VALUE GROUPS, RESIDUE FIELDS AND BAD PLACES OF RATIONAL FUNCTION FIELDS

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Abstract. We classify all possible extensions of a valuation from a ground field K to a rational function field in one or several variables over K. We determine which value groups and residue fields can appear, and we show how to construct extensions having these value groups and residue fields. In particular, we give several constructions of extensions whose corresponding value group and residue field extensions are not finitely generated. In the case of a rational function field $K(x)$ in one variable, we consider the relative algebraic closure of K in the henselization of $K(x)$ with respect to the given extension, and we show that this can be any countably generated separable-algebraic extension of K. In the "tame case", we show how to determine this relative algebraic closure. Finally, we apply our methods to power series fields and the p-adics.

1. Introduction

In this paper, we denote a valued field by (K, v), its value group by vK, and its residue field by Kv. When we write $(L|K, v)$ we mean a field extension $L|K$ endowed with a valuation v on L and its restriction on K.

In many recent applications of valuation theory, valuations on algebraic function fields play a main role. To mention only a short and incomplete list of applications and references: local uniformization and resolution of singularities ([C], [CP], [S], [KU3], [KKU1,2]), model theory of valued fields ([KU1,2,4]), study of curves via constant reduction ([GMP1,2], [PL]), classification of all extensions of an ordering from a base field to a rational function field ([KUKMZ]), Gröbner bases ([SW], [MOSW1,2]).

In many cases, a basic tool is the classification of all extensions of a valuation from a base field to a function field. As the classification of all extensions of a valuation from a field to an algebraic extension is taken care of by general ramification theory (cf. [E], [KU2]), a crucial step in the classification is the case of rational function fields. Among the first papers describing valuations on rational function fields systematically were [M] and [MS]. Since then, an impressive number of papers have been written about the construction of such valuations and about their properties; the following list is by no means exhaustive: [AP], [APZ1-3], [KH1-10], [KHG1-6], [KHPR], [MO1,2], [MOSW1], [O1-3], [PP], [V]. From the paper [APZ3] the reader may get a good idea of how MacLane's original approach has been developed further. Since then, the notion of "minimal pairs" has been adopted and studied by several authors (see, e.g., [KHPR]). In the present paper, we will develop a new approach to this subject. It serves to determine in full generality which value groups

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and which residue fields can possibly occur. This question has recently played a role in two other papers:

1) In [KU3], we prove the existence of "bad places" on rational function fields of transcendence degree ≥ 2. These are places whose value group is not finitely generated, or whose residue field is not finitely generated over the base field. The existence of such places has been shown by MacLane and Schilling ([MS]) and by Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4). However, our approach in [KU3] using Hensel's Lemma seems to be new, and the present paper contains a further refinement of it. The following theorem of [MS] and [ZS] is a special case of a result which we will prove by this refinement:

Theorem 1.1. *Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and k to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation whose restriction to K is trivial, whose value group is Γ and whose residue field is k.*

In particular, every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of K can be realized as value group and residue field of a place of the rational function field $K(x,y)|K$ whose restriction to K is the identity.

The rational rank of an abelian group Γ is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$. We denote it by $\operatorname{rr}\Gamma$. It is equal to the cardinality of any maximal set of rationally independent elements in Γ.

Bad places on function fields are indeed bad: the value group or residue field not being finitely generated constitutes a major hurdle for the attempt to prove local uniformization or other results which are related to resolution of singularities (cf. [CP]). Another hurdle is the phenomenon of **defect** which can appear when the residue characteristic of a valued field is positive, even if the field itself seems to be quite simple. Indeed, we will prove in Section 3.3, and by a different method in Section 3.5:

Theorem 1.2. *Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial, such that $(K(x,y),v)$ admits an infinite chain of immediate Galois extensions of degree p and defect p.*

An extension $(L'|L,v)$ of valued fields is called **immediate** if the canonical embeddings of vL in vL' and of Lv in $L'v$ are surjective (which we will express by writing $vL' = vL$ and $L'v = Lv$). For a finite immediate extension $(L'|L,v)$, its defect is equal to its degree if and only if the extension of v from L to L' is unique (or equivalently, $L'|L$ is linearly disjoint from some (or every) henselization of (L,v)).

One of the examples we shall construct for the proof of the above theorem is essentially the same as in Section 7 of [CP], but we use a different and more direct approach (while the construction in [CP] is more intricate since it serves an additional purpose).

2) In [KUKMZ], the classification of all extensions of an ordering to a rational function field is considered in the context of power series fields, and the above question is partially answered in this setting. In the present paper, we will consider the question without referring to power series fields (see Theorem 1.8 below).

During the preparation of [KUKMZ], we found that the construction of an extension of the valuation v from K to the rational function field $K(x)$ with prescribed value group $vK(x)$ and residue field $K(x)v$ is tightly connected with the determination of the relative algebraic closure of K in a henselization $K(x)^h$ of $K(x)$ with respect to v. In earlier papers, we have introduced the name "henselian function field" for the henselizations of valued function fields (although these are not function fields, unless the valuation is trivial). In the same vein, one can view the relative algebraic closure as being the (exact) constant field of the henselian function field $(K(x)^h|K,v)$. We will call it the **implicit constant field of** $(K(x)|K,v)$ and denote it by $\mathrm{IC}\,(K(x)|K,v)$. Clearly, the henselization $K(x)^h$ depends on the valuation which has been fixed on the algebraic closure $\widetilde{K(x)}$. So whenever we will talk about the implicit constant field, we will do it in a setting where the valuation on $\widetilde{K(x)}$ has been fixed. However, since the henselization L^h of any valued field (L,v) is unique up to valuation preserving isomorphism over L, the implicit constant field is unique up to valuation preserving isomorphism over K. If L_0 is a subfield of L, then L^h contains a (unique) henselization of L_0. Hence, $\mathrm{IC}\,(K(x)|K,v)$ contains a henselization of K and is itself henselian. Further, $L^h|L$ is a separable-algebraic extension; thus, $K(x)^h|K$ is separable. Therefore, $\mathrm{IC}\,(K(x)|K,v)$ is a separable-algebraic extension of K.

In the present paper, we answer the above question on value groups and residue fields by determining which prescribed separable-algebraic extensions of K can be realized as implicit constant fields. The following result shows in particular that every countably generated separable-algebraic extension of a henselian base field can be realized:

Theorem 1.3. *Let $(K_1|K,v)$ be a countable separable-algebraic extension of non-trivially valued fields. Then there is an extension of v from K_1 to the algebraic closure $\widetilde{K_1(x)}=\widetilde{K(x)}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(\widetilde{K(x)},v)$,*

$$K_1^h \;=\; \mathrm{IC}\,(K(x)|K,v)\;. \tag{1.1}$$

In Section 3.1 we will introduce a basic classification ("value-transcendental" – "residue-transcendental" – "valuation-algebraic") of all possible extensions of v from K to $K(x)$. In Section 3.2 we introduce a class of extensions $(K(x)|K,v)$ for which $\mathrm{IC}\,(K(x)|K,v)=K^h$ holds. Building on this, we prove Theorem 1.3 in Section 3.5. In fact, we prove a more detailed version: we show under which additional conditions the extension can be chosen in a prescribed class of the basic classification. This yields the following

Theorem 1.4. *Take any valued field (K,v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0\oplus\mathbb{Z}$ endowed with any extension of the ordering of Γ_0.*

Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0.

Now assume that v is non-trivial on K and that Γ_0/vK and $k_0|Kv$ are countably generated. Suppose that at least one of them is infinite or that (K,v) admits an immediate transcendental extension. Then there is an extension of v from K to $K(x)$ which has value group Γ_0 and residue field k_0.

Here is the converse:

Theorem 1.5. *Let $(K(x)|K,v)$ be a valued rational function field. Then one and only one of the following three cases holds:*
1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$, where $\Gamma_0|vK$ is a finite extension of ordered abelian groups, and $K(x)v|Kv$ is finite;
2) $vK(x)/vK$ is finite, and $K(x)v$ is a rational function field in one variable over a finite extension of Kv;
3) $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic.
In all cases, $vK(x)/vK$ is countable and $K(x)v|Kv$ is countably generated.

In 2), we use a fact which was proved by J. Ohm [O2] and is known as the "Ruled Residue Theorem": *If $K(x)v|Kv$ is transcendental, then $K(x)v$ is a rational function field in one variable over a finite extension of Kv.* For the countability assertion, see Theorem 2.9.

In Section 3.4 we give an explicit description of all possible extensions of v from K to $K(x)$ (Theorem 3.11).

Theorem 1.4 is used in the proof of our next theorem:

Theorem 1.6. *Let (K,v) be any valued field, n,ρ,τ non-negative integers, $n \geq 1$, $\Gamma \neq \{0\}$ an ordered abelian group extension of vK such that Γ/vK is of rational rank ρ, and $k|Kv$ a field extension of transcendence degree τ.*

Part A. *Suppose that $n > \rho + \tau$ and that*
A1) Γ/vK and $k|Kv$ are countably generated,
A2) Γ/vK or $k|Kv$ is infinite.
Then there is an extension of v to the rational function field $K(x_1,\ldots,x_n)$ in n variables such that

$$vK(x_1,\ldots,x_n) \;=\; \Gamma \quad \textit{and} \quad K(x_1,\ldots,x_n)v \;=\; k\;. \tag{1.2}$$

Part B. *Suppose that $n \geq \rho + \tau$ and that*
B1) Γ/vK and $k|Kv$ are finitely generated,
B2) if v is trivial on K, $n = \rho+\tau$ and $\rho = 1$, then k is a simple algebraic extension of a rational function field in τ variables over Kv (or of Kv itself if $\tau = 0$), or a rational function field in one variable over a finitely generated field extension of Kv of transcendence degree $\tau - 1$,
B3) if $n = \tau$, then k is a rational function field in one variable over a finitely generated field extension of Kv of transcendence degree $\tau - 1$,
B4) if $\rho = 0 = \tau$, then there is an immediate extension of (K,v) which is either infinite separable-algebraic linearly disjoint from the henselization of (K,v), or of transcendence degree at least n.
Then again there is an extension of v to $K(x_1,\ldots,x_n)$ such that (1.2) holds.

Theorem 1.1 is the special case of Part A for v trivial on K. The following converse holds:

Theorem 1.7. *Let $n \geq 1$ and v be a valuation on the rational function field $F = K(x_1, \ldots, x_n)$. Set $\rho = \mathrm{rr}\, vF/vK$ and $\tau = \mathrm{trdeg}\, Fv|Kv$. Then $n \geq \rho + \tau$, vF/vK is countable, and $Fv|Kv$ is countably generated.*

If $n = \rho + \tau$, then vF/vK is finitely generated and $Fv|Kv$ is a finitely generated field extension. Assertions B2) and B3) of Theorem 1.6 hold for $k = Fv$, and if $\rho = 0 = \tau$, then there is an immediate extension of $(\tilde{K}, v)$ of transcendence degree n (for any extension of v from K to $\tilde{K}$).

There is a gap between Theorem 1.6 and this converse for the case of $\rho = 0 = \tau$, as the former talks about K and the latter talks about the algebraic closure $\tilde{K}$ of K. This gap can be closed if (K, v) has residue characteristic 0 or is a Kaplansky field; because the maximal immediate extension of such fields is unique up to isomorphism, one can show that $\tilde{K}$ can be replaced by K. But in the case where (K, v) is not such a field, we do not know enough about the behaviour of maximal immediate extensions under algebraic field extensions. This question should be considered in future research.

A valuation on an ordered field is called **convex** if the associated valuation ring is convex. For the case of ordered fields with convex valuations, we can derive from Theorem 1.6 the existence of convex extensions of the valuation with prescribed value groups and residue fields in the frame given by Theorem 1.7, provided that a natural additional condition for the residue fields is satisfied:

Theorem 1.8. *In the setting of Theorem 1.6, assume in addition that K is ordered and that v is convex w.r.t. the ordering. Assume further that k is equipped with an extension of the ordering induced by $<$ on Kv. Then this extension can be lifted through v to $K(x_1, \ldots, x_n)$ in such a way that the lifted ordering extends $<$. It follows that v is convex w.r.t. this lifted ordering on $K(x_1, \ldots, x_n)$.*

In Section 5 we shall introduce "homogeneous sequences". In the "tame case", they can be used to determine the implicit constant field of a valued rational function field, and also to characterize this "tame case". In Section 6 we shall show how to apply our results to power series, in the spirit of [MS] and [ZS] (Theorem 6.1). We will also use our approach to give proofs of two well known facts in p-adic algebra: that the algebraic closure of $\mathbb{Q}_p$ is not complete and that its completion is not maximal.

Finally, let us mention that we use our criteria for $\mathrm{IC}\,(K(x)|K, v) = K^h$ in Section 3.2 to give an example for the following fact: *Suppose that K is relatively algebraically closed in a henselian valued field (L, v) such that vL/vK is a torsion group. Then it is not necessarily true that $vL = vK$, even if v has residue characteristic* 0.

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2. Notation and valuation theoretical preliminaries

For an arbitrary field K, will will denote by K^{sep} the separable-algebraic closure of K, and by $\tilde{K}$ the algebraic closure of K. By $\mathrm{Gal}\, K$ we mean the absolute Galois group $\mathrm{Gal}(\tilde{K}|K) = \mathrm{Gal}(K^{\mathrm{sep}}|K)$. For a valuation v on K, we let $\mathcal{O}_K$ denote the valuation ring of v on K.

Every finite extension $(L|K,v)$ of valued fields satisfies the **fundamental inequality** (cf. [E]):

$$n \;\geq\; \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{2.1}$$

where $n = [L : K]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i = (v_i L : vK)$ are the respective ramification indices and $\mathrm{f}_i = [Lv_i : Kv]$ are the respective inertia degrees. Note that $\mathrm{g} = 1$ if (K,v) is henselian.

In analogy to field theory, an extension $\Gamma \subset \Delta$ of abelian groups will also be written as $\Delta|\Gamma$, and it will be called **algebraic** if Δ/Γ is a torsion group. The fundamental inequality implies the following well known fact:

Lemma 2.1. *If $(L|K,v)$ is finite, then so are vL/vK and $Lv|Kv$. If $(L|K,v)$ is algebraic, then so are vL/vK and $Lv|Kv$.*

Given two subextensions $M|K$ and $L|K$ within a fixed extension $N|K$, the **field compositum** $M.L$ is defined to be the smallest subfield of N which contains both M and L. If $L|K$ is algebraic, the compositum is uniquely determined by taking $N = \tilde{M}$ and specifying a K-embedding of L in $\tilde{M}$.

Lemma 2.2. *Let $(M|K,v)$ be an immediate extension of valued fields, and $(L|K,v)$ a finite extension such that $[L : K] = (vL : vK)[Lv : Kv]$. Then for every K-embedding of L in $\tilde{M}$ and every extension of v from M to $\tilde{M}$, the extension $(M.L|L,v)$ is immediate.*

Proof. Via the embedding, we identify L with a subfield of $\tilde{M}$. Pick any extension of v from M to $\tilde{M}$. This will also be an extension of v from L to $\tilde{M}$ because by the fundamental inequality, the extension of v from K to L is unique. We consider the extension $(M.L|M,v)$. It is clear that $vL \subseteq v(M.L)$ and $Lv \subseteq (M.L)v$; therefore, $(v(M.L) : vK) \geq (vL : vK)$ and $[(M.L)v : Kv] \geq [Lv : Kv]$. Since $(M|K,v)$ is immediate, we have

$$\begin{aligned}
[M.L : M] \;&\geq\; (v(M.L) : vM)[(M.L)v : Mv] \;=\; (v(M.L) : vK)[(M.L)v : Kv] \\
&\geq\; (vL : vK)[Lv : Kv] \;=\; [L : K] \;\geq\; [M.L : M]\,.
\end{aligned}$$

This shows that $(v(M.L) : vK) = (vL : vK)$ and $[(M.L)v : Kv] = [Lv : Kv]$, that is, $v(M.L) = vL$ and $(M.L)v = Lv$. □

2.1. **Pseudo Cauchy sequences.** We assume the reader to be familiar with the theory of pseudo Cauchy sequences as presented in [KA]. Recall that a pseudo Cauchy sequence $\mathbf{A} = (a_\nu)_{\nu<\lambda}$ in (K,v) (where λ is some limit ordinal) is **of transcendental type** if for every $g(x) \in K(x)$, the value $vg(a_\nu)$ is eventually constant, that is, there is some $\nu_0 < \lambda$ such that

$$vg(a_\nu) \;=\; vg(a_{\nu_0}) \quad \text{for all } \; \nu \geq \nu_0\,,\; \nu < \lambda\,. \tag{2.2}$$

Otherwise, $\mathbf{A}$ is **of algebraic type**.

Take a pseudo Cauchy sequence $\mathbf{A}$ in (K,v) of transcendental type. We define an extension $v_{\mathbf{A}}$ of v from K to the rational function field $K(x)$ as follows. For each $g(x)\in K[x]$, we choose $\nu_0<\lambda$ such that (2.2) holds. Then we set

$$v_{\mathbf{A}}\, g(x) \;:=\; vg(a_{\nu_0})\;.$$

We extend $v_{\mathbf{A}}$ to $K(x)$ by setting $v_{\mathbf{A}}(g/h):=v_{\mathbf{A}}g-v_{\mathbf{A}}h$. The following is Theorem 2 of [KA]:

Theorem 2.3. *Let $\mathbf{A}$ be a pseudo Cauchy sequence in (K,v) of transcendental type. Then $v_{\mathbf{A}}$ is a valuation on the rational function field $K(x)$. The extension $(K(x)|K,v_{\mathbf{A}})$ is immediate, and x is a pseudo limit of $\mathbf{A}$ in $(K(x),v_{\mathbf{A}})$. If $(K(y),w)$ is any other valued extension of (K,v) such that y is a pseudo limit of $\mathbf{A}$ in $(K(y),w)$, then $x\mapsto y$ induces a valuation preserving K-isomorphism from $(K(x),v_{\mathbf{A}})$ onto $(K(y),w)$.*

From this theorem we deduce:

Lemma 2.4. *Suppose that in some valued field extension of (K,v), x is the pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type. Then $(K(x)|K,v)$ is immediate and x is transcendental over K.*

Proof. Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) of transcendental type. Then by Theorem 2.3 there is an immediate extension w of v to the rational function field $K(y)$ such that y becomes a pseudo limit of $(a_\nu)_{\nu<\lambda}$; moreover, if also x is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(K(x),v)$, then $x\mapsto y$ induces a valuation preserving isomorphism from $K(x)$ onto $K(y)$ over K. Hence, $(K(x)|K,v)$ is immediate and x is transcendental over K. □

Lemma 2.5. *A pseudo Cauchy sequence of transcendental type in a valued field remains a pseudo Cauchy sequence of transcendental type in every algebraic valued field extension of that field.*

Proof. Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) of transcendental type and that $(L|K,v)$ is an algebraic extension. If $(a_\nu)_{\nu<\lambda}$ were of algebraic type over (L,v), then by Theorem 3 of [KA] there would be an algebraic extension $L(y)|L$ and an immediate extension of v to $L(y)$ such that y is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(L(y),v)$. But then, y is also a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(K(y),v)$. Hence by the foregoing lemma, y must be transcendental over K. This is a contradiction to the fact that $L(y)|L$ and $L|K$ are algebraic. □

2.2. **Valuation independence.** For the easy proof of the following lemma, see [B], chapter VI, §10.3, Theorem 1.

