ of the pseudo Cauchy sequence

$$\left(\sum_{j=1}^{k} a_{E_i(j)}\right)_{k\in\mathbb{N}}\,.$$

In this situation, we can prove the following basic independence lemma:

Lemma 3.1. *Suppose that $k \geq 2$ is an integer and $f \in L[X_1, \ldots, X_i]$ is a polynomial with coefficients in $S_{k-1} \cap \mathcal{O}_L$ such that $\alpha_{E_i(k)} \geq vc_f$ and that f has degree less than k in each variable. Then*

$$vf(y_1, \ldots, y_i) < va_{E_i(k+1)}\,. \tag{13}$$

Proof. We shall prove the lemma by induction on i. We start with $i = 1$ and set

$$u := \sum_{j=1}^{k} a_{E_1(j)} \qquad \text{and} \qquad z := y_1 - u\,.$$

Then $u \in S_{E_1(k)}$ because of (A2) and the fact that the S_k are vector spaces. By (A1), the definition of E_1 and our assumption that $\alpha_{E_1(k)} \geq vc_f$,

$$vz = va_{E_1(k+1)} \geq va_{\varphi(k,E_1(k))+1} > \alpha_{\varphi(k,E_1(k))} \geq k\alpha_{E_1(k)} \geq vc_f + (k-1)\alpha_{E_1(k)}\,. \tag{14}$$

We use the Taylor expansion

$$f(y_1) = f(u+z) = f(u) + zf_1(u) + z^2 f_2(u) + \ldots \tag{15}$$

where $f_j(X) \in \mathcal{O}_L[X]$ is the j-th formal derivative of f as defined in (9). We have that $f_j(u) \in \mathcal{O}_L$ for all j. Hence,

$$v(zf_1(u) + z^2 f_2(u) + \ldots) \geq vz\,. \tag{16}$$

We wish to prove that $vf(u) < vz$. We set

$$u' := \sum_{j=1}^{k-1} a_{E_1(j)} \in S_{E_1(k-1)}$$

so that $u = u' + a_{E_1(k)}$. We use the Taylor expansion

$$f(u) = f(u' + a_{E_1(k)}) = f(u') + f_1(u')a_{E_1(k)} + \ldots + f_m(u')a_{E_1(k)}^m$$

where $m = \deg f < k$. By definition, c_f is the leading coefficient of f, which in turn is equal to the constant $f_m(u') = f_m(X) \in L$. Since f has coefficients in the vector

space S_{k-1}, we know from (9) that also all f_j have coefficients in S_{k-1}. Thus, (A3) and (A2) show that

$$f(u'),\, f_j(u') \,\in\, S_{\varphi(k-1,E_1(k-1))} \,\subseteq\, S_{E_1(k)-1}$$

for each j. Further, $m \leq k-1 \leq E_1(k)-1$. Hence by (A4) and (14),

$$vf(u) \,\leq\, vf_m(u') + m\alpha_{E_1(k)} \,\leq\, vc_f \,+\, (k-1)\alpha_{E_1(k)} \,<\, vz\,.$$

From this together with (15) and (16), we deduce that

$$vf(y_1) \,=\, vf(u) \,<\, vz\,,$$

which gives the assertion of our lemma for the case of $i = 1$.

In the case of $i > 1$ we assume that the assertion of our lemma has been proven for $i-1$ in place of i, and we set

$$u \,:=\, \sum_{j=1}^{k} a_{E_i(j)} \,\in\, S_{E_i(k)}\,, \quad u' \,:=\, \sum_{j=1}^{k-1} a_{E_i(j)} \,\in\, S_{E_i(k-1)}\,, \quad \text{and} \quad z \,:=\, y_i - u\,.$$

Then by (12), (A1) and our assumption that $\alpha_{E_i(k)} \geq vc_f$,

$$vz \,=\, va_{E_i(k+1)} \,\geq\, va_{\varphi(k,E_i(k))+1} \,>\, \alpha_{\varphi(k,E_i(k))} \,\geq\, k\alpha_{E_i(k)} \,\geq\, vc_f + (k-1)\alpha_{E_i(k)}\,.$$

We use the Taylor expansion

$$\begin{aligned} f(y_1,\ldots,y_{i-1},u+z) \;&=\; f(y_1,\ldots,y_{i-1},u) \,+\, zf_1(y_1,\ldots,y_{i-1},u) \\ &+\; z^2 f_2(y_1,\ldots,y_{i-1},u) \,+\, \ldots \end{aligned}$$

where $f_j \in \mathcal{O}_L[X_1,\ldots,X_i]$ is the j-th formal derivative of f with respect to X_i. We obtain the analogue of inequality (16); hence it will suffice to prove that

$$vf(y_1,\ldots,y_{i-1},u) \,<\, vz\,. \tag{17}$$

We set

$$g(X_1,\ldots,X_{i-1}) \,:=\, f\,(X_1,\ldots,X_{i-1},u)$$

so that $g(y_1,\ldots,y_{i-1}) = f(y_1,\ldots,y_{i-1},u)$. Viewing f as a polynomial in the variables $X_1,\ldots,X_{i-1}$ with coefficients in $L[X_i]$, we denote by $h(X_i)$ the coefficient of $X_1^{\mu_1}\cdots X_{i-1}^{\mu_{i-1}}$ in f. Note that h has coefficients in S_{k-1}, its leading coefficient is c_f and its degree is $\mu_i < k$. Again, since h has coefficients in S_{k-1}, (9) shows that the same is true for the j-th formal derivative h_j of h, for all j. Thus, (A3), (12) and (A2) imply that

$$h(u'),\, h_j(u') \,\in\, S_{\varphi(k-1,E_i(k-1))} \,\subseteq\, S_{E_i(k)-1}$$

for each j. As in the first part of our proof we find that

$$vh(u) \,\leq\, vh_{\mu_i}(u') + \mu_i\alpha_{E_i(k)} \,=\, vc_f \,+\, \mu_i\alpha_{E_i(k)}$$

since $h_{\mu_i}(u') = c_f$. In particular, this shows that $h(u) \neq 0$. Hence if $(\mu_1,\ldots,\mu_i)$ is the crucial exponent of f, then $(\mu_1,\ldots,\mu_{i-1})$ is the crucial exponent of g, and

$$c_g \,=\, h(u)\,.$$

We set

$$k' \,:=\, \varphi(k,E_i(k)) \,>\, \max\{k,E_i(k)\}\,.$$

Since $\mu_i \leq k-1$ and $vc_f \leq \alpha_{E_i(k)}$ by assumption, and by virtue of (A1) and (12), it follows that

$$vc_g \,=\, vh(u) \,\leq\, vc_f + (k-1)\alpha_{E_i(k)} \,\leq\, k\alpha_{E_i(k)} \,\leq\, \alpha_{k'} \,<\, \alpha_{E_{i-1}(k')}\,.$$

Since every coefficient of g is of the form $h(u)$ with h a polynomial of degree less than k and coefficients in S_{k-1}, we know from (A3), our conditions on φ and (A2) that the coefficients of g lie in

$$S_{\varphi(k-1,E_i(k))} \subseteq S_{\varphi(k,E_i(k))-1} = S_{k'-1} .$$

Also, its degree in each variable is less than k, hence less than k'. Therefore, we can apply the induction hypothesis to the case of $i-1$, with k' in place of k. We obtain, by (A1) and our choice of the numbers $E_i(k)$:

$$vf(y_1,\ldots,y_{i-1},u) < va_{E_{i-1}(\varphi(k,E_i(k))+1)} \leq va_{E_i(k+1)} = vz .$$

This establishes our lemma. □

By (A2),

$$S_\infty := \bigcup_{k\in\mathbb{N}} S_k$$

contains a_k for all k. We set

$$K_\infty := K(S_\infty) .$$

Further, we note that condition (A1) implies that

$$\Gamma := \{\alpha \in vK_\infty \mid -va_k \leq \alpha \leq va_k \text{ for some } k\}$$

is a convex subgroup of vK_∞.

Corollary 3.2. *Assume that every element of K_∞ with value in Γ can be written as a quotient r/s with $r,s \in S_\infty$ such that $0 \leq vs \in \Gamma$. Then the elements y_i, $i \in \mathbb{N}$, are algebraically independent over K_∞.*

Proof. We have to check that $g(y_1,\ldots,y_i) \neq 0$ for all i and all nonzero polynomials $g(X_1,\ldots,X_i) \in K_\infty[X_1,\ldots,X_i]$. After division by some coefficient of g with minimal value we may assume that g has integral coefficients in K_∞ and at least one of them has value $0 \in \Gamma$. We write all its coefficients which have value in Γ in the form as given in our assumption. We take $\tilde{s}$ to be the product of all appearing denominators. Then $v\tilde{s} \in \Gamma$. After multiplication with $\tilde{s}$, all coefficients of g with value in Γ are elements of S_∞, and there is at least one such coefficient. Now we write $g(X_1,\ldots,X_i) = f(X_1,\ldots,X_i) + h(X_1,\ldots,X_i)$ where every coefficient of f is in S_∞ and has value less than va_k for some k, and every coefficient of h has value bigger than va_k for all k (we allow h to be the zero polynomial). Since g has coefficients of value $v\tilde{s}$, the polynomial f is nonzero. Since $vy_i \geq 0$ for all i, we have that $vh(y_1,\ldots,y_i)$ is bigger than va_k for all k.

