

# Chain union closures and convexity

Franz-Viktor Kuhlmann  
University of Szczecin, Poland  
(joint work with Wiesław Kubiś)

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# The origin of ball spaces

A **ball space**  $(X, \mathcal{B})$  is a nonempty set  $X$  together with any nonempty collection of nonempty subsets of  $X$  which we call **balls**. The motivation is taken from the notion of **ultrametric spaces**. An **ultrametric**  $u$  on a set  $X$  is a function from  $X \times X$  to a partially ordered set  $\Gamma$  with smallest element  $\perp$ , such that for all  $x, y, z \in X$  and all  $\gamma \in \Gamma$ ,

(U1)  $u(x, y) = \perp$  if and only if  $x = y$ ,

(U2) if  $u(x, y) \leq \gamma$  and  $u(y, z) \leq \gamma$ , then  $u(x, z) \leq \gamma$ ,

(U3)  $u(x, y) = u(y, x)$  (symmetry).

Condition (U2) is the ultrametric triangle law; if  $\Gamma$  is totally ordered, it can be replaced by

(UT)  $u(x, z) \leq \max\{u(x, y), u(y, z)\}$ .

We call

$$uX := \{u(x, y) \mid x, y \in X, x \neq y\} \subseteq \Gamma$$

the **value set** of  $(X, u)$ .

# Ultrametric balls and spherical completeness

There are several variants of ultrametric balls. The most convenient for us is the following. Take an ultrametric space  $(X, u)$  and elements  $x, y \in X$ . Then

$$B(x, y) := \{z \in X \mid u(x, z) \leq u(x, y)\}$$

is called a **precise ultrametric ball**. We let  $\mathcal{B}_u$  be the collection of all precise ultrametric balls and call  $(X, \mathcal{B}_u)$  the **ultrametric ball space**. It is **spherically complete** if every chain of precise ultrametric balls (ordered by inclusion) has nonempty intersection. This notion plays an important role in valuation theory. Spherical completeness characterizes the maximal valued fields, which includes all power series fields. Sibylla Prietz-Crampe proved an ultrametric fixed point theorem for them, which she used for an elegant proof of the fact that power series fields are henselian. In collaboration with Paulo Ribenboim she proved many more ultrametric fixed point theorems.

# Spherically complete ball spaces

In analogy to ultrametric spaces, a ball space is called **spherically complete** if every chain of balls has nonempty intersection. By dropping any assumptions what the balls in our ball spaces actually are, the important notion of spherical completeness becomes flexible enough to be applied to different sets of balls. For instance, if in a spherically complete ultrametric space we add all **generalized ultrametric balls**, i.e., unions of chains of precise ultrametric balls, will the larger ball space still be spherically complete? In

[1] W. Kubiś – K: *Chain union closures and convexity*, in preparation

we prove that this holds in the case of ultrametric spaces with totally ordered value sets, and study under which conditions closing a ball space under chain unions preserves spherical completeness.

# A large variety of ball spaces

In

[2] H. Ćmiel – K – K. Kuhlmann: *A generic approach to measuring the strength of completeness/compactness of various types of spaces and ordered structures*, RACSAM **115** (2021), Article No. 156

a large number of different ball spaces is presented, and it is studied what spherical completeness in each case means, and which type of fixed point theorems can be proved from it.

In the present talk, I will concentrate on two of the most striking examples. To this end, let me generalize the notion of “precise ultrametric ball”. If  $(X, \mathcal{B})$  is any ball space, then we call a ball  $B \in \mathcal{B}$  **precise** if there are  $x, y \in B$  such that  $B \subseteq B'$  for every  $B' \in \mathcal{B}$  for which  $x, y \in B'$ . In this case, we write  $B = B(x, y)$ .

# Totally ordered sets, abelian groups and fields

The obvious candidates for subsets in a totally (or partially) ordered set  $(X, \leq)$  to be called precise balls are the closed bounded intervals  $[x, y]$  with  $x, y \in X, x \leq y$ . We call  $(X, \mathcal{B}_\leq)$ , where  $\mathcal{B}_\leq$  is the collection of all such intervals, the **interval ball space** of  $(X, \leq)$ . We write  $B(x, y) = B(y, x) = [x, y]$  if  $x \leq y$ , and  $= [y, x]$  if  $y \leq x$ .