Lemma 2.6. *Let $(L|K,v)$ be an extension of valued fields. Take elements $x_i,y_j\in L$, $i\in I$, $j\in J$, such that the values vx_i , $i\in I$, are rationally independent over vK, and the residues y_jv, $j\in J$, are algebraically independent over Kv. Then the elements x_i,y_j, $i\in I$, $j\in J$, are algebraically independent over K.*

Moreover, if we write

$$f \;=\; \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i,y_j\mid i\in I, j\in J]$$

in such a way that for every $k \neq \ell$ there is some i s.t. $\mu_{k,i} \neq \mu_{\ell,i}$ or some j s.t. $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf \;=\; \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \;=\; \min_k vc_k \,+\, \sum_{i\in I} \mu_{k,i} vx_i \;. \tag{2.3}$$

That is, the value of the polynomial f is equal to the least of the values of its monomials. In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i\in I, j\in J) &= vK \oplus \bigoplus_{i\in I} \mathbb{Z} vx_i \\ K(x_i, y_j \mid i\in I, j\in J)v &= Kv\,(y_jv \mid j\in J)\;. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i\in I, j\in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues y_jv.

Conversely, if (K,v) is any valued field and we assign to the vx_i any values in an ordered group extension of vK which are rationally independent, then (2.3) defines a valuation on L, and the residues y_jv, $j\in J$, are algebraically independent over Kv.

Corollary 2.7. *Let $(L|K,v)$ be an extension of finite transcendence degree of valued fields. Then*

$$\mathrm{trdeg}\, L|K \;\geq\; \mathrm{trdeg}\, Lv|Kv \,+\, \mathrm{rr}\,(vL/vK)\;. \tag{2.4}$$

If in addition $L|K$ is a function field and if equality holds in (2.4), then the extensions $vL|vK$ and $Lv|Kv$ are finitely generated.

Proof. Choose elements $x_1,\dots,x_\rho,y_1,\dots,y_\tau\in L$ such that the values $vx_1,\dots,vx_\rho$ are rationally independent over vK and the residues $y_1v,\dots,y_\tau v$ are algebraically independent over Kv. Then by the foregoing lemma, $\rho+\tau\leq \mathrm{trdeg}\,L|K$. This proves that $\mathrm{trdeg}\,Lv|Kv$ and the rational rank of vL/vK are finite. Therefore, we may choose the elements x_i,y_j such that $\tau=\mathrm{trdeg}\,Lv|Kv$ and $\rho=\mathrm{rr}\,(vL/vK)$ to obtain inequality (2.4).

Assume that equality holds in (2.4). This means that L is an algebraic extension of $L_0 := K(x_1,\dots,x_\rho,y_1,\dots,y_\tau)$. Since $L|K$ is finitely generated, it follows that $L|L_0$ is finite; hence by Lemma 2.1, also vL/vL_0 and $Lv|L_0v$ are finite. Since already $vL_0|vK$ and $L_0v|Kv$ are finitely generated by the foregoing lemma, it follows that also $vL|vK$ and $Lv|Kv$ are finitely generated. □

The algebraic analogue of the transcendental case discussed in Lemma 2.6 is the following lemma (see [R] or [E]):

Lemma 2.8. *Let $(L|K,v)$ be an extension of valued fields. Take $\eta_i\in L$ such that $v\eta_i$, $i\in I$, belong to distinct cosets modulo vK. Further, take $\vartheta_j\in\mathcal{O}_L$, $j\in J$, such that ϑ_jv are Kv-linearly independent. Then the elements $\eta_i\vartheta_j$, $i\in I$, $j\in J$, are K-linearly independent, and for every choice of elements $c_{ij}\in K$, only finitely many of them nonzero, we have that*

$$v\sum_{i,j} c_{ij}\eta_i\vartheta_j \;=\; \min_{i,j} vc_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}\,(vc_{ij}\,+\,v\eta_i)\;.$$

If the elements $\eta_i\vartheta_j$ form a K-basis of L, then $v\eta_i$, $i\in I$, is a system of representatives of the cosets of vL modulo vK, and ϑ_jv, $j\in J$, is a basis of $Lv|Kv$.

The following is an application which is important for our description of all possible value groups and residue fields of valuations on $K(x)$. The result has been proved with a different method in [APZ3] (Corollary 5.2); cf. Remark 3.1 in Section 3.1.

Theorem 2.9. *Let K be any field and v any valuation of the rational function field $K(x)$. Then $vK(x)/vK$ is countable, and $K(x)v|Kv$ is countably generated.*

Proof. Since $K(x)$ is the quotient field of $K[x]$, we have that $vK(x) = vK[x] - vK[x]$. Hence, to show that $vK(x)/vK$ is countable, it suffices to show that the set $\{\alpha + vK \mid \alpha \in vK[x]\}$ is countable. If this were not true, then by Lemma 2.8 (applied with $J = \{1\}$ and $\vartheta_1 = 1$), we would have that $K[x]$ contains uncountably many K-linearly independent elements. But this is not true, as $K[x]$ admits the countable K-basis $\{x^i \mid i \geq 0\}$.

Now assume that $K(x)v|Kv$ is not countably generated. Then by Corollary 2.7, $K(x)v|Kv$ must be algebraic. It also follows that $K(x)v$ has uncountable Kv-dimension. Pick an uncountable set κ and elements $f_i(x)/g_i(x)$, $i \in \kappa$, with $f_i(x), g_i(x) \in K[x]$ and $vf_i(x) = vg_i(x)$ for all i, such that their residues are Kv-linearly independent. As $vK(x)/vK$ is countable, there must be some uncountable subset $\lambda \subset \kappa$ such that for all $i \in \lambda$, the values $vf_i(x) = vg_i(x)$ lie in the same coset modulo vK, say $vh(x) + vK$ with $h(x) \in K[x]$. The residues $(f_i(x)/g_i(x))v$, $i \in \lambda$, generate an algebraic extension of uncountable dimension. Choosing suitable elements $c_i \in K$ such that

$$vc_i f_i(x) \;=\; vh(x) \;=\; vc_i g_i(x)\;,$$

we can write

$$\frac{f_i(x)}{g_i(x)} \;=\; \frac{c_i f_i(x)}{h(x)} \cdot \frac{h(x)}{c_i g_i(x)} \;=\; \frac{c_i f_i(x)}{h(x)} \cdot \left(\frac{c_i g_i(x)}{h(x)}\right)^{-1}$$

for all $i \in \lambda$. Therefore,

$$\frac{f_i(x)}{g_i(x)}\, v \;=\; \left(\frac{c_i f_i(x)}{h(x)}\, v\right) \cdot \left(\frac{c_i g_i(x)}{h(x)}\, v\right)^{-1}$$

for all $i \in \lambda$. In order that these elements generate an algebraic extension of Kv of uncountable dimension, the same must already be true for the elements $(c_i f_i(x)/h(x))v$, $i \in \lambda$, or for the elements $(c_i g_i(x)/h(x))v$, $i \in \lambda$. It follows that at least one of these two sets contains uncountably many Kv-linearly independent elements. But then by Lemma 2.8 (applied with $I = \{1\}$ and $\eta_1 = 1$), there are uncountably many K-linearly independent elements in the set

$$\frac{1}{h(x)}\, K[x]$$

and hence also in $K[x]$, a contradiction. $\square$

Finally, let us mention the following lemma which combines the algebraic and the transcendental case. We leave its easy proof to the reader.

Lemma 2.10. *Let $(L|K, v)$ be an extension of valued fields. Take $x \in L$. Suppose that for some $e \in \mathbb{N}$ there exists an element $d \in K$ such that $vdx^e = 0$ and $dx^e v$*

is transcendental over Kv. Let e be minimal with this property. Then for every $f = c_n x^n + \ldots + c_0 \in K[x]$,

$$vf \;=\; \min_{1\leq i\leq n} vc_i x^i \;.$$

Moreover, $K(x)v = Kv(dx^e v)$ is a rational function field over Kv, and we have

$$vK(x) \;=\; vK + \mathbb{Z}vx \quad \textit{with} \quad (vK(x):vK) \;=\; e\;.$$

2.3. **Construction of valued field extensions with prescribed value groups and residue fields.** In this section, we will deal with the following problem. Suppose that (K,v) is a valued field, $\Gamma|vK$ is an extension of ordered abelian groups and $k|Kv$ is a field extension. Does there exist an extension $(L|K,v)$ of valued fields such that $vL = \Gamma$ and $Lv = k$? We include the case of (K,v) being trivially valued; this amounts to the construction of a valued field with given value group and residue field. Throughout, we use the well known fact that if (K,v) is any valued field and L is any extension field of K, then there is at least one extension of v to L (cf. [E], [R]).

Let us adjust the following notion to our purposes. Usually, when one speaks of an **Artin-Schreier extension** then one means an extension of a field K generated by a root of an irreducible polynomial of the form $X^p - X - c$, provided that $p = \mathrm{char}\, K$. We will replace this by the weaker condition "$p = \mathrm{char}\, Kv$". In fact, such extensions also play an important role in the mixed characteristic case, where $\mathrm{char}\, K = 0$.

Every **Artin-Schreier polynomial** $X^p - X - c$ is separable since its derivative does not vanish. The following is a simple but very useful observation:

Lemma 2.11. *Let (K,v) be a valued field and $c \in K$ such that $vc < 0$. If $a \in \tilde{K}$ such that $a^p - a = c$, then for every extension of v from K to $K(a)$,*

$$0 \;>\; v(a^p - c) \;=\; va \;>\; pva \;=\; vc\;.$$

Proof. Take any extension of v from K to $K(a)$. Necessarily, $va < 0$ since otherwise, $\infty = v(a^p - a - c) = \min\{pva, va, vc\} = vc$, a contradiction. It follows that $va^p = pva < va$ and thus,

$$vc \;=\; \min\{v(a^p - a - c), v(a^p - a)\} \;=\; v(a^p - a) \;=\; \min\{pva, va\} \;=\; pva\;.$$

$\square$

Lemma 2.12. *Let (K,v) be a non-trivially valued field, p a prime and α an element of the divisible hull of vK such that $p\alpha \in vK$, $\alpha \notin vK$. Choose an element $a \in \tilde{K}$ such that $a^p \in K$ and $va^p = p\alpha$. Then v extends in a unique way from K to $K(a)$. It satisfies*

(2.5)

$$va = \alpha\,, \quad [K(a):K] \;=\; (vK(a):vK)\,, \quad vK(a) \;=\; vK + \mathbb{Z}\alpha \quad \textit{and} \quad K(a)v \;=\; Kv\;.$$

If $\mathrm{char}\, K = \mathrm{char}\, Kv = p$, then this extension $K(a)|K$ is purely inseparable. On the other hand, if $\mathrm{char}\, Kv = p$, then there is always an Artin-Schreier extension $K(a)|K$ with properties (2.5); if $\alpha < 0$, then a itself can be chosen to be the root of an Artin-Schreier polynomial over K.

Proof. Take $c \in K$ such that $vc = p\alpha$ and $a \in \tilde{K}$ such that $a^p = c$. Choose any extension of v from K to $K(a)$. Then $pva = vc = p\alpha$. Consequently, $va = \alpha$

and $(vK(a) : vK) \geq (vK + \mathbb{Z}\alpha : vK) = p$. On the other hand, the fundamental inequality (2.1) shows that

$$p = [K(a) : K] \geq (vK(a) : vK) \cdot [K(a)v : Kv] \geq (vK(a) : vK) \geq p\,.$$

Hence, equality holds everywhere, and we find that $(vK(a) : vK) = p$ and $[K(a)v : Kv] = 1$. That is, $vK(a) = vK + \mathbb{Z}\alpha$ and $K(a)v = Kv$. Further, the fundamental inequality implies that the extension of v from K to $K(a)$ is unique.

Now suppose that $\operatorname{char} Kv = p$. Choose $c \in K$ such that $vc = -|p\alpha| < 0$. By the foregoing lemma, every root b of the Artin-Schreier polynomial $X^p - X - c$ must satisfy $pvb = vc$. Now we set $a = b$ if $\alpha < 0$, and $a = 1/b$ if $\alpha > 0$ (but note that then $1/a$ is in general not the root of an Artin-Schreier polynomial). Then as before one shows that (2.5) holds. □

For $f \in \mathcal{O}_K[X]$, we define the **reduction** $fv \in Kv[X]$ to be the polynomial obtained from f through replacing every coefficient by its residue.

Lemma 2.13. *Let (K, v) be a valued field and ζ an element of the algebraic closure of Kv. Choose a monic polynomial $f \in \mathcal{O}_K[X]$ whose reduction fv is the minimal polynomial of ζ over Kv. Further, choose a root $b \in \tilde{K}$ of f. Then there is a unique extension of v from K to $K(b)$ and a corresponding extension of the residue map such that*

(2.6) $bv = \zeta,\quad [K(b) : K] = [Kv(\zeta) : Kv],\quad vK(b) = vK \quad$ *and* $\quad K(b)v = Kv(\zeta)\,.$

In all cases, f can be chosen separable, provided that the valuation v is non-trivial. On the other hand, if $\operatorname{char} K = \operatorname{char} Kv = p > 0$ *and ζ is purely inseparable over Kv, then b can be chosen purely inseparable over K.*

If v is non-trivial, $\operatorname{char} Kv = p > 0$ *and $\zeta^p \in Kv$, $\zeta \notin Kv$, then there is also an Artin-Schreier extension $K(b)|K$ such that (2.6) holds and db is the root of an Artin-Schreier polynomial over K, for a suitable $d \in K$.*

Proof. We choose an extension of v from K to $K(b)$. Since f is monic with integral coefficients, b must also be integral for this extension, and bv must be a root of fv. We may compose the residue map with an isomorphism in $\operatorname{Gal} Kv$ which sends this root to ζ. Doing so, we obtain a residue map (still associated with v) that satisfies $bv = \zeta$. Now $\zeta \in K(b)v$ and consequently, $[K(b)v : Kv] \geq [Kv(\zeta) : Kv] = \deg fv = \deg f$. On the other hand, the fundamental inequality shows that

$$\deg f = [K(b) : K] \geq (vK(b) : vK) \cdot [K(b)v : Kv] \geq [K(b)v : Kv] \geq \deg f\,.$$

Hence, equality holds everywhere, and we find that $[K(b)v : Kv] = [Kv(\zeta) : Kv] = [K(b) : K]$ and $(vK(b) : vK) = 1$. That is, $vK(b) = vK$ and $K(b)v = Kv(\zeta)$. Further, the uniqueness of v on $K(a)$ follows from the fundamental inequality.

If fv is separable, then so is f. Even if fv is not separable but v is nontrivial on K, then f can still be chosen separable since we can add a summand cX with $c \neq 0$, $vc > 0$ (we use that v is non-trivial) without changing the reduction of f. On the other hand, if fv is purely inseparable and hence of the form $X^{p^\nu} - cv$, then we can choose $f = X^{p^\nu} - c$ which also is purely inseparable if $\operatorname{char} K = p$.

Now suppose that $\operatorname{char} Kv = p > 0$ and $\zeta^p \in Kv$, $\zeta \notin Kv$. Choose $c \in K$ such that $cv = \zeta^p$. To construct an Artin-Schreier extension, choose any $d \in K$ with $vd < 0$, and let b_0 be a root of the Artin-Schreier polynomial $X^p - X - d^p c$. Since $vd^p c = pvd < 0$, Lemma 2.11 shows that $v(b_0^p - d^p c) = vb_0 > vb_0^p$. Consequently, $v((b_0/d)^p - c) > v(b_0/d)^p = vc = 0$, whence $(b_0/d)^p v = cv$ and $(b_0/d)v = (cv)^{1/p} =$

ζ. We set $b = b_0/d$; so $K(b) = K(b_0)$. As before, it follows that $vK(b) = vK$ and $K(b)v = Kv(\zeta)$. □

Theorem 2.14. *Let (K, v) be an arbitrary valued field. For every extension $\Gamma|vK$ of ordered abelian groups and every field extension $k|Kv$, there is an extension (L, v) of (K, v) such that $vL = \Gamma$ and $Lv = k$. If $\Gamma|vK$ and $k|Kv$ are algebraic, then $L|K$ can be chosen to be algebraic, with a unique extension of v from K to L. If $\rho = \mathrm{rr}\,\Gamma/vK$ and $\tau = \mathrm{trdeg}\,k|Kv$ are finite, then $L|K$ can be chosen of transcendence degree $\rho + \tau$. If $\Gamma \neq \{0\}$, then L can be chosen to be a separable extension of K.*

If both $\Gamma|vK$ and $k|Kv$ are finite, then $L|K$ can be taken to be a finite extension such that $[L : K] = (\Gamma : vK)[k : Kv]$. If in addition v is non-trivial on K, then $L|K$ can be chosen to be a simple separable-algebraic extension.

If Γ/vK is countable and $k|Kv$ is countably generated, then $L|K$ can be taken to be a countably generated extension.

Proof. For the proof, we assume that $\Gamma \neq \{0\}$ (the other case is trivial). Let α_i , $i \in I$, be a maximal set of elements in Γ rationally independent over vK. Then by Lemma 2.6 there is an extension $(K_1, v) := (K(x_i \mid i \in I), v)$ of (K, v) such that $vK_1 = vK \oplus \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $K_1v = Kv$. Next, choose a transcendence basis ζ_j , $j \in J$, of $k|Kv$. Then by Lemma 2.6 there is an extension $(K_2, v) := (K_1(y_j \mid j \in J), v)$ of (K_1, v) such that $vK_2 = vK_1$ and $K_2v = Kv(\zeta_j \mid j \in J)$. If $\Gamma|vK$ and $k|Kv$ are algebraic, then $I = J = \emptyset$ and $K_2 = K$.

If we have an ascending chain of valued fields whose value groups are subgroups of Γ and whose residue fields are subfields of k, then the union over this chain is again a valued field whose value group is a subgroup of Γ and whose residue field is a subfield of k. So a standard argument using Zorn's Lemma together with the transitivity of separable extensions shows that there are maximal separable-algebraic extension fields of (K_2, v) with these properties. Choose one of them and call it (L, v). We have to show that $vL = \Gamma$ and $Lv = k$. Since already $\Gamma|vK_2$ and $k|K_2v$ are algebraic, the same holds for $\Gamma|vL$ and $k|Lv$. If vL is a proper subgroup of Γ, then there is some prime p and some element $\alpha \in \Gamma \setminus vL$ such that $p\alpha \in vL$. But then, Lemma 2.12 shows that there exists a proper separable-algebraic extension (L', v) of (L, v) such that $vL' = vL + \mathbb{Z}\alpha \subset \Gamma$ and $L'v = Lv \subset k$, which contradicts the maximality of L. If Lv is a proper subfield of k, then there is some element $\zeta \in k \setminus Lv$ algebraic over Lv. But then, Lemma 2.13 shows that there exists a proper separable-algebraic extension (L', v) of (L, v) such that $vL' = vL \subset \Gamma$ and $L'v = Lv(\zeta) \subset k$, which again contradicts the maximality of L (here we have used $\Gamma \neq \{0\}$, which implies that v is not trivial on L). This proves that $vL = \Gamma$ and $Lv = k$, and (L, v) is the required extension of (K, v). Since K_2 is generated over K by a set of elements which are algebraically independent over K, we know that $K_2|K$ is separable. Since also $L|K_2$ is separable, we find that $L|K$ is separable. Since $L|K_2$ is algebraic, $\{x_i\, ,\, y_j \mid i \in I\, ,\, j \in J\}$ is a transcendence basis of $(K_2|K, v)$. If $\Gamma|vK$ and $k|Kv$ are algebraic, then $I = J = \emptyset$ and L is an algebraic extension of $K = K_2$.

If $\Gamma|vK$ and $k|Kv$ are finite, then L can be constructed by a finite number of applications of Lemma 2.12 and Lemma 2.13. Since extension degree, ramification index and inertia degree are multiplicative, we obtain that $[L : K] = (vL : vK)[Lv :$

$Kv] = (\Gamma : vK)[k : Kv]$. If in addition v is non-trivial, then $L|K$ can be chosen to be a separable extension. Since it is finite, it is simple.

