On certain definable coarsenings of valuation rings and their applications

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Introduction

In this talk I will discuss two types of definable coarsenings of valuation rings on (not necessarily henselian) fields and their applications to the presentation of the structure of Galois extensions of prime degree of valued fields. These extensions have been studied in the papers

[1] Cutkosky, S.D. – K – Rzepka, A.: *On the computation of Kähler differentials and characterizations of Galois extensions with independent defect*, Math. Nachrichten **298** (2025), 1549–1577,

[2] Cutkosky, S.D. – K: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, submitted; arxiv:2306.04967.

As our main interest are the applications, we will only deal with definability in a suitable expanson of the language $\mathcal{L}_{\mathrm{val}}$ of valued rings.

Notation

For a valued field (K, v), we denote the value group by vK, the residue field by Kv, the valuation ring by $\mathcal{O}_K$, and its maximal ideal by $\mathcal{M}_K$. Throughout, we will use the convention that $v0 = \infty > \alpha$ for all $\alpha \in vK$.

By $(L|K, v)$ we denote a field extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K, v)$ is called immediate if these embeddings are onto. We call it unibranched if the valuation v has only one extension from K to L.

For $z \in L \setminus K$ we set

$$v(z - K) \ := \ \{v(z - c) \mid c \in K\}.$$

If $(L|K, v)$ is immediate, then $v(z - K)$ has no maximal element.

Ramification ideals

If $L|K$ is Galois, then we denote its Galois group by $\mathrm{Gal}\, L|K$. In this case, a nontrivial $\mathcal{O}_L$-ideal contained in $\mathcal{M}_L$ is called a ramification ideal of $(L|K, v)$ if it is of the form

$$\left(\frac{\sigma b - b}{b} \;\middle|\; \sigma \in H \,,\, b \in L^\times \right) \tag{1}$$

for some subgroup H of $\mathrm{Gal}\, L|K$.

Galois extensions of prime degree

Take a Galois extension $\mathcal{E} = (L|K, v)$ of prime degree q. If $q = \operatorname{char} K$, then $\mathcal{E}$ is an Artin-Schreier extension, that is, generated by an Artin-Schreier generator ϑ which satisfies $\vartheta^p - \vartheta \in K$.

In all other cases, if K contains a primitive q-th root of unity ζ_q, then $\mathcal{E}$ is a Kummer extension, that is, generated by a Kummer generator η which satisfies $\eta^q \in K$.

Let me explain why such extensions are important for us. First we need some definitions.

Kähler differentials

Assume that A is a ring and B is an A-algebra. Then $\Omega_{B|A}$ denotes the module of relative differentials (Kähler differentials), that is, the B-module for which there is a universal derivation

$$d : B \rightarrow \Omega_{B|A}$$

such that for every B-module M and derivation $\delta : B \rightarrow M$ there is a unique B-module homomorphism

$$\phi : \Omega_{B|A} \rightarrow M$$

such that $\delta = \phi \circ d$.

Perfectoid fields

A perfectoid field is a complete nondiscrete rank 1 valued field of residue characteristic $p > 0$ such that $\mathcal{O}_K/p\mathcal{O}_K$ is semiperfect, that is, the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. If char $K = p$, then this is just saying that the Frobenius is surjective on $\mathcal{O}_K$, which is equivalent to K being perfect.

A valued field has rank 1 if its value group is embeddable in the ordered additive group of $\mathbb{R}$.

Neither "complete" nor "rank 1" are elementary properties. A suitable elementary class of valued fields containing the perfectoid fields is that of deeply ramified fields, studied in the article

[3] K – Rzepka, A.: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Transactions Amer. Math. Soc. **376** (2023), 2693–2738.

Deeply ramified fields

We define a nontrivially valued field (K, v) of residue characteristic $p > 0$ to be deeply ramified if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete);

(DRvr) if $\operatorname{char} Kv = p > 0$, then $\mathcal{O}_K/p\mathcal{O}_K$ is semiperfect if $\operatorname{char} K = 0$, and the completion $\widehat{K}$ of (K, v) is perfect if $\operatorname{char} K = p$.

All perfect fields of positive characteristic are deeply ramified.