The question arises: when is the interval ball space of an ordered field  $K$  spherically complete, i.e., when does every chain of closed bounded intervals have a nonempty intersection? This holds when  $K = \mathbb{R}$  because  $\mathbb{R}$  is cut complete. Are there others which do the job even though they will not be cut complete? The point is that with descending chains of closed bounded intervals one can “zoom in” on any Dedekind cut that is not realized by an element in  $K$ . However, can we really zoom in enough to get the intersection empty?

# Asymmetric cuts

Take a Dedekind cut  $(\Lambda^L, \Lambda^R)$ . Let  $\kappa$  be the cofinality of the left cut set  $\Lambda^L$ , and  $\lambda$  the coinitality of the right cut set  $\Lambda^R$ . Let  $[x_i, y_i]$  with  $i \in \mu$  for some regular cardinal  $\mu$  and  $x_i \in \Lambda^L$ ,  $y_i \in \Lambda^R$  form a descending chain. Assume that  $\kappa \neq \lambda$ , say  $\kappa < \lambda$ . If  $\mu > \kappa$ , then eventually  $x_i$  will get fixed. If  $\mu < \lambda$ , then the sequence  $(y_i)_{i \in \mu}$  will not be coinital in  $\Lambda^R$ . Hence in all cases, the chain will have nonempty intersection. This shows that for an ordered field  $K$  in which each Dedekind cut is **asymmetric**, i.e., the cofinality of the left cut set is not equal to the coinitality of the right cut set, the interval ball space is spherically complete. We call such ordered fields **symmetrically complete**.

# Symmetrical completeness

In 2004, Saharon Shelah proved that every ordered field can be embedded in a symmetrically complete ordered field. In [3] K – K. Kuhlmann – S. Shelah: *Symmetrically complete ordered sets, abelian groups and fields*, Israel J. Math. **208** (2015), 261–290 the definition of symmetrical completeness is extended to ordered sets and ordered abelian groups. Taking the balls again to be the nonempty closed bounded intervals, it is then shown that the corresponding ball space is spherically complete if and only if symmetrical completeness holds. Banach's Fixed Point Theorem is generalized to symmetrically complete ordered fields, replacing the usual metric of the reals by a distance function that is derived from the ordering. Further, a characterization and construction of symmetrically complete ordered fields, ordered abelian groups and ordered sets is given.

# Chain unions

We say that a ball space  $(X, \mathcal{B})$  is **chain union closed** if the union of every chain in  $\mathcal{B}$  is a member of  $\mathcal{B}$ . We define  $\text{cu}(\mathcal{B})$  to be the family of all sets of the form  $\bigcup \mathcal{C}$ , where  $\mathcal{C} \subset \mathcal{B}$  is a chain (note that chains of sets are always supposed to be nonempty). Hence  $(X, \mathcal{B})$  is chain union closed if and only if  $\text{cu}(\mathcal{B}) = \mathcal{B}$ .

As mentioned already, we prove in [1] that for a spherically complete ultrametric ball space  $(X, \mathcal{B}_u)$  with totally ordered value set,  $\text{cu}(\mathcal{B}_u)$  is again spherically complete. In fact, this holds for a larger class of ball spaces that we will introduce on the next slide.

In contrast, we also prove: There exists a countable spherically complete ultrametric space with a countable partially ordered value set, whose ultrametric ball space does not admit any **expansion**  $\mathcal{B}'$  of  $\mathcal{B}$  (i.e., a ball space  $\mathcal{B}'$  on  $X$  that contains  $\mathcal{B}$ ) that is simultaneously chain union closed and spherically complete.

# Tree-like ball spaces

A ball space  $(X, \mathcal{B})$  is called **tree-like** if for every  $B_1, B_2 \in \mathcal{B}$  the following implication holds:

$$B_1 \cap B_2 \neq \emptyset \Rightarrow B_1 \subseteq B_2 \text{ or } B_2 \subseteq B_1 .$$

The ball spaces of all ultrametric spaces with totally ordered value set are tree-like.