If $\Gamma|vK$ and $k|Kv$ are countably generated algebraic, then they are unions over a countable chain of algebraic extensions. Hence also L can be constructed as a union over a countable chain of algebraic extensions and will thus be a countably generated extension of K.

If $\Gamma|vK$ and $k|Kv$ are countably generated, then the sets I and J in our above construction are both countable, that is, $K_2|K$ is countably generated. Moreover, the extensions $\Gamma|vK_2$ and $k|K_2v$ are countably generated algebraic. Hence by what we have just shown, L can be taken to be a countably generated extension of K_2, and thus also of K. □

Every ordered abelian group is an extension of the trivial group $\{0\}$ as well as of the ordered abelian group $\mathbb{Z}$. Every field of characteristic 0 is an extension of $\mathbb{Q}$, and every field of characteristic $p > 0$ is an extension of $\mathbb{F}_p$. Let $\Gamma \neq 0$ be an ordered abelian group and k a field. If $\operatorname{char} k = 0$, then $\mathbb{Q}$ endowed with the trivial valuation v will satisfy $v\mathbb{Q} = \{0\} \subset \Gamma$ and $\mathbb{Q}v = \mathbb{Q} \subset k$. If $\operatorname{char} k = p > 0$, then we can choose v to be the p-adic valuation on $\mathbb{Q}$ to obtain that $v\mathbb{Q} = \mathbb{Z} \subset \Gamma$ and $\mathbb{Q}v = \mathbb{F}_p \subset k$. But also $\mathbb{F}_p$ endowed with the trivial valuation v will satisfy $v\mathbb{F}_p = \{0\} \subset \Gamma$ and $\mathbb{F}_pv = \mathbb{F}_p \subset k$. An application of the foregoing theorem now proves:

Corollary 2.15. *Let $\Gamma \neq 0$ be an ordered abelian group and k a field. Then there is a valued field (L, v) with $vL = \Gamma$ and $Lv = k$. If $\operatorname{char} k = p > 0$, then L can be chosen to be of characteristic 0 (mixed characteristic case) or of characteristic p (equal characteristic case).*

For the sake of completeness, we add the following information. From the fundamental inequality it follows that $v\tilde{K}|vK$, $vK^{\mathrm{sep}}|vK$, $\tilde{K}v|Kv$ and $K^{\mathrm{sep}}v|Kv$ are algebraic extensions. On the other hand, Lemma 2.12 shows that the value group of a separable-algebraically closed field must be divisible. Similarly, it follows from Lemma 2.13 that the residue field of a separable-algebraically closed non-trivially valued field must be algebraically closed. This proves:

Lemma 2.16. *Let (K, v) be a non-trivially valued field and extend v to $\tilde{K}$. Then the value groups $v\tilde{K}$ and vK^{sep} are equal to the divisible hull of vK, and the residue fields $\tilde{K}v$ and $K^{\mathrm{sep}}v$ are equal to the algebraic closure of Kv.*

A valued field (K, v) of residue characteristic $p > 0$ is called **Artin-Schreier closed** if every Artin-Schreier polynomial with coefficients in K admits a root in K. Recall that if $\operatorname{char} K = p$, then this means that every Artin-Schreier polynomial with coefficients in K splits into linear factors over K. If K is Artin-Schreier closed, then so is Kv. As a corollary to Lemmas 2.12 and 2.13, we obtain:

Corollary 2.17. *Every Artin-Schreier closed non-trivially valued field of residue characteristic $p > 0$ has p-divisible value group and perfect Artin-Schreier closed residue field.*

2.4. **Orderings and valuations.** We will assume the reader to be familiar with the basic theory of convex valuations, which can be found in [L] and [PR].

Proposition 2.18. *Suppose that $(K, <)$ is an ordered field with convex valuation v, and denote by $<_r$ the ordering induced by $<$ through v on Kv. Let $(L|K, v)$ be* *an extension of valued fields. If $2vL \cap vK = 2vK$, then each extension $<_r'$ of $<_r$ to an ordering of Lv can be lifted through v to an ordering of L which extends the ordering $<$ of K.*

Proof. We fix a section from $vK/2vK$ to $K^\times/K^{\times 2}$. Since $2vL \cap vK = 2vK$, this section can be extended to a section from $vL/2vL$ to $L^\times/L^{\times 2}$. Now there is a bijection between the set of all liftings of $<_r$ through v to orderings of K, and the set of all group characters of $vK/2vK$; see [PR], (7.5) to (7.9). The same construction yields a bijection between the set of all liftings of $<_r'$ through v to orderings of L, and the set of all group characters of $vL/2vL$. Since we use an extension of the section $vK/2vK \to K^\times/K^{\times 2}$ to a section $vL/2vL \to L^\times/L^{\times 2}$, the bijection maps commute with the restriction from L to K of any lifting. That is, if a lifting $<'$ of $<_r'$ to L corresponds to a character χ of $vL/2vL$, then the restriction of $<'$ to K is the unique lifting of $<_r$ to K which corresponds to the restriction of χ to $vK/2vK$.

As the given ordering $<$ of K is a lifting of $<_r$, it corresponds to a unique group character of $vK/2vK$. Since $2vL \cap vK = 2vK$, we can extend it to a group character of $vL/2vL$. Take the lifting of $<_r'$ through v to an ordering $<'$ of L which corresponds to this group character of $vL/2vL$. Then its restriction to K is $<$. □

The following was proved by Knebusch and Wright [KW] and by Prestel (cf. [PR]); see Theorem 5.6 of [L].

Theorem 2.19. *Let v be a convex valuation on the ordered field $(K,<)$, $<_r$ the ordering induced by $<$ on Kv, and R a real closure of $(K,<)$. Then there exists a unique extension of v to a convex valuation of R. Denoting this extension again by v, we have that (R,v) is henselian, vR is divisible, and Rv with the ordering induced by $<$ is a real closure of $(Kv,<_r)$.*

Corollary 2.20. *Let v be a convex valuation on the ordered field $(K,<)$ and R a real closure of $(K,<)$, endowed with the unique convex extension of v. Further, let $\Gamma|vK$ be an algebraic extension of ordered abelian groups, and $k|Kv$ a subextension of some real closure of Kv. Then there is a (separable-algebraic) subextension $(L|K,v)$ of $(R|K,v)$ such that $vL = \Gamma$ and $Lv = k$. If both $\Gamma|vK$ and $k|Kv$ are finite, then $L|K$ can be taken to be a finite simple extension of the form $K(a)|K$ such that $[K(a):K] = (\Gamma : vK)[k:Kv]$.*

We leave the proof of the corollary as an exercise to the reader. It is a straightforward application of Hensel's Lemma, using the fact that (R,v) is henselian. One also uses the fact that all real closures of Kv are isomorphic over Kv, so by passing to an equivalent residue map (place), one passes from Rv to the real closure given in the hypothesis.

2.5. **A version of Krasner's Lemma.** Let (K,v) be any valued field. If $a \in \tilde{K}\setminus K$ is not purely inseparable over K, we choose some extension of v from K to $\tilde{K}$ and define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \text{ and } \tau a \neq \sigma a\} \;\in\; v\tilde{K}$$

and call it the **Krasner constant of a over K**. Since all extensions of v from K to $\tilde{K}$ are conjugate, this does not depend on the choice of the particular extension of v.

For the same reason, over a henselian field (K, v) our Krasner constant $\mathrm{kras}(a, K)$ coincides with the Krasner constant

$$\max\{v(a - \sigma a) \mid \sigma \in \mathrm{Gal}\, K \quad \text{and} \quad a \neq \sigma a\}$$

as defined by S. K. Khanduja in [KH11,12]. The following is a variant of the well-known Krasner's Lemma (cf. [R]).

Lemma 2.21. *Take $K(a)|K$ to be a separable-algebraic extension, and $(K(a,b), v)$ to be any valued field extension of (K, v) such that*

$$v(b - a) \;>\; \mathrm{kras}(a, K)\;. \tag{2.7}$$

Then for every extension of v from $K(a,b)$ to its algebraic closure $\widetilde{K(a,b)} = \widetilde{K(b)}$, the element a lies in the henselization of $(K(b), v)$ in $(\widetilde{K(b)}, v)$.

Proof. Take any extension of v from $K(a,b)$ to $\widetilde{K(b)}$ and denote by $K(b)^h$ the henselization of $(K(b), v)$ in $(\widetilde{K(b)}, v)$. Since a is separable-algebraic over K, it is also separable-algebraic over $K(b)^h$. Since for every $\rho \in \mathrm{Gal}\, K(b)^h$ we have that $\rho a = \rho|_{\tilde{K}} a$ and $\rho|_{\tilde{K}} \in \mathrm{Gal}\, K$, we find that

$$\begin{aligned} &\{v(a - \rho a) \mid \rho \in \mathrm{Gal}\, K(b)^h \text{ and } a \neq \rho a\} \\ &\quad\subseteq\;\; \{v(a - \sigma a) \mid \sigma \in \mathrm{Gal}\, K \text{ and } a \neq \sigma a\} \\ &\quad\subseteq\;\; \{v(\tau a - \sigma a) \mid \sigma, \tau \in \mathrm{Gal}\, K \text{ and } \tau a \neq \sigma a\}\;. \end{aligned}$$

This implies that

$$\mathrm{kras}(a, K(b)^h) \;\leq\; \mathrm{kras}(a, K)\;,$$

and consequently, $v(b - a) \;>\; \mathrm{kras}(a, K(b)^h)$. Now $a \in K(b)^h$ follows from the usual Krasner's Lemma. □

3. Valuations on $K(x)$

3.1. **A basic classification.** In this section, we wish to classify all extensions of the valuation v of K to a valuation of the rational function field $K(x)$. As

$$1 \;=\; \mathrm{trdeg}\, K(x)|K \;\geq\; \mathrm{rr}\, vK(x)/vK \;+\; \mathrm{trdeg}\, K(x)v|Kv \tag{3.1}$$

holds by Lemma 2.6, there are the following mutually exclusive cases:

- $(K(x)|K, v)$ is **valuation-algebraic**:
  $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic,
- $(K(x)|K, v)$ is **value-transcendental**:
  $vK(x)/vK$ has rational rank 1, but $K(x)v|Kv$ is algebraic,
- $(K(x)|K, v)$ is **residue-transcendental**:
  $K(x)v|Kv$ has transcendence degree 1, but $vK(x)/vK$ is a torsion group.

We will combine the value-transcendental case and the residue-transcendental case by saying that

- $(K(x)|K, v)$ is **valuation-transcendental**:
  $vK(x)/vK$ has rational rank 1, or $K(x)v|Kv$ has transcendence degree 1.

A special case of the valuation-algebraic case is the following:

- $(K(x)|K, v)$ is **immediate**:
  $vK(x) = vK$ and $K(x)v = Kv$.

Remark 3.1. It was observed by several authors that a valuation-algebraic extension of v from K to $K(x)$ can be represented as a limit of an infinite sequence of residue-transcendental extensions. See, e.g., [APZ3], where the authors also derive the assertion of our Theorem 2.9 from this fact. A "higher form" of this approach is found in [S]. The approach is particularly important because residue-transcendental extensions behave better than valuation-algebraic extensions: the corresponding extensions of value group and residue field are finitely generated (Corollary 2.7), and they do not generate a defect: see the Generalized Stability Theorem (Theorem 3.1) and its application in [KKU1].

If K is algebraically closed, then the residue field Kv is algebraically closed, and the value group vK is divisible. So we see that for an extension $(K(x)|K,v)$ with algebraically closed K, there are only the following mutually exclusive cases:

$(K(x)|K,v)$ is **immediate**: $vK(x)=vK$ and $K(x)v=Kv$,
$(K(x)|K,v)$ is **value-transcendental**: $\mathrm{rr}\, vK(x)/vK=1$, but $K(x)v=Kv$,
$(K(x)|K,v)$ is **residue-transcendental**: $\mathrm{trdeg}\, K(x)v|Kv=1$, but $vK(x)=vK$.

Let us fix an arbitrary extension of v to $\tilde{K}$. Every valuation w on $K(x)$ can be extended to a valuation on $\tilde{K}(x)$. If v and w agree on K, then this extension can be chosen in such a way that its restriction to $\tilde{K}$ coincides with v. Indeed, if w' is any extension of w to $\tilde{K}(x)$ and v' is its restriction to $\tilde{K}$, then there is an automorphism τ of $\tilde{K}|K$ such that $v'\tau=v$ on $\tilde{K}$. We choose σ to be the (unique) automorphism of $\tilde{K}(x)|K(x)$ whose restriction to $\tilde{K}$ is τ and which satisfies $\sigma x=x$. Then $w'\sigma$ is an extension of w from $K(x)$ to $\tilde{K}(x)$ whose restriction to $\tilde{K}$ is v. We conclude:

Lemma 3.2. *Take any extension of v from K to its algebraic closure $\tilde{K}$. Then every extension of v from K to $K(x)$ is the restriction of some extension of v from $\tilde{K}$ to $\tilde{K}(x)$.*

Now extend v to $\widetilde{K(x)}$. We know that $v\widetilde{K(x)}/vK(x)$ and $v\tilde{K}/vK$ are torsion groups, and also $v\tilde{K}(x)/vK(x)\subset v\widetilde{K(x)}/vK(x)$ is a torsion group. Hence,

$$\mathrm{rr}\, v\tilde{K}(x)/v\tilde{K} \;=\; \mathrm{rr}\, vK(x)/vK\;.$$

Since $v\tilde{K}$ is divisible, $vK(x)/vK$ is a torsion group if and only if $v\tilde{K}(x)=v\tilde{K}$.

Further, the extensions $\widetilde{K(x)}v|K(x)v$ and $\tilde{K}v|Kv$ are algebraic, and also the subextension $\tilde{K}(x)v|K(x)v$ of $\widetilde{K(x)}v|K(x)v$ is algebraic. Hence,

$$\mathrm{trdeg}\, \tilde{K}(x)v|\tilde{K}v \;=\; \mathrm{trdeg}\, K(x)v|Kv\;.$$

Since $\tilde{K}v$ is algebraically closed, $K(x)v|Kv$ is algebraic if and only if $\tilde{K}(x)v=\tilde{K}v$. We have proved:

Lemma 3.3. *$(K(x)|K,v)$ is valuation-algebraic if and only if $(\tilde{K}(x)|\tilde{K},v)$ is immediate. $(K(x)|K,v)$ is valuation-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is not immediate. $(K(x)|K,v)$ is value-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is value-transcendental. $(K(x)|K,v)$ is residue-transcendental if and only if $(\tilde{K}(x)|\tilde{K},v)$ is residue-transcendental.*

The proof can easily be generalized to show:

Lemma 3.4. *Let $(F|K,v)$ be any valued field extension. Then $vF|vK$ and $Fv|Kv$ are algebraic if and only if $(F.\tilde{K}|\tilde{K},v)$ is immediate, for some (or any) extension of v from F to $F.\tilde{K}$.*

3.2. **Pure and weakly pure extensions.** Take $t \in K(x)$. If vt is not a torsion element modulo vK, then t will be called a **value-transcendental element**. If $vt = 0$ and tv is transcendental over Kv, then t will be called a **residue-transcendental element**. An element will be called a **valuation-transcendental element** if it is value-transcendental or residue-transcendental. We will call the extension $(K(x)|K,v)$ **pure (in** x) if one of the following cases holds:

- for some $c,d \in K$, $d \cdot (x-c)$ is valuation-transcendental,
- x is the pseudo limit of some pseudo Cauchy sequence in (K,v) of transcendental type.

We leave it as an exercise to the reader to prove that $(K(x)|K,v)$ is pure in x if and only if it is pure in any other generator of $K(x)$ over K; we will not need this fact in the present paper.

If $(K(x)|K,v)$ is pure in x then it follows from Lemma 2.6 and Lemma 2.4 that x is transcendental over K. If $d \cdot (x-c)$ is value-transcendental, then $vK(x) = vK \oplus \mathbb{Z}v(x-c)$ and $K(x)v = Kv$ by Lemma 2.6 (in this case, we may in fact choose $d = 1$). If $d \cdot (x-c)$ is residue-transcendental, then again by Lemma 2.6, we have $vK(x) = vK$ and that $K(x)v = Kv(d(x-c)v)$ is a rational function field over Kv. If x is the pseudo limit of some pseudo Cauchy sequence in (K,v) of transcendental type, then $(K(x)|K,v)$ is immediate by Lemma 2.4. This proves:

Lemma 3.5. *If $(K(x)|K,v)$ is pure, then vK is pure in $vK(x)$ (i.e., $vK(x)/vK$ is torsion free), and Kv is relatively algebraically closed in $K(x)v$.*

Here is the "prototype" of pure extensions:

Lemma 3.6. *If K is algebraically closed and $x \notin K$, then $(K(x)|K,v)$ is pure.*

Proof. Suppose that the set

$$v(x-K) \;:=\; \{v(x-b) \mid b \in K\} \tag{3.2}$$

has no maximum. Then there is a pseudo Cauchy sequence in (K,v) with pseudo limit x, but without a pseudo limit in K. Since K is algebraically closed, Theorem 3 of [KA] shows that this pseudo Cauchy sequence must be of transcendental type. The extension therefore satisfies the third condition for being pure.

Now assume that the set $v(x-K)$ has a maximum, say, $v(x-c)$ with $c \in K$. If $v(x-c)$ is a torsion element over vK, then $v(x-c) \in vK$ because vK is divisible as K is algebraically closed. It then follows that there is some $d \in K$ such that $vd(x-c) = 0$. If $d(x-c)v$ were algebraic over Kv, then it were in Kv since K and thus also Kv is algebraically closed. But then, there were some $b_0 \in K$ such that $v(d(x-c)-b_0) > 0$. Putting $b := c + d^{-1}b_0$, we would then obtain that $v(x-b) = v((x-c)-d^{-1}b_0) > -vd = v(x-c)$, a contradiction to the maximality of $v(x-c)$. So we see that either $v(x-c)$ is non-torsion over vK, or there is some $d \in K$ such $vd(x-c) = 0$ and $d(x-c)v$ is transcendental over Kv. In both cases, this shows that $(K(x)|K,v)$ is pure. □

We will call the extension $(K(x)|K,v)$ **weakly pure (in** x**)** if it is pure in x or if there are $c,d\in K$ and $e\in\mathbb{N}$ such that $vd(x-c)^e=0$ and $d(x-c)^e v$ is transcendental over Kv.

Lemma 3.7. *Assume that the extension $(K(x)|K,v)$ is weakly pure. If we take any extension of v to $\widetilde{K(x)}$ and take K^h to be the henselization of K in $(\widetilde{K(x)},v)$, then K^h is the implicit constant field of this extension:*

$$K^h \;=\; \mathrm{IC}\,(K(x)|K,v)\;.$$

Proof. As noted already in the introduction, K^h is contained in $\mathrm{IC}\,(K(x)|K,v)$. Since $K(x)^h$ is the fixed field of the decomposition group $G^d_x:=G^d(K(x)^{\mathrm{sep}}|K(x),v)$ in the separable-algebraic closure $K(x)^{\mathrm{sep}}$ of $K(x)$, we know that $\mathrm{IC}\,(K(x)|K,v)$ is the fixed field in K^{sep} of the subgroup

$$G_{\mathrm{res}} \;:=\; \{\sigma|_{K^{\mathrm{sep}}}\mid \sigma\in G^d_x\}$$

of $\mathrm{Gal}\,K$. In order to show our assertion, it suffices to show that $\mathrm{IC}\,(K(x)|K,v)\subseteq K^h$, that is, that the decomposition group $G^d:=G^d(K^{\mathrm{sep}}|K,v)$ is contained in G_{res}. So we have to show: if τ is an automorphism of $K^{\mathrm{sep}}|K$ such that $v\tau=v$ on K^{sep}, then τ can be lifted to an automorphism σ of $K(x)^{\mathrm{sep}}|K(x)$ such that $v\sigma=v$ on $K(x)^{\mathrm{sep}}$. In fact, it suffices to show that τ can be lifted to an automorphism σ of $K^{\mathrm{sep}}(x)|K(x)$ such that $v\sigma=v$ on $K^{\mathrm{sep}}(x)$. Indeed, then we take any extension σ' of σ from $K^{\mathrm{sep}}(x)$ to $K(x)^{\mathrm{sep}}$. Since the extensions $v\sigma'$ and v of v from $K^{\mathrm{sep}}(x)$ to $K(x)^{\mathrm{sep}}$ are conjugate, there is some $\rho\in\mathrm{Gal}\,(K(x)^{\mathrm{sep}}|K^{\mathrm{sep}}(x))$ such that $v\sigma'\rho=v$ on $K(x)^{\mathrm{sep}}$. Thus, $\sigma:=\sigma'\rho\in G^d$ is the desired lifting of τ to $K(x)^{\mathrm{sep}}$.

