Two longstanding open problems in positive characteristic and their relation to valuation theory
Part II: Decidability of Laurent series fields over finite fields

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The second longstanding open problem

The second longstanding open problem we will discuss is the question whether the elementary theory of the Laurent series field $\mathbb{F}_p((t))$ is decidable. Let us first introduce $\mathbb{F}_p((t))$.

Examples of valuations

Well known from number theory are the p-adic valuations v_p on $\mathbb{Q}$, where p is a prime. The completion of $\mathbb{Q}$ with respect to v_p is the field $\mathbb{Q}_p$ of p-adic numbers. On both $\mathbb{Q}$ and $\mathbb{Q}_p$, v_p has value group $\mathbb{Z}$ and residue field $\mathbb{F}_p$.

Take a field K and let $K(t)$ be the rational function field over K. The t-adic valuation v_t on $K(t)$ is the unique valuation trivial on K and such that $v(t) = 1$. Its associated place is the unique place which is the identity on K and sends t to 0.
The completion of $K(t)$ with respect to v_t is the field $K((t))$ of formal Laurent series over K. We are particularly interested in the case of $K = \mathbb{F}_p$. On both $\mathbb{F}_p(t)$ and $\mathbb{F}_p((t))$, v_t has value group $\mathbb{Z}$ and residue field $\mathbb{F}_p$.

$\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

We see that $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ have the same value groups and residue fields. Moreover, both are complete discretely valued fields, and thus are henselian and defectless. They "only" differ in their characteristic.

In 1965, Ax and Kochen [AK 1965] used model theoretic results about henselian valued fields with residue fields of characteristic 0 to prove a corrected version of Artin's Conjecture. Thereafter, they [AK 1965], and independently Ershov [E 1965/66/67], proved that the elementary theory of $\mathbb{Q}_p$ is decidable. Since then, several excellent model theorists have tried their luck on $\mathbb{F}_p((t))$, to no avail. It is not the model theoretic tools that are lacking; we just don't understand the valuation theoretic properties of $\mathbb{F}_p((t))$.

In model theoretic algebra, a basic approach is to talk about algebraic structures (such as groups, rings, valued fields) in an elementary language $\mathcal{L}$ (also called first order language). For example,

$\mathcal{L}_{\text{groups}} = \{1, \cdot, {}^{-1}\}$ or $\{0, +, -\}$,

$\mathcal{L}_{\text{rings}} = \{0, 1, +, \cdot, -\}$.

To talk about valuations, one may use a unary predicate indicating that an element belongs to the valuation ring $\mathcal{O}$, or a binary predicate $|_v$ (valuation divisibility) with "$a|_v b$" interpreted as $v(a) \leq v(b)$.

Elementary formulas and sentences

Elementary $\mathcal{L}$-formulas are syntactically correct strings built from the elements of the language $\mathcal{L}$ together with the logical connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \exists, \forall$, variables, and auxiliary symbols such as $(,)$.

Elementary $\mathcal{L}$-sentences are elementary formulas in which every variable is bound by a quantifier.

Elementary theories

The elementary $\mathcal{L}$-theory of a structure is the collection of all elementary $\mathcal{L}$-sentences that hold in the structure. When it is clear which language we are referring to (when referring to valued fields, the language of valued rings or fields), we just talk of the elementary theory.

Decidability

We say that an elementary $\mathcal{L}$-theory is decidable if, in principle, there is an algorithm which for every $\mathcal{L}$-sentence decides whether it belongs to the $\mathcal{L}$-theory or not.

There are two types of model theorists:

– those who try to find such an algorithm (effective approach),

– those who "just" try to prove the existence, in principle.

A possible way to prove decidability

An elementary theory is decidable if it admits a complete recursive axiomatization.

Complete means that all sentences in the elementary theory follow syntactically from the axioms.

Recursive means that there is an algorithm that generates all axioms.

So the search is on for a complete recursive axiomatization of the elementary theory of $\mathbb{F}_p((t))$.

[K 2001]: straightforward adaptations of the complete recursive axiom system for $(\mathbb{Q}_p, v_p)$ don't do the job, as they are not complete.

A tool for proving completeness

Once we have a candidate for the axiom system, how can we prove completeness? After a reformulation of the problem using basic model theoretic principles we can arrive at an algebraic problem. We have to prove embedding lemmas of the following type.

Embedding lemmas for valued fields

Take any (K, v) that satisfies the given axiom system, and a highly enough saturated elementary extension (K^*, v^*). Here, elementary extension means that (K^*, v^*) satisfies the same elementary sentences, with parameters from K, as (K, v). Saturation can be thought of as some sort of "model theoretic completeness" or "density" (which has no influence on the elementary theory).

Take a valued function field (F, v) over (K, v) and assume that vF can be embedded in v^*K^* over vK and Fv can be embedded in K^*v^* over Kv. The task is to show that (F, v) admits a valuation preserving embedding in (K^*, v^*) over K.

Available tools

Tools we can employ for the construction of such embeddings:

• the embeddings of vF and Fv, via Abhyankar valuations on suitable subfields,

• Hensel's Lemma, which provides criteria for polynomials over henselian fields to admit zeros,

• under certain restrictive conditions, Kaplansky's theory of pseudo Cauchy sequences [Ka 1942].

In general, this is all that is presently available to us. When does it suffice?