In [1] we prove:

## Theorem

*Take a tree-like ball space  $(X, \mathcal{B})$ . Then  $\text{cu}(\mathcal{B})$  is tree-like and the following assertions hold:*

- 1) The ball space  $(X, \text{cu}(\mathcal{B}))$  is chain union closed.*
- 2) If  $(X, \mathcal{B})$  is spherically complete, then so is  $(X, \text{cu}(\mathcal{B}))$ .*

# Preservation of spherical completeness of orderings

In contrast to ultrametric spaces, for ordered fields and ordered abelian groups with interval ball space  $\mathcal{B}_\leq$ ,  $\text{cu}(\mathcal{B}_\leq)$  is never spherically complete. This follows from:

## Proposition

*Take a partially ordered set  $(X, \leq)$  and assume that  $X$  contains a nonempty totally ordered subset  $S$  without a largest element. Then  $(X, \text{cu}(\mathcal{B}_\leq))$  is not spherically complete.*

## Proof.

For each  $x \in S$ , define

$$S_x := \bigcup \{ [x, z] \mid z \in S \text{ with } x \leq z \} \in \text{cu}(\mathcal{B}_\leq).$$

If  $x, y \in S$  with  $x \leq y$ , then  $S_y \subseteq S_x$ , so  $\{S_x \mid x \in S\}$  is a chain in  $\text{cu}(\mathcal{B}_\leq)$ . Since  $\bigcap_{x \in S} S_x = \emptyset$ , we see that  $(X, \text{cu}(\mathcal{B}_\leq))$  is not spherically complete. □

# Chain unions and convexity

In a ball space  $(X, \mathcal{B})$  that contains precise balls  $B(x, y)$  for all  $x, y \in X$  we can define a notion of convexity as follows.

A subset  $S \subset X$  is  $\mathcal{B}$ -convex if  $B(x, y) \subset S$  for all  $x, y \in S$ . In particular, all precise balls are convex. If  $(X, \leq)$  is a partially ordered set, then a subset  $S$  is convex in the usual sense if and only if it is  $\mathcal{B}_{\leq}$ -convex.

Now the question arises: if the ball space  $(X, \mathcal{B})$  contains precise balls  $B(x, y)$  for all  $x, y \in X$ , how can we generate an expansion  $\mathcal{B}'$  of  $\mathcal{B}$  that consists of all  $\mathcal{B}$ -convex subsets of  $X$ ? Note that every union over an increasing chain of  $\mathcal{B}$ -convex sets is again a  $\mathcal{B}$ -convex set.

# Chain union closures

However, closing under such unions just once may not produce all convex sets from the precise balls. Then we have to iterate the process to arrive at what we call the **chain union closure**, i.e., a smallest expansion  $\mathcal{B}'$  of  $\mathcal{B}$  such that  $\text{cu}(\mathcal{B}') = \mathcal{B}'$ ; in this case we call  $\mathcal{B}'$  **chain union closed**. This process and the number of iterations needed to arrive at the chain union closure is studied in detail in [1] for arbitrary ball spaces  $(X, \mathcal{B})$ .

For cardinality reasons, chain union closures exist for all ball spaces, and they are uniquely determined. However, if we start with a ball space consisting of precise balls, it is not a priori clear whether all convex sets lie in the chain union closure.

# Chain union closures of tree-like ball spaces

## Theorem

Take a tree-like ball space  $(X, \mathcal{B})$  consisting of precise balls  $B(x, y)$  for all  $x, y \in X$ . Then its chain union closure consists exactly of all  $\mathcal{B}$ -convex sets in  $X$ .

## Proof.

We know already that all sets in the chain union closure are  $\mathcal{B}$ -convex. Take a  $\mathcal{B}$ -convex subset  $S$  of  $X$  and choose some  $x \in S$ . Then  $B(x, y) \subseteq S$  for all  $y \in S$ , hence

$$S = \bigcup_{y \in S} \{y\} \subseteq \bigcup_{y \in S} B(x, y) \subseteq S,$$

so equality holds everywhere. Since  $(X, \mathcal{B})$  is tree-like,  $\{B(x, y) \mid y \in S\}$  is a chain. Consequently, its union  $S$  lies in the chain union closure. □

# Chain union closures of interval ball spaces

## Theorem

*Take a totally ordered set  $X$ . Then  $\text{cu}(\text{cu}(\mathcal{B}_{\leq}))$  is the chain union closure of  $\mathcal{B}_{\leq}$ , and it consists exactly of all  $\mathcal{B}_{\leq}$ -convex sets in  $X$ .*

Proof.

