SYMMETRICALLY COMPLETE ORDERED SETS, ABELIAN GROUPS AND FIELDS

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Abstract. We characterize linearly ordered sets, abelian groups and fields that are *symmetrically complete*, meaning that the intersection over any chain of closed bounded intervals is nonempty. Such ordered abelian groups and fields are important because generalizations of Banach's Fixed Point Theorem hold in them. We prove that symmetrically complete ordered abelian groups and fields are divisible Hahn products and real closed power series fields, respectively. This gives us a direct route to the construction of symmetrically complete ordered abelian groups and fields, modulo an analogous construction at the level of ordered sets; in particular, this gives an alternative approach to the construction of symmetrically complete fields in [12].

1. Introduction

In the paper [12], the third author introduced the notion of "symmetrically complete" ordered fields and proved that every ordered field can be extended to a symmetrically complete ordered field (see Remark 2 below for some background information). He also proved that an ordered field K is symmetrically complete if and only if every nonempty chain of closed bounded intervals in K (ordered by inclusion) has nonempty intersection. It is this property that is particularly interesting as it allows fixed point theorems to be proved for such fields that generalize Banach's Fixed Point Theorem, replacing the usual metric of the reals by the distance function that is derived from the ordering. For theorems of this type it suffices to consider the ordered additive group underlying the field. So we are led in a natural way to working with ordered abelian groups. As an example for a corresponding fixed point theorem, we cite the following result from [3], where it is Theorem 21:

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Theorem 1. *Take an ordered abelian group $(G,<)$ and a function $f : G \to G$. Assume that every nonempty chain of closed bounded intervals in G has nonempty intersection and that f has the following properties:*

1) f is nonexpanding: $|fx - fy| \le |x - y|$ for all $x, y \in G$,
2) f is contracting on orbits: there is a positive rational number $\frac{m}{n} < 1$ with $m, n \in \mathbb{N}$ such that $n|fx - f^2x| \le m|x - fx|$ for all $x \in G$.

Then f has a fixed point.

In [3] we used the notion "spherically complete w.r.t. the order balls" if the condition on chains of intervals is met. This is because the notion of spherical completeness can provide a general framework for fixed point theorems in many applications; the respective theorems are then obtained simply by choosing the right balls for every application (see [3], [4]). In this paper, however, we will use the equivalent, but more elegant notion of "symmetrically complete" which we will now define. But we will consider spherical completeness later again, then with respect to ultrametric balls when we work with the natural valuations of ordered abelian groups and fields.

A **cut** in a linearly ordered set I is a pair

$$C = (D, E)$$

with a **lower cut set** D and an **upper cut set** E if $I = D \cup E$ and $d < e$ for all $d \in D$, $e \in E$. Throughout this paper, when we talk of cuts we will mean **Dedekind cuts**, that is, cuts with D and E nonempty. The cut C is **principal** (also called **realized**) if either D has a maximal element or E has a minimal element.

By the **cofinality** of the cut C we mean the pair (κ, λ) where κ is the cofinality of D, denoted by $\mathrm{cf}(D)$, and λ is the **coinitiality** of E, denoted by $\mathrm{ci}(E)$. Recall that the coinitiality of a linearly ordered set is the cofinality of this set under the reversed ordering. Recall further that cofinalities and coinitialities of ordered sets are regular cardinals.

The cut C is called **symmetric** if $\kappa = \lambda$, and **asymmetric** otherwise. We will call a linearly ordered set $(I,<)$ **symmetrically complete** if every symmetric cut C in I is principal. Note that the principal symmetric cuts are precisely the cuts with cofinality $(1,1)$. Therefore, in dense linear orderings (and hence in ordered fields) there are no principal symmetric cuts. Consequently, a dense linear ordering is symmetrically complete if and only if all of its cuts are asymmetric. This shows that our definition is a generalization of the definition given for fields in [12].

For example, $\mathbb{Z}$ and $\mathbb{R}$ are symmetrically complete, but $\mathbb{Q}$ is not. In $\mathbb{Z}$ and $\mathbb{R}$, every cut is principal; in $\mathbb{Z}$ all of them have cofinality $(1,1)$, and in $\mathbb{R}$ they have cofinalities $(1,\aleph_0)$ and $(\aleph_0,1)$. In contrast, in $\mathbb{Q}$ the cuts have cofinalities $(1,\aleph_0)$, $(\aleph_0,1)$ and $(\aleph_0,\aleph_0)$.

Remark 2. In [11] the third author proved that any nonstandard model of Peano Arithmetic has a symmetric cut. The motivation was to answer a question on the ideal structure of countable ultraproducts of $\mathbb{Z}$ posed in [9]. In these rings, each prime ideal lies below a unique maximal ideal, and the set of prime ideals below a given maximal ideal is linearly ordered under inclusion. The existence result for symmetric cuts proves that below each maximal ideal there is a prime ideal which is neither a union nor an intersection of countably many principal ideals.

The existence of arbitrarily large symmetrically complete real closed fields proved in [12] and again in the present paper stresses that Peano Arithmetic and the theory of real closed fields are opposite in their behaviour when it comes to the cofinalities of their cuts.

The following characterization of symmetrical completeness will be proved in Section 2:

Proposition 3. *A linearly ordered set I is symmetrically complete if and only if every nonempty chain of nonempty closed bounded intervals in I has nonempty intersection.*

It may come as a surprise to the reader that symmetrically complete fields, other than the reals themselves, do exist. Intuitively, one may believe at first that one can "zoom in" on every cut by a decreasing chain of closed intervals, showing that every symmetrically complete field is cut complete and therefore isomorphic to the reals. But if the cut is asymmetric, then on the side with the smaller cardinality, the endpoints of the intervals must become stationary, so that the intersection over the intervals will contain the stationary endpoint.

The main aim of this paper is to characterize the symmetrically complete ordered abelian groups and fields. This characterization will then be very useful for their construction.

We will make essential use of the fact that ordered abelian groups appear as the value groups of ordered fields w.r.t. their natural field valuations, and, one level lower, linear orderings appear as the value sets of ordered abelian groups w.r.t. their natural group valuations. We will now define natural valuations of ordered abelian groups and fields.

Take an ordered abelian group $(G,<)$. Two elements $a,b\in G$ are called **archimedean equivalent** if there is some $n\in\mathbb{N}$ such that $n|a|\geq|b|$ and $n|b|\geq|a|$. The ordered abelian group $(G,<)$ is archimedean ordered if all nonzero elements are archimedean equivalent. If $0\leq a<b$ and $na<b$ for all $n\in\mathbb{N}$, then we say that "a is infinitesimally smaller than b" and we will write $a\ll b$. We denote by va the archimedean equivalence class of a. The set of archimedean equivalence classes can be ordered by setting $va>vb$ if and only if $|a|<|b|$ and a and b are not archimedean equivalent, that is, if $n|a|<|b|$ for all $n\in\mathbb{N}$. We write $\infty:=v0$; this is the maximal element in the linearly ordered set of equivalence classes. The function $a \mapsto va$ is a group valuation on G, i.e., it satisfies $va = \infty \Leftrightarrow a = 0$ and the ultrametric triangle law

(UT) $v(a-b) \geq \min\{va, vb\}$,

and by definition,

$$0 \leq a \leq b \implies va \geq vb \, .$$

The set $vG := \{vg \mid 0 \neq g \in G\}$ is called the **value set** of the valued abelian group (G, v). For every $\gamma \in vG$, the quotient $\mathcal{C}_\gamma := \mathcal{O}_\gamma / \mathcal{M}_\gamma$, where $\mathcal{O}_\gamma := \{g \in G \mid vg \geq \gamma\}$ and $\mathcal{M}_\gamma := \{g \in G \mid vg > \gamma\}$, is an archimedean ordered abelian group (hence embeddable in the ordered additive group of the reals, by the Theorem of Hölder); it is called an **archimedean component** of G. The natural valuation induces an ultrametric given by $u(a,b) := v(a-b)$.

We define the smallest **ultrametric ball** $B_u(a,b)$ containing the elements a and b to be

$$B_u(a,b) := \{g \mid v(a-g) \geq v(a-b)\} = \{g \mid v(b-g) \geq v(a-b)\}$$

where the last equation holds because in an ultrametric ball, every element is a center. For the basic facts on ultrametric spaces, see [6]. Note that all ultrametric balls are cosets of convex subgroups in G (see [7]). We say that an ordered abelian group (or an ordered field) is **spherically complete w.r.t. its natural valuation** if every nonempty chain of ultrametric balls (ordered by inclusion) has nonempty intersection. The ordered abelian groups that are spherically complete w.r.t. their natural valuation are precisely the Hahn products; see Section 2.2 for the definition and basic properties of Hahn products (cf. also [7] or [8]).

If $(K, <)$ is an ordered field, then we consider the natural valuation on its ordered additive group and define $va + vb := v(ab)$. This turns the set of archimedean classes into an ordered abelian group, with neutral element $0 := v1$ and inverses $-va = v(a^{-1})$. In this way, v becomes a field valuation (with additively written value group). It is the finest valuation on the field K which is compatible with the ordering. The residue field, denoted by Kv, is archimedean ordered, hence by the version of the Theorem of Hölder for ordered fields, it can be embedded in the ordered field $\mathbb{R}$. Via this embedding, we will always identify it with a subfield of $\mathbb{R}$.