We choose k such that the assumptions of Lemma 3.1 hold; note that k exists since by our definition of f, the coefficient c_f has value less than va_k for some k. We obtain that

$$vf(y_1,\ldots,y_i) < va_{E_i(k+1)} < vh(y_1,\ldots,y_i) .$$

This gives that

$$vg(y_1,\ldots,y_i) = v(f(y_1,\ldots,y_i)+h(y_1,\ldots,y_i)) = vf(y_1,\ldots,y_i) < va_{E_i(k+1)} < \infty ,$$

that is, $g(y_1,\ldots,y_i) \neq 0$. □

Now we are able to give the

Proof of Theorem 1.1:

In all cases of the proof, we will choose functions φ that have the previously required properties. We will choose a suitable sequence $(b_k)_{k\in\mathbb{N}}$ of elements in L and a sequence $(c_k)_{k\in\mathbb{N}}$ in K. Then we will set $a_k := c_k b_k$ and choose some values $\alpha_k \geq va_k$.

First, let us consider the valuation-transcendental case. We set

$$\varphi(k,\ell) := k + k\ell \,,$$

and note that equations (11) now read as follows:

$$\begin{aligned} E_1(k+1) &\geq k + kE_1(k) + 1, \\ E_i(k+1) &\geq E_{i-1}(k + kE_i(k) + 1) \,. \end{aligned}$$

Further, we will work with a suitable element $t \in \mathcal{O}_L$ transcendental over K and set, after a suitable choice of the sequence $(c_k)_{k\in\mathbb{N}}$,

$$\begin{aligned} a_k &:= c_k t^k \,, \\ \alpha_k &:= va_k \,, \\ S_k &:= K + Kt + \ldots + Kt^k \,. \end{aligned}$$

Conditions (A2) and (A3) are immediate consequences of our choice of S_k as the set of all polynomials in $K[t]$ of degree at most k.

Suppose that vL/vK is not a torsion group. Then we pick $t \in \mathcal{O}_L$ such that vt is rationally independent over vK (that is, $nvt \notin vK$ for all integers $n > 0$). Further, for all k we set $b_k = t^k$ and $c_k = 1$ so that $a_k = t^k$. Then condition (A1) is satisfied since we have that

$$0 \leq va_k = \alpha_k = vt^k = kvt < (k+1)vt = vt^{k+1} = va_{k+1}$$

and

$$k\alpha_{E_i(k)} = kva_{E_i(k)} = kvt^{E_i(k)} = kE_i(k)vt < (k + kE_i(k))vt = \alpha_{\varphi(k,E_i(k))} \,.$$

Suppose now that vL/vK is a torsion group. In this case, $Kv|Lv$ is transcendental by assumption, and we note that since v is assumed nontrivial on L, it must be nontrivial on K. We pick $t \in \mathcal{O}_L$ such that $vt = 0$ and tv is transcendental over Kv. Further, we choose a sequence $(c_k)_{k\in\mathbb{N}}$ in $\mathcal{O}_K$ such that

$$vc_{k+1} \geq kvc_k$$

for all k. Since $va_k = vc_k + kvt = vc_k$, we obtain that $va_k = \alpha_k < va_{k+1}$ and

$$k\alpha_k = kva_k \leq va_{k+1} \,. \tag{18}$$

Then by (12),

$$k\alpha_{E_i(k)} < E_i(k)\alpha_{E_i(k)} \leq va_{E_i(k)+1} \leq va_{\varphi(k,E_i(k))} \leq \alpha_{\varphi(k,E_i(k))} \,. \tag{19}$$

Hence again, condition (A1) is satisfied.

Now we have to verify (A4), simultaneously for all of the above choices for a_k . Take $d_0, \ldots, d_m \in S_k$, $m \leq k$, and write $d_j = \sum_{\nu=0}^{k} d_{j\nu} t^\nu$ with $d_{j\nu} \in K$. Then

$$d_0 + d_1 a_{k+1} + \ldots + d_m a_{k+1}^m = \sum_{j=0}^{m} \sum_{\nu=0}^{k} c_{k+1}^j d_{j\nu} t^{j(k+1)+\nu} \,.$$

In this sum, each power of t appears only once. So we have, by Lemma 2.5,

$$v(d_0 + d_1 a_{k+1} + \ldots + d_m a_{k+1}^m) \;=\; \min_{j,\nu} v c_{k+1}^j d_{j\nu} t^{j(k+1)+\nu} \;:=\; \beta \;.$$

If this minimum is obtained at $j = j_0$ and $\nu = \nu_0$, then

$$\begin{aligned} \beta \;&=\; v c_{k+1}^{j_0} d_{j_0\nu_0} t^{j_0(k+1)+\nu_0} \;=\; \min_{\nu} v c_{k+1}^{j_0} d_{j_0\nu} t^{j_0(k+1)+\nu} \\ &=\; (\min_{\nu} v d_{j_0\nu} t^{\nu}) + v a_{k+1}^{j_0} \;=\; v d_{j_0} a_{k+1}^{j_0} \;, \end{aligned}$$

where the last equality again holds by Lemma 2.5. For all j,

$$\beta \;\le\; \min_{\nu} v c_{k+1}^j d_{j\nu} t^{j(k+1)+\nu} \;=\; (\min_{\nu} v d_{j\nu} t^{\nu}) + v a_{k+1}^j \;=\; v d_j a_{k+1}^j \;.$$

This gives that

$$v(d_0+d_1 a_{k+1}+\ldots+d_m a_{k+1}^m) = \beta = \min_j v d_j a_{k+1}^j \le v d_m + m v a_{k+1} = v d_m + m\alpha_{k+1} \;,$$

as required. Finally, we have to verify the assumption of Corollary 3.2. Each element in K_∞ can be written as a quotient r/s of polynomials in t with coefficients in K. After multiplying both r and s with a suitable element from K we may assume that s has coefficients in $\mathcal{O}_K$ and one of them is 1. If this is the coefficient of t^i, say, then it follows by Lemma 2.5 that $0 \le vs \le vt^i \le va_i$ and thus, $vs \in \Gamma$.

Now we take any maximal immediate extension (M, v) and y_i as defined preceeding to Lemma 3.1. Then we can infer from Corollary 3.2 that the elements y_i are algebraically independent over K_∞; that is, the transcendence degree of M over K_∞ is infinite. Since the transcendence degree of L over K and thus also that of L over K_∞ is finite, we can conclude that the transcendence degree of M over L is infinite.

Next, we consider the value-algebraic case and the residue-algebraic case. We will assume for now that there is an algebraic subextension $L_0|K$ of $L|K$ such that vL_0/vK contains elements of arbitrarily high order, or $L_0 v$ contains elements of arbitrarily high degree over Kv. The remaining cases will be treated at the end of the proof of our theorem.

For the present case as well as the separable-algebraic case, we work with any function φ that satisfies the conditions outlined in the beginning of this section, and with

$$S_k \;:=\; K(a_1, \ldots, a_k) \;.$$

Then S_∞ is a field and the assumption of Corollary 3.2 are trivially satisfied (taking $s = 1$). Further, condition (A2) is trivially satisfied. To prove that condition (A3) holds, take any $u \in S_\ell = K(a_1, \ldots, a_\ell)$. If $n = \max\{k, \ell\}$, then $d_0, \ldots, d_k, u \in K(a_1, \ldots, a_n) = S_n$ and therefore,

$$d_0 + d_1 u + \ldots + d_k u^k \;\in\; S_n \;\subseteq\; S_{\varphi(k,\ell)} \;.$$

This shows that (A3) holds.

By induction, we define $a_k \in L_0$ as follows, and we always take $\alpha_k = va_k$. We start with $a_1 = 1$ and $\alpha_1 = 0$. Suppose that $a_1, \ldots, a_k$ are already defined. Since $K(a_1, \ldots, a_k)|K$ is a finite extension, also $vK(a_1, \ldots, a_k)/vK$ and

$K(a_1,\dots,a_k)v|Kv$ are finite. Hence by our assumption in the algebraic case, there is some $b_{k+1}\in L_0$ such that

(20) $0, vb_{k+1}, 2vb_{k+1},\dots,kvb_{k+1}$ lie in distinct cosets modulo $vK(a_1,\dots,a_k)$, or

(21) $1, b_{k+1}v, (b_{k+1}v)^2,\dots,(b_{k+1}v)^k$ are $K(a_1,\dots,a_k)v$-linearly independent.

If L_0v contains elements of arbitrarily high degree over Kv, we always choose b_{k+1} such that (21) holds; in this case, $vb_{k+1}=0$ and we choose the elements c_k as in the residue-transcendental case above. Otherwise, vL_0/vK contains elements of arbitrarily high order, and we always choose b_{k+1} such that (20) holds. In this case, we choose c_{k+1} such that for $a_{k+1}:=c_{k+1}b_{k+1}$ we obtain $k\alpha_k = kva_k \le va_{k+1}$; this is possible since the values of b_k and hence of all a_k lie in the convex hull of vK in vL. As in the residue-transcendental case above, we obtain (18) and (19), showing that condition (A1) is satisfied.