Again, we know that all sets in the chain union closure are  $\mathcal{B}_{\leq}$ -convex. Take a  $\mathcal{B}_{\leq}$ -convex subset  $S$  of  $X$  and choose some  $x \in S$ . Then  $[x, y] = B(x, y) \subseteq S$  for all  $y \in S$  with  $x \leq y$ , hence  $\{[x, y] \mid y \in S, x \leq y\}$  is a chain. Therefore,

$$S_x := \{y \in S \mid x \leq y\} = \bigcup_{y \in S, x \leq y} [x, y] \in \text{cu}(\mathcal{B}_{\leq}).$$

# Chain union closures of interval ball spaces

Observe that  $\{S_x \mid x \in S\}$  is a chain in  $\text{cu}(\mathcal{B}_{\leq})$ . Consequently,

$$S = \bigcup_{x \in S} S_x \in \text{cu}(\text{cu}(\mathcal{B}_{\leq})).$$

The union over each chain in  $\text{cu}(\text{cu}(\mathcal{B}_{\leq}))$  is again convex, so it already lies in  $\text{cu}(\text{cu}(\mathcal{B}_{\leq}))$ . This shows that  $\text{cu}(\text{cu}(\mathcal{B}_{\leq}))$  is chain union closed. □

Note that it can happen that  $\text{cu}(\text{cu}(\mathcal{B}_{\leq})) = \text{cu}(\mathcal{B}_{\leq})$ , for instance, if  $X = \mathbb{N}$ .

Take any ball space  $(X, \mathcal{B})$ . Using transfinite recursion, we define  $\text{cu}_\alpha(\mathcal{B})$  for each ordinal  $\alpha$ , as follows:

$$\text{cu}_0(\mathcal{B}) := \mathcal{B}, \quad \text{cu}_\alpha(\mathcal{B}) := \text{cu} \left( \bigcup_{\beta < \alpha} \text{cu}_\beta(\mathcal{B}) \right) \text{ for } \alpha > 0.$$

Further, we define the **chain union rank** of  $\mathcal{B}$ , denoted by  $\text{cur}(\mathcal{B})$ , to be the smallest ordinal  $\alpha$  such that  $\text{cu}_{\alpha+1}(\mathcal{B}) = \text{cu}_\alpha(\mathcal{B})$ , i.e.,  $\text{cu}_\alpha(\mathcal{B})$  is chain union closed. Thus,  $\text{cur}(\mathcal{B}) = 0$  if and only if  $\mathcal{B}$  is chain union closed, while  $\text{cur}(\mathcal{B}) \leq 1$  means that in order to make  $\mathcal{B}$  chain union closed, it suffices to expand it by adding all unions of chains.

# Chain union rank of some ball spaces

- For every ordinal  $\alpha \geq 1$  there is a ball space  $(X, \mathcal{B})$  such that  $\text{cur}(\mathcal{B}) = \alpha$ . To construct it, set  $X := \aleph_\alpha$  and let  $\mathcal{B}$  be the collection of all nonempty countable subsets of  $\aleph_\alpha$ . We note that  $X \notin \mathcal{B}$  since  $\alpha \geq 1$ . A standard transfinite induction shows that for every  $\beta \leq \alpha$ ,  $\text{cu}_\beta(\mathcal{B})$  is the collection of all nonempty subsets of  $\aleph_\alpha$  of cardinality smaller than or equal to  $\aleph_\beta$ .

Since the subsets of  $\aleph_\alpha$  have cardinality at most  $\aleph_\alpha$ , it follows that  $\text{cu}_\alpha(\mathcal{B}) = \mathcal{P}(\aleph_\alpha) \setminus \{\emptyset\} = \mathcal{P}(X) \setminus \{\emptyset\}$ . Therefore,  $\text{cu}_\alpha(\mathcal{B})$  is chain union closed. On the other hand,  $\text{cu}_\beta(\mathcal{B})$  is not chain union closed for any  $\beta < \alpha$ . Hence,  $\text{cur}(\mathcal{B}) = \alpha$ .

# Chain union rank of ultrametric spaces

- As mentioned before, for ultrametric ball spaces  $(X, \mathcal{B}_u)$  with totally ordered value set,  $\text{cu}(\mathcal{B}_u)$  is chain union closed, hence  $\text{cur}(\mathcal{B}_u) = 1$ .
- Under certain additional conditions, this also holds for ultrametric spaces with partially ordered value sets:

## Theorem

*Assume that  $(X, u)$  is an ultrametric space with a partially ordered value set that does not contain any uncountable strictly increasing sequence. Then  $\text{cu}(\mathcal{B}_u)$  is chain union closed, hence  $\text{cur}(\mathcal{B}_u) = 1$ .*