Remark 4. In contrast to the notation for the natural valuation (in the Baer tradition) that we used in [3], we use here the Krull notation because it is more compatible with our constructions in Section 5. In this notation, two elements in an (additively written) ordered abelian group or field are close to each other when the value of their difference is large.

In [3], we have already proved that if an ordered abelian group $(G, <)$ is spherically complete w.r.t. the order balls, then it is spherically complete w.r.t. its natural valuation v. Moreover, if G is an ordered field, then we proved that in addition, it has residue field $\mathbb{R}$. Using Proposition 3, we can reformulate these results as follows:

Proposition 5. *If an ordered abelian group is symmetrically complete, then it is spherically complete w.r.t. its natural valuation. If an ordered field is symmetrically complete, then it is spherically complete w.r.t. its natural valuation v and has residue field $Kv = \mathbb{R}$.*

In the present paper, we wish to extend these results. It turns out that for an ordered abelian group G to be symmetrically complete, the same must be true for the value set vG, and in fact, it must have an even stronger property. We will call a cut with cofinality (κ, λ) in a linearly ordered set $(I, <)$ **strongly asymmetric** if $\kappa \neq \lambda$ and at least one of κ, λ is uncountable. We call $(I, <)$ **strongly symmetrically complete** if every cut in I has cofinality $(1, 1)$ or is strongly asymmetric, and we call it **extremely symmetrically complete** if in addition, the coinitiality and cofinality of I are both uncountable. Note that I is strongly symmetrically complete if and only if it is symmetrically complete and does not admit cuts of cofinality $(1, \aleph_0)$ or $(\aleph_0, 1)$. The reals are not strongly symmetrically complete.

In Section 4, we will prove the following results:

Theorem 6. *A nontrivial densely ordered abelian group $(G, <)$ is symmetrically complete if and only if it is spherically complete w.r.t. its natural valuation v, has a dense strongly symmetrically complete value set vG, and all archimedean components $\mathcal{C}_\gamma$ are isomorphic to $\mathbb{R}$. It is strongly symmetrically complete if and only if in addition, vG has uncountable cofinality, and it is extremely symmetrically complete if and only if in addition, vG is extremely symmetrically complete.*

Now we turn to ordered fields.

Theorem 7. *An ordered field K is symmetrically complete if and only if it is spherically complete w.r.t. its natural valuation v, has residue field $\mathbb{R}$ and a dense strongly symmetrically complete value group vK. Further, the following are equivalent:*

a) K is strongly symmetrically complete,

b) K is extremely symmetrically complete,

c) K is spherically complete w.r.t. its natural valuation v, has residue field $\mathbb{R}$ and a dense extremely symmetrically complete value group vK.

Note that the natural valuation of a symmetrically complete ordered field K can be trivial, in which case K is isomorphic to $\mathbb{R}$. But if K is strongly symmetrically complete, then vK and hence also v must be nontrivial.

Every ordered field that is spherically complete w.r.t. its natural valuation is maximal, in the sense of [2]. In that paper Kaplansky showed that under certain conditions, which in particular hold when the residue field has characteristic 0, every such field is isomorphic to a power series field. In general, a nontrivial factor system is needed on the power series field, but it is not needed for instance when the residue field is $\mathbb{R}$. From the previous two theorems, we derive:

Corollary 8. *Every dense symmetrically complete ordered abelian group is divisible and isomorphic to a Hahn product. Every symmetrically complete ordered field is real closed and isomorphic to a power series field with residue field $\mathbb{R}$ and divisible value group.*

These results give us a natural way to construct symmetrically complete and extremely symmetrically complete ordered fields K, which is an alternative to the construction given in [12]. For the former type of fields, construct a strongly symmetrically complete linearly ordered set I with uncountable coinitiality. Then take G to be the Hahn product with index set I and all archimedean components equal to $\mathbb{R}$. Finally, take $K = \mathbb{R}((G))$, the power series field with coefficients in $\mathbb{R}$ and exponents in G. To obtain an extremely symmetrically complete ordered field K, construct I such that in addition, also its cofinality is uncountable. See Section 5 for the detailed construction of such orders I.

In Section 6, we will use our theorems to prove the following result, which extends the corresponding result of [12] by a more direct method:

Theorem 9. *Every ordered abelian group can be extended to an extremely symmetrically complete ordered abelian group. Every ordered field can be extended to an extremely symmetrically complete ordered field.*

For the proof of this theorem, we need to extend any given ordered set I to an extremely symmetrically complete ordered set J. We do this by constructing suitable lexicographic products of ordered sets.

Remark 10. Already in the years 1906-8 Hausdorff has constructed ordered sets with prescribed cofinalities for all of its cuts, see [1]. (This paper was brought to our attention by Salma Kuhlmann after the completion of the present paper.) However, he did not discuss how to embed arbitrary orders I in such ordered sets. Moreover, the constructions we present in Section 5 leed more directly to the ordered sets we need.

Let us describe the most refined result that we achieve, which gives us the best control of the cofinalities of cuts in the constructed ordered set J. We denote by Reg the class of all infinite regular cardinals, and for any ordinal λ, by

$$\mathrm{Reg}_{<\lambda} \;=\; \{\kappa < \lambda \mid \aleph_0 \leq \kappa = \mathrm{cf}(\kappa)\}$$

the set of all infinite regular cardinals $< \lambda$. We define:

$$\begin{aligned}\mathrm{Coin}(I) \;&:=\; \{\mathrm{ci}(S) \mid S \subseteq I \text{ such that } \mathrm{ci}(S) \text{ is infinite}\} \,\subset\, \mathrm{Reg}\,,\\ \mathrm{Cofin}(I) \;&:=\; \{\mathrm{cf}(S) \mid S \subseteq I \text{ such that } \mathrm{cf}(S) \text{ is infinite}\} \,\subset\, \mathrm{Reg}\,.\end{aligned}$$

We choose any $\mu, \kappa_0, \lambda_0 \in \mathrm{Reg}$. Then we set

$$\begin{aligned} R_{\mathrm{left}} &:= \mathrm{Cofin}(I) \cup \mathrm{Reg}_{<\kappa_0} \cup \mathrm{Reg}_{<\mu} \subset \mathrm{Reg}\,, \\ R_{\mathrm{right}} &:= \mathrm{Coin}(I) \cup \mathrm{Reg}_{<\lambda_0} \cup \mathrm{Reg}_{<\mu} \subset \mathrm{Reg}\,. \end{aligned}$$

All of the subsets we have defined here are initial segments of Reg in the sense that if they contain κ, then they also contain every infinite regular cardinal $< \kappa$.

Further, we assume that functions

$$\varphi_{\mathrm{left}} : \{1\} \cup \mathrm{Reg} \to \mathrm{Reg} \ \text{ and } \ \varphi_{\mathrm{right}} : \{1\} \cup \mathrm{Reg} \to \mathrm{Reg}$$

are given. We prove in Section 5:

Theorem 11. *Assume that μ is uncountable and that*

$$(1) \qquad \varphi_{\mathrm{left}}(\{1\} \cup R_{\mathrm{right}}) \subset R_{\mathrm{left}} \ \text{ and } \ \varphi_{\mathrm{right}}(\{1\} \cup R_{\mathrm{left}}) \subset R_{\mathrm{right}}$$

with $\varphi_{\mathrm{left}}(\kappa) \neq \kappa \neq \varphi_{\mathrm{right}}(\kappa)$ for all $\kappa \in R_{\mathrm{left}} \cup R_{\mathrm{right}}$. Then I can be extended to a strongly symmetrically complete ordered set J of cofinality κ_0 and coinitiality λ_0, in which the cuts have the following cofinalities:

$$\{(1,\mu),(\mu,1)\} \cup \{(\kappa,\varphi(\kappa)) \mid \kappa \in R_{\mathrm{left}}\} \cup \{(\varphi(\lambda),\lambda) \mid \lambda \in R_{\mathrm{right}}\}\,.$$

If in addition κ_0 and λ_0 are uncountable, then J is extremely symmetrically complete.

Among the value groups of valued fields, not only the dense, but also the discretely ordered groups play an important role. The value groups of formally p-adic fields are discretely ordered, and the value groups of p-adically closed fields are $\mathbb{Z}$-groups, that is, ordered abelian groups G that admit (an isomorphic image of) $\mathbb{Z}$ as a convex subgroup such that $G/\mathbb{Z}$ is divisible. We wish to prove a version of Theorem 6 for discretely ordered abelian groups.

Theorem 12. *Take a nontrivial discretely ordered abelian group $(G,<)$. Then the following are equivalent:*

a) $(G,<)$ is symmetrically complete,

b) $(G,<)$ is strongly symmetrically complete,

c) $(G,<)$ is a $\mathbb{Z}$-group such that $G/\mathbb{Z}$ is strongly symmetrically complete.

Further, $(G,<)$ is extremely symmetrically complete if and only if $(G,<)$ is a $\mathbb{Z}$-group and $G/\mathbb{Z}$ is extremely symmetrically complete.

Again, this gives us a natural way for our construction. To obtain a symmetrically complete discretely ordered abelian group G, construct a divisible strongly symmetrically complete ordered abelian group H and then take the lexicographic product $H \times \mathbb{Z}$. If in addition the cofinality of H is uncountable, then G will even be extremely symmetrically complete.

Note that G is isomorphic to a Hahn product if and only if $G/\mathbb{Z}$ is. Therefore, if G is symmetrically complete, then it is a Hahn product.