To prove that (A4) holds, take any $k\ge 1$ and $d_0,\dots,d_k\in S_k=K(a_1,\dots,a_k)$. By Lemma 2.7 applied to b_{k+1},

$$\begin{aligned} v(d_0+d_1a_{k+1}+\ldots+d_ka_{k+1}^k) &= v(d_0+d_1c_{k+1}b_{k+1}+\ldots+d_kc_{k+1}^kb_{k+1}^k)\\ &= \min_i vd_ic_{k+1}^ib_{k+1}^i \;=\; \min_i vd_ia_{k+1}^i\,. \end{aligned}$$

This shows that (A4) holds.

As in the valuation-transcendental case, we can now deduce our assertion about the transcendence degree.

Next, we consider the separable-algebraic case. In this case, we can w.l.o.g. assume that (K,v) is henselian. Indeed, each maximal immediate extension of (L,v) contains a henselization L^h of (L,v) and hence also a henselization K^h of (K,v), and our assumption on L_0 implies that the subfield $L_0.K^h$ of L^h is an infinite separable-algebraic extension of K^h. (Here, $L_0.K^h$ denotes the field compositum, i.e., the smallest subfield of L^h which contains L_0 and K^h.)

We take S_k and $\varphi(k,\ell)$ as in the previous case, so that again, (A2), (A3) and the assumption of Corollary 3.2 hold. Then we take $a_1=b_1$ to be any element in $\mathcal{O}_{L_0}\setminus K$ and choose some $\alpha_1\in vK$ such that $\alpha_1\ge \operatorname{kras}(a_1,K)\in v\tilde{K}$; this is possible since vK is cofinal in its divisible hull, which is equal to $v\tilde{K}$. Inequality (4) of Lemma 2.2 shows that $\operatorname{kras}(a_1,K)\ge va_1$, so that $\alpha_1\ge va_1$. Suppose we have chosen $a_1,\dots,a_k\in\mathcal{O}_{L_0}$. Since $L_0|K$ is infinite and separable-algebraic, the same is true for $L_0|K(a_1,\dots,a_k)$. By the Theorem of the Primitive Element, we can therefore find an element $b_{k+1}\in L_0$ such that

$$[K(a_1,\dots,a_k,b_{k+1}):K(a_1,\dots,a_k)] \;\ge\; k+1\,.$$

We choose $c_{k+1}\in K$ such that for $a_{k+1}:=c_{k+1}b_{k+1}$ we have that $k\alpha_k\le va_{k+1}$. Finally, we choose $\alpha_{k+1}\in vK$ such that

$$\alpha_{k+1} \;\ge\; \operatorname{kras}(a_{k+1},K) \;\ge va_{k+1}\,.$$

Again, we obtain that (19) and (A1) hold.

It only remains to show that (A4) holds. But this follows readily from inequality (5) of Lemma 2.2, where we take $K(a_1,\dots,a_k)$ in place of K and $a=a_{k+1}$, together with the fact that $\operatorname{kras}(a_{k+1},K(a_1\dots,a_k))\le\operatorname{kras}(a_{k+1},K)$.

As before, we now obtain our assertion about the transcendence degree.

It remains to prove the value-algebraic case and the residue-algebraic case for transcendental valued field extensions $(L|K,v)$ of finite transcendence degree. We assume that vL/vK is a torsion group containing elements of arbitrarily high order or the extension $Lv|Kv$ is algebraic and such that Lv contains elements of arbitrarily high degree over Kv.

Take any subextension $E|K$ of $L|K$. Then $(L|K,v)$ satisfies the above assumption if and only if at least one of the extensions $(L|E,v)$ and $(E|K,v)$ satisfies the assumption. Choose a transcendence basis $(x_1,\ldots,x_n)$ of $L|K$ and set

$$F \;:=\; K(x_1,\ldots,x_n)\,.$$

Then $L|F$ is algebraic. By what we have already proved, if vL/vF contains elements of arbitrarily high order or Lv contains elements of arbitrarily high degree over Fv, then any maximal immediate extension of (L,v) has infinite transcendence degree over L.

Suppose now that $(F|K,v)$ satisfies the assumption on the value group or the residue field extension. Take $s\in\mathbb{N}$ minimal such that $vK(x_1,\ldots,x_s)/vK$ contains elements of arbitrarily high order or $K(x_1,\ldots,x_s)v$ contains elements of arbitrarily high degree over Kv. Then the assertion holds also for the value group or the residue field extension of $(K(x_1,\ldots,x_s)|K(x_1,\ldots,x_{s-1}),v)$. We can replace K by $K(x_1,\ldots,x_{s-1})$ and we will write x in place of x_s so that now we have a subextension $(K(x)|K,v)$ that satisfies the assertion for its value group or its residue field extension.

In both the value-algebraic and the residue-algebraic case we define $b_k\in K[x]$ by induction on k and set

$$S_k \;:=\; K[x]_{N_k} \quad \text{with } N_k := \deg b_k.$$

Assume that $vK(x)$ contains elements of arbitrarily high order modulo vK. Then such elements can be already chosen from $vK[x]$. We set $b_1=1$. Suppose that $b_1,\ldots,b_k$ are already chosen with $\deg b_{i-1}<\deg b_i$ for $1<i\le k$. From Corollary 2.8 we know that vS_k contains only finitely many values that represent distinct cosets modulo vK. Since all of these values are torsion modulo vK, the subgroup $\langle vS_k\rangle$ of $vK(x)$ generated by vS_k satisfies $(\langle vS_k\rangle : vK)<\infty$. By assumption, there is $b_{k+1}\in K[x]$ for which the order of vb_{k+1} modulo vK is at least $(k+1)(\langle vS_k\rangle : vK)$; this forces $0, vb_{k+1}, 2vb_{k+1},\ldots,kvb_{k+1}$ to lie in distinct cosets modulo $\langle vS_k\rangle$. Since $b_{k+1}\notin K[x]_{N_k}$, we have that $N_{k+1}=\deg b_{k+1}>N_k$.

Assume now that $K(x)v$ contains elements of arbitrarily high degree over Kv. Without loss of generality we can assume that $vK(x)/vK$ is then a torsion group with a finite exponent N. Otherwise, vL/vK is not a torsion group and we are in the valuation-transcendental case or $vK(x)/vK$ contains elements of arbitrarily high order and we are in the value-algebraic case.

The elements of arbitrarily high degree over Kv can be chosen from $K[x]v$. Indeed, suppose there is $m\in\mathbb{N}$ such that $[Kv(fv):Kv]\le m$ for every polynomial f of nonnegative value. Take any $r=\frac{h}{g}$, where $g,h\in K[x]$ and $vr=0$. By the assumption on $vK(x)/vK$ we have that $nvh=vd$ for some natural number $n\le N$ and $d\in K$. Then

$$r \;=\; \frac{d^{-1}h^n}{d^{-1}h^{n-1}g}$$

and $vd^{-1}h^{n-1}g = vd^{-1}h^n = 0$, since $vh = vg$. Therefore we may assume that $vh = vg = 0$. Hence,

$$[Kv(rv) : Kv] \;\leq\; [Kv(rv, gv) : Kv] \;=\; [Kv(hv, gv) : Kv] \leq m^2$$

for every $r \in K(x)$ with $vr = 0$, a contradiction to our assumption.

As in the value-algebraic case, we set $b_1 = 1$. Suppose that $b_1, \ldots, b_k$ are already chosen with $\deg b_{i-1} < \deg b_i$ for $1 < i \leq k$. By Corollary 2.8, there are at most $NN_k + 1$ many Kv-linearly independent elements in $K[x]_{NN_k}v$, and as all of them are algebraic over Kv, it follows that the extension $Kv\,(K[x]_{NN_k}v)\,|Kv$ is finite. By assumption, there is $b_{k+1} \in K[x]$ such that $vb_{k+1} = 0$ and the degree of $b_{k+1}v$ over Kv is at least $(k+1)[Kv\,(K[x]_{NN_k}v) : Kv]$, which forces the elements $1, b_{k+1}v, (b_{k+1}v)^2, \ldots, (b_{k+1}v)^k$ to be $Kv\,(K[x]_{NN_k}v)$-linearly independent. Since $b_{k+1} \notin K[x]_{NN_k}$, we have that $N_{k+1} = \deg b_{k+1} > NN_k \geq N_k$.

For the value-algebraic as well as for the residue-algebraic case we set

$$\varphi(k, l) := N_k + N_k N_l.$$

Since in both cases $(N_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, φ has the required properties. As in the first part of the proof of the value-algebraic and the residue-algebraic case, one can show that the elements $c_k \in K$ can be chosen in such a way that condition (A1) holds for $a_k := c_k b_k$ and $\alpha_k := va_k$. Since $(N_k)_{k\in\mathbb{N}}$ is strictly increasing, condition (A2) is trivially satisfied. Moreover, $N_k \geq k$ for every $k \in \mathbb{N}$. Hence for any $d_0, \ldots, d_k \in S_k$ and $u \in S_l$,

$$\deg(d_0 + d_1u + \cdots + d_ku^k) \leq N_k + kN_l \;\leq\; \varphi(k, l) \;\leq\; N_{\varphi(k,l)}\,.$$

Thus, $d_0 + d_1u + \cdots + d_ku^k \in S_{\varphi(k,l)}$. This shows that (A3) holds.