# Chain union rank of ordered sets

- Here is an upper bound for the chain union rank of the interval ball space  $\mathcal{B}_{\leq}$  of any totally ordered set  $X$ : according to a theorem we already proved,  $\text{cu}(\text{cu}(\mathcal{B}_{\leq}))$  is chain union closed, hence  $\text{cur}(\mathcal{B}_{\leq}) \leq 2$ .
- And here is a lower bound for the chain union rank of the interval ball space  $\mathcal{B}_{\leq}$  of certain partially ordered sets:

## Proposition

*Take a partially ordered set  $(X, \leq)$  and assume that  $X$  contains a nonempty totally ordered subset  $S$  whose initiality and cofinality are distinct infinite cardinals. Then  $\text{cu}(\mathcal{B}_{\leq})$  is not chain union closed. Hence  $\text{cur}(\mathcal{B}_{\leq}) \geq 2$ .*

The proof uses a similar idea as the proof of the fact that decreasing chains of closed bounded intervals zooming in on asymmetric Dedekind cuts always have nonempty intersection.

# Chain union rank of ordered fields

- The case of ordered fields:

## Theorem

*If  $K$  is any subfield of  $\mathbb{R}$ , then  $\text{cur}(\mathcal{B}_K) = 1$ . If  $K$  is an ordered field allowing an unrealized asymmetric Dedekind cut, and in particular if  $K$  is a symmetrically complete ordered field properly containing  $\mathbb{R}$ , then  $\text{cur}(\mathcal{B}_K) = 2$ .*

The same holds for ordered abelian groups in place of ordered fields.

Proof.

All coinitalities and cofinalities in subfields  $K$  of  $\mathbb{R}$  are equal to 1 or  $\aleph_0$ , and it follows that  $\text{cu}(\mathcal{B}_K)$  is precisely the set of all intervals in  $K$ . Hence for such  $K$  we have that  $(K, \text{cu}(\mathcal{B}_K))$  is chain union closed.

# Chain union rank of ordered fields

Assume that  $K$  is an ordered field allowing an unrealized asymmetric Dedekind cut  $(\Lambda^L, \Lambda^R)$ . Let  $(\Lambda_1^L, \Lambda_1^R)$  be the cut  $(\Lambda^L, \Lambda^R)$  shifted by 1 to the left. Since the cofinality of  $\Lambda^L$  is not equal to the coinitality of  $\Lambda^R$  and the shift preserves cofinality and coinitality, the coinitality of the set  $\Lambda_1^R \cap \Lambda^L$  between the two cuts is not equal to its cofinality. Since  $(\Lambda^L, \Lambda^R)$  and hence also  $(\Lambda_1^L, \Lambda_1^R)$  are not realized, neither the cofinality nor the coinitality of  $\Lambda_1^R \cap \Lambda^L$  is equal to 1. Hence it follows from the previous proposition that for those  $K$ ,  $\text{cu}(\mathcal{B}_K)$  is not chain union closed. On the other hand, the previous lemma shows that  $\text{cu}(\text{cu}(\mathcal{B}_K))$  is chain union closed.

Finally, every symmetrically complete ordered field properly containing  $\mathbb{R}$  will have unrealized Dedekind cuts and they must have distinct cofinality and coinitality. □

# Products of ball spaces

Assume that  $(X_j, \mathcal{B}_j)_{j \in J}$  is a family of ball spaces. We set  $X = \prod_{j \in J} X_j$  and define the **box product**  $(X_j, \mathcal{B}_j)_{j \in J}^{\text{bpr}}$  of the family to be  $(X, (\mathcal{B}_j)_{j \in J}^{\text{bpr}})$ , where

$$(\mathcal{B}_j)_{j \in J}^{\text{bpr}} := \left\{ \prod_{j \in J} B_j \mid \forall j \in J : B_j \in \mathcal{B}_j \right\}.$$

Let us note that the following theorem is proven in [2]:

## Theorem

*The following assertions are equivalent:*

- the ball spaces  $(X_j, \mathcal{B}_j)$ ,  $j \in J$ , are spherically complete,*
- their box product is spherically complete.*

# Products of interval ball spaces

Take totally ordered sets  $(X_j, \leq)$ ,  $j \in J$ , and consider the associated family  $(X_j, \mathcal{B}_{\leq}^j)_{j \in J}$  of interval ball spaces.