2. Preliminaries and notations

2.1. **Proof of Proposition 3.** A **quasicut** in a linearly ordered set I is a pair $C = (D, E)$ of subsets D and E of I such that $I = D \cup E$ and $d \leq e$ for all $d \in D$, $e \in E$. In this case, $D \cap E$ is empty or a singleton; if it is empty, then (D, E) is a cut.

Assume that I is symmetrically complete, so every symmetric cut in I is principal. Every nonempty chain of nonempty closed bounded intervals has a cofinal subchain $([d_\nu, e_\nu])_{\nu<\mu}$ indexed by a regular cardinal μ, either equal to 1 or infinite. We set $D := \{d \in I \mid d \leq d_\nu \text{ for some } \nu < \mu\}$ and $E := \{e \in I \mid e \geq e_\nu \text{ for some } \nu < \mu\}$. Then $d \leq e$ for all $d \in D$ and $e \in E$. If $D \cap E \neq \emptyset$, then (D, E) is a quasicut and the unique element of $D \cap E$ lies in the intersection of the chain. If (D, E) is a cut, then it must be principal by our assumption on I since $\mathrm{cf}(D) = \mu = \mathrm{ci}(E)$. That is, $\mu = 1$ and $\{d_0, e_0\} = [d_0, e_0]$ is contained in the intersection of the chain. If $D \cap E = \emptyset$ but (D, E) is not a cut, then the set $\{c \in I \mid d < c < e \text{ for all } d \in D,\ e \in E\}$ is nonempty and contained in the intersection of the chain. So in all cases, the intersection of the chain is nonempty.

Now assume that every nonempty chain of nonempty closed bounded intervals in I has nonempty intersection. Suppose that (D, E) is a cut with $\kappa := \mathrm{cf}(D) = \mathrm{ci}(E)$. Then we can choose a cofinal strictly increasing sequence $(d_\nu)_{\nu<\kappa}$ in D and a coinitial strictly decreasing sequence $(e_\nu)_{\nu<\kappa}$ in E. By assumption, the chain $([d_\nu\,,\, e_\nu])_{\nu<\kappa}$ of intervals has nonempty intersection. Take an element a in the intersection. Then $d_\nu \leq a \leq e_\nu$ for all ν. If $a \in D$, then the former inequalities imply that $\kappa = 1$ and $d_1 = a$ is the largest element of D. If $a \in E$, then the latter inequalities imply that $\kappa = 1$ and $e_1 = a$ is the smallest element of E. So we find that (D, E) is principal. This proves that every symmetric cut in I is principal.

2.2. **Hahn products.** Given a linearly ordered index set I and for every $\gamma \in I$ an arbitrary abelian group C_γ, we define a group called the **Hahn product**, denoted by $\mathbf{H}_{\gamma\in I}\, C_\gamma$. Consider the product $\prod_{\gamma\in I} C_\gamma$ and an element $c = (c_\gamma)_{\gamma\in I}$ of this group. Then the **support** of c is the set $\mathrm{supp}\, c := \{\gamma \in I \mid c_\gamma \neq 0\}$. As a set, the Hahn product is the subset of $\prod_{\gamma\in I} C_\gamma$ containing all elements whose support is a wellordered subset of I, that is, every nonempty subset of the support has a minimal element. In particular, the support of every nonzero element c in the Hahn product has a minimal element γ_0, which enables us to define a group valuation by setting $vc = \gamma_0$ and $v0 = \infty$. The Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two wellordered sets is again wellordered.

We leave it to the reader to show that a Hahn product is divisible if and only if all of its components are.

If the components C_γ are (not necessarily archimedean) ordered abelian groups, we obtain the **ordered Hahn product**, also called **lexicographic product**, where the ordering is defined as follows. Given a nonzero element $c = (c_\gamma)_{\gamma\in I}$, let γ_0 be the minimal element of its support. Then we take $c > 0$ if and only if $c_{\gamma_0} > 0$. If all C_γ are archimedean ordered, then the valuation v of the Hahn product coincides with the natural valuation of the ordered Hahn product. Every ordered abelian group G can be embedded in the Hahn product with its set of archimedean classes as index sets and its archimedean components as components. Then G is spherically complete w.r.t. the ultrametric balls if and only if the embedding is onto.

2.3. **Some facts about cofinalities and coinitialities.** Take a nontrivial ordered abelian group G and define

$$G^{>0} := \{g \in G \mid g > 0\} \quad \text{and} \quad G^{<0} := \{g \in G \mid g < 0\}\,.$$

Since $G \ni g \mapsto -g \in G$ is an order inverting bijection,

$$\mathrm{ci}(G) = \mathrm{cf}(G) \quad \text{and} \quad \mathrm{cf}(G^{<0}) = \mathrm{ci}(G^{>0})\,.$$

Further, we have:

Lemma 13. *1) The cofinality of G is equal to* $\max\{\aleph_0, \mathrm{ci}(vG)\}$. *Hence it is uncountable if and only if the coinitiality of vG is uncountable.*

2) If G is discretely ordered, then $\mathrm{ci}(G^{>0}) = \mathrm{cf}(vG) = 1$. *Otherwise,* $\mathrm{ci}(G^{>0}) = \max\{\aleph_0, \mathrm{cf}(vG)\}$.

3) Take $\gamma \in vG$, not the largest element of vG, and let κ be the coinitiality of the set $\{\delta \in vG \mid \delta > \gamma\}$. Then $\mathrm{cf}(\mathcal{M}_\gamma) = \max\{\aleph_0, \kappa\}$.

Proof: 1): Since a nontrivial ordered abelian group has no maximal element, its cofinality is at least $\aleph_0$. If vG has a smallest element, then take a positive $g \in G$ whose value is this smallest element. Then the sequence $(ng)_{n\in\mathbb{N}}$ is cofinal in G, so its cofinality is $\aleph_0$.

If $\kappa := \mathrm{ci}(vG)$ is infinite, then take a sequence $(\gamma_\nu)_{\nu<\kappa}$ which is coinitial in vG, and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu<\kappa}$ is cofinal in G and therefore, $\mathrm{cf}(G) \leq \mathrm{ci}(vG)$. On the other hand, for every sequence $(g_\nu)_{\nu<\lambda}$ cofinal in G, the sequence of values $(vg_\nu)_{\nu<\lambda}$ must be coinitial in vG, which shows that $\mathrm{cf}(G) \geq \mathrm{ci}(vG)$.

2) If G is discretely ordered, then it has a smallest positive element g and hence, $\mathrm{ci}(G^{>0}) = 1$. Further, vg must be the largest element of vG, so $\mathrm{cf}(vG) = 1$.

If G is not discretely ordered, then it is densely ordered and the coinitiality of $G^{>0}$ is at least $\aleph_0$. If vG has a largest element γ, then we take a positive $g \in G$ with $vg = \gamma$. Then $\mathcal{M}_\gamma = \{0\}$ and $\mathcal{O}_\gamma$ is an archimedean ordered convex subgroup of G. This implies that $\aleph_0 \leq \mathrm{ci}(G^{>0}) = \mathrm{ci}(\mathcal{O}_\gamma^{>0}) \leq \aleph_0$, and we have equality everywhere.

If $\kappa := \mathrm{cf}(vG)$ is infinite, then take a sequence $(\gamma_\nu)_{\nu<\kappa}$ which is cofinal in vG, and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu<\kappa}$ is coinitial in $G^{>0}$ and therefore, $\mathrm{ci}(G^{>0}) \leq \mathrm{cf}(vG)$. On the other hand, for every sequence $(g_\nu)_{\nu<\lambda}$ coinitial in $G^{>0}$, the sequence of values $(vg_\nu)_{\nu<\lambda}$ must be coinitial in vG, which shows that $\mathrm{ci}(G^{>0}) \geq \mathrm{cf}(vG)$.

3): By our condition on γ, $\mathcal{M}_\gamma$ is a nontrivial subgroup of G and therefore, its cofinality is at least $\aleph_0$. If $v\mathcal{M}_\gamma = \{\delta \in vG \mid \delta > \gamma\}$ has a smallest element, then take a positive $g \in G$ whose value is this smallest element. Then the sequence $(ng)_{n\in\mathbb{N}}$ is cofinal in $\mathcal{M}_\gamma$, so $\mathrm{cf}(\mathcal{M}_\gamma) = \aleph_0$.

Assume that $\kappa = \mathrm{ci}(v\mathcal{M}_\gamma)$ is infinite. Take a sequence $(\gamma_\nu)_{\nu<\kappa}$ which is coinitial in $v\mathcal{M}_\gamma = \{\delta \in vG \mid \delta > \gamma\}$ and take positive elements $g_\nu \in G$, $\nu < \kappa$, with $vg_\nu = \gamma_\nu$. Then the sequence $(g_\nu)_{\nu<\kappa}$ is cofinal in $\mathcal{M}_\gamma$ and therefore, $\mathrm{cf}(\mathcal{M}_\gamma) \leq \kappa$. On the other hand, for every sequence $(g_\nu)_{\nu<\lambda}$ cofinal in $\mathcal{M}_\gamma$, the sequence of values $(vg_\nu)_{\nu<\lambda}$ must be coinitial in $v\mathcal{M}_\gamma$, which shows that $\mathrm{cf}(\mathcal{M}_\gamma) \geq \kappa$. □

3. Analysis of cuts in ordered abelian groups

We will first discuss principal cuts.