To verify (A4), we take any $k, m \in \mathbb{N}$ with $m \leq k$, and $d_0, \ldots, d_m \in S_k$. We wish to estimate the value of the element $d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m$. We discuss first the value-algebraic case. Note that the values $v(d_ia_{k+1}^i)$, $0 \leq i \leq m$, lie in distinct cosets modulo vK. Indeed, $vd_ia_{k+1}^i = vd_ic_{k+1}^i + ivb_{k+1}$, where $d_ic_{k+1}^i \in S_k$. Therefore, if

$$vd_ia_{k+1}^i + vK \;=\; vd_ja_{k+1}^j + vK$$

for some $0 \leq i \leq j \leq m$, then also

$$ivb_{k+1} + \langle vS_k\rangle \;=\; jvb_{k+1} + \langle vS_k\rangle\,,$$

which by our choice of b_{k+1} yields that $i = j$. Hence, from Lemma 2.7 it follows that

$$v(d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m) \;=\; \min_i vd_ia_{k+1}^i \;\leq\; vd_m + mva_{k+1}\,.$$

We obtain the same assertion also in the residue-algebraic case. If $d_i = 0$ for all i, then it is trivially satisfied. If not, take i_0 so that

$$vd_{i_0}c_{k+1}^{i_0} \;=\; \min_i vd_ic_{k+1}^i \;=\; \min_i vd_ia_{k+1}^i\,.$$

We have that $vd_{i_0}^N = vc$ for some $c \in K$. Setting $d := c^{-1}c_{k+1}^{-i_0}d_{i_0}^{N-1}$, we obtain that

$$v(d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m) \;= -vd + v\xi$$

with $\xi := dd_0 + dd_1c_{k+1}b_{k+1} + \cdots + dd_mc_{k+1}^mb_{k+1}^m$. Note that $dd_i \in K[x]_{NN_k}$ for $0 \leq i \leq m$, and that

$$vdd_ic_{k+1}^i \;\geq\; vdd_{i_0}c_{k+1}^{i_0} \;=\; vc^{-1}d_{i_0}^N \;=\; 0.$$

In particular, $v\xi \geq 0$, and

$$\xi v = (dd_0)v + (dd_1c_{k+1})vb_{k+1}v + \cdots + (dd_mc_{k+1}^m)v(b_{k+1}v)^m$$

is a linear combination of $1, b_{k+1}v, (b_{k+1}v)^2, \ldots, (b_{k+1}v)^m$ with coefficients from $Kv\,(K[x]_{N\cdot N_k}v)$. Since at least one of them, the element $dd_{i_0}c_{k+1}^{i_0}v$, is nonzero, also the linear combination is nontrivial by our choice of b_{k+1}. Hence $v\xi = 0$ and

$$v(d_0 + d_1a_{k+1} + \cdots d_ma_{k+1}^m) \;=\; -vd \;=\; vd_{i_0}c_{k+1}^{i_0} \;\leq\; vd_m + mva_{k+1}\,.$$

Therefore, condition (A4) is satisfied in both cases.

It suffices now to verify the assumptions of Corollary 3.2. Take any element $\frac{h}{g}$ of $K_\infty = K(x)$, where $g, h \in S_\infty = K[x]$. In both the value-algebraic and the residue-algebraic case we assumed that $vK(x)/vK$ is a torsion group. Therefore, as in the residue-algebraic case above one can multiply h and g by a suitable polynomial to obtain that $vg = 0 \in \Gamma$. Hence the assumptions of the corollary are satisfied.

Since the transcendence degree of the extension $L|K(x)$ is finite, we can now deduce the assertion about about the transcendence degree as in the previous cases.

In the value-algebraic case, we still have to deal with the case where there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv. We may assume that $Lv|Kv$ is algebraic since otherwise, the assertion of our theorem follows from the valuation-transcendental case. Since every maximal immediate extension of (L, v) contains a henselization of (L, v), we may assume that both (L, v) and (K, v) are henselian. We take L' to be the relative separable-algebraic closure of K in L. Then by Lemma 2.4, vL/vL' is a p-group, which yields that $\Gamma \subseteq vL'$. In view of the fundamental inequality, we find that $L'|K$ must be an infinite extension. Now the assertion of our theorem follows from the separable-algebraic case.

Finally, we have to deal with our additional assertion about the completion. Since the transcendence degree of $L|K$ is finite, we know that vL/vK has finite rational rank. Therefore, vK is cofinal in vL or there exists some $\alpha \in vL$ such that the sequence $(i\alpha)_{i\in\mathbb{N}}$ is cofinal in vL. In the latter case (which always holds if vL contains an element γ such that $\gamma > vK$), we are in the value-transcendental case and we choose the element t such that $vt = \alpha$. In the former case, provided that the cofinality of vL is countable, we choose the elements c_i such that the sequence $(vc_ib_i)_{i\in\mathbb{N}}$ is cofinal in vL. In all of these cases, the sequence $(va_i)_{i\in\mathbb{N}}$ will be cofinal in vL and the elements y_i will lie in the completion of (L, v). □

4. Extensions of maximal fields

We start with the

Proof of Theorem 1.2:

Take a maximal field (K, v) which satisfies (1), and denote by p the characteristic exponent of Kv. Further, take an infinite algebraic extension $(L|K, v)$. Denote the relative separable-algebraic closure of K in L by L'. Assume that $L'|K$ is infinite. Since K is henselian, the separable-algebraic case of Theorem 1.1 shows that any maximal immediate extension of (L, v) has infinite transcendence degree over L.

Assume now that $L'|K$ is a finite extension. Then the field (L', v) is maximal, $(vL' : pvL')[L'v : L'v^p] = (vK : pvK)[Kv : Kv^p] < \infty$, and $L|L'$ is an infinite

purely inseparable extension. Therefore at least one of the extensions $vL|vL'$ or $Lv|L'v$ is infinite. Indeed, suppose that $(vL : vL')$ and $[Lv : L'v]$ were finite. Take any finite subextension $E|L'$ of $L|L'$ such that $[E : L'] > (vL : vL')[Lv : L'v]$. Then

$$[E : L'] \ > \ (vL : vL')[Lv : L'v] \ \geq \ (vE : vL')[Ev : L'v] \, ,$$

which contradicts the fact that L' as a maximal field is defectless by Theorem 2.1. If vL/vL' contains elements of arbitrarily high order or Lv contains elements of arbitrarily high degree over $L'v$, then from the value-algebraic or residue-algebraic case of Theorem 1.1 we deduce that any maximal immediate extension of L is of infinite transcendence degree over L. Otherwise, vL/vL' is a p-group of finite exponent, $Lv|L'v$ is a purely inseparable extension with $(Lv)^{p^n} \subseteq L'v$ for some natural number n, and vL/vL' or $Lv|L'v$ is infinite. But this is not possible if $[L'v : L'v^p](vL' : pvL') < \infty$. □

For the proof of Theorem 1.4 we will need the following result:

Lemma 4.1. *If (K, v) is a maximal field of characteristic $p > 0$, then also $K^{1/p}$ with the unique extension of the valuation v is a maximal field.*

Proof. If (a_ν) is a pseudo Cauchy sequence in L, then (a_ν^p) is a pseudo Cauchy sequence in K. Since (K, v) is maximal, it has a pseudo limit $b \in K$. But then, $a = b^{1/p} \in L$ is a pseudo limit of (a_ν). □

After this preparation, we can give the

Proof of Theorem 1.4:

Part a) follows immediately from Lemma 4.1.

To prove assertions b) and c) we consider the following subsets of K. We take A to be a set of elements of K such that the cosets $\frac{1}{p}va + vK$, $a \in A$, form a basis of the $\mathbb{Z}/p\mathbb{Z}$-vector space $\frac{1}{p}vK/vK$. Similarly, we take B to be a set of elements of the valuation ring of (K, v) such that the residues $(bv)^{1/p}$, $b \in B$, form a basis of $(Kv)^{1/p}|Kv$. Then

$$\frac{1}{p}vK \ = \ vK + \sum_{a \in A} \frac{1}{p}va\mathbb{Z} \quad \text{and} \quad (Kv)^{1/p} \ = \ Kv((bv)^{1/p} \mid b \in B) \, .$$

In order to prove assertion b) of our theorem, we set

$$L_\infty \ := \ K(a^{1/p}, b^{1/p} \mid a \in A, \, b \in B) \ \subseteq \ K^{1/p}$$

and obtain that $vL_\infty = \frac{1}{p}vK$ and $L_\infty v = (Kv)^{1/p}$. So the extension $(K^{1/p}|L_\infty, v)$ is immediate. Lemma 4.1 shows that $(K^{1/p}, v)$ is a maximal immediate extension of (L_∞, v). Our goal is now to show that under the assumptions of the theorem, this extension is of degree at least κ. Once this is proved, we can take $X \subseteq K^{1/p}$ to be a minimal set of generators of the extension $K^{1/p}|L_\infty$. Then the elements of X are p-independent over L_∞. Take any natural number n. As X is infinite, we can choose $x_1, \ldots, x_n \in X$ and set $L_n := L_\infty(X \setminus \{x_1, \ldots, x_n\})$. Then $K^{1/p}|L_n$ is an immediate extension of degree p^n. Similarly, for λ any infinite cardinal $\leq \kappa$, take $Y \subseteq X$ of cardinality λ and set $L_\lambda := L_\infty(X \setminus Y)$. Then $K^{1/p}|L_\lambda$ is an immediate algebraic extension of degree λ.