Inertial generation

These tools suffice when the valued function field (F, v) is inertially generated over K, i.e., it admits a transcendence basis T such that F lies in the absolute inertia field of $(K(T), v)$. This is what our decidability problem has in common with the local uniformization problem, which shows that it meets the same obstruction: the defect.

A different connection between local uniformization and decidability will be discussed later.

How close are we?

The best approximation to the $\mathbb{F}_p((t))$ problem to date is the model theory of tame fields [K 2016] and of separably tame fields [KP 2016]. However, while all tame fields are perfect, $\mathbb{F}_p((t))$ is not. With many questions about valuation theory in positive characteristic, we are running up against the wall of imperfection. On the other hand, tame fields are crucial in the proof of our theorem on local uniformization by alteration.

Tame and separably tame fields

A valued field (K, v) is separably tame if it is henselian and

(T1) vK is p-divisible if $\mathrm{char}\, Kv = p > 0$,

(T2) Kv is perfect,

(T3) no finite separable-algebraic extension of (K, v) has nontrivial defect.

A separably tame field is tame if it is defectless.

Immediate extensions

The next theorem will deal with immediate function fields. An extension (F, v) of (K, v) is called immediate if $vF = vK$ and $Fv = Kv$.

The henselization of a valued field (K, v) is the smallest algebraic extension that is henselian. (It is its absolute decomposition field and lies inside its absolute inertia field.) The henselization is an immediate separable-algebraic extension of (K, v); we denote it by $(K, v)^h$ or (K^h, v).

The Henselian Rationality Theorem

Theorem ([K 2019])

Let (K, v) be a separably tame field and (F, v) an immediate function field over (K, v). Assume that $F|K$ is a separable extension of transcendence degree 1. Then there is $x \in F$ such that $F \subset K(x)^h$.

The theorem is trivial if char $Kv = 0$, but hard to prove otherwise.

Together with the Generalized Stability Theorem, the Henselian Rationality Theorem is the main ingredient in the proof of our theorem on local uniformization by alteration, and in proofs of model theoretic results for tame and separably tame fields.

The reason why the Henselian Rationality Theorem is hard to prove in positive characteristic is because we have to work around the defect. Take an immediate function field (F, v) over (K, v) of transcendence degree 1 and pick any element $x \in F$ transcendental over K. Since (F, v) is an immediate extension of (K, v), it is an immediate algebraic extension of $(K(x), v)$. Observe that $(K(x), v)^h$ is an immediate extension of $(K(x), v)$ and $(F, v)^h$ is an immediate extension of (F, v). By a result from ramification theory, F^h is the compositum of F and $K(x)^h$. Being an immediate extension is transitive, hence $(F, v)^h$ is an immediate extension of $(K(x), v)$. Consequently, $(F, v)^h$ is an immediate extension of $(K(x), v)^h$.

Henselian rationality and defect

If the extension is trivial, then we have proved that $F \subset K(x)^h$. If it is not, then we have nontrivial defect: since $(vF^h : vK(x)^h) = 1$ and $[F^h v : K(x)^h v] = 1$, the defect of the extension is equal to its degree. Unfortunately, this case can always happen in positive characteristic, indicating that we have chosen the wrong transcendental element x. The method of proof of the Henselian Rationality Theorem is to decrease the degree $[F^h : K(x)^h]$ step by step until we reach degree 1.

Relative decidability of tame fields

Finally, let us present some decidability results in positive characteristic that have some similarity with what we want for $\mathbb{F}_p((t))$.

Theorem ([K 2016])

Let (K, v) be a tame field of positive characteristic. Assume that the elementary theories of vK and and of Kv admit recursive elementary axiomatizations. Then also the elementary theory of (K, v) admits a recursive elementary axiomatization and is thus decidable.

As a consequence, for certain ordered abelian groups Γ larger than $\mathbb{Z}$, for example, $\Gamma = \mathbb{Q}$, the elementary theory of the power series field $\mathbb{F}_p((t^\Gamma))$, which has exponents in Γ, is decidable.

Applications of the theory of tame fields

The results on tame fields have been applied in [K 2004] to the structure theory of spaces of places of function fields and to model theoretic questions related to rational places, large fields and local uniformization.

Another application was given in [AF 2016], which we will discuss now.

Connection between local uniformization and decidability

We have worked out one connection between local uniformization and decidability. Now we describe another connection.

The existential theory of a structure is the subset of its elementary theory that consists of all existential sentences, i.e., elementary sentences in which all quantifiers, when appearing only at the start (prenex normal form), are existential.

Connection between local uniformization and decidability

Denef and Schoutens [DS 2003] showed that if resolution of singularities holds in positive characteristic, then the existential theory of $\mathbb{F}_p((t))$ is decidable, which implies that it can be decided whether a given polynomial with coefficients in $\mathbb{F}_p(t)$ has a root in $\mathbb{F}_p((t))$. Recently, Anscombe, Dittmann and Fehm [ADF 2022] showed that local uniformization in positive characteristic suffices.

A weaker, but unconditional result had already been shown by Anscombe and Fehm [AF 2016]. Since a weaker language is used, the existence of roots in $\mathbb{F}_p((t))$ can only be decided for polynomials with coefficients in $\mathbb{F}_p$. The proof crucially uses the results on tame fields.

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More detailed information

This presentation can be found on the web page

https://math.usask.ca/fvk/Fvkslides.html,

and a lecture series on valued function fields and the defect can be found on the web page

https://math.usask.ca/fvk/Fvkls.html.

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.