

Higher ramification theory

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*Dedicated to Professor Sudesh Khanduja on the occasion of her
75th birthday*

Ramification theory describes the properties of algebraic extensions of valued fields. While one origin of ramification theory is number theory, where the scope up until recently was restricted to discretely valued fields, or at best valued fields with archimedean ordered value groups (which can be represented as subfields of the reals - the **rank one** case), another origin is the theory of ordered fields. They can have associated valuations with arbitrarily large value groups. In his seminal paper *Allgemeine Bewertungstheorie* in 1931 Wolfgang Krull says that this was the motivation for writing the paper.

Basic ramification theory

When we apply ramification theory to the separable algebraic closure of a valued field (K, v) , we obtain:

- the **absolute decomposition field**, also called the **henselization** of (K, v) , from where the extension of the valuation v to every algebraic extension of K is uniquely determined,
- the **absolute inertia field** of (K, v) , which has the same value group as K , but whose residue field is the separable-algebraic closure of the one of K , and
- the **absolute ramification field** of (K, v) , which has the same residue field as the absolute inertia field, but whose value group is the divisible closure of that of K under all primes different from the characteristic of the residue field of K .

Longstanding open problems

If the characteristic of the residue field is 0, then the absolute ramification field is algebraically closed, but if it is a prime $p > 0$, then it may not be algebraically closed and the Galois group of its extension to the separable algebraic closure is a p -group. This is where the **wild ramification** appears, including the **defect** (to be defined later), which is a serious obstacle in several longstanding open problems about valued fields with positive residue characteristic, such as:

- local uniformization, the local form of resolution of singularities,
- decidability of the Laurent series field over the field with p elements.

Higher ramification theory

Therefore, it is very important to study the algebraic extensions of the absolute ramification field, and this is the subject of higher ramification theory.

For a long time, basic ramification theory (with Ribenboim's 1968 book *Théorie des Valuations* as a valuable source), together with the fact that Galois extensions of the absolute ramification field have Galois groups that are p -groups, were the basic tools in my study of the above mentioned open problems. I knew about the notion of **higher ramification groups** from Ribenboim's writings, but was not particularly interested. Recently, that changed when I noticed the importance of the associated **ramification ideals**.

Some notation

Before we define ramification ideals, we fix some notation. For a valued field (K, v) , we denote its value group by vK , its residue field by Kv , and its valuation ring by $\mathcal{O}_K$ with its maximal ideal $\mathcal{M}_K$. For $a \in K$, we write va for its value in vK .

We say that (K, v) is of **equal characteristic** if $\text{char } K = \text{char } Kv$, and that it is of **mixed characteristic** if $\text{char } K = 0$ and $\text{char } Kv$ is positive.

By $(L|K, v)$ we denote a field extension $L|K$ with a valuation v on L , where K is endowed with the restriction of v . In this case, there are induced embeddings of vK in vL and of Kv in Lv . The extension $(L|K, v)$ is called **immediate** if these embeddings are onto. Further, $(L|K, v)$ is called **unbranched** if the extension of v from K to L is unique.

Definition of ramification ideals

Take a unibranched Galois extension $(L|K, v)$ with Galois group G . For each subgroup H of G we define the $\mathcal{O}_L$ -ideal

$$I_H := \left(\frac{\sigma b}{b} - 1 \mid \sigma \in H, b \in L^\times \right) \quad (1)$$

(this is an $\mathcal{O}_L$ -ideal since the extension is assumed unibranched). Note that $I_{\{\text{id}\}} = (0)$. If I_H is nonzero, then it is called it a **ramification ideal**. If all ramification ideals are contained in $\mathcal{M}_L$, then the extension $(L|K, v)$ is called **purely wild**. For a more general definition of this notion and for background, see

[1] K – Matthias Pank – Peter Roquette: *Immediate and purely wild extensions of valued fields*, manuscripta math. **55** (1986), 39–67.

All algebraic extensions of an absolute ramification field are purely wild.

Importance of ramification ideals

Ramification ideals play a central role in the presentation of the **Kähler differentials** (aka **modules of relative differentials**) that appear in the papers

[2] Cutkosky, S.D. – K – Rzepka, A.: *On the computation of Kähler differentials and characterizations of Galois extensions with independent defect*, Math. Nachrichten **298** (2025), 1549–1577,

[3] Cutkosky, S.D. – K: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, submitted.