Deeply ramified fields and Kähler differentials

In the book

[4] Gabber, O. – Ramero, L.: *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003,

deeply ramified fields are characterized by means of Kähler differentials:

Theorem (Gabber and Ramero 2003)

Take a valued field (K, v). Choose any extension of v to the separable-algebraic closure K^{sep} of K. Then (K, v) is a deeply ramified field if and only if

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} = 0 .$$

Note that $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K}$ does not depend on the chosen extension of v from K to K^{sep}.

Deeply ramified fields and Kähler differentials

Gabber and Ramero use heavy machinery to prove their theorem. An alternate proof is given in the paper [2]. Its two main parts are:

• computation of the Kähler differentials of Artin-Schreier extensions and of Kummer extensions of prime degree of K, and then

• putting these together to obtain $\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K}$.

On the way, the following characterization is proven:

Theorem

Let (K, v) be a valued field of residue characteristic $p > 0$. If K has characteristic 0, then assume in addition that it contains all p-th roots of unity. Then (K, v) is a deeply ramified field if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ for all unibranched Galois extensions $(L|K, v)$ of degree p.

The case of defect extensions

In [1] we compute the Kähler differentials of Artin-Schreier extensions and of Kummer extensions with defect. Take a finite extension $(L|K,v)$ of valued fields. By the Lemma of Ostrowski,

$$[L:K] \;=\; \tilde{p}^{\nu} \cdot (vL:vK)[Lv:Kv]\,, \tag{2}$$

where ν is a non-negative integer and $\tilde{p}$ the characteristic exponent of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K,v) := \tilde{p}^{\nu}$ is the defect of the extension $(L|K,v)$. If $d(L|K,v) = 1$, then the extension $(L|K,v)$ is called defectless; otherwise we call it a defect extension. Note that if $\operatorname{char} Kv = 0$, then every finite extension of (K,v) is defectless.

By use of ramification theory, the study of the defect can be reduced to the study of normal extensions of prime degree, where the separable ones among them are Artin-Schreier and Kummer extensions.

Higher ramification invariants

Take a valued field (K, v) with char $Kv = p > 0$, and a Galois defect extension $\mathcal{E} = (L|K, v)$ of degree p. For id$\neq \sigma \in \mathrm{Gal}\,(L|K)$, we set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma b - b}{b}\right) \middle|\; b \in L^\times,\, \sigma b \neq b \right\} . \tag{3}$$

It is shown in [1] that this set is a final segment of vK and independent of the choice of σ; so we denote it by $\Sigma_{\mathcal{E}}$. It is also shown that

$$I_{\mathcal{E}} \;=\; (b \in L \mid vb \in \Sigma_{\mathcal{E}}) \tag{4}$$

is the unique ramification ideal of $\mathcal{E}$.

Classification of the defect

The following classification is introduced in [3]. We say that $\mathcal{E}$ has independent defect if

$$\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}$$

for some proper convex subgroup $H_{\mathcal{E}}$ of vK such that $vK/H_{\mathcal{E}}$ has no smallest positive element; in this case, we call $H_{\mathcal{E}}$ the associated convex subgroup and note that it is a convex subgroup of both vK and vL since the extension $\mathcal{E}$ is immediate. Otherwise we say that $\mathcal{E}$ has dependent defect.

The following is a consequence of part 1) of Theorem 1.10 of [1]:

Theorem

Assume that (K, v) is a deeply ramified field. Then every Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree $p = \mathrm{char}\, Kv > 0$ has independent defect.

The equal characteristic case

Let us first discuss the case where (K, v) is of equal positive characteristic, that is, $\operatorname{char} K = \operatorname{char} Kv = p > 0$. Then a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p is an Artin-Schreier extension. As shown in [1],

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta - K) \tag{5}$$

for every Artin-Schreier generator $\vartheta \in L \setminus K$ with $\vartheta^p - \vartheta \in K$. By Theorem 1.7 of [1], $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$ if and only if

$$v(\vartheta^p - \vartheta - \wp(K)) \;=\; \{\alpha \in pvK \mid \alpha < H_{\mathcal{E}}\}\,, \tag{6}$$

where $\wp(X) := X^p - X$.