Lemma 14. *Take any ordered abelian group G. Every principal cut in G is asymmetric if and only it G is densely ordered. Every principal cut in G is strongly asymmetric if and only if* $\mathrm{cf}(vG)$ *is uncountable.*

Proof: Take a principal cut $C = (D, E)$ with cofinality (κ, λ) in the ordered abelian group G. If D has largest element g, then the set $g + G^{>0}$ is coinitial in E, showing that C has cofinality $(1, \mathrm{ci}(G^{>0}))$. Symmetrically, if E has smallest element g, then the set $g + G^{<0}$ is cofinal in D, showing that C has cofinality $(\mathrm{ci}(G^{>0}), 1)$ since $\mathrm{cf}(G^{<0}) = \mathrm{ci}(G^{>0})$. This immediately proves the second statement.

We will now apply part 2) of Lemma 13 repeatedly. If G is densely ordered, then $\mathrm{ci}(G^{>0}) > 1$ and consequently, C is asymmetric. If G is not densely ordered, then it is discretely ordered and admits symmetric principal cuts; in fact, every principal cut has cofinality $(1, 1)$. This proves the first statement.

If $\mathrm{cf}(vG)$ is uncountable, then $\mathrm{ci}(G^{>0}) = \mathrm{cf}(vG)$ and consequently, C is strongly asymmetric. If $\mathrm{cf}(vG) \leq \aleph_0$, then $\mathrm{ci}(G^{>0}) \leq \aleph_0$ and no principal cut is strongly asymmetric. This proves the second statement. □

From now on we will discuss nonprincipal cuts. We start with a simple but useful observation. The only countable cardinality that can appear as coinitiality or cofinality in a nonprincipal cut is $\aleph_0$. This shows:

Lemma 15. *If a nonprincipal cut is asymmetric, then it is strongly asymmetric.*

We will now classify cuts by considering the ultrametric balls $B_u(d,e)$ for all $d \in D$, $e \in E$. Any two of them have nonempty intersection since this intersection will contain both a final segment of D and an initial segment of E. Since two ultrametric balls with nonempty intersection are already comparable by inclusion, it follows that these balls form a nonempty chain. Now there are two cases:
(I) the chain contains a smallest ball,
(II) the chain does not contain a smallest ball.

First, we discuss cuts of type (I).

Lemma 16. *Take any nontrivial ordered abelian group G. Then every nonprincipal cut of type (I) is (strongly) asymmetric if and only if the following conditions are satisfied:*
a) $\mathcal{C}_\gamma \simeq \mathbb{R}$ for all $\gamma \in vG$, or
$\mathcal{C}_\gamma \simeq \mathbb{Z}$ if γ is the largest element of vG and $\mathcal{C}_\gamma \simeq \mathbb{R}$ otherwise.
b) for every cut in vG of cofinality $(1,\lambda)$, λ is uncountable.

Proof: Take a nonprincipal cut $C=(D,E)$ of type (I) in G. We have to start our proof with some preparations.

We choose $d_0 \in D$, $e_0 \in E$ such that $B_u(d_0,e_0)$ is the smallest ball. The shifted cut

$$C - d_0 \;:=\; (\{d-d_0 \mid d\in D\}\,,\,\{e-d_0 \mid e\in E\})$$

has the same cofinality as C. Moreover,

$$B_u(d_0,e_0)-d_0 \;:=\; \{b-d_0 \mid b\in B_u(d_0,e_0)\} \;=\; B_u(0,e_0-d_0)$$

remains the smallest ball in the new situation. Therefore, we can assume that $d_0=0$. Set $\gamma := ve_0$ and $I:=[0,e_0]$. Then $vh \geq \gamma$ for all $h\in I$, that is, $h\in\mathcal{O}_\gamma$. The images D' of $D\cap I$ and E' of $E\cap I$ in $\mathcal{C}_\gamma=\mathcal{O}_\gamma/\mathcal{M}_\gamma$ are convex and satisfy $D'\leq E'$. If there were $d'\in D'\cap E'$, then it would be the image of elements $d\in D\cap I$ and $e\in E\cap I$ with $\gamma<v(e-d)$, and $B_u(d,e)$ would be a ball properly contained in $B_u(0,e_0)$, contrary to our minimality assumption. Hence, $D'<E'$. If there were an element strictly between D' and E', then it would be the image of an element h strictly between D and E, which is impossible. So we see that (D',E') defines a cut C' in $\mathcal{C}_\gamma$, with D' a final segment of the left cut set and E' an initial segment of the right cut set.

Since $\mathcal{C}_\gamma$ is archimedean ordered, it can be embedded in $\mathbb{R}$ and therefore, the cofinality of C' can only be $(1,1)$, $(1,\aleph_0)$, $(\aleph_0,1)$, or $(\aleph_0,\aleph_0)$. Lifting cofinal sequences in D' back into D, we see that if the cofinality of D' is $\aleph_0$, then so is the cofinality of D. Similarly, if the coinitiality of E' is $\aleph_0$, then so is the coinitiality of E. However, if D' contains a last element a', and if $a \in D \cap I$ is such that a has image a' in $\mathcal{C}_\gamma$, then the set of all elements in G that are sent to a' is exactly the coset $a + \mathcal{M}_\gamma$. This set has empty intersection with E since $a' \notin E'$. This together with a' being the last element of D' shows that $a + \mathcal{M}_\gamma$ is a final segment of D and therefore, the cofinality of D is equal to that of $\mathcal{M}_\gamma$. Similarly, if E' has a first element b' coming from an element $b \in E \cap I$, then $b + \mathcal{M}_\gamma$ is an initial segment of E and therefore, the coinitiality of E is equal to that of $\mathcal{M}_\gamma$, which in turn is equal to the cofinality of $\mathcal{M}_\gamma$. We see that the cofinality of C is

α) $(\mathrm{cf}(\mathcal{M}_\gamma), \mathrm{cf}(\mathcal{M}_\gamma))$ if C' has cofinality $(1,1)$,
β) $(\mathrm{cf}(\mathcal{M}_\gamma), \aleph_0)$ or $(\aleph_0, \mathrm{cf}(\mathcal{M}_\gamma))$ if C' has cofinality $(1, \aleph_0)$ or $(\aleph_0, 1)$, and
γ) $(\aleph_0, \aleph_0)$ if C' has cofinality $(\aleph_0, \aleph_0)$.

If γ is not the last element of vG, then by part 3) of Lemma 13, $\mathrm{cf}(\mathcal{M}_\gamma) = \max\{\aleph_0, \lambda\}$ where λ is the coinitiality of the set $\{\delta \in vG \mid \delta > \gamma\}$.

Assume first that conditions a) and b) of the lemma are satisfied. Then by condition a), C' cannot have cofinality $(\aleph_0, \aleph_0)$, so case γ) cannot appear. Further, C' can have cofinality $(1,1)$ only if γ is the largest element of vG. Hence in this case, which is case α), we have that $\mathcal{M}_\gamma = \{0\}$ and thus, $\mathrm{cf}(\mathcal{M}_\gamma) = 1$. But as we have taken C to be nonprincipal, this case cannot appear.

Finally, if we are in case β) then γ cannot be the largest element of vG since otherwise, $\mathcal{M}_\gamma = \{0\}$ and as in case α), C would be principal. So $\mathrm{cf}(\mathcal{M}_\gamma) = \lambda$ is uncountable by condition b), which yields that C is strongly asymmetric. We have now shown that conditions a) and b) together imply that every nonprincipal cut of type (I) is strongly asymmetric and hence also asymmetric.

We will prove the converse by contraposition. We will have to lift, for any $\gamma \in vG$, a given cut $C'_1 = (D'_1, E'_1)$ in $\mathcal{C}_\gamma$ to a cut in G. We set

$$\begin{aligned} D &:= \{d \in G \mid d \leq d_1 \text{ for some } d_1 \in \mathcal{O}_\gamma \text{ with } d_1 + \mathcal{M}_\gamma \in D'_1\}\,, \\ E &:= \{e \in G \mid e \geq e_1 \text{ for some } e_1 \in \mathcal{O}_\gamma \text{ with } e_1 + \mathcal{M}_\gamma \in E'_1\}\,. \end{aligned}$$

This defines a cut $C = (D, E)$ in G which is of type (I) since $\gamma = \min\{v(e - d) \mid d \in D,\, e \in E\}$. With the notation as above, we obtain that $D' = D'_1$ and $E' = E'_1$.

Suppose that condition a) is violated. Then there is $\gamma \in vG$ with $\mathcal{C}_\gamma$ not isomorphic to $\mathbb{R}$, and also not to $\mathbb{Z}$ if γ is the largest element of vG. If $\mathcal{C}_\gamma$ is neither isomorphic to $\mathbb{R}$ nor to $\mathbb{Z}$, then there is a cut C'_1 in $\mathcal{C}_\gamma$ of cofinality $(\aleph_0, \aleph_0)$. So we are in case γ) and C is nonprincipal and symmetric. If $\mathcal{C}_\gamma$ is isomorphic to $\mathbb{Z}$ with γ not the largest element of vG, then we are in case α) with $\mathrm{cf}(\mathcal{M}_\gamma) = \max\{\aleph_0, \lambda\} \geq \aleph_0$, which yields that C is nonprincipal and symmetric.