Moreover, ramification ideals can be used to classify defects. All of this and more is worked out in detail in

[4] K: *On certain definable coarsenings of valuation rings and their applications*, submitted.

Early articles by Ribenboim on higher ramification

Among the early articles on ramification ideals for not necessarily discrete valuations are:

[5] Paulo Ribenboim: *Higher ramification groups for rank one valuations*, Math. Ann. **173** (1967), 253–259,

[6] Paulo Ribenboim: Corrections to: “Higher ramification groups for rank one valuations”, Math. Ann. **185** (1970), 22–24.

In [6], Ribenboim presents his method to compute ramification ideals. However, he restricts the scope to rank one valuations, that is, valuations whose value groups can be embedded in the reals. Ribenboim uses the fact that after such an embedding, every nonempty set of values that is bounded from below has an infimum in $\mathbb{R}$. In particular, when I is a nonzero (possibly fractional) ideal of the valuation ring of a valued field (K, v) , then it can be associated in a unique way with the infimum of

$$vI := \{va \mid 0 \neq a \in I\}. \quad (2)$$

Final segments

Note that even when the value group vK is not embeddable in the reals, the set vI is a **final segment** of vK , i.e., if $\alpha \in vI$ and $\alpha \leq \beta \in vK$, then $\beta \in vI$. Hence there is always a bijection between the nonzero (possibly fractional) ideals of the valuation ring and the (nonempty) final segments of vK . In the preprint

[7] K: *Topics in higher ramification theory, I: ramification ideals*, submitted,

I show that Ribenboim's construction can easily be generalized to valuations with arbitrarily large value groups, using final segments in place of real numbers that occur as infima.

Arithmetic of ideals over valuation rings

In the same way, results about the arithmetic of ideals over valuation rings proven by Ribenboim in

[8] Paulo Ribenboim: *Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation*, Summa Brasil. Math. **4** (1958), 1–64

for rank one valuations have been generalized in

[9] K – Katarzyna Kuhlmann: *Arithmetic of cuts in ordered abelian groups and of ideals over valuation rings*, Pacific J. Math. **340** (2026), 339–374

to valuations with arbitrarily large value groups.

Arithmetic of ideals over valuation rings

Paper [9] deals with the existence and uniqueness of solutions J to equations $I_1 J = I_2$ and inequalities $I_1 J \subseteq I_2$ for given ideals I_1 and I_2 . This uses the translation (2) to the question of existence and uniqueness of solutions T to equations $S_1 + T = S_2$ and inequalities $S_1 + T \subseteq S_2$ for given final segments S_1 and S_2 .

In contrast, Ribenboim's paper [8] only deals with equations and is restricted to rank one.

The defect

If $(L|K, v)$ is a finite unbranched extension, then by the Lemma of Ostrowski,

$$[L : K] = \tilde{p}^v \cdot (vL : vK)[Lv : Kv], \quad (3)$$

where v is a nonnegative integer and $\tilde{p}$ the **characteristic exponent** of Kv , that is, $\tilde{p} = \text{char } Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K, v) := \tilde{p}^v$ is the **defect** of the extension $(L|K, v)$. We call $(L|K, v)$ a **defect extension** if $d(L|K, v) > 1$, and a **defectless extension** if $d(L|K, v) = 1$. If $(L|K, v)$ is a defect extension of prime degree p , then it follows from (3) that $[L : K] = p = \text{char } Kv$ and that $(vL : vK) = 1$ and $[Lv : Kv] = 1$, that is, $(L|K, v)$ is an immediate extension. Nontrivial defect only appears when $\text{char } Kv = p > 0$, in which case $\tilde{p} = p$.

Ramification ideals and the defect

In [5] Ribenboim claims that for all unbranched Galois extensions $(L|K, v)$ with valuations v of rank one, all ramification ideals are principal, but this is false. The mistake was noticed by J. L. Chabert. In [6] Ribenboim then gives a correct proof of the statement under the additional assumption that the extension is defectless. Curiously, his false statement from [5] is later repeated, with exactly the same incorrect proof, in his book

[10] Paulo Ribenboim: *The theory of classical valuations*, Springer Monographs in Mathematics, 1998.