## Theorem

*Assume that for some  $j_0 \in J$ ,  $X_{j_0}$  contains a nonempty totally ordered subset  $S$  without a largest element. If  $(X, (\mathcal{B}_{\leq}^j)_{j \in J}^{\text{bpr}})$  is the box product of the family  $(X_j, \mathcal{B}_{\leq}^j)_{j \in J}$ , then  $(X, \text{cu}((\mathcal{B}_j)_{j \in J}^{\text{bpr}}))$  is not spherically complete.*

## Proof.

By a previous proposition, there is a chain  $(B_{j_0, i})_{i \in I}$  in  $(X_{j_0}, \mathcal{B}_{\leq})$  with empty intersection. Fix any balls  $B_{j, i} = B_j \in \mathcal{B}_{\leq}^j$  for  $j \neq j_0$  and consider the boxes  $C_i := \prod_{j \in J} B_{j, i}$ . As the  $(C_i)_{i \in I}$  form a chain with empty intersection, we see that the box product is not spherically complete. □

# Chain union rank of products of ordered sets

It is not a straightforward task to compute the chain union rank of a product of ordered sets from the chain union ranks of these sets. Here we present a useful special case.

## Proposition

Take distinct infinite regular cardinals  $\kappa_1, \dots, \kappa_n$ . Consider each  $\kappa_j$  as a totally ordered set with its associated interval ball space  $(\kappa_j, \mathcal{B}_{\leq}^j)$ . Then the box product  $(\kappa_1 \times \dots \times \kappa_n, (\mathcal{B}_{\leq}^j)_{1 \leq j \leq n}^{\text{bpr}})$  has chain union rank  $n$ .

Note that if  $(X, \leq)$  is a partially ordered set containing another partially ordered set  $(Y, \leq)$ , then the chain union rank of  $(X, \mathcal{B}_{\leq})$  is no less than the chain union rank of  $(Y, \mathcal{B}_{\leq})$ . Thus the above result can be used to lower bounds of suitable partially ordered sets.

From the previous proposition, we deduce:

## Corollary

*For each  $n \in \mathbb{N}$  there exist ordered fields  $K$  for which the partially ordered vector space  $(K^n, \leq)$  has chain union rank  $\text{cur}(\mathcal{B}_{\leq}) \geq n$ .*

For this one needs ordered fields admitting Dedekind cuts with lower cut sets of cofinalities  $\kappa_1, \dots, \kappa_n$ . The existence of symmetrically complete ordered fields with this property follows from results in [3]. For the case that one does not insist in  $K$  being symmetrically complete, a short direct proof of the existence of the required ordered fields is given in [1].

# Not always can the convex sets be generated from the precise balls

Take  $n \geq 2$  and consider the vector space  $\mathbb{R}^n$  together with the ball space consisting of all straight lines with endpoints. These are precise balls, and the usual convex subsets of  $\mathbb{R}^n$  are again exactly the subsets that for any two distinct points  $x, y$  in them also contain the straight line with endpoints  $x$  and  $y$ . However, the chain union closure of the ball space of precise balls is far from containing all convex sets. Indeed, it just consists of all convex subsets of straight lines in  $\mathbb{R}^n$ .

# Suggestions for further research

There exist various generalizations of the classical notion of valuation, e.g. valuations on rings or lattice-ordered groups, lattice valuations on fields, globally valued fields, etc. Here are some questions that may be worth investigating:

- In which cases where the notion of valuation is generalized is there also a corresponding generalization of the underlying ultrametric space?
- If that is the case, are there precise balls, and what are the corresponding convex sets?
- Which generalized ultrametric spaces are spherically complete, and can meaningful and interesting fixed point theorems be proven using this?
- For a given generalized notion of valuation, are there invariants (like value group and residue field) and properties (like henselian) that can be used to characterize algebraically closed models (in the model theoretic sense), or determine when two models are isomorphic or elementarily equivalent?

# More papers on ball spaces

K – K. Kuhlmann – M. Paulsen: *The Caristi-Kirk Fixed Point Theorem from the point of view of ball spaces*, J. Fixed Point Theory Appl. **20** (2018), Paper No. 107

P. Błaszkwicz – H. Ćmiel – A. Linzi – P. Szewczyk: *Caristi-Kirk and Oettli-Théra ball spaces, and applications*, J. Fixed Point Theory Appl. **21** (2019), Paper No. 98

W. Kubiś – K: *Chain intersection closures*, Topology Appl. **262** (2019), 11–19

R. Bartsch – K – K. Kuhlmann: *Construction of ball spaces and the notion of continuity*, New Zealand J. Mathematics **51** (2021), 49–64

**THE END**

**Thank you for your attention!**

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