Definable coarsening of $\mathcal{O}_K$

If $\mathcal{E}$ has independent defect, then we obtain the following $\mathcal{L}_{\mathrm{val}}$-definable coarsening of $\mathcal{O}_K$:

$$\mathcal{O}_{\mathcal{E},K} \;:=\; \{b \in K \mid \forall c \in K : v(\vartheta^p - \vartheta - c^p + c) < vb\}\,, \qquad (7)$$

whose value group is $vK/H_{\mathcal{E}}$. For this definition it is not needed that (K,v) be henselian. (Note that deeply ramified fields are not required to be henselian.) For the case of henselian (K,v), this definition is used in

[5] Ketelsen, M. – Ramello, S. – Szewczyk, P.: *Definable henselian valuations in positive residue characteristic*, J. Symb. Logic, DOI: https://doi.org/10.1017/jsl.2024.55

to define corresponding henselian valuations on K already in the language of rings.

Definable coarsening of $\mathcal{O}_L$

For our applications, we are more interested in the coarsening of $\mathcal{O}_L$ corresponding to $H_{\mathcal{E}}$. By the definition of independent defect combined with Equation (5), $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$ if and only if

$$v(\vartheta - K) \;=\; \{\alpha \in vK \mid \alpha < H_{\mathcal{E}}\}\,. \tag{8}$$

As a consequence, in the language $\mathcal{L}_{\mathrm{val},K}$ of valued fields with a predicate for membership in K, the following coarsening of $\mathcal{O}_L$ corresponding to $H_{\mathcal{E}}$ can be defined:

$$\begin{aligned}\mathcal{O}_{\mathcal{E}} \;:=\; \{b \in L \quad\mid\quad & \forall x \in L \setminus K \;\; \forall c \in K : \\ & x^p - x \in K \to \; v(x-c) < vb\}\,.\end{aligned}$$

The maximal ideal of $\mathcal{O}_{\mathcal{E}}$ can be defined as follows:

$$\mathcal{M}_{\mathcal{E}} \;:=\; \{b \in L \mid \exists x \in L \setminus K : x^p - x \in K \wedge \exists c \in K : -v(x-c) \leq vb\}\,.$$

The mixed characteristic case

Now we discuss the case where (K, v) is of mixed characteristic, that is, $\operatorname{char} K = 0$ and $\operatorname{char} Kv = p > 0$. We assume in addition that K contains a primitive p-th root of unity ζ_p. Then a Galois defect extension $\mathcal{E} = (L|K, v)$ of prime degree p is a Kummer extension. As shown in [1],

$$\Sigma_{\mathcal{E}} \;=\; v(\zeta_p - 1) - v(\eta - K) \;=\; \frac{1}{p-1}vp - v(\eta - K) \qquad (9)$$

for every 1-unit $\eta \in L \setminus K$ with $\eta^p \in K$.
By Theorem 1.8 of [1], $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$ if and only if

$$v(\eta^p - K^p) \;=\; \frac{p}{p-1}vp + \{\alpha \in pvK \mid \alpha < H_{\mathcal{E}}\}\,, \qquad (10)$$

and we can define the following coarsening of $\mathcal{O}_K$:

$$\mathcal{O}_{\mathcal{E},K} \;:=\; \{b \in K \mid \forall c \in K : v(\eta^p - c^p) - \frac{p}{p-1}vp < vb\}\,.$$

The mixed characteristic case

By the definition of independent defect combined with Equation (9), $\mathcal{E}$ has independent defect with associated convex subgroup $H_{\mathcal{E}}$ if and only if

$$v(\eta - K) - v(\zeta_p - 1) \;=\; \{\alpha \in vK \mid \alpha < H_{\mathcal{E}}\}\,. \tag{11}$$