Ramification ideals and the defect

In [7], I prove without the rank one assumption:

Proposition

- 1) *For all defectless unibranched Galois extensions $(L|K, v)$, all ramification ideals are principal.*
- 2) *If $(L|K, v)$ is a defect extension of prime degree p , then its unique ramification ideal is nonprincipal.*

In view of statement 2), the question arises whether the converse of statement 1) also holds. That is, if all ramification ideals are principal, does it follow that $(L|K, v)$ is defectless?

A counterexample

I prove the following in [7]:

Proposition

There are Galois extensions of degree p^2 of valued fields of equal characteristic $p > 0$ and rank one that have only one ramification ideal, and this ramification ideal is principal although the extension is not defectless.

Since every ramification ideal is of the form I_H with H a nontrivial subgroup of the Galois group G , the number of ramification ideals does not exceed the number of nontrivial subgroups of G . However, our example shows that this upper bound is not always assumed.

Construction of the counterexample

Every Galois extension $L|K$ of degree p of a field of characteristic $p > 0$ is an [Artin-Schreier extension](#), i.e., it is generated by an element $\vartheta \in L$ such $\vartheta^p - \vartheta \in K$. Then ϑ is called an [Artin-Schreier generator](#).

Our counterexample is a tower of two Artin-Schreier extensions which satisfies a certain condition that guarantees that this tower is itself a Galois extension. The upper extension is taken to be a defectless valued field extension, while the lower is taken to be a defect extension. The values of the respective generators ϑ are taken in such a way that the defectless upper extension overrules the lower defect extension.

What about mixed characteristic?

Open problem: Is there a counterexample in mixed characteristic?

Every Galois extension $L|K$ of degree p of a field of characteristic 0 which contains a primitive p -th root of unity, is a **Kummer extension**, i.e., it is generated by an element $\eta \in L$ such $\eta^p \in K$. Then η is called a **Kummer generator**.

The problem is that we know much less about towers of Kummer extensions than about towers of Artin-Schreier extensions. As to the latter, look out for recent papers by Josnei Novacoski and coauthors.

Defect extensions of prime degree

Take a defect extension $\mathcal{E} := (L|K, v)$ of degree $p = \text{char } Kv > 0$. As already mentioned above, such an extension is immediate, i.e., $vL = vK$ and $Lv = Kv$. It follows that for $z \in L \setminus K$,

$$v(z - K) := \{v(z - c) \mid c \in K\}$$

has no maximal element. In

[11] K – Anna Rzepka: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738,

we show the following:

Defect extensions of prime degree

Proposition

Assume that $\mathcal{E}$ is an Artin-Schreier extension with Artin-Schreier generator $\vartheta \in L$. Then the unique ramification ideal of $\mathcal{E}$ is

$$I_{\mathcal{E}} := \left(\frac{1}{\vartheta - c} \mid c \in K \right)$$

with

$$vI_{\mathcal{E}} = -v(\vartheta - K).$$

Defect extensions of prime degree

An element a in a valued field is a **1-unit** if $v(a - 1) > 0$.

Proposition

Assume that K contains a primitive p -th root of unity ζ_p . Then $\mathcal{E}$ is a Kummer extension with a Kummer generator η which is a 1-unit, and the unique ramification ideal of $\mathcal{E}$ is

$$I_{\mathcal{E}} := \left(\frac{\zeta_p - 1}{\eta - c} \mid c \in K \text{ a 1-unit} \right)$$

with

$$vI_{\mathcal{E}} = v(\zeta_p - 1) - v(\eta - K) = \frac{vp}{p-1} - v(\eta - K).$$

Assume that A is a ring and B is an A -algebra. Then $\Omega_{B|A}$ denotes the module of relative differentials (Kähler differentials), that is, the B -module for which there is a universal derivation

$$d : B \rightarrow \Omega_{B|A}$$

such that for every B -module M and derivation $\delta : B \rightarrow M$ there is a unique B -module homomorphism

$$\phi : \Omega_{B|A} \rightarrow M$$

such that $\delta = \phi \circ d$.

In [2], we prove:

Theorem

Take a defect extension $\mathcal{E} = (L|K, v)$ of degree p . If $\text{char } K = 0$, then assume that K contains a primitive p -th root of unity. Then there is an $\mathcal{O}_L$ -module isomorphism

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \simeq I_{\mathcal{E}}/I_{\mathcal{E}}^p.$$

For defectless extensions of prime degree, the ramification ideal is also essential, but the story is more involved. See [4].