We take σ on $K^{\mathrm{sep}}(x)$ to be the unique automorphism which satisfies $\sigma x=x$ and $\sigma|_{K^{\mathrm{sep}}}=\tau$. Using that $(K(x)|K,v)$ is weakly pure, we have to show that $v\sigma=v$ on $K^{\mathrm{sep}}(x)$. Assume first that for some $c,d\in K$ and $e\in\mathbb{N}$, $d(x-c)^e$ is valuation-transcendental. Since $K(x-c)=K(x)$, we may assume w.l.o.g. that $c=0$. Every element of $K^{\mathrm{sep}}(x)$ can be written as a quotient of polynomials in x with coefficients from K^{sep}. For every polynomial $f(x)=a_nx^n+\ldots+a_1x+a_0\in K^{\mathrm{sep}}[x]$,

$$\begin{aligned} v\sigma f(x) &= v\,(\sigma(a_n)x^n+\ldots+\sigma(a_1)x+\sigma(a_0))\\ &= \min_i(v\sigma(a_i)+ivx) \;=\; \min_i(v\tau(a_i)+ivx)\\ &= \min_i(va_i+ivx) \;=\; vf(x)\;, \end{aligned}$$

where the second equality holds by Lemma 2.6 and Lemma 2.10. This shows that $v\sigma=v$ on $K^{\mathrm{sep}}(x)$.

Now assume that x is the pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type. By Lemma 2.5, this pseudo Cauchy sequence is also of transcendental type over (K^{sep},v). Observe that x is still a pseudo limit of this pseudo Cauchy sequence in $(K^{\mathrm{sep}}(x),v\sigma)$, because $v\sigma(x-a)=v(\sigma x-\sigma a)=v(x-a)$ for all $a\in K$. But $v\sigma=v\tau=v$ on K^{sep}, and the extension of v from K^{sep} to $K^{\mathrm{sep}}(x)$ is uniquely determined by the pseudo Cauchy sequence (cf. Theorem 2.3). Consequently, $v\sigma=v$ on $K^{\mathrm{sep}}(x)$. □

3.3. **Construction of nasty examples.** We are now able to give the

Proof of Theorem 1.2:

Let K be any algebraically closed field of characteristic $p>0$. On $K(x)$, we take v to be the x-adic valuation. We work in the power series field $K((\frac{1}{p^\infty}\mathbb{Z}))$ of all power series in x with exponents in $\frac{1}{p^\infty}\mathbb{Z}$, the p-divisible hull of $\mathbb{Z}$. We choose y to be a power series

$$y = \sum_{i=1}^{\infty} x^{-p^{-e_i}} \tag{3.3}$$

where e_i is any increasing sequence of natural numbers such that $e_{i+1} \geq e_i + i$ for all i. We then restrict the canonical (x-adic) valuation of $K((\frac{1}{p^\infty}\mathbb{Z}))$ to $K(x,y)$ and call it again v. We show first that $vK(x,y) = \frac{1}{p^\infty}\mathbb{Z}$. Indeed, taking p^{e_j}-th powers and using that the characteristic of K is p, we find

$$y^{p^{e_j}} - \sum_{i=1}^{j} x^{-p^{e_j - e_i}} = \sum_{i=j+1}^{\infty} x^{-p^{e_j - e_i}} .$$

Since $e_j - e_i \geq 0$ for $i \leq j$, the left hand side is an element in $K(x,y)$. The right hand side has value

$$-p^{e_j - e_{j+1}} vx \; ;$$

since $e_j - e_{j+1} \leq -j$, we see that $\frac{1}{p^j} vx$ lies in $vK(x,y)$. Hence, $\frac{1}{p^\infty}\mathbb{Z} \subseteq vK(x,y)$. On the other hand, $vK(x,y) \subseteq vK((\frac{1}{p^\infty}\mathbb{Z})) = \frac{1}{p^\infty}\mathbb{Z}$ and therefore, $vK(x,y) = \frac{1}{p^\infty}\mathbb{Z}$.

By definition, y is a pseudo limit of the pseudo Cauchy sequence

$$\left(\sum_{i=1}^{\ell} x^{-p^{-e_i}} \right)_{\ell \in \mathbb{N}}$$

in the field $L = K(x^{1/p^i} \mid i \in \mathbb{N}) \subset K((\frac{1}{p^\infty}\mathbb{Z}))$. Suppose it were of algebraic type. Then by [KA], Theorem 3, there would exist some algebraic extension $(L(b)|L, v)$ with b a pseudo limit of the sequence. But then b would also be algebraic over $K(x)$ and hence the extension $K(x,b)|K(x)$ would be finite. On the other hand, since b is a pseudo limit of the above pseudo Cauchy sequence, it can be shown as before that $vK(x,b) = \frac{1}{p^\infty}\mathbb{Z}$ and thus, $(vK(x,b) : vK(x)) = \infty$. This contradiction to the fundamental inequality shows that the sequence must be of transcendental type. Hence by Lemma 3.7, L^h is relatively algebraically closed in $L(y)^h$. Since $L^h = L.K(x)^h$ is a purely inseparable extension of $K(x)^h$ and $K(x,y)^h|K(x)^h$ is separable, this shows that $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$.

Now we set $\eta_0 := \frac{1}{x}$, and by induction on i we choose $\eta_i \in \widetilde{K(x)}$ such that $\eta_i^p - \eta_i = \eta_{i-1}$. Then we have

$$v\eta_i = -\frac{1}{p^i} vx$$

for every i. Since $vK(x)^h = vK(x) = \mathbb{Z}vx$, this shows that $K(x)^h(\eta_i)|K(x)^h$ has ramification index at least p^i. On the other hand, it has degree at most p^i and therefore, it must have degree and ramification index equal to p^i. Note that for all $i \geq 0$, $K(x,\eta_i) = K(x,\eta_1,\ldots,\eta_i)$ and every extension $K(x,\eta_{i+1})|K(x,\eta_i)$ is a Galois extension of degree and ramification index p. By what we have shown, the chain of these extensions is linearly disjoint from $K(x)^h|K(x)$. Since $K(x)^h$ is relatively algebraically closed in $K(x,y)^h$ and the extensions are separable, it follows that the chain is also linearly disjoint from $K(x,y)^h|K(x)$.

We will now show that all extensions $K(x,y,\eta_i)|K(x,y)$ are immediate. First, we note that $K(x,y)v = K$ since $K \subset K(x,y) \subset K((\frac{1}{p^\infty}\mathbb{Z}))$ and $K((\frac{1}{p^\infty}\mathbb{Z}))v = K$. Since K is algebraically closed, the inertia degree of the extensions must be 1.

Further, as the ramification index of a Galois extension is always a divisor of the extension degree, the ramification index of these extensions must be a power of p. But the value group of $K(x,y)$ is p-divisible, which yields that the ramification index of the extensions is 1. By what we have proved above, they are linearly disjoint from $K(x,y)^h|K(x,y)$, that is, the extension of the valuation is unique. This shows that the defect of each extension $K(x,y,\eta_i)|K(x,y)$ is equal to its degree p^i. □

Remark 3.8. Instead of defining y as in (3.3), we could also use any power series

$$y \;=\; \sum_{i=1}^{\infty} x^{n_i p^{-e_i}} \tag{3.4}$$

where $n_i \in \mathbb{Z}$ are prime to p and the sequence $n_i p^{-e_i}$ is strictly increasing. The example in [CP] is of this form. But in this example, the field $K(x,y)$ is an extension of degree p^2 of a field $K(u,v)$ such that the extension of the valuation from $K(u,v)$ to $K(x,y)$ is unique. Since the value group of $K(x,y)$ is $\frac{1}{p^\infty}\mathbb{Z}$, it must be equal to that of $K(u,v)$. Since K is algebraically closed, both have the same residue field. Therefore, the extension has defect p^2. This shows that we can also use subfields instead of field extensions to produce defect extensions, in quite the same way.

A special case of (3.4) is the power series

$$y \;=\; \sum_{i=1}^{\infty} x^{i-p^{-e_i}} \;=\; \sum_{i=1}^{\infty} x^i x^{-p^{-e_i}} \tag{3.5}$$

which now has a support that is cofinal in $\frac{1}{p^\infty}\mathbb{Z}$.

To conclude this section, we use Lemma 3.7 to construct an example about relatively algebraically closed subfields in henselian fields. The following fact is well known:
Suppose that K is relatively algebraically closed in a henselian valued field (L,v) of residue characteristic 0 and that $Lv|Kv$ is algebraic. Then vL/vK is torsion free.
We show that the assumption "$Lv|Kv$ is algebraic" is necessary.

Example 3.9. On the rational function field $\mathbb{Q}(x)$, we take v to be the x-adic valuation. We extend v to the rational function field $\mathbb{Q}(x,y)$ in such a way that $vy = 0$ and yv is transcendental over $\mathbb{Q}(x)v = \mathbb{Q}$. So by Lemma 2.6 we have $v\mathbb{Q}(x,y) = v\mathbb{Q}(x) = \mathbb{Z}vx$ and $\mathbb{Q}(x,y)v = \mathbb{Q}(yv)$. We pick any integer $n > 1$. Then $v\mathbb{Q}(x^n) = \mathbb{Z}nvx$ and $\mathbb{Q}(x^n)v = \mathbb{Q}$. Further, $v\mathbb{Q}(x^n,xy) = \mathbb{Z}vx$ since $vx = vxy \in v\mathbb{Q}(x^n,xy) \subseteq \mathbb{Z}vx$. Also, $\mathbb{Q}(x^n,xy)v = \mathbb{Q}(y^n v)$ by Lemma 2.10. From Lemma 3.7 we infer that $\mathbb{Q}(x^n)^h$ is relatively algebraically closed in $\mathbb{Q}(x^n,xy)^h$. But

$$v\mathbb{Q}(x^n,xy)^h/v\mathbb{Q}(x^n)^h \;=\; v\mathbb{Q}(x^n,xy)/v\mathbb{Q}(x^n) \;=\; \mathbb{Z}vx/\mathbb{Z}nvx \;\simeq\; \mathbb{Z}/n\mathbb{Z}$$

is a non-trivial torsion group.

3.4. **All valuations on $K(x)$.** In this section, we will explicitly define extensions of a given valuation on K to a valuation on $K(x)$. First, we define valuation-transcendental extensions, using the idea of valuation independence. Let (K,v) be an arbitrary valued field, and x transcendental over K. Take $a \in K$ and an element γ in some ordered abelian group extension of vK. We define a map $v_{a,\gamma} :\, K(x)^\times \to$

$vK + \mathbb{Z}\gamma$ as follows. Given any $g(x) \in K[x]$ of degree n, we can write

$$g(x) = \sum_{i=0}^{n} c_i (x-a)^i \ . \tag{3.6}$$

Then we set

$$v_{a,\gamma}\, g(x) \ := \ \min_{0\leq i\leq n} vc_i + i\gamma \ . \tag{3.7}$$

We extend $v_{a,\gamma}$ to $K(x)$ by setting $v_{a,\gamma}(g/h) := v_{a,\gamma}g - v_{a,\gamma}h$.

For example, the valuation $v_{0,0}$ is called **Gauß valuation** or **functional valuation** and is given by

$$v_{0,0}\,(c_n x^n + \ldots + c_1 x + c_0) \ = \ \min_{0\leq i\leq n} vc_i \ .$$

Lemma 3.10. *$v_{a,\gamma}$ is a valuation which extends v from K to $K(x)$. It satisfies:*
1) If γ is non-torsion over vK, then $v_{a,\gamma}K(x) = vK \oplus \mathbb{Z}\gamma$ and $K(x)v_{a,\gamma} = Kv$.
2) If γ is torsion over vK, e is the smallest positive integer such that $e\gamma \in vK$ and $d \in K$ is some element such that $vd = -e\gamma$, then $d(x-a)^e v_{a,\gamma}$ is transcendental over Kv, $K(x)v_{a,\gamma} = Kv(d(x-a)^e v_{a,\gamma})$ and $v_{a,\gamma}K(x) = vK + \mathbb{Z}\gamma$. In particular, if $\gamma = 0$ then $(x-a)v_{a,\gamma}$ is transcendental over Kv, $K(x)v_{a,\gamma} = Kv\,((x-a)v_{a,\gamma})$ and $v_{a,\gamma}K(x) = vK$.

Proof. It is a straightforward exercise to prove that $v_{a,\gamma}$ is a valuation and that 1) and 2) hold. However, one can also deduce this from Lemma 2.6. It says that if we assign a non-torsion value γ to $x-a$ then we obtain a unique valuation which satisfies (3.7). Since this defines a unique map $v_{a,\gamma}$ on $K[x]$, we see that $v_{a,\gamma}$ must coincide with the valuation given by Lemma 2.6, which in turn satisfies assertion 1). Similarly, if $\gamma \in vK$, $d \in K$ with $vd = -\gamma$ and we assign a transcendental residue to $d(x-a)$, then Lemma 2.6 gives us a valuation on $K(x)$ which satisfies (3.7) and hence must coincide with $v_{a,\gamma}$. This shows that $v_{a,\gamma}$ is a valuation and satisfies 2).

If $e > 1$, then we can first use Lemma 2.6 to see that $v_{a,\gamma}$ is a valuation on the subfield $K(d(x-a)^e)$ of $K(x)$ and that $v_{a,\gamma}K(d(x-a)^e) = vK$ and $K(d(x-a)^e)v_{a,\gamma} = Kv(d(x-a)^e v_{a,\gamma})$ with $d(x-a)^e v_{a,\gamma}$ transcendental over Kv. We know that there is an extension w of $v_{a,\gamma}$ to $K(x)$. It must satisfy $w(x-a) = -vd/e = \gamma$. So $0, w(x-a), w(x-a)^2, \ldots, w(x-a)^{e-1}$ lie in distinct cosets modulo vK. From Lemma 2.8 it follows that w satisfies (3.7) on $K(x)$, hence it must coincide with the valuation $v_{a,\gamma}$ on $K(x)$. Assertion 2) for this case follows from Lemma 2.8. □

Now we are able to prove:

Theorem 3.11. *Take any valued field (K,v). Then all extensions of v to the rational function field $\tilde{K}(x)$ are of the form*

- *$\tilde{v}_{a,\gamma}$ where $a \in \tilde{K}$ and γ is an element of some ordered group extension of vK, or*
- *$\tilde{v}_{\mathbf{A}}$ where $\mathbf{A}$ is a pseudo Cauchy sequence in $(\tilde{K},\tilde{v})$ of transcendental type,*

where $\tilde{v}$ runs through all extensions of v to $\tilde{K}$. The extension is of the form $\tilde{v}_{a,\gamma}$ with $\gamma \notin \tilde{v}\tilde{K}$ if and only if it is value-transcendental, and with $\gamma \in \tilde{v}\tilde{K}$ if and only if it is residue-transcendental. The extension is of the form $\tilde{v}_{\mathbf{A}}$ if and only if it is valuation-algebraic.

All extensions of v to $K(x)$ are obtained by restricting the above extensions, already from just one fixed extension $\tilde{v}$ of v to $\tilde{K}$.

Proof. By Lemma 3.10 and Theorem 2.3, $\tilde{v}_{a,\gamma}$ and $\tilde{v}_{\mathbf{A}}$ are extensions of $\tilde{v}$ to $\tilde{K}(x)$. For the converse, let w be any extension of v to $\tilde{K}(x)$ and set $\tilde{v} = w|_{\tilde{K}}$. From Lemma 3.6 we know that $(\tilde{K}(x)|\tilde{K},w)$ is always pure. Hence, either $d(x-c)$ is valuation-transcendental for some $c,d \in \tilde{K}$, or x is the pseudo limit of some pseudo Cauchy sequence $\mathbf{A}$ in $(\tilde{K},\tilde{v})$ of transcendental type. In the first case, Lemma 2.6 shows that

$$w\sum_{i=0}^{n} d_i(d(x-c))^i \;=\; \min_{0\leq i\leq n} vd_i + iwd(x-c) \;=\; \min_{0\leq i\leq n} vd_id^i + iw(x-c)$$

for all $d_i \in \tilde{K}$. This shows that $w = \tilde{v}_{c,\gamma}$ for $\gamma = w(x-c)$. If this value is not in $\tilde{v}\tilde{K}$, then it is non-torsion over $\tilde{v}\tilde{K}$ and thus, the extension of $\tilde{v}$ to $\tilde{K}(x)$, and hence also the extension of v to $K(x)$, is value-transcendental. If it is in $\tilde{v}\tilde{K}$, then the residue of $d(x-c)$ is not in $\tilde{K}\tilde{v}$, and the extension of $\tilde{v}$ to $\tilde{K}(x)$, and hence also the extension of v to $K(x)$, is residue-transcendental.

In the second case, we know from Theorem 2.3 that $\mathbf{A}$ induces an extension $\tilde{v}_{\mathbf{A}}$ of $\tilde{v}$ to $\tilde{K}(x)$ such that x is a pseudo limit of $\mathbf{A}$ in $(\tilde{K}(x),\tilde{v}_{\mathbf{A}})$. Since x is also a pseudo limit of $\mathbf{A}$ in $(\tilde{K}(x),w)$, we can infer from Lemma 2.4 that $w = \tilde{v}_{\mathbf{A}}$. It also follows from Theorem 2.3 that $(\tilde{K}(x)|\tilde{K},\tilde{v}_{\mathbf{A}})$ is immediate and consequently, $(\tilde{K}(x)|K,\tilde{v}_{\mathbf{A}})$ is valuation-algebraic.

For the last assertion, we invoke Lemma 3.2. Now it just remains to show that it suffices to take the restrictions of the valuations $\tilde{v}_{a,\gamma}$ and $\tilde{v}_{\mathbf{A}}$ for one fixed $\tilde{v}$. Suppose that $\tilde{w}$ is another extension of v to $\tilde{K}$. Since all such extensions are conjugate, there is $\sigma \in \operatorname{Gal} K$ such that $\tilde{w} = \tilde{v}\circ\sigma$. Let $g(x) \in K[x]$ be given as in (3.6). Extend σ to an automorphism of $\tilde{K}(x)$ which satisfies $\sigma x = x$. Since g has coefficients in K, we then have

$$g(x) \;=\; \sigma g(x) \;=\; \sum_i \sigma c_i(x-\sigma a)^i$$

and therefore,

$$\tilde{w}_{a,\gamma}\, g(x) \;=\; \min_i(\tilde{w}c_i + i\gamma) \;=\; \min_i(\tilde{v}\sigma c_i + i\gamma) \;=\; \tilde{v}_{\sigma a,\gamma}\, g(x)\;.$$

This shows that $\tilde{w}_{a,\gamma} = \tilde{v}_{\sigma a,\gamma}$ on $K(x)$.