Suppose that condition b) is violated. Then there is $\gamma \in vG$ which is not the largest element of vG such that the coinitiality λ of the set $\{\delta \in vG \mid \delta > \gamma\}$ is countable. We then obtain that $\mathrm{cf}(\mathcal{M}_\gamma) = \max\{\aleph_0, \lambda\} = \aleph_0$.

Since $\mathcal{C}_\gamma$ always admits cuts of cofinality $(1,1)$, $(1,\aleph_0)$ or $(\aleph_0,1)$, case α) or β) will appear, leading to a nonprincipal symmetric cut C with cofinality $(\aleph_0,\aleph_0)$.

In all cases where conditions a) or b) are violated, we have obtained a nonprincipal symmetric cut of type (I). This completes our proof. $\square$

Now we discuss nonprincipal cuts C of type (II).

Lemma 17. *Take any ordered abelian group G which is spherically complete w.r.t. its natural valuation v. Then every nonprincipal cut of type (II) is (strongly) asymmetric if and only if every cut in vG of cofinality (κ,λ) with κ infinite is strongly asymmetric.*

Proof: Take a nonprincipal cut $C=(D,E)$ of type (II) in G. Since G is assumed to be spherically complete w.r.t. its natural valuation v, there is some $g\in G$ such that

$$g\in \bigcap_{d\in D,\, e\in E} B_u(d,e)\;.$$

Replacing the cut C by the shifted cut $C-g$ as we have done in the last proof, we can assume that $g=0$. Since C is nonprincipal by asssumption, there must be $d_0\in D$, $e_0\in E$ such that $d_0\leq 0\leq e_0$ does not hold, and we have two cases:

A) $e_0<0\,$,

B) $0<d_0$.

Again, we set $I:=[d_0,e_0]$. We set $\tilde{D}=\{vd\mid d\in D\cap I\}\subseteq vG$ and $\tilde{E}=\{ve\mid e\in E\cap I\}\subseteq vG$.

Let us first discuss case A). We claim that $\tilde{D}<\tilde{E}$. We observe that "$\leq$" holds since $d<e<0$ for $d\in D\cap I$ and $e\in E\cap I$. Suppose that $\tilde{D}\cap\tilde{E}\neq\emptyset$, that is, $vd=ve$ for some $d\in D\cap I$ and $e\in E\cap I$. Then $v(e-d)\geq vd$ by the ultrametric triangle law, and since there is no smallest ball by assumption, we can even choose d,e such that $v(e-d)>vd$. But then, 0 would not lie in $B_u(d,e)$, a contradiction. We have proved our claim. Now if there were an element α stricly between the two sets, then there would be some $a\in I$ with $va=\alpha$ and $a<0$. This would yield that $d<a<e$ for all $d\in D\cap I$ and $e\in E\cap I$ and thus, $D<a<E$, a contradiction.

We conclude that $(\tilde{D},\tilde{E})$ defines a cut $\tilde{C}$ in vG, with $\tilde{D}$ a final segment of the left cut set, and $\tilde{E}$ an initial segment of the right cut set. Denote by $(\tilde{\kappa},\tilde{\lambda})$ its cofinality. We have that $vd<ve$ and consequently $vd=v(e-d)$ for all $d\in D\cap I$ and $e\in E\cap I$. Since by assumption there is no smallest ball, there is no largest value $v(e-d)$. This shows that $\tilde{D}$ has no largest element and therefore, $\tilde{\kappa}$ is infinite. Lifting cofinal sequences in $\tilde{D}$ to coinitial sequences in D, we see that $\kappa=\tilde{\kappa}$. By the same argument, if $\tilde{\lambda}$ is infinite, then $\lambda=\tilde{\lambda}$. If on the other hand $\tilde{\lambda}=1$, then we take $\gamma\in G$ to be the

smallest element of $\tilde{E}$. The preimage of γ under the valuation is $\mathcal{O}_\gamma \setminus \mathcal{M}_\gamma$, and this set is coinitial in E. The cofinality of $\mathcal{O}_\gamma \setminus \mathcal{M}_\gamma$ is equal to the cofinality of $\mathcal{O}_\gamma$, which in turn is equal to the cofinality $\aleph_0$ of the archimedean ordered group $\mathcal{C}_\gamma$. Hence in this case, $\lambda = \aleph_0$. In all cases, $\lambda = \max\{\aleph_0, \tilde{\lambda})$.

If the nonprincipal cut C is asymmetric, then $\kappa \neq \lambda$ and one of them is uncountable. If $\tilde{\lambda}$ is countable, then $\lambda = \aleph_0$ and $\tilde{\kappa} = \kappa > \lambda \geq \tilde{\lambda}$, that is, $\tilde{C}$ is strongly asymmetric. If $\tilde{\lambda}$ is uncountable, then $\tilde{\lambda} = \lambda \neq \kappa = \tilde{\kappa}$, hence again, $\tilde{C}$ is strongly asymmetric. Conversely, suppose that $\tilde{C}$ is strongly asymmetric. If $\tilde{\lambda}$ is countable, then $\tilde{\kappa}$ is uncountable, so $\kappa = \tilde{\kappa} > \aleph_0 = \lambda$, showing that C is asymmetric. If $\tilde{\lambda}$ is uncountable, then $\lambda = \tilde{\lambda} \neq \tilde{\kappa} = \kappa$, hence again, C is asymmetric. We have now proved that C is asymmetric if and only if $\tilde{C}$ is strongly asymmetric.

Now we consider case B). Since $0 < d < e$ for $d \in D \cap I$ and $e \in E \cap I$, we now obtain that $\tilde{E} \leq \tilde{D}$. It is proven as in case A) that $(\tilde{E}, \tilde{D})$ defines a cut $\tilde{C}$ in vG, and that $\tilde{E}$ has no largest element. In this case the argument is the same as before, but with $\tilde{D}$ and $\tilde{E}$ interchanged, and the conclusion is the same as in case A). We note that in both cases, the cofinality of the left cut set of $\tilde{C}$ must be infinite.

Putting both cases together, we have now proved that if every cut in vG of cofinality $(\tilde{\kappa}, \tilde{\lambda})$ with $\tilde{\kappa}$ infinite is strongly asymmetric, then every nonprincipal cut of type (II) in G is asymmetric.

Let us prove the converse. If we have a cut $\tilde{C} = (\tilde{D}, \tilde{E})$ in vG of cofinality $(\tilde{\kappa}, \tilde{\lambda})$ with $\tilde{\kappa}$ infinite, then we can associate to it a nonprincipal cut of type (II) as follows. We set $D = \{d \in G \mid d < 0 \text{ and } vd \in \tilde{D}\}$ and $E = \{e \in G \mid e > 0 \text{ or } ve \in \tilde{E}\}$. This is a cut in G, and it is nonprincipal: $\tilde{D}$ and hence also D has no largest element, and E has no smallest element because for every $e \in E$ with $e < 0$ we have $2e < e$ with $v(2e) = ve$. Further, it is of type (II) since for all $d \in D$ and $e \in E$, $e < 0$, we have that $vd < ve$ and hence $v(e - d) = vd \in \tilde{D}$, which has no largest element. Now C induces the cut $\tilde{C}$ in the way described under case A). From our previous discussion we see that if $\tilde{C}$ is not strongly asymmetric, then C is not asymmetric. This completes the proof of the converse and of our lemma. □

4. Proofs of the main theorems

Proof of Theorem 6:
Take any densely ordered abelian group G. Assume first that G is symmetrically complete. Then by Proposition 5, G is spherically complete w.r.t. its natural valuation. G cannot have an archimedean component $\mathcal{C}_\gamma \simeq \mathbb{Z}$ with γ the largest element of vG because otherwise, it would have a convex subgroup isomorphic to $\mathbb{Z}$ and would then be discretely ordered. Hence by

Lemma 16, every archimedean component of G is isomorphic to $\mathbb{R}$ and for every cut in vG of cofinality $(1,\kappa)$, κ is uncountable. Finally by Lemma 17, every cut in vG of cofinality (λ,κ) with λ infinite is strongly asymmetric. Altogether, every cut in vG is strongly asymmetric. This proves that vG is dense and strongly symmetrically complete.

Conversely, if G is spherically complete w.r.t. its natural valuation, every archimedean component of G is isomorphic to $\mathbb{R}$ and vG is dense and strongly symmetrically complete, then it follows from Lemmas 14, 16 and 17 that G is symmetrically complete. This proves the first assertion of the theorem.

We have already remarked in the introduction that for a symmetrically complete ordered gtoup G to be strongly symmetrically complete it suffices that every principal cut is strongly asymmetric. By Lemma 14, this holds if and only if in addition to the other conditions, the cofinality of vG is uncountable. This proves the second assertion.

Finally, a strongly symmetrically complete ordered group G is extremely symmetrically complete if and only if in addition, its cofinality (which is equal to its coinitiality) is uncountable. By part 1) of Lemma 13, this holds if and only if the coinitiality of vG is uncountable. Hence by what we have just proved before, a symmetrically complete G is extremely symmetrically complete if and only if in addition to the other conditions, vG is extremely symmetrically complete. $\square$

Proof of Theorem 7:
Considering the additive ordered abelian group of the ordered field K, which is always dense, the first assertion of Theorem 7 follows readily from that of Theorem 6 if one takes into account that through multiplication, all archimedean components are isomorphic to the ordered additive group of the residue field.