Hence in this case the $\mathcal{L}_{\mathrm{val},K}$-definition of $\mathcal{O}_{\mathcal{E}}$ is

$$\begin{aligned}\mathcal{O}_{\mathcal{E}} \;:=\; \{b \in L \quad | \quad & \forall x \in L \setminus K \;\; \forall c \in K : x^p \in K \wedge v(x-1) > 0 \\ & \to \;\; v(x-c) - \frac{1}{p-1} vp < vb\}\,,\end{aligned}$$

and maximal ideal of $\mathcal{O}_{\mathcal{E}}$ can be defined as

$$\begin{aligned}\mathcal{M}_{\mathcal{E}} \;:=\; \{b \in L \quad | \quad & \exists x \in L \setminus K : x^p \in K \wedge v(x-1) > 0 \\ & \wedge \exists c \in K : -v(x-c) + \frac{1}{p-1} vp \le vb\}\,.\end{aligned}$$

The following is proven in [1]:

Proposition

Take a Galois extension $\mathcal{E} = (L|K, v)$ of prime degree p with independent defect. If $\operatorname{char} K = 0$, then assume in addition that K contains a primitive p-th root of unity. Then the following assertions hold.

1) The ideal $\mathcal{M}_\mathcal{E}$ is equal to the ramification ideal $I_\mathcal{E}$, and $\mathcal{O}_\mathcal{E}$ is the largest of all coarsenings $\mathcal{O}'$ of $\mathcal{O}_L$ such that $I_\mathcal{E}$ is an $\mathcal{O}'$-ideal.

2) The trace $\operatorname{Tr}_{L|K}(\mathcal{M}_L)$ is equal to $\mathcal{M}_\mathcal{E} \cap K$.

The ramification ideal $I_{\mathcal{E}}$

For an $\mathcal{O}_L$-ideal I we define its invariance valuation ring to be

$$\mathcal{O}(I) := \{b \in L \mid bI \subseteq I\}.$$

This is a coarsening of $\mathcal{O}_L$ and is the largest of all coarsenings $\mathcal{O}'$ of $\mathcal{O}_L$ such that I is an $\mathcal{O}'$-ideal. If I is definable in some expansion $\mathcal{L}$ of the ring language, then so is $\mathcal{O}(I)$.

Based on part 1) of the previous proposition, we can extend the definition of $\mathcal{O}_{\mathcal{E}}$ from the independent defect case by setting $\mathcal{O}_{\mathcal{E}} := \mathcal{O}(I_{\mathcal{E}})$ and denoting its maximal ideal again by $\mathcal{M}_{\mathcal{E}}$.

Proposition

Under the assumptions of the previous proposition, the ramification ideal $I_{\mathcal{E}}$ and its invariance valuation ring are always $\mathcal{L}_{\mathrm{val},K}$-definable in (L, v).

The Kähler differentials in the defect case

The following is proven in [1]:

Theorem

Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. If $\mathrm{char}\, K = 0$*, then assume in addition that K contains a primitive p-th root of unity. Then*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_{\mathcal{E}}/I_{\mathcal{E}}^p$$

as $\mathcal{O}_L$-modules. This is zero if and only if $\mathcal{E}$ has independent defect.

In the case of independent defect, we have $I_{\mathcal{E}} = \mathcal{M}_{\mathcal{E}}$ and that $\mathcal{M}_{\mathcal{E}}$ is nonprincipal since $vL/H_{\mathcal{E}} = vK/H_{\mathcal{E}}$ has no smallest positive element. It follows that $\mathcal{M}_{\mathcal{E}}^p = \mathcal{M}_{\mathcal{E}}$, so that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

The defectless case

Take a defectless extension $\mathcal{E} = (L|K, v)$, not necessarily Galois, of prime degree q, not necessarily equal to char Kv. Then either $[L : K] = (vL : vK)$ or $[L : K] = [Lv : Kv]$. I will discuss the more interesting case of $[L : K] = (vL : vK)$, which I will assume throughout. I will now outline results from paper [2].

We define $H_{\mathcal{E}}$ to be the largest convex subgroup of vL which is also a convex subgroup of vK; it exists since unions over any non-empty collections of convex subgroups are again convex subgroups. We take $\mathcal{O}_{\mathcal{E}}$ to be the coarsening of $\mathcal{O}_L$ associated with $H_{\mathcal{E}}$ so that its value group is $vL/H_{\mathcal{E}}$, and denote its maximal ideal by $\mathcal{M}_{\mathcal{E}}$.

The convex subgroup $H_{\mathcal{E}}$ defined here has important similarities with the convex subgroup $H_{\mathcal{E}}$ defined in the defect case.