Given a pseudo Cauchy sequence $\mathbf{A}$ in $(\tilde{K},\tilde{w})$, we set $\mathbf{A}_\sigma = (\sigma a_\nu)_{\nu<\lambda}$. This is a pseudo Cauchy sequence in $(\tilde{K},\tilde{v})$ since $\tilde{v}(\sigma a_\mu - \sigma a_\nu) = \tilde{v}\sigma(a_\mu - a_\nu) = \tilde{w}(a_\mu - a_\nu)$. For every polynomial $f(x) \in \tilde{K}[x]$, we have $\tilde{v}f(\sigma a_\nu) = \tilde{w}\sigma^{-1}(f(\sigma a_\nu)) = \tilde{w}(\sigma^{-1}(f))(a_\nu)$, where $\sigma^{-1}(f)$ denotes the polynomial obtained from $f(x)$ by applying σ^{-1} to the coefficients. So we see that $\mathbf{A}_\sigma$ is of transcendental type if and only if $\mathbf{A}$ is. If $g(x) \in K[x]$, then $\sigma^{-1}(g) = g$ and the above computation shows that $\tilde{v}g(\sigma a_\nu) = \tilde{w}g(a_\nu)$. This implies that $\tilde{w}_{\mathbf{A}} = \tilde{v}_{\mathbf{A}_\sigma}$ on $K(x)$. □

Remark 3.12. If v is trivial on K, hence $Kv = K$ (modulo an isomorphism), and if we choose $\gamma > 0$, then the restriction w of $\tilde{v}_{a,\gamma}$ to $K(x)$ will satisfy $xw = aw = a$. It follows that $K(x)w = K(a)$. Further, $wK(x) \subseteq \mathbb{Z}\gamma$ and thus, $wK(x) \simeq \mathbb{Z}$.

3.5. **Prescribed implicit constant fields.** If not stated otherwise, we will always assume that (K,v) is any valued field. The following is an immediate consequence of our version of Krasner's Lemma:

Lemma 3.13. *Assume that $K(a)|K$ is a separable-algebraic extension. Assume further that $K(x)|K$ is a rational function field and v is a valuation on $\widetilde{K(x)}$ such that*

$$v(x-a) \;>\; \mathrm{kras}(a,K)\;. \tag{3.8}$$

Then $K(a)\subseteq \mathrm{IC}\,(K(x)|K,v)$, and consequently,

$$vK(a)\subseteq vK(x) \quad \text{and} \quad K(a)v\subseteq K(x)v\;.$$

Proof. By Lemma 2.21, condition (3.8) implies that $K(a)\subseteq K(x)^h$, the henselization being chosen in $(\widetilde{K(x)},v)$. Consequently, $K(a)\subseteq \mathrm{IC}\,(K(x)|K,v)$, $vK(a)\subseteq vK(x)^h = vK(x)$ and $K(a)v\subseteq K(x)^h v = K(x)v$. □

Proposition 3.14. *Assume that $(K(a)|K,v)$ is a separable-algebraic extension of valued fields. Further, take Γ to be the abelian group $vK(a)\oplus\mathbb{Z}$ endowed with any extension of the ordering of $vK(a)$, and take k to be the rational function field in one variable over $K(a)v$. Then there exists an extension v_1 of v from $K(a)$ to $\widetilde{K(x)}$ such that*

$$v_1K(x) \;=\; \Gamma \quad \text{and} \quad K(x)v_1 \;=\; K(a)v \quad \text{and} \quad K(a)^h \;=\; \mathrm{IC}\,(K(x)|K,v_1)\;. \tag{3.9}$$

If v is non-trivial on K, then there exists an extension v_2 of v from $K(a)$ to $\widetilde{K(x)}$ such that

$$v_2K(x) \;=\; vK(a) \quad \text{and} \quad K(x)v_2 \;=\; k \quad \text{and} \quad K(a)^h \;=\; \mathrm{IC}\,(K(x)|K,v_2)\;. \tag{3.10}$$

If $(K(a),v)$ admits a transcendental immediate extension, then there is also an extension v_3 such that

$$v_3K(x) = vK(a) \quad \text{and} \quad K(x)v_3 = K(a)v\;. \tag{3.11}$$

If in addition $(K(a),v)$ admits a pseudo Cauchy sequence of transcendental type, then v_3 can be chosen such that $K(a)^h = \mathrm{IC}\,(K(x)|K,v_3)$.

Suppose that $(K(a),<)$ is an ordered field and that v is convex. Denote by $<_r$ the ordering induced by $<$ on the residue field $K(a)v$. Then $<_r$ can be lifted through v_1 and through v_3 to $K(x)$ in such a way that the lifted orderings extend $<$ (from K). If $<'_r$ is an extension of $<_r$ to k, then $<'_r$ can be lifted through v_2 to $K(x)$ in such a way that the lifted ordering extends $<$ (from K).

Proof. If $K(a)=K$, then we do the following. We take a generator γ of Γ over vK and let v_1 be any extension of $v_{0,\gamma}$ to $\widetilde{K(x)}$. Further, we take v_2 to be any extension of $v_{0,0}$ to $\widetilde{K(x)}$. Then the desired properties of v_1 and v_2 follow from Lemma 3.10 and Lemma 3.7. To construct v_3, we send x to some transcendental element in the given immediate extension of (K,v). The so obtained embedding induces an extension v_3 of v to $K(x)$ such that $(K(x)|K,v_3)$ is immediate. We extend v_3 further to $\widetilde{K(x)}$. If (K,v) admits a pseudo Cauchy sequence of transcendental type, then we can use Theorem 2.3 to construct an immediate extension v_3 of v to $K(x)$ such that x is a pseudo limit of this pseudo Cauchy sequence. Then by Lemma 3.7, $K^h \;=\; \mathrm{IC}\,(K(x)|K,v_3)$.

Now assume that $a \notin K$. If v' is any extension of v to $\widetilde{K(x)}$ such that $v'(x-a) > \mathrm{kras}(a,K)$, then Lemma 3.13 shows that $a \in K(x)^h$, where the henselization is taken in $(\widetilde{K(x)}, v')$. Thus,

$$K(x)^h \;=\; K(a,x)^h \;, \tag{3.12}$$

and consequently,

$$\left.\begin{array}{rcccccl} v'K(x) & = & v'K(x)^h & = & v'K(a,x)^h & = & v'K(a,x) \\ K(x)v' & = & K(x)^h v' & = & K(a,x)^h v' & = & K(a,x)v' \;. \end{array}\right\} \tag{3.13}$$

If in addition we know that

$$K(a)^h \;=\; \mathrm{IC}\,(K(a,x)|K(a),v') \;, \tag{3.14}$$

then

$$K(a)^h \;\subseteq\; \mathrm{IC}\,(K(x)|K,v') \;\subseteq\; \mathrm{IC}\,(K(a,x)|K(a),v') \;=\; K(a)^h \;,$$

which yields that $K(a)^h = \mathrm{IC}\,(K(x)|K,v')$.

As vK is cofinal in its divisible hull $v\tilde{K}$, we can choose some $\alpha \in vK$ such that $\alpha \geq \mathrm{kras}(a,K)$.

To construct v_1, we choose any positive generator β of Γ over $vK(a)$. Then also $\gamma := \alpha + \beta$ is a generator of Γ over $vK(a)$. Now we take v_1 to be any extension of $v_{a,\gamma}$ to $\widetilde{K(x)}$. From Lemma 3.10 we know that $v_{a,\gamma}K(a,x) = vK(a) \oplus \mathbb{Z}\gamma = \Gamma$ and $K(a,x)v_{a,\gamma} = K(a)v$. Since

$$v_{a,\gamma}(x-a) \;=\; \gamma \;>\; \mathrm{kras}(a,K) \;, \tag{3.15}$$

equations (3.13) hold for $v' = v_{a,\gamma}$, and we obtain:

$$\begin{array}{rcccccl} v_1K(x) & = & v_{a,\gamma}K(a,x) & = & vK(a) + \mathbb{Z}\gamma & = & \Gamma \\ K(x)v_1 & = & K(a,x)v_{a,\gamma} & = & K(a)v \;. & & \end{array}$$

From Lemma 3.7 we infer that (3.14) holds for $v' = v_{a,\gamma}$. Consequently, $K(a)^h = \mathrm{IC}\,(K(x)|K,v_1)$.

To construct v_2, we make use of our assumption that v is non-trivial on K and choose some positive $\beta \in vK$. We set $\gamma := \alpha + \beta$ and take v_2 to be any extension of $v_{a,\gamma}$ to $\widetilde{K(x)}$. From Lemma 3.10 we know that $v_{a,\gamma}K(a,x) = vK(a) = \Gamma$ and $K(a,x)v_{a,\gamma}$ is a rational function field in one variable over $K(a)v$; in this sense, $K(a,x)v_{a,\gamma} = k$. Again we have (3.15), and equations (3.13) hold for $v' = v_{a,\gamma}$; so we obtain

$$\begin{array}{rcccl} v_2K(x) & = & v_{a,\gamma}K(a,x) & = & vK(a) \\ K(x)v_2 & = & K(a,x)v_{a,\gamma} & = & k \;. \end{array}$$

From Lemma 3.7 we infer that (3.14) holds for $v' = v_{a,\gamma}$. Consequently, $K(a)^h = \mathrm{IC}\,(K(x)|K,v_2)$.

To construct v_3, we take $(L|K(a),v)$ to be the transcendental immediate extension which exists by hypothesis. We choose some $y \in L$, transcendental over $K(a)$ and such that $\gamma := vy > \mathrm{kras}(a,K)$. Then $(K(a,y)|K(a),v)$ is immediate. Now the isomorphism $K(a,y) \simeq K(a,x)$ induced by $y \mapsto x-a$ induces on $K(a,x)$ a valuation v' such that $(K(a,x)|\widetilde{K(a)},v')$ is immediate and $v'(x-a)=\gamma$. We take v_3 to be any extension of v' to $\widetilde{K(x)}$. Again, we have (3.13), and we obtain

$$\begin{array}{rcl} v_3K(x) &=& v'K(a,x) \;=\; vK(a) \\ K(x)v_3 &=& K(a,x)v' \;=\; K(a)v\;. \end{array}$$

If $(K(a),v)$ admits a pseudo Cauchy sequence of transcendental type, then we can use Theorem 2.3 to construct an immediate extension $(K(a,z)|K(a),v)$. Multiplying z with a suitable element from K will give us y such that $vy>\mathrm{kras}(a,K)$. By our above construction, also x will be a pseudo limit of a pseudo Cauchy sequence of transcendental type. Then by Lemma 3.7, $K(a)^h \;=\; \mathrm{IC}\,(K(x)|K,v_3)$.

Finally, suppose that $(K(a),<)$ is an ordered field and that $<_r$ is the ordering induced by $<$ on $K(a)v$. We have that $vK(a)=v_2K(a,x)=v_3K(a,x)$, so it is trivially true that $2v_2K(a,x)\cap vK(a)=2vK(a)=2v_3K(a,x)\cap vK(a)$. Further, $v_1K(a,x)=vK(a)\oplus\mathbb{Z}\gamma$, hence also in this case, $2v_1K(a,x)\cap vK(a)=2vK(a)$. Therefore, Proposition 2.18 shows that the given orderings on $K(a)v$ and k can be lifted through v_1, v_2 and v_3 respectively, to orderings on $K(a,x)$ which extend the ordering $<$ of $K(a)$. □

This proposition proves Theorem 1.3 in the case where $K_1^h|K^h$ is finite since then, there is some $a\in K_1$ such that $K_1^h=K(a)^h$. Further, we obtain:

Proposition 3.15. *Take any finite ordered abelian group extension Γ_0 of vK and any finite field extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0\oplus\mathbb{Z}$ endowed with any extension of the ordering of Γ_0, and take k to be the rational function field in one variable over k_0. If v is trivial on K, then assume in addition that $k_0|Kv$ is separable. Then there exists an extension v_1 of v from K to $K(x)$ such that*

$$v_1K(x) \;=\; \Gamma \quad\textit{and}\quad K(x)v_1 \;=\; k_0\;. \tag{3.16}$$

If v is non-trivial on K, then there exists an extension v_2 of v from K to $K(x)$ such that

$$v_2K(x) \;=\; \Gamma_0 \quad\textit{and}\quad K(x)v_2 \;=\; k\;. \tag{3.17}$$

If (K,v) admits a transcendental immediate extension, then there is also an extension v_3 such that

$$v_3K(x)=\Gamma_0 \quad\textit{and}\quad K(x)v_3=k_0\;. \tag{3.18}$$

Suppose that $(K,<)$ is an ordered field and that v is convex. Suppose further that k_0 and k are equipped with extensions of the ordering induced by $<$ on Kv. Then these extensions can be lifted through v_1, v_2, v_3 to $K(x)$ in such a way that the lifted orderings extend $<$.

Proof. We choose any finite separable extension $(K(a)|K,v)$ such that $vK(a)=\Gamma_0$, $K(a)v=k_0$ and $[K(a):K]=(vK(a):vK)[K(a)v:Kv]$. If v is non-trivial on K, then such an extension exists by Theorem 2.14; otherwise, $K(a)$ is just equal to k_0, up to the isomorphism induced by the residue map of the trivial valuation v. If (K,v) admits a transcendental immediate extension, then by Lemma 2.2, also $(K(a),v)$ admits a transcendental immediate extension. Now the first part of our theorem follows from Proposition 3.14.

Suppose that $(K,<)$ is an ordered field with v convex. Then by Corollary 2.20 we can choose the field $K(a)$ to be a subfield of a real closure $(R,<)$ of K, equipped with a convex extension of v, in such a way that the given ordering on k_0 is induced by $<$ through this extension of v. Now again, our assertion for the ordered case follows from Proposition 3.14. $\square$

We turn to the realization of countably infinite separable-algebraic extensions as implicit constant fields.

Proposition 3.16. *Let $(K_1|K,v)$ be a countably infinite separable-algebraic extension of non-trivially valued henselian fields. Then (K_1,v) admits a pseudo Cauchy sequence of transcendental type. In particular, there is an extension v_3 of v to $\widetilde{K(x)}$ such that $(K_1(x)|K_1,v)$ is immediate, with x being the pseudo limit of this pseudo Cauchy sequence. Moreover, $K_1 = \mathrm{IC}\,(K(x)|K,v_3)$.*

Proof. $K_1|K$ is a countably infinite union of finite subextensions. Thus, we can choose a sequence a_i , $i\in\mathbb{N}$ such that $K(a_i)|K$ is separable-algebraic and $K(a_i)\subsetneqq K(a_{i+1})$ for all i, and such that $K_1=\bigcup_{i\in\mathbb{N}}K(a_i)$. Through multiplication with elements from K it is possible to choose each a_i in such a way that

$$va_{i+1} \;>\; \max\{va_i,\mathrm{kras}(a_i,K(a_1,\dots,a_{i-1}))\}\;. \tag{3.19}$$

We set

$$b_i \;:=\; \sum_{j=1}^{i} a_j\;.$$

By construction, $(b_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence. Suppose that x is a pseudo limit of it, for some extension of v from K_1 to $K_1(x)$. Then by induction on i, we show that $a_i\in K(x)^h$, where $K(x)^h$ is chosen in some henselization of $(K_1(x),v)$. First, by Lemma 2.21, $v(x-a_1)=v(x-b_1)=va_2>\mathrm{kras}(a_1,K)$ implies that $a_1\in K(x)^h$. If we have already shown that $a_1,\dots,a_i\in K(x)^h$, then $b_i\in K(x)^h$ and $v(x-b_i-a_{i+1})=v(x-b_{i+1})=va_{i+2}>\mathrm{kras}(a_{i+1},K(a_1,\dots,a_i))$ implies that $a_{i+1}\in K(a_1,\dots,a_i)(x-b_i)^h=K(a_1,\dots,a_i)(x)^h=K(x)^h$.

This proves that $K_1\subseteq K(x)^h$. Since $K_1|K$ is infinite, this also proves that x must be transcendental. As a pseudo Cauchy sequence of algebraic type would admit an algebraic pseudo limit ([KA], Theorem 3), this yields that $(b_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in (K_1,v).

By Theorem 2.3 we can now extend v to $K_1(x)$ such that $(K_1(x)|K_1,v)$ is immediate and x is a pseudo limit of $(b_i)_{i\in\mathbb{N}}$. We choose any extension v_3 of v from $K_1(x)$ to $\widetilde{K(x)}$. By Lemma 3.7 we know that $K_1=\mathrm{IC}\,(K_1(x)|K_1,v_3)$. Hence,

$$K_1\;\subseteq\;K(x)^h\;\subseteq\;\mathrm{IC}\,(K(x)|K,v_3)\;\subseteq\;\mathrm{IC}\,(K_1(x)|K_1,v_3)\;=\;K_1\;,$$

which shows that $K_1=\mathrm{IC}\,(K(x)|K,v_3)$. $\square$

Since $\mathrm{IC}\,(K(x)|K,v)=\mathrm{IC}\,(K^h(x)|K^h,v)$, this proposition proves Theorem 1.3 in the case where $K_1^h|K^h$ is countably infinite.

Proposition 3.17. *Take any non-trivially valued field (K,v), any countably generated ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and any countably generated algebraic field extension k_0 of Kv. Assume that Γ_0/vK or $k_0|Kv$ is infinite. Then there exists an extension v_3 of v from K to $K(x)$ such that equations (3.18) hold.*

Suppose that $(K,<)$ is an ordered field and that v is convex. If k_0 is equipped with an extension of the ordering induced by $<$ on Kv, then this can be lifted through v_3 to an ordering of $K(x)$ which extends $<$.

Proof. We fix an extension of v to $\tilde{K}$. By Theorem 2.14 there is a countably generated separable-algebraic extension $K_1|K^h$ such that $vK_1 = \Gamma_0$ and $K_1v = k_0$. Since at least one of the extensions $\Gamma_0|vK$ and $k_0|Kv$ is infinite, $K_1^h|K^h$ is infinite too, taking the henselizations in $(\tilde{K},v)$. Hence by Proposition 3.16 there is an extension v_3 of v to $K_1(x)$ such that $K_1 \subset K(x)^h$ and that $(K_1(x)|K_1,v_3)$ is immediate. Consequently, $K_1(x)^h = K(x)^h$, which gives us $v_3K(x) = v_3K(x)^h = v_3K_1(x) = vK_1 = \Gamma_0$ and $K(x)v_3 = K(x)^h v_3 = K_1(x)v_3 = K_1v = k_0$.

Suppose that $(K,<)$ is ordered, v is convex and k_0 is equipped with an extension of the ordering induced by $<$ on Kv. Take any real closure $(R,<)$ of $(K,<)$ with an extension of v to a convex valuation of R. By Corollary 2.20 we can choose the extension $K_1|K$ as a subextension of $R|K$, with the ordering on k_0 induced by $<$ through v. Since $(K_1(x)|K_1,v_3)$ is immediate, Proposition 2.18 shows that there is a lifting of the ordering of k_0 through v to an ordering of $K_1(x)$ which extends the ordering $<$ of K_1. Its restriction to $K(x)$ is an extension of the ordering $<$ of K. □

To conclude this section, we now give an alternative

Proof of Theorem 1.2:

Let K be an algebraically closed field of characteristic $p>0$. On $K(x)$, we again take v to be the x-adic valuation. We assume that K contains an element t which is transcendental over the prime field of K. Then it can be proved that $K(x)^h$ admits two infinite linearly disjoint towers of Galois extensions of degree p and ramification index p. They can be defined as follows. For the first tower, we set $\eta_0 = x^{-1}$, take η_{i+1} to be a root of $X^p - X - \eta_i$, and set $L_i := K(\eta_i)$. For the second tower, we set $\vartheta_0 = tx^{-1}$, take ϑ_{i+1} to be a root of $X^p - X - t\vartheta_i$, and set $N_i := K(\vartheta_i)$. Note that $L_0 = N_0 = K(x)$, $\eta_i \in L_{i+1}$ and $\vartheta_i \in N_{i+1}$. For each $i \geq 0$, we have that $v\eta_i = v\vartheta_i = -\frac{1}{p^i}vx$, and the extensions $L_{i+1}|L_i$ and $N_{i+1}|N_i$ are Galois of degree p with ramification index p. Consequently, the same is true for the extensions $L_{i+1}^h|L_i^h$ and $N_{i+1}^h|N_i^h$ (note that $L_i^h = L_i.K(x)^h$ and $N_i^h = N_i.K(x)^h$).