We assume first that $\kappa = (vK : pvK)$, so the set A is infinite. Then we take a partition of A into κ many countably infinite sets A_τ, $\tau < \kappa$. We choose enumerations

$$A_\tau \;=\; \{a_{\tau,i} \mid i \in \mathbb{N}\}\,.$$

For every $\mu < \kappa$ we set $\mathcal{A}_\mu := \bigcup_{\tau<\mu} A_\tau$ and

$$K_\mu \;:=\; K(a^{1/p} \mid a \in \mathcal{A}_\mu)\,.$$

Note that $\mathcal{A}_0 = \emptyset$ and $K_0 = K$. We claim that

$$vK_\mu \;=\; vK + \sum_{a\in\mathcal{A}_\mu} \frac{1}{p} va\mathbb{Z} \quad\text{and}\quad K_\mu v \;=\; Kv\,. \tag{22}$$

The inclusions "$\supseteq$" are clear. For the converses, we observe that value group and residue field of K_μ are the unions of the value groups and residue fields of all finite subextensions of $K_\mu|K$. Such subextensions can be written in the form $F = K(a_1, \ldots, a_k)$ with distinct $a_1, \ldots, a_k \in \mathcal{A}_\mu$, and we have that

$$p^k \;\geq\; [F : K] \;\geq\; (vF : vK)[Fv : Kv] \;\geq\; p^k \cdot 1\,,$$

so equality holds everywhere. Consequently, $vF = vK + \sum_{i=1}^k va_i\mathbb{Z}$ and $Fv = Kv$. This proves our claim.

For every $\tau < \kappa$ we choose a sequence $(c_{\tau,i})_{i\in\mathbb{N}}$ of elements in K such that the sequence of values

$$(vc_{\tau,i}a_{\tau,i}^{1/p})_{i\in\mathbb{N}} \tag{23}$$

is strictly increasing. If the cofinality of vK is countable, then the elements $c_{\tau,i}$ can be chosen in such a way that the sequence (23) is cofinal in $\frac{1}{p}vK$. For every $n \in \mathbb{N}$, we set

$$\xi_{\tau,n} \;:=\; \sum_{i=1}^{n} c_{\tau,i}a_{\tau,i}^{1/p} \;\in\; K_{\tau+1}\,. \tag{24}$$

Then $(\xi_{\tau,n})_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence, hence it admits a pseudo limit ξ_τ in the maximal field $(K^{1/p}, v)$. In order to show that the degree of $K^{1/p}|L_\infty$ is at least κ, we prove by induction that for every $\mu < \kappa$ and each K' such that $K_{\mu+1} \subseteq K' \subseteq L_\infty$, the pseudo Cauchy sequence $(\xi_{\mu,n})_{n\in\mathbb{N}}$ admits no pseudo limit in $K'(\xi_\tau \mid \tau < \mu)$ and the extension

$$(K'(\xi_\tau \mid \tau \leq \mu)|K', v) \tag{25}$$

is immediate.

Take $\mu < \kappa$ and assume that our assertions have already been shown for all $\mu' < \mu$. If $\mu = \mu' + 1$ is a successor ordinal, then from (25) we readily get that the extension

$$(K'(\xi_\tau \mid \tau < \mu)|K', v) \tag{26}$$

is immediate for every K' such that $K_\mu \subseteq K' \subseteq L_\infty$. If μ is a limit ordinal, then (26) follows from the induction hypothesis since $K_{\mu'} \subseteq K_\mu \subseteq K'$ for each $\mu' < \mu$ and since the union over the increasing chain of immediate extensions $K'(\xi_\tau \mid \tau \leq \mu')$, $\mu' < \mu$, of (K', v) is again an immediate extension of (K', v).

In order to prove the induction step, suppose towards a contradiction that $(\xi_{\tau,n})_{n\in\mathbb{N}}$ admits a pseudo limit η_μ in $K'(\xi_\tau \mid \tau < \mu)$ for some K' such that $K_{\mu+1} \subseteq K' \subseteq L_\infty$. Then η_μ lies already in a finite extension

$$E := K_\mu(\xi_\tau \mid \tau < \mu)(a_1^{1/p}, \ldots, a_k^{1/p}, b_1^{1/p}, \ldots, b_\ell^{1/p}) \tag{27}$$

of $K_\mu(\xi_\tau \mid \tau < \mu)$ in $L_\infty(\xi_\tau \mid \tau < \mu)$, with distinct elements $a_1, \ldots, a_k \in A \setminus \mathcal{A}_\mu$ and $b_1, \ldots, b_\ell \in B$. We claim that

$$vE = vK_\mu + \sum_{i=1}^{k} \frac{1}{p} va_i\mathbb{Z} \quad \text{and} \quad Ev = K_\mu v((b_1 v)^{1/p}, \ldots, (b_\ell v)^{1/p})\,. \tag{28}$$

As the extension (26) is immediate for K_μ in place of K', the inclusions "$\supseteq$" are clear. Conversely, from these inclusions together with the equations in (22) and our assumption on the a_i, it follows that $(vE : vK) \geq p^k$ as well as $[Ev : Kv] \geq p^\ell$. Therefore, we have that $p^k \cdot p^\ell \geq [E : K] \geq (vE : vK)[Ev : Kv] \geq p^k \cdot p^\ell$, so equality holds everywhere. Consequently, $(vE : vK) = p^k$ and $[Ev : Kv] = p^\ell$, which proves that the inclusions are equalities.

Now we take n to be the minimum of all $i \in \mathbb{N}$ such that $a_{\mu,i}$ is not among the $a_1, \ldots, a_k$. We set $\xi_E := 0$ if $n = 1$, and $\xi_E := \xi_{\mu,n-1}$ otherwise. Then $\eta_\mu - \xi_E \in E$. In contrast, the fact that η_μ is a pseudo limit, together with the first equation of (28), yields that

$$v(\eta_\mu - \xi_E) = v(\xi_{\mu,n} - \xi_E) = vc_{\mu,n} + \frac{1}{p} va_{\mu,n} \notin vK_\mu + \sum_{i=1}^{k} \frac{1}{p} va_i\mathbb{Z} = vE\,.$$

This contradiction proves that $(\xi_{\mu,n})_{n\in\mathbb{N}}$ admits no pseudo limit in $K'(\xi_\tau \mid \tau < \mu)$. Thus in particular, $\xi_\mu \notin K'(\xi_\tau \mid \tau < \mu)$. Since $K_{\mu+1} \subseteq K'$, $(\xi_{\mu,n})_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence in $(K'(\xi_\tau \mid \tau < \mu), v)$. As $[K'(\xi_\tau \mid \tau < \mu)(\xi_\mu) : K'(\xi_\tau \mid \tau < \mu)] = p$ is a prime, Lemma 2.12 shows that the extension $(K'(\xi_\tau \mid \tau \leq \mu)|K'(\xi_\tau \mid \tau < \mu), v)$ is immediate. As also the extension (26) is immediate, we find that the extension $(K'(\xi_\tau \mid \tau \leq \mu)|K', v)$ is immediate. This completes our induction step. Because every extension $L_\infty(\xi_\tau \mid \tau \leq \mu)|L_\infty(\xi_\tau \mid \tau < \mu)$ is nontrivial, it follows that the degree of $K^{1/p}|L_\infty$ is at least κ.

A simple modification of the above arguments allows us to show the assertion of part b) of the theorem in the case of $\kappa = [Kv : (Kv)^p]$, in which the set B is infinite. Let us describe these modifications.