A case distinction

We distinguish three mutually exclusive cases describing how vK extends to vL:

(DL2a): there is no smallest convex subgroup of vL that properly contains $H_{\mathcal{E}}$,

(DL2b): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is dense,

(DL2c): there is a smallest convex subgroup $\tilde{H}_{\mathcal{E}}$ of vL that properly contains $H_{\mathcal{E}}$, and the archimedean quotient $\tilde{H}_{\mathcal{E}}/H_{\mathcal{E}}$ is discrete.

Our goal is to find an element $x \in L$ with $vx \notin vK$ such that

$$\mathcal{O}_L = \bigcup_{c \in K \text{ with } vcx > 0} \mathcal{O}_K[cx] \,. \tag{12}$$

If $c, c' \in K$ with $vc \geq vc'$, then $cx = \frac{c}{c'} c'x \in \mathcal{O}_K[c'x]$, hence $\mathcal{O}_K[cx] \subseteq \mathcal{O}_K[c'x]$.

Theorem

Take an extension $\mathcal{E} = (L|K, v)$ of prime degree $q = (vL : vK)$, with $x_0 \in L$ such that $vx_0 \notin vK$. Then the following assertions hold.

1) If $\mathcal{E}$ is of type (DL2a) or (DL2b), then (12) holds for $x = x_0$.

2) If $\mathcal{E}$ is of type (DL2c), then (12) holds for $x = x_0^j$ with suitable $j \in \{1, \ldots, q-1\}$. If in addition $H_{\mathcal{E}} = \{0\}$, then $\mathcal{O}_L = \mathcal{O}_K[cx]$ for suitable $c \in K$.

The assumption of part 1) always holds when every archimedean component of vK is dense (which is the case for deeply ramified fields).

Defining $\mathcal{O}_\mathcal{E}$ and $\mathcal{M}_\mathcal{E}$

Proposition

The $\mathcal{O}_L$-ideal $\mathcal{M}_\mathcal{E}$ is equal to the $\mathcal{O}_L$-ideal

$$I_x \;:=\; (cx \mid c \in K \text{ with } vcx > 0)\,. \tag{13}$$

Corollary

The set $\{vcx \mid c \in K \text{ with } vcx > 0\}$ is coinitial in $vK^{>0} \setminus H_\mathcal{E}$.

From the proposition we obtain the following $\mathcal{L}_{\mathrm{val},K}$-definition:

$$\begin{aligned}\mathcal{M}_\mathcal{E} \quad := \quad & \{b \in L \mid \exists x \in L \setminus K : \forall y \in K : \; vx \neq vy \\ & \wedge \exists c \in K : \; vb \geq vcx > 0\}\,.\end{aligned}$$

From this, we can define $\mathcal{O}_\mathcal{E}$ by including the units of $\mathcal{O}_\mathcal{E}$:

$$\mathcal{O}_\mathcal{E} \;:=\; \{b \in L \mid \forall x \in \mathcal{M}_\mathcal{E} : \; -vx < vb\}\,.$$

Applications of $\mathcal{O}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ in the defectless case

I will now summarize the results for defectless Galois extensions $\mathcal{E} = (L|K, v)$ which will demonstrate the importance of the ideal $\mathcal{M}_{\mathcal{E}}$. If $[L : K] = q \neq \operatorname{char} K$, then we assume that K contains a q-th root of unity, so that $L|K$ is a Kummer extension. If $[L : K] = p = \operatorname{char} K$, then $L|K$ is an Artin-Schreier extension. If $[L : K] = p = \operatorname{char} Kv$, then we denote by $I_{\mathcal{E}}$ the ramification ideal of $\mathcal{E}$.