We set $L := \bigcup_{i\in\mathbb{N}} L_i$ and $N := \bigcup_{i\in\mathbb{N}} N_i$. By the above, L and N are linearly disjoint from $K(x)^h$ over $K(x)$. Thus, $L^h = L.K(x)^h$ and $N^h = N.K(x)^h$ are countably infinite separable-algebraic extensions of $K(x)^h$, and it can be proved that they are linearly disjoint over $K(x)^h$. We use Proposition 3.16 (with K replaced by $K(x)^h$ and x replaced by y) to obtain an extension of v from $K(x)^h$ to $K(x)^h(y)$ such that $vK(x)^h(y) = vL^h = \frac{1}{p^\infty}\mathbb{Z}$ and $K(x)^h(y)v = L^hv = K(x)^hv = K$, and such that the extension $(K(x)^h(y)|K(x)^h,v)$ has implicit constant field L^h (i.e., L^h is relatively algebraically closed in $K(x,y)^h = (K(x)^h(y))^h$). Since $K(x)$ is relatively algebraically closed in $K(x,y)$, we see that N is linearly disjoint from $K(x,y)$ over $K(x)$ and therefore, $N.K(x,y)|K(x,y)$ is again an infinite tower of Galois extensions of degree p. Since N is linearly disjoint from $K(x)^h$ over $K(x)$ and $N^h = N.K(x)^h$ is linearly disjoint from L^h over $K(x)^h$, we see that N is linearly disjoint from L^h over $K(x)$. Since L^h is relatively algebraically closed in $K(x,y)^h$, this implies that N is linearly disjoint from $K(x,y)^h$ over $K(x)$ and hence, $N.K(x,y)$ is linearly disjoint from $K(x,y)^h$ over $K(x,y)$. Therefore, the extension of v from $K(x,y)$ to $N.K(x,y)$ is unique. Since $(K(x)^h(y))^h = K(x,y)^h$, we see that $(K(x)^h(y)|K(x,y),v)$ is immediate. So we have $vK(x,y) = \frac{1}{p^\infty}\mathbb{Z}$, which is p-divisible, and $K(x,y)v = K$, which is algebraically closed. Hence, the extension $(N.K(x,y)|K(x,y),v)$ is immediate, and so it is an infinite tower of Galois extensions of degree p and defect p. □

Remark 3.18. For the above defined η_i, we have that the roots of $X^p - X - \eta_{i-1}$ are $\eta_i, \eta_i + 1, \ldots, \eta_i + p - 1$. Therefore,

$$\mathrm{kras}(\eta_i, K(\eta_{i-1})) \;=\; 0$$

for all i. Setting $a_i \;:=\; x^i\eta_i$, we obtain that

$$\begin{aligned}\mathrm{kras}(a_i, K(x)^h(a_1,\ldots,a_{i-1})) &= \mathrm{kras}(a_i, K(x)^h(\eta_{i-1}))\\ &= ivx + \mathrm{kras}(\eta_i, K(x)^h(\eta_{i-1})) \;=\; ivx\end{aligned}$$

and $va_{i+1} = (i+1)vx - \frac{1}{p^{i+1}}vx > ivx$. This shows that $va_{i+1} > va_i$ and that (3.19) is satisfied. So we can take y to be a pseudo limit of the Cauchy sequence $(\sum_{j=1}^{i} a_j)_{i\in\mathbb{N}}$. That is,

$$y \;=\; \sum_{i=1}^{\infty} x^i\eta_i\;. \tag{3.20}$$

Comparing this with (3.5), we see that we have replaced the term $x^{-p^{-e_i}}$ by η_i which has an infinite expansion in powers of x, starting with $x^{-p^{-i}}$.

4. Rational function fields of higher transcendence degree

This section is devoted to the proof of Theorems 1.6 and 1.8. We will make use of the following theorem which we prove in [KU5]:

Theorem 4.1. *Let $(L|K,v)$ be a valued field extension of finite transcendence degree ≥ 0, with v non-trivial on L. Assume that one of the following two cases holds:*

transcendental case: vL/vK has rational rank at least 1 or $Lv|Kv$ is transcendental;

separable-algebraic case: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ is infinite.

Then each maximal immediate extension of (L,v) has infinite transcendence degree over L.

Note: The assertion need not be true for an infinite purely inseparable extension $L|K$.

Now we can give the

Proof of Theorem 1.6: Assume that (K,v) is a valued field, n,ρ,τ,ℓ are non-negative integers, Γ is an ordered abelian group extension of vK such that Γ/vK is of rational rank ρ, and $k|Kv$ is a field extension of transcendence degree τ. We pick a maximal set of elements $\alpha_1,\ldots,\alpha_\rho$ in Γ rationally independent over vK, and a transcendence basis $\zeta_1,\ldots,\zeta_\tau$ of $k|Kv$.

We prove **Part A** of Theorem 1.6 first. So assume further that $n > \rho+\tau$, $\Gamma|vK$ and $k|Kv$ are countably generated, and at least one of them is infinite. By

Lemma 2.6 there is a unique extension of v from K to $K(x_1,\ldots,x_{\rho+\tau})$ such that $vx_i=\alpha_i$ for $1\leq i\leq\rho$, and $x_{\rho+i}v=\zeta_i$ for $1\leq i\leq\tau$. Since $\Gamma|vK$ and $k|Kv$ are countably generated, $vK(x_1,\ldots,x_{\rho+\tau})$ contains $\alpha_1,\ldots,\alpha_\rho$ and $K(x_1,\ldots,x_{\rho+\tau})v$ contains $\zeta_1,\ldots,\zeta_\tau$, the extensions $\Gamma|vK(x_1,\ldots,x_{\rho+\tau})$ and $k|K(x_1,\ldots,x_{\rho+\tau})v$ are countably generated and algebraic.

Suppose first that at least one of these extensions is infinite. From our assumption that $\Gamma\neq\{0\}$ it follows that v is non-trivial on $K(x_1,\ldots,x_{\rho+\tau})$. Hence we can use Proposition 3.17 to find an extension of v from $K(x_1,\ldots,x_{\rho+\tau})$ to $K(x_1,\ldots,x_{\rho+\tau+1})$ such that $vK(x_1,\ldots,x_{\rho+\tau+1})=\Gamma$ and $K(x_1,\ldots,x_{\rho+\tau+1})v=k$, and that the implicit constant field of $(K(x_1,\ldots,x_{\rho+\tau+1})|K(x_1,\ldots,x_{\rho+\tau}),v)$ is an infinite separable-algebraic extension of $K(x_1,\ldots,x_{\rho+\tau})^h$. That means that $K(x_1,\ldots,x_{\rho+\tau+1})^h|K(x_1,\ldots,x_{\rho+\tau})^h$ contains an infinite separable-algebraic subextension. Hence by the separable-algebraic case of Theorem 4.1, each maximal immediate extension of $K(x_1,\ldots,x_{\rho+\tau+1})^h$ is of infinite transcendence degree. This shows that we can find an immediate extension of v from $K(x_1,\ldots,x_{\rho+\tau+1})^h$ to $K(x_1,\ldots,x_{\rho+\tau+1})^h(x_{\rho+\tau+2},\ldots,x_n)$. Its restriction to $K(x_1,\ldots,x_n)$ is an immediate extension of v from $K(x_1,\ldots,x_{\rho+\tau+1})$.

Now suppose that both extensions $\Gamma|vK(x_1,\ldots,x_{\rho+\tau})$ and $k|K(x_1,\ldots,x_{\rho+\tau})v$ are finite. Then from our hypothesis that at least one of the extensions $\Gamma|vK$ and $k|Kv$ is infinite, it follows that $\rho>0$ or $\tau>0$. Hence by the transcendental case of Theorem 4.1, any maximal immediate extension of $(K(x_1,\ldots,x_{\rho+\tau}),v)$ is of infinite transcendence degree. In combination with Proposition 3.15, this shows that there is an extension of v from $K(x_1,\ldots,x_{\rho+\tau})$ to $K(x_1,\ldots,x_{\rho+\tau+1})$ such that $vK(x_1,\ldots,x_{\rho+\tau+1})=\Gamma$ and $K(x_1,\ldots,x_{\rho+\tau+1})v=k$. By the transcendental case of Theorem 4.1, every maximal immediate extension of $(K(x_1,\ldots,x_{\rho+\tau+1}),v)$ is of infinite transcendence degree. So we can find an extension of v to $K(x_1,\ldots,x_n)$ as in the previous case.

In both cases we have that $vK(x_1,\ldots,x_n)=\Gamma$ and $K(x_1,\ldots,x_n)v=k$, as required.

Now we prove **Part B**. So assume that $n\geq\rho+\tau$, and that $\Gamma|vK$ and $k|Kv$ are finitely generated.

I) First, we consider the case of $\rho>0$. By Lemma 2.6 there is an extension of v from K to $K(x_1,\ldots,x_{\rho-1+\tau})$ such that $vx_i=\alpha_i$ for $1\leq i\leq\rho-1$, and $x_{\rho-1+i}v=\zeta_i$ for $1\leq i\leq\tau$. Since $\Gamma|vK$ and $k|Kv$ are finitely generated, $vK(x_1,\ldots,x_{\rho-1+\tau})$ contains $\alpha_1,\ldots,\alpha_{\rho-1}$ and $K(x_1,\ldots,x_{\rho-1+\tau})v$ contains $\zeta_1,\ldots,\zeta_\tau$, the extension $\Gamma|vK(x_1,\ldots,x_{\rho-1+\tau})$ is finitely generated of rational rank 1 (and thus, Γ is of the form $\Gamma_0\oplus\mathbb{Z}$ with $\Gamma_0/vK(x_1,\ldots,x_{\rho-1+\tau})$ finite), and the extension $k|K(x_1,\ldots,x_{\rho-1+\tau})v$ is finite.

If v is not trivial on $K(x_1,\ldots,x_{\rho-1+\tau})$, then we can apply Proposition 3.15 to obtain an extension of v from $K(x_1,\ldots,x_{\rho-1+\tau})$ to $K(x_1,\ldots,x_{\rho+\tau})$ such that $vK(x_1,\ldots,x_{\rho+\tau})=\Gamma$ and $K(x_1,\ldots,x_{\rho+\tau})v=k$. From the transcendental case of Theorem 4.1 we see that any maximal immediate extension of $K(x_1,\ldots,x_{\rho+\tau}),v)$ is of infinite transcendence degree. This shows that we can find an immediate extension of v from $K(x_1,\ldots,x_{\rho+\tau})$ to $K(x_1,\ldots,x_n)$.

Now suppose that v is trivial on $K(x_1,\ldots,x_{\rho-1+\tau})$. Then $\rho=1$ and v is trivial on K. It follows that $\Gamma\simeq\mathbb{Z}$, and we pick a generator γ of Γ.

If $n > 1+\tau$, we modify the above procedure in such a way that we use Lemma 2.6 to choose an extension of v from $K(x_1,\ldots,x_\tau)$ to $K(x_1,\ldots,x_{1+\tau})$ so that the latter has value group $\mathbb{Z}\gamma = \Gamma$ and residue field $Kv(\zeta_1,\ldots,\zeta_\tau)$. Then we use Proposition 3.15 in combination with the transcendental case of Theorem 4.1 to find an extension of v to $K(x_1,\ldots,x_n)$ with value group Γ and residue field k.

If $n = 1+\tau$, then by our assumption B2), k is a simple algebraic extension of a rational function field k' in τ variables over Kv (or of Kv itself if $\tau = 0$), or a rational function field in one variable over a finitely generated field extension k_0 of Kv of transcendence degree $\tau - 1$.

Assume the first case holds. The extension of v to the rational function field $K(x_1,\ldots,x_\tau)$ can be chosen such that $k' = Kv(x_1v,\ldots,x_\tau v) = K(x_1,\ldots,x_\tau)v$. Then the extension $k|K(x_1,\ldots,x_\tau)v$ is simple algebraic, and according to Remark 3.12 there is an extension of v to the rational function field $K(x_1,\ldots,x_\tau)(x_{1+\tau})$ which satisfies $vK(x_1,\ldots,x_{1+\tau}) = \mathbb{Z}\gamma = \Gamma$ and $K(x_1,\ldots,x_{1+\tau})v = k$.

In the second case, we know that $\tau \geq 1$. The elements $\zeta_i \in k$ can be chosen in such a way that $\zeta_1,\ldots,\zeta_{\tau-1}$ form a transcendence basis of $k_0|Kv$. We pick $\gamma \in \Gamma$, $\gamma \neq 0$. By Lemma 2.6 there is an extension of v from K to $K(x_1,\ldots,x_\tau)$ such that $x_iv = \zeta_i$ for $1 \leq i \leq \tau - 1$, and $vx_\tau = \gamma$. Since $K(x_1,\ldots,x_\tau)v = Kv(\zeta_1,\ldots,\zeta_{\tau-1})$, we see that k is a rational function field in one variable over a finite extension of $K(x_1,\ldots,x_\tau)v$. Further, $vK(x_1,\ldots,x_\tau) = \Gamma$. Now we use Proposition 3.15 to find an extension of v to $K(x_1,\ldots,x_{1+\tau})$ with value group Γ and residue field k.

II) Second, suppose that $\rho = 0$. Then because of $\Gamma \neq \{0\}$, we know that v is non-trivial on K. If $\tau > 0$, then we proceed as follows. The case of $n > \tau$ is covered by Part A. So we assume that $n = \tau$, and that k is a rational function field in one variable over a finitely generated field extension k_0 of Kv of transcendence degree $\tau - 1$. Again we choose $\zeta_1,\ldots,\zeta_{\tau-1}$ to form a transcendence basis of $k_0|Kv$, and use Lemma 2.6 to find an extension of v from K to $K(x_1,\ldots,x_{\tau-1})$ such that $x_iv = \zeta_i$ for $1 \leq i \leq \tau - 1$. Again we obtain that k is a rational function field in one variable over a finite extension of $K(x_1,\ldots,x_{\tau-1})v$. By assumption, $\Gamma/vK(x_1,\ldots,x_{\tau-1}) = \Gamma/vK$ is finite. Hence by Proposition 3.15 there is an extension of v from $K(x_1,\ldots,x_{\tau-1})$ to $K(x_1,\ldots,x_\tau)$ such that $vK(x_1,\ldots,x_\tau) = \Gamma$ and $K(x_1,\ldots,x_\tau)v = k$.

Finally, suppose that $\rho = 0 = \tau$ and that there is an immediate extension (K',v) of (K,v) which is either infinite separable-algebraic or of transcendence degree at least n. If the former holds, then we obtain from the separable-algebraic case of Theorem 4.1 that every maximal immediate extension of (K',v) has infinite transcendence degree. But a maximal immediate extension of (K',v) is also a maximal immediate extension of (K,v). Thus in all cases, we have the existence of immediate extensions of (K,v) of transcendence degree at least n. Therefore, we can choose an immediate extension of v from K to $K(x_1,\ldots,x_{n-1})$ such that $(K(x_1,\ldots,x_{n-1}),v)$ still admits a transcendental immediate extension. The extensions $\Gamma|vK = \Gamma|vK(x_1,\ldots,x_{n-1})$ and $k|Kv = k|K(x_1,\ldots,x_{n-1})v$ are both finite since they are finitely generated and algebraic. Hence by Proposition 3.15 there is an extension of v from $K(x_1,\ldots,x_{n-1})$ to $K(x_1,\ldots,x_n)$ such that $vK(x_1,\ldots,x_n) = \Gamma$ and $K(x_1,\ldots,x_n)v = k$. □

Now we give the

Proof of Theorem 1.7: Corollary 2.7 shows that $n \geq \rho + \tau$. If $n = \rho + \tau$, then Corollary 2.7 tells us that $vF|vK$ and $Fv|Kv$ are finitely generated extensions.

The fact that $vF|vK$ and $Fv|Kv$ are always countably generated follows from Theorem 2.9 by induction on the transcendence degree n.

Suppose that F admits a transcendence basis $x_1,\dots,x_n$ such that the residue field extension $Fv|K(x_1,\dots,x_{n-1})v$ is of transcendence degree 1. (This is in particular the case when $n=\tau$.) Then by Ohm's Ruled Residue Theorem, Fv is a rational function field in one variable over a finite extension of $K(x_1,\dots,x_{n-1})v$. Whenever equality holds in (2.4) for an extension $L|K$ of finite transcendence degree, it will hold in the respective inequality for any subextension of $L|K$. Hence if $n=\rho+\tau$ then by Corollary 2.7, $K(x_1,\dots,x_{n-1})v|Kv$ is a finitely generated field extension, and if $n=\tau$, this extension is of transcendence degree $\tau-1$. This yields that B3) holds for $k=Fv$.

Now suppose that there does not exist such a transcendence basis, and that $n=\rho+\tau$ with $\rho=1$ and v is trivial on K. We pick any transcendence basis $x_1,\dots,x_n$. Since $Fv|K(x_1,\dots,x_{n-1})v$ is algebraic and $n=\rho+\tau$, we must have that the rational rank of $vF/vK(x_1,\dots,x_{n-1})$ is 1. Since $\rho=1$ and v is trivial on K, this means that v is also trivial on $K(x_1,\dots,x_{n-1})$. Hence, $Kv=K$ and $K(x_1,\dots,x_{n-1})v=K(x_1,\dots,x_{n-1})$ (modulo an isomorphism). Remark 3.12 now shows that Fv is a simple algebraic extension of $K(x_1,\dots,x_{n-1})$, which in turn is a rational function field in τ variables over $K=Kv$, or equal to Kv if $\tau=0$. Together with what we have shown before, this proves that B2) holds for $k=Fv$.

To conclude this proof, assume that $\rho=0=\tau$. Then by Lemma 3.4, $(F.\tilde{K}|\tilde{K},v)$ is immediate, for any extension of v from F to $F.\tilde{K}$. This shows that $(\tilde{K},v)$ admits an immediate extension of transcendence degree n. $\square$

Finally, we give the

Proof of Theorem 1.8: We have to show that whenever we construct an extension of the form $(L_2|L_1,v)$ with $Kv\subseteq L_1v\subseteq L_2v\subseteq k$ in the previous proof, then the ordering of L_2v induced by the given ordering of k can be lifted to an extension of the lifting that we have already obtained on L_1. Whenever we apply Propositions 3.14, 3.15 and 3.17, we obtain this already from the assertions of these propositions. Whenever we apply Lemma 2.6, we obtain this from Proposition 2.18 because in this case, we always have that vL_2 is generated over vL_1 by rationally independent values and therefore, $2vL_2\cap vL_1=2vL_1$. Finally, whenever $(L_2|L_1,v)$ is an immediate extension, we can also apply Proposition 2.18 because $2vL_2\cap vL_1=2vL_1\cap vL_1=2vL_1$. $\square$

5. Homogeneous sequences

In this section, we will develop special sequences which under certain tameness conditions can be used to determine implicit constant fields.