Similarly, the equivalence of b) and c) follows from the third case of Theorem 6. Since vK is an ordered abelian group, its cofinality is equal to its coinitiality, so the condition that it is strongly symmetrically complete with uncountable cofinality already implies that it is extremely symmetrically complete. Hence, by the second case of Theorem 6, a) is equivalent to c). $\square$

Proof of Corollary 8:
The assertion for ordered abelian groups follows from the facts that have been mentioned before. For ordered fields, it remains to show that a power series field with residue field $\mathbb{R}$ and divisible value group is real closed. Since every power series field is henselian under its canonical valuation, this follows from [10, Theorem (8.6)]. $\square$

Proof of Theorem 12:

Since G is discretely ordered, vG must have a largest element vg (where g can be chosen to be the smallest positive element of G) with archimedean component $\mathcal{O}_{vg} \simeq \mathcal{C}_{vg} \simeq \mathbb{Z}$. We identify the convex subgroup $\mathcal{O}_{vg}$ with $\mathbb{Z}$.

Take any cut (D,E) in G. Since the canonical epimorphism $G \to G/\mathbb{Z}$ preserves $\leq$, the image $(\overline{D},\overline{E})$ of (D,E) in $G/\mathbb{Z}$ is a quasicut. If $\overline{D}$ and $\overline{E}$ have a common element $\overline{d}$, then there is $d \in D$ and $z \in \mathbb{Z}$ such that $d+z \in E$. In this case, the cofinality of (D,E) is $(1,1)$. Now suppose that $\overline{D}$ and $\overline{E}$ have no common element. Then for all $d \in D$ and $e \in E$, we have that $d+\mathbb{Z} = \{d+z \mid d \in D,\, z \in \mathbb{Z}\} \subset D$ and $e+\mathbb{Z} \subset E$. Hence if $D' \subset D$ is a set of representatives for $\overline{D}$ and $E' \subset E$ is a set of representatives for $\overline{E}$, then $D = D'+\mathbb{Z} = \{d+z \mid d \in D',\, z \in \mathbb{Z}\}$ and $E = E'+\mathbb{Z}$. This yields that

$$\text{(2)} \qquad \left\{ \begin{array}{lclcl} \mathrm{cf}(D) & = & \max\{\mathrm{cf}(D'),\aleph_0\} & = & \max\{\mathrm{cf}(\overline{D}),\aleph_0\}\,, \\ \mathrm{ci}(E) & = & \max\{\mathrm{ci}(E'),\aleph_0\} & = & \max\{\mathrm{ci}(\overline{E}),\aleph_0\}\,. \end{array} \right.$$

We assume first that G is a $\mathbb{Z}$-group and $G/\mathbb{Z}$ is strongly symmetrically complete. We take a cut (D,E) in G of cofinality $\neq (1,1)$. Then by what we have just shown, $\overline{D}$ and $\overline{E}$ have no common element, and $(\mathrm{cf}(D),\mathrm{ci}(E)) = (\max\{\mathrm{cf}(\overline{D}),\aleph_0\},\max\{\mathrm{ci}(\overline{E}),\aleph_0\})$. By our assumption on $G/\mathbb{Z}$, $(\overline{D},\overline{E})$ is strongly asymmetric, which yields that $\mathrm{cf}(D)$ and $\mathrm{ci}(E)$ are not equal and at least one of them is uncountable. This proves that G is strongly symmetrically complete, hence also symmetrically complete.

For the converse, we assume that G is symmetrically complete. We take any cut $(\overline{D},\overline{E})$ in $G/\mathbb{Z}$. Then we pick a set $D' \subset G$ of representatives for $\overline{D}$ and a set $E' \subset E$ of representatives for $\overline{E}$. With $D = D'+\mathbb{Z}$ and $E = E'+\mathbb{Z}$ we obtain a nonprincipal cut (D,E) in G with image $(\overline{D},\overline{E})$ in $G/\mathbb{Z}$. By our assumption on G, the cut (D,E) is asymmetric. Now (2) yields that at least one of $\mathrm{cf}(\overline{D})$ and $\mathrm{ci}(\overline{E})$ is uncountable and that if both are, then they are not equal. This shows that $(\overline{D},\overline{E})$ is strongly asymmetric, which proves that $G/\mathbb{Z}$ is strongly symmetrically complete and does not admit any cuts of cofinality $(1,1)$. Hence, $G/\mathbb{Z}$ is densely ordered, and Corollary 8 now shows that it is divisible. This proves that G is a $\mathbb{Z}$-group.

The last equivalence in the theorem is seen as follows. If G is extremely symmetrically complete, then it cannot be isomorphic to $\mathbb{Z}$ and hence, $G/\mathbb{Z}$ is nontrivial. But then, the cofinality of G is equal to that of $G/\mathbb{Z}$. $\square$

5. Construction of symmetrically complete linearly ordered sets

In this section, we will use "+" in a different way than before. If I and J are ordered sets, then $I+J$ denotes the sum of I and J in the sense of order theory, that is, the disjoint union $I \,\dot\cup\, J$ with the extension of the orderings of I and J given by $i<j$ for all $i \in I$, $j \in J$.

For any linearly ordered set $I = (I, <)$, we denote by I^c its completion. Note that $\mathrm{Coin}(I^c) = \mathrm{Coin}(I)$ and $\mathrm{Cofin}(I^c) = \mathrm{Cofin}(I)$. Further, we denote by I^* the set I endowed with the inverted ordering $<^*$, where $i <^* j \Leftrightarrow j < i$.

We choose some ordered set I (where $I = \emptyset$ is allowed) and infinite regular cardinals μ and κ_ν , λ_ν for all $\nu < \mu$. We define

$$I_0 \;:=\; \lambda_0^* + I^c + \kappa_0 \quad \text{and} \quad I_\nu := \lambda_\nu^* + \kappa_\nu \text{ for } 0 < \nu < \mu.$$

Note that all I_ν , $\nu < \mu$, are cut complete. Note further that if C is a cut in I_0 with cofinality (κ, λ), then $\kappa \in \mathrm{Cofin}(I) \cup \mathrm{Reg}_{<\kappa_0}$ and $\lambda \in \mathrm{Coin}(I) \cup \mathrm{Reg}_{<\lambda_0}$.

We define J to be the lexicographic product over the I_ν with index set μ; that is, J is the set of all sequences $(\alpha_\nu)_{\nu<\mu}$ with $\alpha_\nu \in I_\nu$ for all $\nu < \mu$, endowed with the following ordering: if $(\alpha_\nu)_{\nu<\mu}$ and $(\beta_\nu)_{\nu<\mu}$ are two different sequences, then there is a smallest $\nu_0 < \mu$ such that $\alpha_{\nu_0} \neq \beta_{\nu_0}$ and we set $(\alpha_\nu)_{\nu<\mu} < (\beta_\nu)_{\nu<\mu}$ if $\alpha_{\nu_0} < \beta_{\nu_0}$.

Theorem 18. *The cofinalities of the cuts of J are:*
$(1, \mu)$, $(\mu, 1)$,
(κ_1, λ) , (κ, λ_1) *for* $\lambda \in \mathrm{Coin}(I) \cup \mathrm{Reg}_{<\lambda_0}$, $\kappa \in \mathrm{Cofin}(I) \cup \mathrm{Reg}_{<\kappa_0}$,
$(\kappa_{\nu+1}, \lambda)$, $(\kappa, \lambda_{\nu+1})$ *for* $0 < \nu < \mu$ *and* $\kappa < \kappa_\nu$, $\lambda < \lambda_\nu$ *regular cardinals,*
$(\kappa_\nu, \lambda_\nu)$ *for* $\nu < \mu$ *a successor ordinal, and*
(κ_ν, μ') , (μ', λ_ν) *for* $\nu < \mu$ *a limit ordinal and* $\mu' < \mu$ *its cofinality.*
Further, the cofinality of J is κ_0 and its coinitiality is λ_0 .

Proof: Take any cut (D, E) in J. Assume first that D has a maximal element $(\alpha_\nu)_{\nu<\mu}$. By our choice of the linearly ordered sets I_ν we can choose, for every $\nu < \mu$, some $\beta_\nu \in I_\nu$ such that $\beta_\nu > \alpha_\nu$. For $\rho < \mu$ we define $\beta^\rho_\rho := \beta_\rho$ and $\beta^\rho_\nu := \alpha_\nu$ for $\nu \neq \rho$. Then the elements $(\beta^\rho_\nu)_{\nu<\mu}$, $\rho < \mu$, form a strictly decreasing coinitial sequence of elements in E. Since μ was chosen to be regular, this shows that the cofinality of (D, E) is $(1, \mu)$. Similarly, it is shown that if E has a minimal element, then the cofinality of (D, E) is $(\mu, 1)$.

Now assume that (D, E) is nonprincipal. Take S to be the set of all $\nu' < \mu$ for which there exist $a_{\nu'} = (\alpha_{\nu',\nu})_{\nu<\mu} \in D$ and $b_{\nu'} = (\beta_{\nu',\nu})_{\nu<\mu} \in E$ such that $\alpha_{\nu',\nu} = \beta_{\nu',\nu}$ for all $\nu \leq \nu'$. Note that S is a proper initial segment of the set μ. We claim that $\nu_1 < \nu_2 \in S$ implies that

$$\alpha_{\nu_1,\nu} \;=\; \alpha_{\nu_2,\nu} \quad \text{for all } \nu \leq \nu_1 \,,$$

or in other words, $(\alpha_{\nu_1,\nu})_{\nu\leq\nu_1}$ is a truncation of a_{ν_2} . Indeed, suppose that this were not the case. Then there would be some $\nu_0 < \nu_1$ such that

$$\beta_{\nu_1,\nu_0} = \alpha_{\nu_1,\nu_0} \neq \alpha_{\nu_2,\nu_0} = \beta_{\nu_2,\nu_0} \,.$$

Suppose that ν_0 is minimal with this property and that the left hand side is smaller. But then, $(\beta_{\nu_1,\nu})_{\nu<\mu} < (\alpha_{\nu_2,\nu})_{\nu<\mu}$, so $b_{\nu_1} \in D$, a contradiction. A similar contradiction is obtained if the right hand side is smaller.