We take a partition of B into κ many countably infinite sets B_τ, $\tau < \kappa$, and choose enumerations

$$B_\tau = \{b_{\tau,i} \mid i \in \mathbb{N}\}\,.$$

For every $\mu < \kappa$ we set $\mathcal{B}_\mu := \bigcup_{\tau<\mu} B_\tau$ and

$$K_\mu := K(b^{1/p} \mid b \in \mathcal{B}_\mu)\,.$$

Similarly as before, it is shown that

$$vK_\mu = vK \quad \text{and} \quad K_\mu v = Kv((bv)^{1/p}) \mid b \in \mathcal{B}_\mu)\,. \tag{29}$$

We choose a sequence $(c_i)_{i\in\mathbb{N}}$ of elements in K with strictly increasing values. Again, if the cofinality of vK is countable, then the elements c_i can be chosen in

such a way that the sequence of their values is cofinal in vK. For every $\tau < \kappa$ and $n \in \mathbb{N}$, we set

$$\xi_{\tau,n} \;:=\; \sum_{i=1}^{n} c_i b_{\tau,i}^{1/p} \;\in\; K_{\tau+1}\,. \tag{30}$$

Now the only further part of the proof that needs to be modified is the one that shows that $\eta_\mu \in E$, where η_μ is a pseudo limit of $(\xi_{\mu,n})_{n\in\mathbb{N}}$, leads to a contradiction. In the present case, we take n to be the minimum of all $i \in \mathbb{N}$ such that $b_{\mu,i}$ is not among the $b_1, \ldots, b_\ell$. As before, we set $\xi_E := 0$ if $n = 1$, and $\xi_E := \xi_{\mu,n-1}$ otherwise. Then $\eta_\mu - \xi_E \in E$. In contrast, the fact that η_μ is a pseudo limit, together with the second equation of (28), yields that

$$\begin{aligned} c_n^{-1}(\eta_\mu - \xi_E)v \;&=\; c_n^{-1}(\xi_{\mu,n} - \xi_E)v \;=\; (b_{\mu,n}^{1/p})v \;=\; (b_{\mu,n}v)^{1/p} \\ &\notin\; K_\mu v((b_1 v)^{1/p}, \ldots, (b_\ell v)^{1/p}) \;=\; Ev\,, \end{aligned}$$

a contradiction. This completes our modification and thereby the proof that the extension $(K^{1/p}|L_\infty, v)$ is of degree at least κ.

We now turn to part c) of the theorem. Again, we consider separately the cases of $\kappa = (vK : pvK)$ and of $\kappa = [Kv : (Kv)^p]$.

We assume first that $\kappa = (vK : pvK)$ and take a partition of A as in the proof of part b). Further, we set $s(1) = 0$ and $s(m) = 1 + 2 + \cdots + (m-1)$ for $m > 1$. For every $\tau < \mu$ and every $m \in \mathbb{N}$, we set

$$z_{\tau,m} \;:=\; \sum_{i=1}^{m} d_{\tau,s(m)+i} a_{\tau,s(m)+i}^{p^{-i}} \in K^{1/p^\infty}\,,$$

where $d_{\tau,j}$ are elements from K such that for every $m \in \mathbb{N}$,

1) the sequence $(vd_{\tau,s(m)+i} a_{\tau,s(m)+i}^{p^{-i}})_{1\le i\le m}$ is strictly increasing,

2) $vd_{\tau,s(m)+m} a_{\tau,s(m)+m}^{p^{-m}} < vd_{\tau,s(m+1)+1} a_{\tau,s(m+1)+1}^{p^{-1}}$.

If the cofinality of vK is countable, then the elements $d_{\tau,i}$ can be chosen in such a way that the sequence that results from the above is cofinal in vK.

We note that $z_{\tau,m}^{p^m} \in K$ with

$$[K(z_{\tau,m}) : K] \;=\; p^m \quad\text{and}\quad \frac{1}{p}va_{\tau,s(m)+1}, \ldots, \frac{1}{p}va_{\tau,s(m)+m} \in vK(z_{\tau,m})\,. \tag{31}$$

We set

$$L_\mu \;:=\; K(z_{\tau,m} \mid \tau < \mu,\, m \in \mathbb{N}) \;\text{ for } \mu \le \kappa\,, \text{ and } \quad L \;:=\; L_\kappa\,.$$

Further, we fix a maximal immediate extension (M, v) of (L, v). We claim that

$$vL_\mu \;=\; vK + \sum_{a\in\mathcal{A}_\mu} \frac{1}{p}va \quad\text{and}\quad L_\mu v \;=\; Kv\,. \tag{32}$$

In particular, this shows that

$$vL = \frac{1}{p}vK \quad\text{and}\quad Lv \;=\; Kv\,. \tag{33}$$

To prove our claim, we observe that the first inclusion "$\supseteq$" in (32) follows from (31). We choose any $\mu < \kappa$, $k \in \mathbb{N}$, $\tau_1, \ldots, \tau_k < \mu$ and $m_1, \ldots, m_k \in \mathbb{N}$ such that the pairs (τ_i, m_i), $1 \leq i \leq k$, are distinct. Then we compute, using (31):

$$\begin{aligned} p^{m_1} \cdot \ldots \cdot p^{m_k} &\geq [K(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k}) : K] \\ &\geq (vK(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k}) : vK)[K(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k})v : Kv] \\ &\geq (vK(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k}) : vK) \\ &\geq (vK + \sum_{j=1}^{k} \sum_{i=1}^{m_j} \frac{1}{p} v a_{\tau_j, s(m_j)+i}\mathbb{Z} : vK) \;\geq\; p^{m_1} \cdot \ldots \cdot p^{m_k}\,, \end{aligned}$$

showing that equality holds everywhere. Therefore,

$$vK(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k}) \;=\; vK + \sum_{j=1}^{k} \sum_{i=1}^{m_j} \frac{1}{p} v a_{\tau_j, s(m_j)+i}\mathbb{Z} \;\subseteq\; vK + \sum_{a \in \mathcal{A}_\mu} \frac{1}{p} va$$

and

$$K(z_{\tau_1,m_1}, \ldots, z_{\tau_k,m_k})v \;=\; Kv\,.$$

Since the value group and residue field of L_μ are the unions of the value groups and residue fields of all subfields of the above form, this proves our claim.

For every $\tau < \kappa$ and $n \in \mathbb{N}$, we set

$$\zeta_{\tau,n} := \sum_{m=1}^{n} z_{\tau,m} \in L.$$

Then $(\zeta_{\tau,n})_{n\in\mathbb{N}}$ is a pseudo Cauchy sequence in (L, v), hence it admits a pseudo limit ζ_τ in the maximal field (M, v). In order to show that the transcendence degree of $M|L$ is at least κ, we prove by induction that for every $\mu < \kappa$ and every field L' such that $L_{\mu+1} \subseteq L' \subseteq L$, the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ is of transcendental type over $L'(\zeta_\tau \mid \tau < \mu)$, so that the extension $(L'(\zeta_\tau \mid \tau \leq \mu)|L'(\zeta_\tau \mid \tau < \mu), v)$ is immediate and transcendental and then also the extension

$$(L'(\zeta_\tau \mid \tau \leq \mu)|L', v) \tag{34}$$

is immediate.

Take $\mu < \kappa$ and assume that our assertions have already been shown for all $\mu' < \mu$. If $\mu = \mu' + 1$ is a successor ordinal, then from (34) we readily get that the extension

$$(L'(\zeta_\tau \mid \tau < \mu)|L', v) \tag{35}$$

is immediate for every L' such that $L_\mu \subseteq L' \subseteq L$. If μ is a limit ordinal, then (35) follows from the induction hypothesis since $L_{\mu'} \subseteq L_\mu \subseteq L'$ for each $\mu' < \mu$ and since the union over an increasing chain of immediate extensions of (L', v) is again an immediate extension of (L', v).

In order to prove the induction step, take any L' such that $L_{\mu+1} \subseteq L' \subseteq L$. Suppose towards a contradiction that the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ in $L_{\mu+1}$ is of algebraic type over $(L'(\zeta_\tau \mid \tau < \mu), v)$ (which includes the case where it has a pseudo limit in $L'(\zeta_\tau \mid \tau < \mu)$). Then by Theorem 2.10 there exists an immediate algebraic extension $(L'(\zeta_\tau \mid \tau < \mu)(d)|L'(\zeta_\tau \mid \tau < \mu), v)$ with d a pseudo limit of the sequence. The element d is also algebraic over $L_\mu(\zeta_\tau \mid \tau < \mu)$. On the other hand, we will now show that from the fact that d is a pseudo limit of

$(\zeta_{\mu,n})_{n\in\mathbb{N}}$ it follows that the value group $vL_\mu(\zeta_\tau \mid \tau<\mu)(d)$ is an infinite extension of $vL_\mu(\zeta_\tau \mid \tau<\mu)$. Take $n\in\mathbb{N}$ and define

$$\eta_{\mu,n} := \zeta_{\mu,n}^{p^{n-1}} - d_{\mu,s(n)+n}^{p^{n-1}} a_{\mu,s(n)+n}^{1/p} = \zeta_{\mu,n-1}^{p^{n-1}} + \sum_{i=1}^{n-1} d_{\mu,s(n)+i}^{p^{n-1}} a_{\mu,s(n)+i}^{p^{n-1-i}} \in K\,.$$

Since d is a pseudo limit of the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$, we deduce that

$$v(d-\zeta_{\mu,n}) = vz_{\mu,n+1} = vd_{\mu,s(n+1)+1}a_{\mu,s(n+1)+1}^{1/p} > vd_{\mu,s(n)+n}a_{\mu,s(n)+n}^{p^{-n}}\,. \tag{36}$$