5.1. **Homogeneous approximations.** Let (K,v) be any valued field and a,b elements in some valued field extension (L,v) of (K,v). We will say that a is **strongly homogeneous over** (K,v) if $a\in K^{\rm sep}\setminus K$, the extension of v from K to $K(a)$ is unique (or equivalently, $K(a)|K$ is linearly disjoint from all henselizations of (K,v)), and

$$va \;=\; \mathrm{kras}(a,K)\,. \tag{5.1}$$

Note that in this case, $va=v(\sigma a-a)$ for all automorphisms such that $\sigma a\neq a$; indeed, we have that $v\sigma a=va$ and therefore, $va\geq v(\sigma a-a)\geq va$.

We will say that a is **homogeneous over** (K,v) if there is some $d\in K$ such that $a-d$ is strongly homogeneous over (K,v), i.e.,

$$v(a-d) \;=\; \mathrm{kras}(a-d,K) \;=\; \mathrm{kras}(a,K)\;.$$

We call $a\in L$ a **homogeneous approximation of** b **over** K if there is some $d\in K$ such that $a-d$ is strongly homogeneous over K and $v(b-a)>v(b-d)\geq vb$. It then follows that $va=vb$ and $v(a-d)=v(b-d)$.

Lemma 5.1. *If $a\in L$ is a homogeneous approximation of b then a lies in the henselization of $K(b)$ w.r.t. every extension of the valuation v from $K(a,b)$ to $\widetilde{K(b)}$.*

Proof. From Corollary 2.21 we obtain that $a-d$ and hence also a lies in the henselization of $K(b-d)=K(b)$ w.r.t. every extension of the valuation v from $K(a,b)$ to $\widetilde{K(b)}$. □

We will also exploit the following easy observation:

Lemma 5.2. *Let (K',v) be any henselian extension field of (K,v) such that $a\notin K'$. If a is homogeneous over (K,v), then it is also homogeneous over (K',v), and* $\mathrm{kras}(a,K)=\mathrm{kras}(a,K')$. *If a is strongly homogeneous over (K,v), then it is also strongly homogeneous over (K',v).*

Proof. Suppose that $a-d$ is separable-algebraic over K and $v(a-d)=\mathrm{kras}(a-d,K)$ for some $d\in K$. Then $a-d$ is also separable-algebraic over K'. Further, $\mathrm{kras}(a-d,K')\leq\mathrm{kras}(a-d,K)$ since restriction to $\tilde{K}$ is a map sending $\{\sigma\in\mathrm{Gal}\,K'\mid\sigma a\neq a\}$ into $\{\sigma\in\mathrm{Gal}\,K\mid\sigma a\neq a\}$. Hence, $v(a-d)\leq\mathrm{kras}(a-d,K')\leq\mathrm{kras}(a-d,K)=v(a-d)$, which shows that equality holds everywhere. Thus, $\mathrm{kras}(a,K')=\mathrm{kras}(a-d,K')=\mathrm{kras}(a-d,K)=\mathrm{kras}(a,K)$. Since (K',v) is henselian by assumption, the extension of v from K' to $K'(a)$ is unique. This shows that $a-d$ is strongly homogeneous over (K',v), and concludes the proof of our assertions. □

The following gives the crucial criterion for an element to be (strongly) homogeneous over (K,v):

Lemma 5.3. *Suppose that $a\in\tilde{K}$ and that there is some extension of v from K to $K(a)$ such that if e is the least positive integer for which $\mathrm{e}va\in vK$, then*
a) e is not divisible by $\mathrm{char}\,Kv$,
b) there exists some $c\in K$ such that $vca^{\mathrm{e}}=0$, $ca^{\mathrm{e}}v$ is separable-algebraic over Kv, and the degree of ca^{e} over K is equal to the degree f of $ca^{\mathrm{e}}v$ over Kv.
Then $[K(a):K]=\mathrm{ef}$ and if $a\notin K$, then a is strongly homogeneous over (K,v).

Proof. We have

$$\begin{aligned}\mathrm{ef} &\;\geq\; [K(a):K(a^{\mathrm{e}})]\cdot[K(a^{\mathrm{e}}):K] \;=\; [K(a):K]\\ &\;\geq\; (vK(a):vK)\cdot[K(a)v:Kv] \;\geq\; \mathrm{ef}\;.\end{aligned}$$

So equality holds everywhere, and we obtain $[K(a):K]=\mathrm{ef}$, $(vK(a):vK)=\mathrm{e}$ and $[K(a)v:Kv]=\mathrm{f}$. By the fundamental inequality, the latter implies that the extension of v from K to $K(a)$ is unique.

Now assume that $a\notin K$. Take two distinct conjugates $\sigma a\neq\tau a$ of a and set $\eta:=\sigma a/\tau a\neq 1$. If $\sigma a^e\neq\tau a^e$, then $c\sigma a^e=\sigma ca^e$ and $c\tau a^e=\tau ca^e$ are distinct conjugates of ca^e. By hypothesis, their residues are also distinct and therefore, the residue of $\sigma a^e/\tau a^e = \eta^e$ is not 1. It follows that the residue of η is not 1. If $\sigma a^e = \tau a^e$, then η is an e-th root of unity. Since e is not divisible by the residue characteristic, it again follows that the residue of η is not equal to 1. Hence in both cases, we obtain that $v(\eta - 1) = 0$, which shows that $v(\sigma a - \tau a) = v\tau a = va$. We have now proved (5.1). □

Lemma 5.4. *Assume that b is an element in some algebraically closed valued field extension (L, v) of (K, v). Suppose that there is some* $\mathrm{e} \in \mathbb{N}$ *not divisible by* $\mathrm{char}\, Kv$*, and some $c \in K$ such that $vcb^{\mathrm{e}} = 0$ and $cb^{\mathrm{e}}v$ is separable-algebraic over Kv. If the smallest possible* $\mathrm{e} \in \mathbb{N}$ *is bigger than 1 or if $cb^{\mathrm{e}}v \notin Kv$, then we can find $a \in L$, strongly homogeneous over K and such that $v(b - a) > vb$. In particular, a is a homogeneous approximation of b over K.*

Proof. Take a monic polynomial g over K with v-integral coefficients whose reduction modulo v is the minimal polynomial of $cb^{\mathrm{e}}v$ over Kv. Then let $a_0 \in \tilde{K}$ be the root of g whose residue is $cb^{\mathrm{e}}v$. The degree of a_0 over K is the same as that of $cb^{\mathrm{e}}v$ over Kv. We have that $v(\frac{a_0}{cb^{\mathrm{e}}} - 1) > 0$. So there exists $a_1 \in \tilde{K}$ with residue 1 and such that $a_1^{\mathrm{e}} = \frac{a_0}{cb^{\mathrm{e}}}$. Then for $a := a_1 b$, we find that $v(a - b) = vb + v(a_1 - 1) > vb$ and $ca^{\mathrm{e}} = a_0$. It follows that $va = vb$ and $ca^{\mathrm{e}}v = cb^{\mathrm{e}}v$. By the foregoing lemma, this shows that a is strongly homogeneous over K. □

5.2. **Homogeneous sequences.** Let $(K(x)|K, v)$ be any extension of valued fields. We fix an extension of v to $\widetilde{K(x)}$.

Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i \in S}$$

of elements in $\tilde{K}$ will be called a **homogeneous sequence for** x if the following conditions are satisfied for all $i \in S$ (where we set $a_0 := 0$):

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0, \ldots, a_{i-1})$.

Recall that then by the definition of "strongly homogeneous", $a_i \notin K(a_0, \ldots, a_{i-1})^h$. We call S the **support** of the sequence $\mathfrak{S}$. We set

$$K_{\mathfrak{S}} \;:=\; K(a_i \mid i \in S)\;.$$

If $\mathfrak{S}$ is the empty sequence, then $K_{\mathfrak{S}} = K$.

From this definition, we obtain:

Lemma 5.5. *If $i, j \in S$ with $1 \leq i < j$, then*

$$v(x - a_j) \;>\; v(x - a_i) \;=\; v(a_{i+1} - a_i)\;. \tag{5.2}$$

If $S = \mathbb{N}$ then $(a_i)_{i \in S}$ is a pseudo Cauchy sequence in $K_{\mathfrak{S}}$ with pseudo limit x.

Proof. If $1 \leq i \in S$, then $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$. Hence by definition,

$$v(x - a_i) \;=\; v(x - a_{i-1} - (a_i - a_{i-1})) \;>\; v(x - a_{i-1})\;,$$

whence $v(a_i - a_{i-1}) = \min\{v(x - a_i), v(x - a_{i-1})\} = v(x - a_{i-1})$. If $i < j \in S$, then by induction, $v(x - a_j) > v(x - a_i)$.

Suppose that $S = \mathbb{N}$. Then it follows by induction that for all $k > j > i \geq 1$,

$$v(x - a_k) \;>\; v(x - a_j) \;>\; v(x - a_i)$$

and therefore,

$$\begin{aligned} v(a_k - a_j) &= \min\{v(x-a_k), v(x-a_j)\} \;=\; v(x-a_j) > v(x-a_i) \\ &= \min\{v(x-a_j), v(x-a_i)\} \;=\; v(a_j - a_i)\;. \end{aligned}$$

This shows that $(a_i)_{i\in S}$ is a pseudo Cauchy sequence. The equality in (5.2) shows that x is a pseudo limit of this sequence. □

Let us also observe the following:

Lemma 5.6. *Take $x, x' \in L$.*
1) If $a \in L$ is a homogeneous approximation of x over K and if $v(x-x') \geq v(x-a)$, then a is also a homogeneous approximation of x' over K.
2) Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. If $v(x-x') > v(x-a_k)$ for all $k \in S$, then $(a_i)_{i\in S}$ is also a homogeneous sequence for x' over K.
In particular, for each $k \in S$ such that $k > 1$, $(a_i)_{i<k}$ is a homogeneous sequence for a_k over K.

Proof. 1): Suppose that a is a homogeneous approximation of x over K, with $v(x-a) > v(x-d) \geq vx$ and $a-d$ strongly homogeneous over K. If in addition $v(x-x') \geq v(x-a) > v(x-d)$, then $v(x'-d) = \min\{v(x-x'), v(x-d)\} = v(x-d)$ and $v(x'-a) \geq \min\{v(x-x'), v(x-a)\} \geq v(x-a) > v(x-d) = v(x'-d)$. This yields the assertion.
2): Since $a_k - a_{k-1}$ is a homogeneous approximation of $x - a_{k-1}$ over $K(a_0, \ldots, a_{k-1})$ and

$$v(x-a_{k-1}-(x'-a_{k-1})) \;=\; v(x-x') \;>\; v(x-a_k) \;=\; v(x-a_{k-1}-(a_k-a_{k-1}))\;,$$

it follows from part 1) that $a_k - a_{k-1}$ is also a homogeneous approximation of $x' - a_{k-1}$ over $K(a_0, \ldots, a_{k-1})$. Hence, $(a_i)_{i\in S}$ is a homogeneous sequence for x' over K.
If $k > 1$, then by (5.2), $v(x-a_k) > v(x-a_i)$ for $0 \leq i < k$. Hence the last assertion follows from what we have just proved. □

What is special about homogeneous sequences is described by the following lemma:

Lemma 5.7. *Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then*

$$K_{\mathfrak{S}} \;\subset\; K(x)^h\;. \tag{5.3}$$

For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$. If $S = \{1, \ldots, n\}$, then

$$K_{\mathfrak{S}}^h \;=\; K(a_n)^h\;. \tag{5.4}$$

Proof. We use induction on $k \in S$. Suppose that we have already shown that $a_{k-1} \in K(x)^h$. As $a_k - a_{k-1}$ is a homogeneous approximation of $x - a_{k-1}$, we know from Lemma 5.1 that

$$a_k - a_{k-1} \;\in\; K(x-a_{k-1})^h \subseteq K(x)^h$$

and hence also $a_k \in K(x)^h$. This proves (5.3). Now all other assertions follow when we replace x by a_k in the above argument, using the fact that by the previous lemma, $(a_i)_{i<n}$ is a homogeneous sequence for a_n over K. □

Proposition 5.8. *Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with support $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x, and $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate and pure.*

Proof. By Lemma 5.5, $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence with pseudo limit x. Suppose it were of algebraic type. Then by [KA], Theorem 3, there would exist some algebraic extension $(K_{\mathfrak{S}}(b)|K_{\mathfrak{S}}, v)$ with b a pseudo limit of the sequence. But then $v(x-b) > v(x-a_k)$ for all $k \in S$ and by Lemma 5.6, $(a_i)_{i\in S}$ is also a homogeneous sequence for b over K. Hence by Lemma 5.7, $K^h_{\mathfrak{S}} \subset K(b)^h = K^h(b)$. Since b is algebraic over K, the extension $K^h(b)|K^h$ is finite. On the other hand, $K^h_{\mathfrak{S}}|K^h$ is infinite since by the definition of homogeneous elements, $a_k \notin K(a_i \mid 1 \le i < k)^h$ for every $k \in \mathbb{N}$ and therefore, each extension $K(a_i \mid 1 \le i \le k+1)^h|K(a_i \mid 1 \le i \le k)^h$ is non-trivial. This contradiction shows that the sequence is of transcendental type. Hence by definition, $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is pure. Further, it follows from Lemma 2.4 that $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate. □

This proposition leads to the following definition. A homogeneous sequence $\mathfrak{S}$ for x over K will be called **(weakly) pure homogeneous sequence** if $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is (weakly) pure in x. Hence if $S = \mathbb{N}$, then $\mathfrak{S}$ is always a pure homogeneous sequence. The empty sequence is a (weakly) pure homogeneous sequence for x over K if and only if already $(K(x)|K, v)$ is (weakly) pure in x.

Theorem 5.9. *Suppose that $\mathfrak{S}$ is a (weakly) pure homogeneous sequence for x over K. Then*

$$K^h_{\mathfrak{S}} \;=\; \mathrm{IC}\,(K(x)|K, v)\;.$$

Further, $K_{\mathfrak{S}}v$ is the relative algebraic closure of Kv in $K(x)v$, and the torsion subgroup of $vK(x)/vK_{\mathfrak{S}}$ is finite. If $\mathfrak{S}$ is pure, then $vK_{\mathfrak{S}}$ is the relative divisible closure of vK in $vK(x)$.

Proof. The assertions follow from Lemma 3.5, Lemma 2.10 and Lemma 3.7, together with the fact that because $K_{\mathfrak{S}}|K$ is algebraic, the same holds for $vK_{\mathfrak{S}}|vK$ and $K_{\mathfrak{S}}v|Kv$ by Lemma 2.1. □

5.3. **Conditions for the existence of homogeneous sequences.** Now we have to discuss for which extensions $(K(x)|K, v)$ there exist homogeneous sequences.

An algebraic extension $(L|K, v)$ of henselian fields is called **tame** if the following conditions hold:

(TE1) $Lv|Kv$ is separable,
(TE2) if $\mathrm{char}\,Kv = p > 0$, then the order of each element in vL/vK is prime to p,
(TE3) $[K' : K] = (vK' : vK)[K'v : Kv]$ holds for every finite subextension $K'|K$ of $L|K$.

Condition (TE3) means that equality holds in the fundamental inequality (2.1). If $L'|K$ is any subextension of $L|K$, then $(L|K, v)$ is a tame extension if and only if $(L|L', v)$ and $(L'|K, v)$ are (this is easy to prove if $L|K$ is finite). Further, it is well known that for (K, v) henselian, the ramification field of the extension $(K^{\mathrm{sep}}|K, v)$ is the unique maximal tame extension of (K, v) (cf. [E]). A henselian valued field (K, v) is called a **tame field** if all its algebraic extensions are tame, or equivalently, the following conditions hold:

(T1) Kv is perfect,
(T2) if $\mathrm{char}\,Kv = p > 0$, then vK is p-divisible,

(T3) for every finite extension $K'|K$, $[K':K] = (vK':vK)[K'v:Kv]$.

Note that every valued field with a residue field of characteristic zero is tame; this is a consequence of the Lemma of Ostrowski (cf. [R]). It follows directly from the definition together with the multiplicativity of ramification index and inertia degree that every finite extension of a tame field is again a tame field. If (K,v) is a tame field, then condition (T3) shows that (K,v) does not admit any proper immediate algebraic extensions; hence by Theorem 3 of [KA], every pseudo Cauchy sequence in (K,v) without a pseudo limit in K must be of transcendental type.

If an element $a \in \tilde{K}$ satisfies the conditions of Lemma 5.3, then $(K(a)|K,v)$ is a tame extension. The following implication is also true, as was noticed by Sudesh K. Khanduja (cf. [KH11], Theorem 1.2):

Proposition 5.10. *Suppose that (K,v) is henselian. If a is homogeneous over (K,v), then $(K(a)|K,v)$ is a tame extension. If $\mathfrak{S}$ is a homogeneous sequence over (K,v), then $K_{\mathfrak{S}}$ is a tame extension of K.*

Proof. Since $K(a-d) = K(a)$ for $d \in K$, we may assume w.l.o.g. that a is strongly homogeneous over (K,v). If $(K(a)|K,v)$ were not a tame extension, then a would not lie in the ramification field of the extension $(K^{\mathrm{sep}}|K,v)$. So there would exist an automorphism σ in the ramification group such that $\sigma a \neq a$. But by the definition of the ramification group,

$$v(\sigma a - a) \;>\; va \;=\; \mathrm{kras}(a,K)\;,$$

a contradiction.

The second assertion is proved using the first assertion and the fact that a (possibly infinite) tower of tame extensions is itself a tame extension. □

In fact, it can also be shown that if a is separable over K, then $va = \mathrm{kras}(a,K)$ implies that a satisfies the conditions of Lemma 5.3.

We can give the following characterization of elements in tame extensions:

Proposition 5.11. *An element $b \in \tilde{K}$ belongs to a tame extension of the henselian field (K,v) if and only if there is a finite homogeneous sequence $a_1,\ldots,a_k$ for b over (K,v) such that $b \in K(a_k)$.*

Proof. Suppose that such a sequence exists. By the foregoing proposition, $K_{\mathfrak{S}}$ is a tame extension of K. Since $b \in K(a_k) \subseteq K_{\mathfrak{S}}$, it contains b.

For the converse, let b be an element in some tame extension of (K,v). Since $K(b)|K$ is finite, also the extensions $vK(b)|vK$ and $K(b)v|Kv$ are finite. Take $\eta_i \in K(b)$ with $\eta_1 = 1$ such that $v\eta_i$, $1 \leq i \leq \ell$, belong to distinct cosets modulo vK. Further, take $\vartheta_j \in \mathcal{O}_{K(b)}$ with $\vartheta_1 = 1$ such that $\vartheta_j v$, $1 \leq j \leq m$, are Kv-linearly independent. Then by Lemma 2.8, the elements $\eta_i\vartheta_j$, $1 \leq i \leq \ell$, $1 \leq j \leq m$, are K-linearly independent. By (TE3), $[K(b):K] = \ell m$, so these elements form a basis of $K(b)|K$. Now we write

$$b \;=\; \sum_{i,j} c_{ij}\eta_i\vartheta_j$$

with $c_{ij} \in K$. Again by Lemma 2.8,

$$vb \;=\; v\sum_{i,j} c_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}\, vc_{ij}\eta_i\vartheta_j \;=\; \min_{i,j}\,(vc_{ij} \,+\, v\eta_i)\;.$$

If $c_{11}\eta_1\vartheta_1 = c_{11} \in K$ happens to be the unique summand of minimal value, then we set $d = c_{1,1}$ and consider $b - d$ in place of b; otherwise, we set $d = 0$.