The presentation (12) of $\mathcal{O}_L$ as a union over simple ring extensions of $\mathcal{O}_K$ is used to prove our results about $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

Theorem

Take an Artin-Schreier extension $\mathcal{E} = (L|K, v)$ of degree $p = \operatorname{char} Kv = (vL : vK)$. Then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^p \tag{14}$$

as $\mathcal{O}_L$-modules; in particular, $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

The case of a Kummer extension

Let $\mathcal{E} = (L|K, v)$ be a Kummer extension of prime degree q with $(vL : vK) = q$. We distinguish two cases:

i) there is a Kummer generator $\eta \in L$ such that $vL = vK + \mathbb{Z}v\eta$ and (12) holds for $x = \eta$;

ii) there is a Kummer generator $\eta \in L$ which is a 1-unit such that for

$$\xi := \frac{\eta - 1}{\zeta_q - 1}, \tag{15}$$

we have that $v\xi < 0,\ vL = vK + \mathbb{Z}v\xi$ and (12) holds for $x = \xi^j$ with suitable $j \in \{1, \ldots, q-1\}$.

The case of a Kummer extension

In case i),

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \mathcal{M}_{\mathcal{E}}/q\mathcal{M}_{\mathcal{E}}^q \tag{16}$$

as $\mathcal{O}_L$-modules. If $q \neq \operatorname{char} Kv$, then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; \mathcal{M}_{\mathcal{E}}/\mathcal{M}_{\mathcal{E}}^q\,. \tag{17}$$

If $q = \operatorname{char} Kv$, then

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^q\,. \tag{18}$$

We have that $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ if and only if $q \notin \mathcal{M}_{\mathcal{E}}$ and $\mathcal{M}_{\mathcal{E}}$ is a nonprincipal $\mathcal{O}_{\mathcal{E}}$-ideal. The condition $q \notin \mathcal{M}_{\mathcal{E}}$ always holds when $q \neq \operatorname{char} Kv$.

In case ii),

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}}/(I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}})^q \tag{19}$$

and $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$.

The annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$

The annihilator of an $\mathcal{O}_L$-module M is the largest $\mathcal{O}_L$-ideal J such that $JM = 0$; we denote it by $\operatorname{ann} M$.

The proofs of our results about $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ proceed by showing it to be isomorphic to quotients U/UV for certain $\mathcal{O}_L$-ideals U and V. The annihilator of U/UV certainly contains V, but can it be larger? The annihilator of U/UV is the largest ideal J such that $JU \subseteq UV$, commonly denoted by $UV : U$. So how do we compute $UV : U$? This and related questions are answered in

[6] K – Kuhlmann, K.: *Arithmetic of cuts in ordered abelian groups and of ideals over valuation rings*, to appear in Pacific J. Math.; arXiv:2406.10545.

The annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$

Since $\mathcal{E}$ is defectless, $I_{\mathcal{E}}$ is a principal $\mathcal{O}_L$-ideal. Hence we can choose $a \in L$ such that $I_{\mathcal{E}} = a\mathcal{O}_L$ to obtain that $I_{\mathcal{E}}\mathcal{M}_{\mathcal{E}} = a\mathcal{M}_{\mathcal{E}}$. Now we apply Proposition 3.21 of [6]:

Proposition

Let $\mathcal{E}$ be an Artin-Schreier extension or a Kummer extension of degree $p = \operatorname{char} Kv$. Assume that $[L:K] = (vL:vK)$. If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$, then

$$\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \begin{cases} (a\mathcal{M}_{\mathcal{E}})^{p-1} & \text{if } \mathcal{M}_{\mathcal{E}} \text{ is a principal } \mathcal{O}_{\mathcal{E}}\text{-ideal,} \\ (a\mathcal{O}_{\mathcal{E}})^{p-1} = a^{p-1}\mathcal{O}_{\mathcal{E}} & \text{if } \mathcal{M}_{\mathcal{E}} \text{ is a nonprincipal } \mathcal{O}_{\mathcal{E}}\text{-ideal.} \end{cases}$$

Further, $\operatorname{ann}\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ if and only if $p = 2$, $a \notin \mathcal{M}_L$ and $\mathcal{M}_{\mathcal{E}} = \mathcal{M}_L$ is a principal $\mathcal{O}_L$-ideal.

