Ball Spaces: a generic approach to measuring the strength of completeness/compactness of various types of spaces and ordered structures

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Banach's Fixed Point Theorem

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In joint work with P. Ribenboim, Prieß-Crampe later extended this theorem to the case of generalized ultrametric spaces which have partially ordered value sets Γ.

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To every ultrametric space (X, u) we can associate a ball space $(X, \mathcal{B})$ by taking $\mathcal{B}$ to be the collection of all closed ultrametric balls. Then (X, u) is spherically complete if and only if $(X, \mathcal{B})$ is.

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The ultrametric FPT is a direct consequence of the second part of this theorem.

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Ordered sets, abelian groups and fields

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Does the spherical completeness of $(X, \mathcal{B})$ imply cut completeness? Then it would not be interesting for ordered abelian groups and fields since all cut complete densely ordered abelian groups and ordered fields are isomorphic to $\mathbb{R}$.

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The cut (D, E) is called asymmetric if the cofinality of D is not equal to the coinitiality of E. By what we have seen above, nests of intervals $[a_\nu, b_\nu]$ over asymmetric cuts will always have nonempty intersection.

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Theorem

$(X, \mathcal{B})$ is spherically complete if and only if X is symmetrically complete.

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- (2004) S. Shelah introduced the notion of "symmetrically complete ordered field" and proved that every ordered field can be extended to a symmetrically complete ordered field.
- (2013) In joint work with S. Shelah we extended his result to ordered sets and abelian groups, characterized all symmetrically complete ordered abelian groups and fields, and proved an analogue of the Banach FPT. (Israel J. Math. **208** (2015))

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A poset is chain complete if every chain in $(T, <)$ has a least upper bound.

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1) The ball space $(T, \mathcal{B})$ *is spherically complete if and only if* $(T, <)$ *is inductively ordered. If this is the case, then the intersection of every nest in* $(T, \mathcal{B})$ *contains a ball.*

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Metric spaces

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Is there an alternative for our choice of balls in the metric case?

The Caristi-Kirk FPT

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Take a complete metric space (X, d) and a lower semicontinuous function $\varphi : X \to \mathbb{R}$ which is bounded from below.

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then f has a fixed point on X.

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If φ is lower semicontinuous and bounded from below, then we will call $(X, \mathcal{B}_\varphi)$ a Caristi-Kirk ball space associated with (X, d).

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There is a generic FPT for ball spaces which together with this theorem proves the Caristi-Kirk FPT.

What we got so far

spaces	balls	completeness property
ultrametric spaces	all closed ultrametric balls	spherically complete
topological spaces	all nonempty closed sets	compact
linearly ordered sets, ordered abelian groups and fields	all intervals $[a, b]$ with $a \leq b$	symmetrically complete
posets	all intervals $[a, \infty)$	inductively ordered
metric spaces	metric balls with radii in suitable sets of positive real numbers Caristi-Kirk balls *	complete

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Moreover, observe that in a topological space arbitrary intersections of closed sets are again closed. What about ball spaces in which all (nonempty) intersections of nests, directed systems, or centered systems, are again balls?

Hierarchy of spherical completeness

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We will also write $\mathbf{S}^*$ for $\mathbf{S}_5^c$ because this turns out to be the "star" (the strongest) among the ball spaces:

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Summary of $\mathbf{S}^*$ ball spaces

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- A poset is a complete lattice if and only if it has a bottom and a top element and the ball space defined by its nonempty closed intervals $[a, b]$ is S^* .

Spherical closure in **S*** ball spaces

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$\mathrm{scl}_{\mathcal{B}}(S) \in \mathcal{B}$ *is the smallest ball containing* S.

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Proposition

The associated ball space of $\mathbb{R}$ is $\mathbf{S}^*$.

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Assume that $(T, <)$ is a totally ordered set whose associated ball space is $\mathbf{S}_1^d$. Then $(T, <)$ is cut complete.

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Let $\mathrm{Fix}(f)$ be the set of fixed points of f, and set

$$\mathcal{B}^f_{\mathrm{Fix}} := \{B \cap \mathrm{Fix}(f) \mid B \in \mathcal{B}^f\}.$$

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From this theorem one can derive a Tychonoff theorem for generalized ultrametric spaces.

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The answer is yes, as we will now see, so the Tychonoff theorem can indeed be deduced from its ball spaces analogue.

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

Lemma

If $\mathcal{S}$ is a maximal centered system of balls in $\mathcal{B}'$ (that is, no subset of $\mathcal{B}'$ properly containing $\mathcal{S}$ is a centered system),

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From the previous two theorems we obtain:

Theorem

Take a ball space $(X, \mathcal{B})$. If $\mathcal{B}'$ is obtained from $\mathcal{B}$ by first closing under finite unions and then under arbitrary nonempty intersections, then:

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c) if $(X, \mathcal{B})$ is $\mathbf{S}_1^c$, then $(X, \mathcal{B}')$ is an $\mathbf{S}^*$ ball space.

The topology associated to a ball space

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Theorem

This topology associated to $\mathcal{B}$ is compact if and only if $(X, \mathcal{B})$ is an $\mathbf{S}_1^c$ ball space.

Ball continuous functions

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Theorem

Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$, and a function $f : X \to X'$. If f is ball continuous and $(X, \mathcal{B})$ is spherically complete, then so is $(X', \mathcal{B}')$.

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Theorem

Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$, and a function $f : X \to X'$. If f is ball continuous and $(X, \mathcal{B})$ is spherically complete, then so is $(X', \mathcal{B}')$. If f is ball closed and finite-to-one,

Ball continuous functions

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Theorem

Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$, and a function $f : X \to X'$. If f is ball continuous and $(X, \mathcal{B})$ is spherically complete, then so is $(X', \mathcal{B}')$. If f is ball closed and finite-to-one, and if $(X', \mathcal{B}')$ is spherically complete, Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ and a function $f : X \to X'$. We will call f ball continuous if the preimage of every ball in $\mathcal{B}'$ is a ball in $\mathcal{B}$, and ball closed if the image of every ball in $\mathcal{B}$ is a ball in $\mathcal{B}'$.

Theorem

Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$, and a function $f : X \to X'$. If f is ball continuous and $(X, \mathcal{B})$ is spherically complete, then so is $(X', \mathcal{B}')$. If f is ball closed and finite-to-one, and if $(X', \mathcal{B}')$ is spherically complete, then so is $(X, \mathcal{B})$.

Categories of ball spaces

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The categories of spherically complete ball spaces, of $\mathbf{S}_2$ *ball spaces,*

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The end

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Thank you for your attention — and stay tuned for further developments!