Choose i_0 such that vb is in the coset of $v\eta_{i_0}$. If $vc_{i_1j_1}\eta_{i_1} = vc_{i_2j_2}\eta_{i_2}$ then since the η's are in distinct cosets modulo vK, we must have that $i_1 = i_2$. So we can list the summands of minimal value as $c_{i_0j_r}\eta_{i_0}\vartheta_{j_r}$, $1 \leq r \leq t$, for some $t \leq m$. We obtain that

$$v\left(b - d - \sum_{r=1}^{t} c_{i_0j_r}\eta_{i_0}\vartheta_{j_r}\right) \;>\; v(b-d)\,. \tag{5.5}$$

Take e to be the least positive integer such that $ev(b-d) \in vK$. Choose $c \in K$ such that $vc(b-d)^{\mathrm{e}} = 0$. Then by (TE1), $c(b-d)^{\mathrm{e}}v$ is separable-algebraic over Kv. If $i_0 > 1$, then $va = v(b-d) \notin vK$, and by (TE2), e is not divisible by $\mathrm{char}\, Kv$. If $i_0 = 1$, then $\mathrm{e} = 1$ and $\eta_{i_0} = 1$, and in view of (5.5),

$$c(b-d)v \;=\; \sum_{r=1}^{t}(cc_{i_0j_r}v)\cdot\vartheta_{j_r}v\,.$$

This is not in Kv since by our choice of d, some $j_k > 1$ must appear in the sum and the residues $\vartheta_{j_r}v$, $1 \leq r \leq t$, are linearly independent over Kv.

We conclude that $b-d$ satisfies the assumptions of Lemma 5.4. Hence there is an element $a \in \tilde{K}$, strongly homogeneous over K and such that $v(b-d-a) > v(b-d)$. We set $a_1 := a + d$ to obtain that a_1 is a homogeneous approximation of b over K. By the foregoing proposition, $K(a_1)$ is a tame extension of K and therefore, by the general facts we have noted following the definition of tame extensions, $K(a_1, b-a_1)$ is a tame extension of $K(a_1)$.

We repeat the above construction, replacing b by $b - a_1$. By induction, we build a homogeneous sequence for b over K. It cannot be infinite since b is algebraic over K (cf. Proposition 5.8). Hence it stops with some element a_k. Our construction shows that this can only happen if $b \in K(a_1, \ldots, a_k)$, which by Lemma 5.7 is equal to $K(a_k)$. □

Proposition 5.12. *Assume that (K, v) is a henselian field. Then (K, v) is a tame field if and only if for every element x in any extension (L, v) of (K, v) there exists a weakly pure homogeneous sequence for x over K, provided that x is transcendental over K.*

Proof. First, let us assume that (K, v) is a tame field and that x is an element in some extension (L, v) of (K, v), transcendental over K. We set $a_0 = 0$. We assume that $k \geq 0$ and that a_i for $i \leq k$ are already constructed. Like K, also the finite extension $K_k := K(a_0, \ldots, a_k)$ is a tame field. Therefore, if x is the pseudo limit of a pseudo Cauchy sequence in K_k, then this pseudo Cauchy sequence must be of transcendental type, and $K_k(x)|K_k$ is pure and hence weakly pure in x.

If $K_k(x)|K_k$ is weakly pure in x, then we take a_k to be the last element of $\mathfrak{S}$ if $k > 0$, and $\mathfrak{S}$ to be empty if $k = 0$.

Assume that this is not the case. Then x cannot be the pseudo limit of a pseudo Cauchy sequence without pseudo limit in K_k. So the set $v(x - a_k - K_k)$ must have a maximum, say $x - a_k - d$ with $d \in K_k$. Since we assume that $K_k(x)|K_k$ is not weakly pure in x, there exist $\mathrm{e} \in \mathbb{N}$ and $c \in K_k$ such that $vc(x - a_k - d)^{\mathrm{e}} = 0$ and $c(x - a_k - d)^{\mathrm{e}}v$ is algebraic over K_kv. Conditions (T1) and (T2) yield that e can be chosen to be prime to $\mathrm{char}\, Kv$ and that $c(x - a_k - d)^{\mathrm{e}}v$ is separable-algebraic over $K_k v$. Since $v(x - a_k - d)$ is maximal in $v(x - a_k - K_k)$, we must have that $\mathrm{e} > 1$ or $c(x - a_k - d)^{\mathrm{e}} v \notin K_k v$.

Now Lemma 5.4 shows that there exists $a \in \tilde{K}$, strongly homogeneous over K_k and such that $v(x - a_k - d - a) > v(x - a_k - d)$. So $a + d$ is a homogeneous approximation of $x - a_k$ over K_k, and we set $a_{k+1} := a_k + a + d$. This completes our induction step. If our construction stops at some k, then $K_k(x)|K_k$ is weakly pure in x and we have obtained a weakly pure homogeneous sequence. If the construction does not stop, then $S = \mathbb{N}$ and the obtained sequence is pure homogeneous.

For the converse, assume that (K, v) is not a tame field. We choose an element $b \in \tilde{K}$ such that $K(b)|K$ is not a tame extension. On $K(b, x)$ we take the valuation $v_{b,\gamma}$ with γ an element in some ordered abelian group extension such that $\gamma > vK$. Choose any extension of v to $\tilde{K}(x)$. Since vK is cofinal in $v\tilde{K}$, we have that $\gamma > v\tilde{K}$. Since $b \in \tilde{K}$, we find $\gamma \in v\tilde{K}(x)$. Hence, $(\tilde{K}(x)|\tilde{K}, v)$ is value-transcendental.

Now suppose that there exists a weakly pure homogeneous sequence $\mathfrak{S}$ for x over K. By Lemma 3.3, also $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is value-transcendental. Since $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is also weakly pure, it follows that there must be some $c \in K_{\mathfrak{S}}$ such that $x - c$ is a value-transcendental element (all other cases in the definition of "weakly pure" lead to immediate or residue-transcendental extensions). But if $c \neq b$ then $v(b - c) \in v\tilde{K}$ and thus, $v(c - b) < \gamma$. This implies $v(x - c) = \min\{v(x-b), v(b-c)\} = v(b-c) \in v\tilde{K}$, a contradiction. This shows that $b = c \in K_{\mathfrak{S}}$. On the other hand, $K_{\mathfrak{S}}$ is a tame extension of K by Proposition 5.10 and cannot contain b. This contradiction shows that there cannot exist a weakly pure homogeneous sequence for x over K. □

6. Applications

Let us show how to apply our results to power series fields. We denote by $k((G))$ the field of power series with coefficients in the field k and exponents in the ordered abelian group G.

Theorem 6.1. *Let (K, v) be a henselian subfield of a power series field $k((G))$ such that v is the restriction of the canonical valuation of $k((G))$. Suppose that K contains all monomials of the form ct^γ for $c \in Kv \subseteq k$ and $\gamma \in vK \subseteq G$. Consider a power series*

$$z = \sum_{i \in \mathbb{N}} c_i t^{\gamma_i}$$

where $(\gamma_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence in G, all $c_i \in k$ are separable-algebraic over Kv, and for each i there is an integer $\mathrm{e}_i > 0$ prime to the characteristic of Kv and such that $\mathrm{e}_i \gamma_i \in vK$. Then, upon taking the henselization of $K(z)$ in $k((G))$, we obtain that $K(c_i t^{\gamma_i} \mid i \in \mathbb{N}) \subseteq K(z)^h$. Consequently, $vK(z)$ contains all γ_i, and if $\gamma_i \in vK$ for all $i \in \mathbb{N}$, then $K(z)v$ contains all c_i.

If $vK + \sum_{i=1}^{\infty} \mathbb{Z}\gamma_i / vK$ or $Kv(c_i \mid i \in \mathbb{N})|Kv$ is infinite, then z is transcendental over K, we have $\mathrm{IC}\,(K(z)|K, v) = K(c_i t^{\gamma_i} \mid i \in \mathbb{N})$, and

a) $vK(z)$ is the group generated over vK by the elements γ_i,

b) if $\gamma_i \in vK$ for all $i \in \mathbb{N}$, then $K(z)v = Kv(c_i \mid i \in \mathbb{N})$.

Proof. We derive a homogeneous sequence $\mathfrak{S}$ from z as follows. We set $a_0 = 0$. If all c_i are in Kv and all γ_i are in vK, then we take $\mathfrak{S}$ to be the empty sequence.

Otherwise, having chosen $i_j \in \mathbb{N}$ and defined

$$a_j \;:=\; \sum_{1\leq i\leq i_j} c_i t^{\gamma_i}$$

for all $j < m$, we proceed as follows. We let ℓ be the first index in the power series $z - a_{m-1}$ for which $c_\ell t^{\gamma_\ell} \notin K(a_1,\ldots,a_{m-1})$; if such an index does not exist, we let a_{m-1} be the last element of $\mathfrak{S}$. Otherwise, we set $i_m := \ell$ and $a_m := \sum_{1\leq i\leq i_m} c_i t^{\gamma_i}$. We have that

$$c_\ell t^{\gamma_\ell} \;=\; a_m - a_{m-1} - d \quad \text{ with } \quad d \;=\; \sum_{i_{m-1}<i<\ell} c_i t^{\gamma_i} \;\in\; K(a_1,\ldots,a_{m-1})\;.$$

By assumption, $\mathrm{e}_\ell v c_\ell t^{\gamma_\ell} = \mathrm{e}_\ell \gamma_\ell \in vK$ and hence, $c := t^{-\mathrm{e}_\ell\gamma_\ell} \in K$. We have that $c(c_\ell t^{\gamma_\ell})^{\mathrm{e}_\ell} = c_\ell^{\mathrm{e}_\ell}$. Since c_ℓ is separable-algebraic over Kv, the same holds for $c_\ell^{\mathrm{e}_\ell}$. Since v is trivial on k, the degree of $c_\ell^{\mathrm{e}_\ell} v$ over Kv is equal to that of $c_\ell^{\mathrm{e}_\ell}$ over K. It now follows by Lemma 5.3 that $c_\ell t^{\gamma_\ell}$ is strongly homogeneous over K. By Lemma 5.2 it follows that it is also strongly homogeneous over the henselian field $K(a_1,\ldots,a_{m-1})$. Further, $v(z - a_{m-1} - (a_m - a_{m-1})) = v(z - a_m) = \gamma_{\ell+1} > \gamma_\ell = v(z - a_{m-1} - d) \geq v(z - a_{m-1})$. This proves that $a_m - a_{m-1}$ is a homogeneous approximation of $z - a_{m-1}$ over $K(a_0,\ldots,a_{m-1})$. By induction, we obtain a homogeneous sequence $\mathfrak{S}$ for z in $k((G))$. It now follows from Lemma 5.7 that $K(c_i t^{\gamma_i} \mid i \in \mathbb{N}) = K(a_j \mid j \in S) \subseteq K(z)^h$; thus, $\gamma_i = v c_i t^{\gamma_i} \in vK(z)$ for all $i \in \mathbb{N}$. If $\gamma_i \in vK$ and hence $t^{\gamma_i} \in K$, then $c_i \in K(z)^h$; since the residue map is the identity on elements of k, this implies that $c_i \in K(z)v$.

Now assume that $vK + \sum_{i=1}^\infty \mathbb{Z}\gamma_i/vK$ or $Kv(c_i \mid i \in \mathbb{N})|Kv$ is infinite. Then $\mathfrak{S}$ must be infinite, and it follows from Proposition 5.8 that z is a pseudo limit of the pseudo Cauchy sequence $(a_i)_{i\in\mathbb{N}}$ of transcendental type. Thus, z is transcendental over K by Lemma 2.4. Theorem 5.9 now shows that $\mathrm{IC}\,(K(z)|K,v) = K(c_i t^{\gamma_i} \mid i \in \mathbb{N})$, $vK(z) = vK(c_i t^{\gamma_i} \mid i \in \mathbb{N})$ and $K(z)v = K(c_i t^{\gamma_i} \mid i \in \mathbb{N})v$. Since $K(c_i t^{\gamma_i} \mid i \in \mathbb{N}) \subseteq K(c_i, t^{\gamma_i} \mid i \in \mathbb{N})$, for the proof of assertions a) and b) it now suffices to show that the value group of the latter field is generated over vK by the γ_i , and that its residue field is generated over Kv by the c_i . As $t^\gamma \in K$ for every $\gamma \in vK$, we have that

$$(vK + \sum_{i=1}^{\ell} \mathbb{Z}\gamma_i : vK) \;=\; [K(t^{\gamma_i} \mid 1 \leq i \leq \ell) : K]\;.$$

On the other hand,

$$[K(t^{\gamma_i} \mid 1 \leq i \leq \ell) : K] \;\geq\; (vK(t^{\gamma_i} \mid 1 \leq i \leq \ell) : vK) \geq (vK + \sum_{i=1}^{\ell} \mathbb{Z}\gamma_i : vK)\;,$$

where the last inequality holds since $\gamma_i \in vK(t^{\gamma_i} \mid 1 \leq i \leq \ell)$ for $i = 1,\ldots,\ell$. We obtain that $vK(t^{\gamma_i} \mid 1 \leq i \leq \ell) = vK + \sum_{i=1}^\ell \mathbb{Z}\gamma_i$ and that $K(t^{\gamma_i} \mid 1 \leq i \leq \ell)v = Kv$ for all $\ell \in \mathbb{N}$. This implies that $vK(t^{\gamma_i} \mid i \in \mathbb{N}) = vK + \sum_{i=1}^\infty \mathbb{Z}\gamma_i$ and $K(t^{\gamma_i} \mid i \in \mathbb{N})v = Kv$.

Set $K' := K(t^{\gamma_i} \mid i \in \mathbb{N})$. As $K'v = Kv \subset K \subset K'$, we have that

$$[K'v(c_i \mid 1 \leq i \leq \ell) : K'v] \;\geq\; [K'(c_i \mid 1 \leq i \leq \ell) : K']\;.$$

On the other hand,

$$[K'(c_i \mid 1 \leq i \leq \ell) : K'] \geq [K'(c_i \mid 1 \leq i \leq \ell)v : K'v] \geq [K'v(c_i \mid 1 \leq i \leq \ell) : K'v] ,$$

where the last inequality holds since $c_i \in K'(c_i \mid 1 \leq i \leq \ell)v$ for $i = 1, \ldots, \ell$. We obtain that $K'(c_i \mid 1 \leq i \leq \ell)v = K'v(c_i \mid 1 \leq i \leq \ell)$ and that $vK'(c_i \mid 1 \leq i \leq \ell) = vK'$. This implies that $vK(c_i, t^{\gamma_i} \mid i \in \mathbb{N}) = vK'(c_i \mid i \in \mathbb{N}) = vK' = vK + \sum_{i=1}^{\infty} \mathbb{Z}\gamma_i$ and $K(c_i, t^{\gamma_i} \mid i \in \mathbb{N})v = K'(c_i \mid i \in \mathbb{N})v = K'v(c_i \mid i \in \mathbb{N}) = Kv(c_i \mid i \in \mathbb{N})$. This proves assertions a) and b). □

Remark 6.2. Assertions a) and b) of the previous theorem will also hold if $K = Kv((vK))$. Indeed, if G_0 denotes the subgroup of G generated by the γ_i over Kv, and $k_0 = Kv(c_i \mid i \in \mathbb{N})$, then $K(z) \subset k_0((G_0))$ and therefore, $vK(z) \subseteq G_0$ and $K(z)v \subseteq k_0$.

Our methods also yield an alternative proof of the following well known fact:

Theorem 6.3. *The algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers is not complete.*

Proof. Choose any compatible system of n-th roots $p^{1/n}$ of p, that is, such that $(p^{1/mn})^m = p^{1/n}$ for all $m, n \in \mathbb{N}$. For $i \in \mathbb{N}$, choose any $n_i \in \mathbb{N}$ not divisible by p, and set $\gamma_i := i + \frac{1}{n_i}$ if $n_i > 1$, and $\gamma_i = i$ otherwise. Further, choose c_i in some fixed set of representatives in $\widetilde{\mathbb{Q}_p}$ of its residue field $\widetilde{\mathbb{F}_p}$ such that the degree of $c_i^{n_i}$ over $\mathbb{Q}_p$ is equal to the degree of $c_i^{n_i} v_p$ over $\mathbb{Q}_p v_p = \mathbb{F}_p$. Then set

$$b_i := \sum_{1 \leq j \leq i} c_j p^{\gamma_j} . \tag{6.1}$$

Since $\gamma_i < \gamma_{i+1}$ for all i and the sequence $(\gamma_i)_{i \in \mathbb{N}}$ is cofinal in the value group $\mathbb{Q}$ of $\widetilde{\mathbb{Q}_p}$, the sequence $(b_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in $(\widetilde{\mathbb{Q}_p}, v_p)$. If this field were complete, it would contain a pseudo limit z to every such Cauchy sequence. On the other hand, as in the proof of Theorem 6.1 one shows that $\mathbb{Q}_p(c_i p^{\gamma_i} \mid i \in \mathbb{N}) \subseteq \mathbb{Q}_p(z)^h$, and that

a) $v\mathbb{Q}_p(z)$ is the group generated over $\mathbb{Z}$ by the elements γ_i ,

b) if $\gamma_i \in \mathbb{Z}$ for all $i \in \mathbb{N}$, then $\mathbb{Q}_p(z)v = \mathbb{F}_p(c_i v_p \mid i \in \mathbb{N})$,

c) if $v\mathbb{Q}_p(z)/\mathbb{Z}$ or $\mathbb{F}_p(c_i v_p \mid i \in \mathbb{N})|\mathbb{F}_p$ is infinite, then z is transcendental over $\mathbb{Q}_p$.

Hence, if we choose $(n_i)_{i \in \mathbb{N}}$ to be a strictly increasing sequence and $c_i = 1$ for all i, or if we choose $n_i = 1$ for all i and the elements c_i of increasing degree over $\mathbb{Q}_p$, then z will be transcendental over $\mathbb{Q}_p$. Since z lies in the completion of $\widetilde{\mathbb{Q}_p}$, we have now proved that this completion is transcendental over $\widetilde{\mathbb{Q}_p}$. □

With the same method, we can also prove another well known result:

Theorem 6.4. *The completion $\mathbb{C}_p$ of $\widetilde{\mathbb{Q}_p}$ admits a pseudo Cauchy sequence without a pseudo limit in $\mathbb{C}_p$. Hence, $\mathbb{C}_p$ is not maximal and not spherically complete.*

Proof. In the same setting as in the foregoing proof, we now choose $c_i = 1$ for all i. Further, we choose $(n_i)_{i \in \mathbb{N}}$ to be a strictly increasing sequence and set $\gamma_i := 1 - \frac{1}{n_i}$. Then $(b_i)_{i \in \mathbb{N}}$ is a pseudo Cauchy sequence. Suppose it would admit a pseudo limit y in $\mathbb{C}_p$. Then, using that $\widetilde{\mathbb{Q}_p}$ is dense in $\mathbb{C}_p$, we could choose $z \in \widetilde{\mathbb{Q}_p}$ such that $v_p(y - z) \geq 1$. Since $1 > \gamma_i$ for all i, it would follow that also z is a pseudo limit of $(b_i)_{i\in\mathbb{N}}$. But as in the foregoing proof one shows that z must be transcendental over $\mathbb{Q}_p$. This contradiction shows that $(b_i)_{i\in\mathbb{N}}$ cannot have a pseudo limit in $\mathbb{C}_p$. By the results of [KA], this implies that $\mathbb{C}_p$ admits a proper immediate extension, which shows that $\mathbb{C}_p$ is not maximal. With the elements b_i defined as in (6.1), we also find that the intersection of the nest $(\{a \in \mathbb{C}_p \mid v(a-b_i) \geq \gamma_{i+1}\})_{i\in\mathbb{N}}$ of balls is empty. Hence, $\mathbb{C}_p$ is not spherically complete. □

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