Classification of the defect

Take $\mathcal{O}_{\mathcal{E}}$ to be the largest of all overrings $\mathcal{O}'$ of $\mathcal{O}_L$ such that $I_{\mathcal{E}}$ is still an $\mathcal{O}'$ -ideal. Since $\mathcal{O}_L$ is a valuation ring, the same holds for all overrings; all of them are called coarsenings of $\mathcal{O}_L$.

Hence $\mathcal{O}_{\mathcal{E}}$ is again a valuation ring on L . We denote its maximal ideal by $\mathcal{M}_{\mathcal{E}}$. In rank one, $\mathcal{M}_{\mathcal{E}}$ is just $\mathcal{M}_L$.

In [4] I give the following definition:

$\mathcal{E}$ has **independent defect** if

$$I_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}} \quad \text{and } \mathcal{M}_{\mathcal{E}} \text{ is a nonprincipal } \mathcal{O}_{\mathcal{E}}\text{-ideal}, \quad (4)$$

otherwise we say that $\mathcal{E}$ has **dependent defect**. This is equivalent to the definition given in [11].

Classification of the defect

In [2], we prove:

Proposition

Take a defect extension $\mathcal{E} = (L|K, v)$ of degree p . If $\text{char } K = 0$, then assume that K contains a primitive p -th root of unity. Then the following assertions are equivalent:

- a) $\mathcal{E}$ has independent defect,*
- b) $I_{\mathcal{E}}^p = I_{\mathcal{E}}$,*
- c) $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.*

Dependent defect

Initially, the notion of independent defect was only given for valued fields of positive characteristic in

[12] K: *A classification of Artin–Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448.

There, the name “dependent defect” was chosen because in positive characteristic, defect extensions of prime degree with dependent defect can be derived from suitable purely inseparable defect extensions via a transformation. For mixed characteristic, in the absence of nontrivial purely inseparable extensions, a suitably generalized definition had to be given in [11].

Dependent defect appears to be more harmful for the above mentioned open problems than independent defect.

Perfectoid fields

A **perfectoid field** is a complete nondiscrete rank one valued field of residue characteristic $p > 0$ such that $\mathcal{O}_K/p\mathcal{O}_K$ is **semiperfect**, that is, the Frobenius $x \mapsto x^p$ is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. If $\text{char } K = p$, then this is just saying that the Frobenius is surjective on $\mathcal{O}_K$, which is equivalent to K being perfect. The importance of perfectoid fields is that they allow Scholze's tilting construction which connects their Galois theory to that of fields of positive characteristic.

In general, perfectoid fields admit nontrivial defect extensions, but all of them have independent defect. This is an indication that independent defect is less harmful than dependent defect.

Deeply ramified fields

Neither “complete” nor “rank one” are elementary (aka first order) properties. A suitable elementary class of valued fields containing the perfectoid fields is that of [deeply ramified fields](#), studied in [11]. Again, all of their nontrivial defect extensions have independent defect.

James Ax is famous, in particular among model theorists, since together with Simon Kochen he proved in 1965, using model theory (!), that Artin's conjecture holds for $\mathbb{Q}_p$ for all but a finite number of primes p .

A valued field (K, v) is called **henselian** if it satisfies Hensel's Lemma, or equivalently, the extension of v to its algebraic closure is unique. We will then denote this extension again by v . In the paper

[13] James Ax: *Zeros of polynomials over local fields—The Galois action*, J. Algebra **15** (1970), 417–428,

Ax claims the following:

If (K, v) is a perfect henselian rank one valued field of positive characteristic, then for each element a in its algebraic closure there is an element $b \in K$ such that

$$v(a - b) \geq \min\{v(a - a') \mid a' \text{ } K\text{-conjugate of } a\}.$$

Taking on James Ax

In the paper

Sudesh Khanduja: *A note on a result of J. Ax*, J. Algebra **140** (1991), 360–361,

Professor Khanduja gives a counterexample to the statement of Ax. Then in

Sudesh Khanduja: *On a result of J. Ax*, J. Algebra **172** (1995), 147–151,

she proves that the statement is true if and only if (K, v) is a [defectless field](#), i.e., all finite extensions of (K, v) are defectless.

THE END

Thank you for your attention!

More detailed information

Preprints and further information:

<https://www.valth.eu/>

The slides of this talk can be found on my new personal homepage:

<https://fvkuhlmann.de/>

The most recent versions of my submitted papers can be found at:

<https://fvkuhlmann.de/Fvkprepr.html>