Now take μ_0 to be the minimum of $\mu \setminus S$; in fact, S is is equal to the set μ_0. We define

$$\begin{aligned} D_{\mu_0} &:= \{\alpha \in I_{\mu_0} \mid \exists\, (\alpha_\nu)_{\nu<\mu} \in D :\ \alpha_{\mu_0} = \alpha \text{ and } \alpha_\nu = \alpha_{\nu,\nu} \text{ for } \nu < \mu_0\}\,, \\ E_{\mu_0} &:= \{\beta \in I_{\mu_0} \mid \exists (\beta_\nu)_{\nu<\mu} \in E :\ \beta_{\mu_0} = \beta \text{ and } \beta_\nu = \alpha_{\nu,\nu} \text{ for } \nu < \mu_0\}\,. \end{aligned}$$

By our definition of μ_0, these two sets are disjoint, and it is clear that their union is I_{μ_0} and every element in D_{μ_0} is smaller than every element in E_{μ_0}. However, one of the sets may be empty, and we will first consider this case. Suppose that $E_{\mu_0} = \emptyset$. Then $D_{\mu_0} = I_{\mu_0}$ and since this has no last element, the cofinality of D is the same as that of I_{μ_0}, which is κ_{μ_0}. In order to determine the coinitiality of E, we proceed as in the beginning of this proof. Observe that since $E_{\mu_0} = \emptyset$, for an element $(\beta_\nu^\rho)_{\nu<\mu}$ to lie in E it is necessary that $\beta_\nu^\rho > \alpha_{\nu,\nu}$ for some $\nu < \mu_0$. For all $\nu < \mu_0$, we choose some $\beta_\nu \in I_\nu$ such that $\beta_\nu > \alpha_{\nu,\nu}$; then for all $\rho < \mu_0$ we define $\beta_\rho^\rho := \beta_\rho$, $\beta_\nu^\rho := \alpha_{\nu,\nu}$ for $\nu < \rho$, and choose β_ν^ρ arbitrarily for $\rho < \nu < \mu$. Then the elements $(\beta_\nu^\rho)_{\nu<\mu}$, $\rho < \mu_0$, form a strictly decreasing coinitial sequence in E. If μ' denotes the cofinality of μ_0, this shows that the coinitiality of E is μ', and the cofinality of (D,E) is (κ_{μ_0}, μ'). Since μ was chosen to be regular, we have that $\mu' < \mu$.

Similarly, it is shown that if $D_{\mu_0} = \emptyset$, then the cofinality of (D,E) is (μ', λ_{μ_0}) for some regular cardinal $\mu' < \mu$. Note that D_{μ_0} or E_{μ_0} can only be empty if μ_0 is a limit ordinal. Indeed, if $\mu_0 = 0$ and $(\alpha_\nu)_{\nu<\mu} \in D$, $(\beta_\nu)_{\nu<\mu} \in E$, then $\alpha_0 \in D_0$ and $\beta_0 \in E_0$; if $\mu_0 = \mu' + 1$, then with $(\alpha_{\mu',\nu})_{\nu<\mu} \in D$ and $(\beta_{\mu',\nu})_{\nu<\mu} \in E$ chosen as before, it follows that $\alpha_{\mu',\mu_0} \in D_{\mu_0}$ and $\beta_{\mu',\mu_0} \in E_{\mu_0}$.

From now on we assume that both D_{μ_0} and E_{μ_0} are nonempty. Since I_{μ_0} is complete, D_{μ_0} has a maximal element or E_{μ_0} has a minimal element.

Suppose that D_{μ_0} has a maximal element $\tilde{\alpha}$. Then for all $\rho \in \kappa_{\mu_0+1} \subset I_{\mu_0+1}$, we define $\alpha_\nu^\rho = \alpha_{\nu,\nu}$ for $\nu < \mu_0$, $\alpha_{\mu_0}^\rho = \tilde{\alpha}$, $\alpha_{\mu_0+1}^\rho = \rho$, and choose an arbitrary element of I_ν for α_ν^ρ when $\mu_0 + 1 < \nu < \mu$. Then the elements $(\alpha_\nu^\rho)_{\nu<\mu}$, $\rho \in \kappa_{\mu_0+1}$, form a strictly increasing cofinal sequence in D. Since κ_{μ_0+1} was chosen to be a regular cardinal, this shows that the cofinality of D is κ_{μ_0+1}.

Suppose that E_{μ_0} has a minimal element $\tilde{\beta}$. Then for every $\sigma \in \lambda^*_{\mu_0+1} \subset I_{\mu_0+1}$, we define $\beta_\nu^\sigma = \alpha_{\nu,\nu}$ for $\nu \le \mu_0$, $\beta_{\mu_0}^\sigma = \tilde{\beta}$, $\beta_{\mu_0+1}^\sigma = \sigma$, and choose an arbitrary element of I_ν for β_ν^σ when $\mu_0 + 1 < \nu < \mu$. Then the elements $(\beta_\nu^\sigma)_{\nu<\mu}$, $\sigma \in \lambda^*_{\mu_0+1}$, form a strictly decreasing coinitial sequence in E. Since λ_{μ_0+1} was chosen to be a regular cardinal, this shows that the coinitiality of E is λ_{μ_0+1}.

If D_{μ_0} has a maximal element and E_{μ_0} has a minimal element, then we obtain that the cofinality of (D,E) is $(\kappa_{\mu_0+1}, \lambda_{\mu_0+1})$.

Now we deal with the case where D_{μ_0} does not have a maximal element. Since I_{μ_0} is complete, E_{μ_0} must then have a smallest element, and by what we have already shown, we find that E has coinitiality λ_{μ_0+1}. Denote the cofinality of D_{μ_0} by κ. We choose a sequence of elements $\alpha^\rho_{\mu_0}$, $\rho<\kappa$, cofinal in D_{μ_0}. For all $\rho<\kappa$, we define $\alpha^\rho_\nu=\alpha_{\nu,\nu}$ for $\nu<\mu_0$ and choose an arbitrary element of I_ν for α^ρ_ν when $\mu_0+1<\nu<\mu$. Then the elements $(\alpha^\rho_\nu)_{\nu<\mu}$, $\rho<\kappa$, form a strictly increasing cofinal sequence in D. Hence, (D,E) has cofinality $(\kappa,\lambda_{\mu_0+1})$ with κ the cofinality of a lower cut set in I_{μ_0}, i.e., $\kappa\in\mathrm{Cofin}(I)\cup\mathrm{Reg}_{<\kappa_0}$ if $\mu_0=0$, and $\kappa\in\mathrm{Reg}_{<\kappa_{\mu_0}}$ otherwise.

If E_{μ_0} does not have a minimal element, then a symmetrical argument shows that the cofinality of (D,E) is $(\kappa_{\mu_0+1},\lambda)$ for some λ the coinitiality of an upper cut set in I_{μ_0}, i.e., $\lambda\in\mathrm{Coin}(I)\cup\mathrm{Reg}_{<\lambda_0}$ if $\mu_0=0$, and $\lambda\in\mathrm{Reg}_{<\lambda_{\mu_0}}$ otherwise.

We have now proved that the cofinalities of the cuts in J are all among those listed in the statement of the theorem. By our arguments it is also clear that all listed cofinalities do indeed appear.

Finally, the easy proof of the last statement of the theorem is left to the reader. $\square$

The following result is an immediate consequence of the theorem:

Corollary 19. *Assume that*

a) $\kappa_1\notin\mathrm{Coin}(I)\cup\mathrm{Reg}_{<\lambda_0}$ *and* $\lambda_1\notin\mathrm{Cofin}(I)\cup\mathrm{Reg}_{<\kappa_0}$*,*
b) $\kappa_{\nu+1}\geq\lambda_\nu$ *and* $\lambda_{\nu+1}\geq\kappa_\nu$ *for all* $\nu<\mu$*,*
c) $\kappa_\nu\neq\lambda_\nu$ *for* $\nu<\mu$ *a successor ordinal, and*
d) $\kappa_\nu\geq\mu$ *and* $\lambda_\nu\geq\mu$ *for* $\nu<\mu$ *a limit ordinal.*

Then J is symmetrically complete. If in addition μ is uncountable, then J is strongly symmetrically complete, and if also κ_0 and λ_0 are uncountable, then J is extremely symmetrically complete.

It is easy to choose our cardinals by transfinite induction in such a way that all conditions of this corollary are satisfied. We choose

- κ_0 and λ_0 to be arbitrary uncountable regular cardinals,
- $\mu>\max\{\kappa_0,\lambda_0,\mathrm{card}(I)\}$,
- $\kappa_\nu=\mu$ and $\lambda_\nu=\mu^+$ for $\nu=1$ or $\nu<\mu$ a limit ordinal,
- $\kappa_{\nu+1}=\kappa_\nu^{++}$ and $\lambda_{\nu+1}=\lambda_\nu^{++}$ for $0<\nu<\mu$.