Note that if (DRvg) holds (and in particular, if (K, v) is a deeply ramified field), then the maximal ideals of coarsenings of $\mathcal{O}_L$ are never principal, so $\mathcal{M}_L$ is never the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

The annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$

In the case of a Kummer extension of prime degree $q = (vL : vK) \neq \mathrm{char}\, Kv$, we have $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong \mathcal{M}_\mathcal{E} / \mathcal{M}_\mathcal{E}^q$, and we set $a = 1$. Then we obtain:

Proposition

Let $\mathcal{E}$ be a Kummer extension of degree $q = (vL : vK) \neq \mathrm{char}\, Kv$. If $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \neq 0$, then $\mathcal{M}_\mathcal{E}$ is a principal $\mathcal{O}_\mathcal{E}$-ideal and

$$\mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_\mathcal{E}^{q-1}\,.$$

Further, $\mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{M}_L$ if and only if $q = 2$ and $\mathcal{M}_\mathcal{E} = \mathcal{M}_L$.

The ramification ideal $I_{\mathcal{E}}$

Theorem

1) If $\mathcal{E}$ is an Artin-Schreier extension, then it admits an Artin-Schreier generator ϑ such that $0 \geq v\vartheta \notin vK$. For every such ϑ,

$$I_{\mathcal{E}} = \left(\frac{1}{\vartheta}\right) . \tag{20}$$

2) If $\mathcal{E}$ is a Kummer extension, then $\mathcal{E}$ admits a Kummer generator η such that $0 < v\eta \notin vK$, or a Kummer generator η such that η is a 1-unit with $v(\eta - 1) \leq v(\zeta_p - 1)$. For every such η,

$$I_{\mathcal{E}} = \left(\frac{\zeta_p - 1}{\eta - 1}\right) . \tag{21}$$

Defining the ramification ideal $I_{\mathcal{E}}$

Let us show that under our above assumptions, $I_{\mathcal{E}}$ is always $\mathcal{L}_{\mathrm{val},K}$-definable. If $\mathcal{E}$ is an Artin-Schreier extension, then we can define

$$\begin{aligned} I_{\mathcal{E}} \;:=\; \{\, b\in L \;\;|\;\; & \exists x\in L:\; x^p-x\in K \,\wedge\, vx\leq 0 \\ & \wedge\,(\forall y\in K:\; vx\neq vy)\,\wedge\, vb\geq -vx \,\}\,. \end{aligned}$$

If $\mathcal{E}$ is a Kummer extension, then we can define

$$\begin{aligned} I_{\mathcal{E}} \;:=\; \{\, b\in L \;\;|\;\; & \exists x\in L:\; x^p\in K \,\wedge\, vp\geq (p-1)v(x-1) \\ & \wedge\,(\forall y\in K:\; 0<vx\neq vy \,\vee\, 0<v(x-1)\neq vy) \\ & \wedge\,(p-1)vb\geq vp-(p-1)v(x-1)\,\}\,. \end{aligned}$$

Take an ordered abelian group Γ and denote by $\mathcal{C}$ the chain of its proper convex subgroups. Let p be any prime.

If H is a convex subgroup of Γ that is largest among all convex subgroups that do not contain a given element $\gamma \in \Gamma$, then we call it a subprincipal convex subgroup. A subprincipal convex subgroup may or may not be principal.

Prescribing associated convex subgroups

Theorem

Take any subset $\mathcal{C}_0^{\mathrm{sp}} \subseteq \mathcal{C}$ containing only subprincipal convex subgroups of Γ. Further, take any subset $\mathcal{C}_0^{\cap} \subseteq \mathcal{C}$ containing only convex subgroups of Γ that are intersections of countably infinite descending chains of elements in $\mathcal{C} \setminus \mathcal{C}_0^{\mathrm{sp}}$.

a) There exists a henselian deeply ramified field of characteristic p for which the convex subgroups associated with Galois defect extensions of prime degree are exactly the elements of $\mathcal{C}_0^{\mathrm{sp}} \cup \mathcal{C}_0^{\mathrm{sp}}$.

b) Assume in addition that Γ has a largest proper convex subgroup. Then there exists a henselian deeply ramified field of characteristic 0 and residue characteristic p for which the associated convex subgroups are exactly the elements of $\mathcal{C}_0^{\mathrm{sp}} \cup \mathcal{C}_0^{\mathrm{sp}}$.

THE END

Thank you for your attention!

More detailed information

Preprints and further information:

https://www.valth.eu/Valth.html.

My new personal homepage:

https://www.fvkuhlmann.de/