Therefore,

$$\begin{aligned} v\left(d^{p^{n-1}} - \eta_{\mu,n}\right) &= v\left(d^{p^{n-1}} - \zeta_{\mu,n}^{p^{n-1}} + d_{\mu,s(n)+n}^{p^{n-1}} a_{\mu,s(n)+n}^{1/p}\right) \\ &= p^{n-1} v\left(d - \zeta_{\mu,n} + d_{\mu,s(n)+n} a_{\mu,s(n)+n}^{p^{-n}}\right) \\ &= p^{n-1}\min\left\{v(d-\zeta_{\mu,n})\,,\, v\left(d_{\mu,s(n)+n}a_{\mu,s(n)+n}^{p^{-n}}\right)\right\} \\ &= p^{n-1} v\left(d_{\mu,s(n)+n}a_{\mu,s(n)+n}^{p^{-n}}\right) \\ &= p^{n-1} v d_{\mu,s(n)+n} + \frac{1}{p} v a_{\mu,s(n)+n}\,, \end{aligned}$$

which shows that

$$\frac{1}{p} v a_{\mu,s(n)+n} \in vL_\mu(\zeta_\mu \mid \mu<\tau)(d)$$

for all $n\in\mathbb{N}$. In view of (32), these values are not in vL_μ. Since the extension (35) is immediate for L_μ in place of L', they are also not in $vL_\mu(\zeta_\tau \mid \tau<\mu)$. It follows that the index $(vL_\mu(\zeta_\tau \mid \tau<\mu)(d) : vL_\mu(\zeta_\tau \mid \tau<\mu))$ is infinite. This contradicts the fact that the extension $L_\mu(\zeta_\tau \mid \tau<\mu)(d)|L_\mu(\zeta_\tau \mid \tau<\mu)$ is finite. This contradiction proves that the pseudo Cauchy sequence $(\zeta_{\mu,n})_{n\in\mathbb{N}}$ is of transcendental type over $L'(\zeta_\tau \mid \tau<\mu)$. From Theorem 2.9 it follows that $(L'(\zeta_\tau \mid \tau\le\mu)|L'(\zeta_\tau \mid \tau<\mu), v)$ is an immediate transcendental extension. Since the extension (35) is immediate, we obtain that also $(L'(\zeta_\tau \mid \tau\le\mu)|L', v)$ is immediate.

This completes our induction step. By induction on μ we have therefore shown that $(L(\zeta_\tau \mid \tau<\mu), v)$ is an immediate extension of (L,v) for each $\mu<\kappa$, which yields that also the union $(L(\zeta_\tau \mid \tau<\kappa), v)$ of these fields is an immediate extension of (L,v). As every extension $L(\zeta_\tau \mid \tau\le\mu)|L(\zeta_\tau \mid \tau<\mu)$ is transcendental, the transcendence degree of $L(\zeta_\tau \mid \tau<\kappa)$ over L is at least κ.

A simple modification of the above arguments allows us to show the assertion of part c) of the theorem in the case of $\kappa=[Kv:(Kv)^p]$. We take the partition of B as in the proof of part b). We now list the modifications.

Since the $vb=0$ for all $b\in B$, the only requirement for the elements $d_{\tau,i}$ that we need is that $vd_{\tau,i}<vd_{\tau,j}$ for $i<j$. If the cofinality of vK is countable, then the elements $d_{\tau,i}$ can be chosen in such a way that the sequence of their values is cofinal in vK. We set

$$z_{\tau,m} := \sum_{i=1}^{m} d_{\tau,s(m)+i} b_{\tau,s(m)+i}^{p^{-i}} \in K^{1/p^\infty}\,,$$

Equation (31) is replaced by

$$[K(z_{\tau,m}):K] = p^m \quad\text{and}\quad (b_{\tau,s(m)+1}v)^{1/p},\dots,(b_{\tau,s(m)+m}v)^{1/p} \in K(z_{\tau,m})v\,. \tag{37}$$

One proves in a similar way as before that

$$vL_\mu \;=\; vK \quad \text{and} \quad L_\mu v \;=\; Kv((bv)^{1/p} \mid b \in \mathcal{B}_\mu)\,. \tag{38}$$

In particular, this shows that

$$vL = vK \quad \text{and} \quad Lv \;=\; (Kv)^{1/p}\,. \tag{39}$$

Now the only further part of the proof that needs to be modified is the one that shows that the extension $L_\mu(\zeta_\tau \mid \tau < \mu)(d)|L_\mu(\zeta_\tau \mid \tau < \mu)$ cannot be finite. We define $\eta_{\mu,n}$ as before, with "b" in place of "a". Also (36) holds with "b" in place of "a", whence

$$v\, d_{\mu,s(n)+n}^{-p^{n-1}} \left(d^{p^{n-1}} - \zeta_{\mu,n}^{p^{n-1}}\right) \;=\; p^{n-1} v d_{\mu,s(n)+n}^{-1}\,(d - \zeta_{\mu,n}) \;>\; 0\,.$$

This leads to

$$\begin{aligned} d_{\mu,s(n)+n}^{-p^{n-1}} \left(d^{p^{n-1}} - \eta_{\mu,n}\right) v \;&=\; d_{\mu,s(n)+n}^{-p^{n-1}} \left(d^{p^{n-1}} - \zeta_{\mu,n}^{p^{n-1}} + d_{\mu,s(n)+n}^{p^{n-1}} b_{\mu,s(n)+n}^{1/p}\right) v \\ &=\; \left(d_{\mu,s(n)+n}^{-p^{n-1}} (d^{p^{n-1}} - \zeta_{\mu,n}^{p^{n-1}}) + b_{\mu,s(n)+n}^{1/p}\right) v \\ &=\; (b_{\mu,s(n)+n}^{1/p}) v \;=\; (b_{\mu,s(n)+n} v)^{1/p}\,, \end{aligned}$$

which shows that

$$(b_{\mu,s(n)+n} v)^{1/p} \;\in\; L_\mu(\zeta_\tau \mid \tau < \mu)(d)v$$

for all $n \in \mathbb{N}$. In view of (38), these residues are not in $L_\mu v$. As before, this is shown to contradict d being algebraic over $L_\mu(\zeta_\tau \mid \tau < \mu)$. This completes our modification and thereby the proof that $(M|L, v)$ is of transcendence degree at least κ. □

We now come to the proof of

Proof of Theorem 1.5:

Note that a field (K, v) which satisfies the assumptions of Theorem 1.5 also satisfies the assumptions of Theorem 1.4. We choose the sets $A, B \subseteq K^{1/p}$ and define $L := L_\infty$ as in the proof of part b) of Theorem 1.4. Then, as we have already seen, $(K^{1/p}, v)$ is a maximal immediate extension of (L, v).

To show the existence of an immediate extension of L of infinite transcendence degree over L, we consider separately the cases i) and ii) of the theorem. We assume first that the conditions of case i) hold. Then the set A can be chosen so as to contain an infinite countable subset A' such that the set of values $S = \{va \mid a \in A'\}$ is bounded. It must contain a bounded infinite strictly increasing or a bounded infinite strictly decreasing sequence. If it does not contain the former, we replace A' by $\{a^{-1} \mid a \in A'\}$, thereby passing from S to $-S$. Note that in our proof we will not need that $A' \subseteq A$; we will only use that $A' \subseteq L$. Now we can choose a sequence $(a_j)_{j\in\mathbb{N}}$ of elements in A' such that the sequence $(va_j)_{j\in\mathbb{N}}$ is strictly increasing and bounded by some $\gamma \in vK$. We partition the sequence $(a_j)_{j\in\mathbb{N}}$ into countably many subsequences

$$(a_{N,i})_{i\in\mathbb{N}} \qquad (N \in \mathbb{N})\,.$$

As in the proof of Theorem 1.4, we define $K_N := K(a_{n,i} \mid n < N\,,\, i \in \mathbb{N}) \subseteq K^{1/p}$.

For every $N \in \mathbb{N}$ we consider the pseudo Cauchy sequence $(\xi_{N,m})_{m\in\mathbb{N}}$ defined by

$$\xi_{N,m} \;:=\; \sum_{i=1}^{m} a_{N,i}^{1/p} \;\in\; K_{N+1}\,.$$

and the pseudo limits ξ_N of the sequences in the maximal immediate extension $(K^{1/p}, v)$ of (L, v). We show that for every N the pseudo limit ξ_N does not lie in the completion L^c of (L, v). Fix $N \in \mathbb{N}$ and take any $d \in L$. Then d lies already in some finite extension

$$E \;:=\; K(a_1^{1/p}, \ldots, a_k^{1/p}, b_1^{1/p}, \ldots, b_l^{1/p})$$

of K in L. Choosing ξ_E as in the proof of Theorem 1.4, we obtain that $\xi_E - d \in E$. But from equalities (28) with $\mu = 0$ it follows that $v(\xi_N - \xi_E) = \frac{1}{p} v a_{N,n} \notin vE$. Thus,

$$v(\xi_N - d) \;=\; \min\{v(\xi_N - \xi_E), v(\xi_E - d)\} \leq \frac{1}{p}\, v a_{N,n} < \frac{1}{p}\, \gamma\,.$$

Hence the values $v(\xi_N - d)$, $d \in L$, are bounded by $\frac{1}{p}\gamma$ and consequently, $\xi_N \notin L^c$.

Again from the proof of Theorem 1.4 it follows that for every field K' such that $K_1 \subseteq K' \subseteq L$ the extension $(K'(\xi_1)|K', v)$ is immediate and purely inseparable of degree p. Since $\xi_1 \notin L^c$, from Proposition 2.13 we deduce that for an element $d_1 \in K^\times$ satisfying inequality (8) with $\eta = \xi_1$, a root ϑ_1 of the polynomial

$$f_1 \;:=\; X^p - X - \left(\frac{\xi_1}{d_1}\right)^p$$

generates an immediate Galois extension $(L(\vartheta_1)|L, v)$ of degree p with a unique extension of the valuation v from L to $L(\vartheta_1)$. Take any field K' such that $K_1 \subseteq K' \subseteq L$. Then $\xi_1 \notin K'^c$ and the element d_1 satisfies inequality (8) with every element $c \in K'$. Therefore also $(K'(\vartheta_1)|K', v)$ is an immediate extension of degree p with a unique extension of v from K' to $K'(\vartheta_1)$.

Take any $m > 1$. Suppose that we have shown that for every $l < m$ there is $d_l \in K^\times$ such that a root ϑ_l of the polynomial

$$f_l \;:=\; X^p - X - \left(\frac{\xi_l}{d_l}\right)^p$$

generates, for any field K' with $K_{l+1} \subseteq K' \subseteq L$, an immediate Galois extension $(K'(\vartheta_1, \ldots, \vartheta_l)|K'(\vartheta_1, \ldots, \vartheta_{l-1}), v)$ of degree p with a unique extension of the valuation v from $K'(\vartheta_1, \ldots, \vartheta_{l-1})$ to $K'(\vartheta_1, \ldots, \vartheta_l)$. Then in particular, the extension $(K_{m+1}(\vartheta_1, \ldots, \vartheta_{m-1})|K_{m+1}, v)$ is immediate. Take any field K' such that $K_{m+1} \subseteq K' \subseteq L$. Replacing in the argumentation of the proof of part b) of Theorem 1.4 the field $K'(\xi_l \mid l < m)$ by $K'(\vartheta_l \mid l < m)$, we deduce that $(K'(\vartheta_1, \ldots, \vartheta_{m-1})(\xi_m)|K'(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ is an immediate purely inseparable extension of degree p. Since $\xi_m \notin L^c$ and $L(\vartheta_1, \ldots, \vartheta_{m-1})^c = L^c(\vartheta_1, \ldots, \vartheta_{m-1})$ is a separable extension of L, linearly disjoint from the purely inseparable extension $L^c(\xi_m)|L^c$, we obtain that

$$[L^c(\vartheta_1, \ldots, \vartheta_{m-1})(\xi_m) : L^c(\vartheta_1, \ldots, \vartheta_{m-1})] \;=\; p.$$

Therefore, ξ_m does not lie in $L(\vartheta_1, \ldots, \vartheta_{m-1})^c$. Thus, from Proposition 2.13 it follows that for an element $d_m \in K^\times$ satisfying inequality (8) with $\eta = \xi_m$, a root ϑ_m of the polynomial $f_m := X^p - X - (\xi_m/d_m)^p$ generates an immediate Galois extension $(L(\vartheta_1, \ldots, \vartheta_m)|L(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ of degree p with a unique extension of the valuation v from $L(\vartheta_1, \ldots, \vartheta_{m-1})$ to $L(\vartheta_1, \ldots, \vartheta_m)$. As in the case of $m = 1$ we deduce that also the extension $(K'(\vartheta_1, \ldots, \vartheta_m)|K'(\vartheta_1, \ldots, \vartheta_{m-1}), v)$ is immediate and the valuation v of $K'(\vartheta_1, \ldots, \vartheta_{m-1})$ extends uniquely to $K'(\vartheta_1, \ldots, \vartheta_m)$.

By induction, we obtain an infinite immediate separable-algebraic extension $F := L(\vartheta_m \mid m \in \mathbb{N})$ of L with the unique extension of the valuation v of L to F. Thus, the extension $F|L$ is linearly disjoint from $L^h|L$. From the separable algebraic case of Theorem 1.1 it follows that each maximal immediate extension (M, v) of (F, v) has infinite transcendence degree over F. Since $(F|L, v)$ is immediate, M is also a maximal immediate extension of L.

Similar arguments allow us to prove the assertion in the case of an infinite residue field extension $Kv|(Kv)^p$ when the value group vK is not discrete. Let us describe the modifications.

Take an infinite countable subset B' of B and an infinite partition of B into infinite sets

$$B_N \;=\; \{b_{N,i} \mid i \in \mathbb{N}\} \qquad (N \in \mathbb{N})\,.$$

Since vK is not discrete, we can choose elements $c_i \in K$ such that the sequence $(vc_i)_{i\in\mathbb{N}}$ of their values is strictly increasing and bounded by some element $\gamma \in vK$. For every N we consider the pseudo Cauchy sequence $(\xi_{N,m})_{m\in\mathbb{N}}$ defined by (30).

The only further part of the proof that needs to be modified is the one that shows that $\xi_N \notin L^c$. More precisely, we need to show that for any element $d \in E$ we have that $v(\xi_N - d) < \gamma$. Take ξ_E as in the second case of the proof of part b) of Theorem 1.4. From the equalities (28) with $\mu = 0$ we deduce that $c_n^{-1}(\xi_N - \xi_E)v = (b_{N,n}v)^{1/p} \notin Ev$. Suppose that $v(\xi_N - d) > v(\xi_n - \xi_E)$. Then

$$v\left(c_n^{-1}(\xi_N - \xi_E) - c_n^{-1}(\xi_E - d)\right) \;>\; vc_n^{-1}(\xi_N - \xi_E) \;=\; 0.$$

It follows that $c_n^{-1}(\xi_N - \xi_E)v = c_n^{-1}(\xi_E - d)v \in Ev$, a contradiction. Consequently,

$$v(\xi_N - d) \;\leq\; v(\xi_N - \xi_E) \;=\; vc_n \;<\; \gamma\,.$$

This completes our modification and thereby the proof that (L, v) admits a maximal immediate extension of infinite transcendence degree over L. □

Finally, we give the

Proof of Theorem 1.6:

Take an extension $(L|K, v)$ as in the assumptions of the theorem. In view of the value-algebraic and residue-algebraic cases of Theorem 1.1, it suffices to show that at least one of the extensions $vL|vK$ or $Lv|Kv$ is infinite.

Take K' to be the relative algebraic closure of K in L^h. By the assumptions on the residue field and value group extensions of $(L|K, v)$, it follows from Lemma 2.4 that $vK' = vL^h = vL$ and $K'v = L^hv = Lv$. Therefore, $(L^h|K', v)$ is an immediate transcendental extension.

Suppose that the value group extension and the residue field extension of $(L|K, v)$ and hence of $(K'|K, v)$ were finite. But since K is henselian and a defectless field by Theorem 2.1, the degree $[K' : K]$ is equal to $(vK' : vK)[K'v : Kv]$ and hence would be finite, so (K', v) would again be a maximal field, which contradicts the fact that $(L^h|K', v)$ is a nontrivial immediate extension. □

References

[1] Bourbaki, N.: *Commutative algebra*, Paris (1972)
[2] Endler, O.: *Valuation theory*, Springer, Berlin (1972)
[3] Engler, A.J. – Prestel, A.: *Valued fields*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005
[4] Gravett, K. A. H.: *Note on a result of Krull*, Cambridge Philos. Soc. Proc. **52** (1956), 379
[5] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321
[6] Kuhlmann, F.–V.: *Valuation theoretic and model theoretic aspects of local uniformization*, in: Resolution of Singularities - A Research Textbook in Tribute to Oscar Zariski. Herwig Hauser, Joseph Lipman, Frans Oort, Adolfo Quiros (eds.), Progress in Mathematics Vol. **181**, Birkhäuser Verlag Basel (2000), 381–456
[7] Kuhlmann, F.-V.: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. **356** (2004), 4559-4600
[8] Kuhlmann, F.-V.: *A classification of Artin-Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448
[9] Kuhlmann, F.–V.: *Valuation Theory*, book in preparation. Preliminary versions of several chapters are available on the web site: http://math.usask.ca/˜fvk/Fvkbook.htm.
[10] Kuhlmann, F.-V., Pank, M., Roquette, P.: *Immediate and purely wild extensions of valued fields*, Manuscripta math., **55** (1986), 39–67
[11] Krull, W.: *Allgemeine Bewertungstheorie*, J. reine angew. Math. **167** (1931), 160–196
[12] MacLane, S. – Schilling, O.F.G.: *Zero-dimensional branches of rank 1 on algebraic varieties*, Annals of Math. **40** (1939), 507–520
[13] Nagata, M.: *Local rings*, Wiley Interscience, New York (1962)
[14] Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal, 1st ed. (1964), 2nd ed. (1968)
[15] Warner, S.: *Nonuniqueness of immediate maximal extensions of a valuation*, Math. Scand. **56** (1985), 191–202
[16] Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam (1989)
[17] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin (1960)

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