Sending an element $\alpha\in I$ to an arbitrary element $(\alpha_\nu)_{\nu<\mu}\in J$ with $\alpha_0=\alpha$ induces an order preserving embedding of I in J. So we obtain the following result:

Corollary 20. *Every linearly ordered set I can be embedded in an extremely symmetrically complete ordered set J.*

Our above construction can be seen as a "brute force" approach. We will now present a construction that offers more choice for the prescribed cofinalities.

If an index set I is not well ordered, then the lexicographic product of ordered abelian groups G_i, $i \in I$, is defined to be the subset of the product consisting of all elements $(g_i)_{i\in I}$ with well ordered support $\{i \in I \mid g_i \neq 0\}$. Likewise, the lexicographic sum is defined to be the subset consisting of all elements $(g_i)_{i\in I}$ with finite support $\{i \in I \mid g_i \neq 0\}$. The problem with ordered sets is that they ususally do not have distinguished elements (like neutral elements for an operation). The remedy used in [5] is to fix distinguished elements in all linear orderings we wish to use for our lexicographic sum. Hausdorff ([1]) does this in quite an elegant way: he observes that the full product is still partially ordered. Singling out one element in the product then determines the distinguished elements in the ordered sets (being the corresponding components of the chosen element), and in this manner one obtains an associated maximal linearly ordered subset of the full product.

While the index sets we use here are ordinals and hence well ordered, which makes a condition on the support unnecessary for the work with lexicographic products, we will use the idea (as apparent in the definition of the lexicographic sum) that certain elements can be singled out by means of their support.

We choose infinite regular cardinals μ, κ_0 and λ_0. Further, we denote by On the class of all ordinals and set

$$I_0 \;:=\; \lambda_0^* + I^c + \kappa_0 \quad \text{and} \quad I_\nu := \mathrm{On}^* + \mu + \{\underline{0}\} + \mu^* + \mathrm{On} \quad \text{for } 0 < \nu < \mu,$$

assuming that $\underline{0}$ does not appear in I^c or any ordinal or reversed ordinal. Note that On can be replaced by a large enough cardinal; its minimal size depends on the choice of I, μ, κ_0 and λ_0. But the details are not essential for our construction, so we skip them.

We define J° to consist of all elements of the lexicographic product over the I_ν with index set μ whose support

$$\mathrm{supp}\,(\alpha_\nu)_{\nu<\mu} \;=\; \{\nu \mid \nu < \mu \text{ and } \alpha_\nu \neq \underline{0}\}$$

is an initial segment of μ (i.e., an ordinal $\leq \mu$).

A further refinement of our construction uses the idea to define suitable subsets of J° by restricting the choice of the coefficient α_ν in dependence on the truncated sequence $(\alpha_\rho)_{\rho<\nu}$.

For every $\nu < \mu$ we consider the following set of truncations:

$$J_\nu^\circ \;:=\; \{(\alpha_\rho)_{\rho\leq\nu} \mid (\alpha_\rho)_{\rho<\mu} \in J^\circ\}\,.$$

By induction on $\nu < \mu$ we define subsets

$$J_\nu \;\subset\; J_\nu^\circ$$

as follows:

(J1) $J_0 := I_0$.

(J2) If $\nu > 0$ and $J_{\nu'}$ for all $\nu' < \nu$ are already constructed, then we first define the auxiliary set

$$J_{<\nu} := \{(\alpha_\rho)_{\rho<\nu} \mid (\alpha_\rho)_{\rho\le\nu} \in J^\circ_\nu \text{ and } (\alpha_\rho)_{\rho\le\nu'} \in J_{\nu'} \text{ for all } \nu' < \nu\} .$$

For $a = (\alpha_\rho)_{\rho<\nu} \in J_{<\nu}$ we set

$$\kappa_a := \mathrm{cf}(\{b \in J_{<\nu} \mid b < a\}) \text{ and } \lambda_a := \mathrm{ci}(\{b \in J_{<\nu} \mid b > a\}) ,$$

and

$$I_\nu(a) := \varphi_{\mathrm{right}}(\kappa_a)^* + \mu + \{\underline{0}\} + \mu^* + \varphi_{\mathrm{left}}(\lambda_a) \subset I_\nu . \tag{3}$$

Now we let $(\alpha_\rho)_{\rho\le\nu} \in J^\circ_\nu$ be an element of J_ν if and only if $a = (\alpha_\rho)_{\rho<\nu} \in J_{<\nu}$ and $\alpha_\nu \in I_\nu(a)$. Note that by our definition, each $(\alpha_\rho)_{\rho\le\nu} \in J_\nu$ is a truncation of an element in J°, hence its support must be an initial segment of μ. Thus if $\alpha_\rho = 0$ for some $\rho < \nu$, then we must have $\alpha_\nu = 0$; otherwise, α_ν can be any element of $I_\nu(a)$.

After having defined J_ν for all $\nu < \mu$, we set

$$J := \{(\alpha_\rho)_{\rho<\mu} \in J^\circ \mid (\alpha_\rho)_{\rho\le\nu} \in J_\nu \text{ for all } \nu < \mu\} .$$

The following is our first step towards the proof of Theorem 11:

Theorem 21. *With the sets R_{left} and R_{right} defined as in the introduction, assume that (1) holds. Then the cofinalities of the cuts of J are:*

$$\{(1,\mu),(\mu,1)\} \cup \{(\kappa,\varphi_{\mathrm{right}}(\kappa)) \mid \kappa \in R_{\mathrm{left}}\} \cup \{(\varphi_{\mathrm{left}}(\lambda),\lambda) \mid \lambda \in R_{\mathrm{right}}\} .$$

Further, the cofinality of J is κ_0 and its coinitiality is λ_0.

Proof: First, we observe that for each $\nu < \mu$ we obtain an embedding

$$\iota_\nu : J_\nu \hookrightarrow J$$

by sending $(\alpha_\rho)_{\rho\le\nu}$ to $(\beta_\rho)_{\rho\le\mu}$, where $\beta_\rho = \alpha_\rho$ for $\rho \le \nu$ and $\beta_\rho = \underline{0}$ for $\nu < \rho < \mu$.

We start by proving that the principal cuts in J have cofinalities $(1,\mu)$ and $(\mu,1)$. Take $(\alpha_\rho)_{\rho\le\mu} \in J$ and assume first that its support is smaller than μ. Set $\nu := \min\{\rho < \mu \mid \alpha_\rho = \underline{0}\} \ge 1$. Then by our definition of J, α_ν can be any element of $I_\nu(a)$. Since the cofinalities of the principal cuts generated by $\underline{0}$ in I_ν are $(1,\mu)$ and $(\mu,1)$, the cofinalities of the principal cuts generated by $(\alpha_\rho)_{\rho\le\nu}$ in J_ν are also $(1,\mu)$ and $(\mu,1)$. By means of the embeddings ι_ν it follows that the cofinalities of the principal cuts generated by $(\alpha_\rho)_{\rho<\mu}$ in J are again $(1,\mu)$ and $(\mu,1)$.

Now assume that the support of $a := (\alpha_\rho)_{\rho<\mu}$ is μ. For each $\nu < \mu$ there are elements $\beta_\nu, \gamma_\nu \in I_\nu((\alpha_\rho)_{\rho<\nu})$ with $\beta_\nu < \alpha_\nu < \gamma_\nu$. We set $\beta_\rho := \gamma_\rho := \alpha_\rho$ for $\rho < \nu$, and define

$$b_\nu := \iota_\nu((\beta_\rho)_{\rho\le\nu}) \text{ and } c_\nu := \iota_\nu((\gamma_\rho)_{\rho\le\nu}) .$$

Whenever $\nu < \nu' < \mu$, it follows that

$$b_\nu < b_{\nu'} < a < a_{\nu'} < a_\nu .$$

This proves that again, the cofinalities of the principal cuts generated by $(\alpha_\rho)_{\rho<\mu}$ in J are $(1,\mu)$ and $(\mu,1)$.

Now take any nonprincipal cut (D,E) in J. By restricting the elements to index set $\nu+1=\{\rho\mid\rho\leq\nu\}$, this cut induces a quasicut (D_ν,E_ν) in J_ν.

Assume that $\nu<\mu$ is such that $\iota_\nu(D_\nu)$ is not a cofinal subset of D and $\iota_\nu(E_\nu)$ is not a coinitial subset of E. Then we have one of the following cases:

- $\iota_\nu(D_\nu)\cap E\neq\emptyset$ or $\iota_\nu(E_\nu)\cap D\neq\emptyset$,
- there are $d_\nu\in D$ and $e_\nu\in E$ such that $\iota_\nu(D_\nu)<d_\nu<e_\nu<\iota_\nu(E_\nu)$, which yields that the restrictions of d_ν and e_ν to index set $\nu+1$ are equal and lie in $D_\nu\cap E_\nu$.

In both cases, $D_\nu\cap E_\nu\neq\emptyset$. This implies that also $D_{\nu'}\cap E_{\nu'}\neq\emptyset$ for all $\nu'<\nu$, with the element in $D_{\nu'}\cap E_{\nu'}$ being the restriction of the element in $D_\nu\cap E_\nu$.

Now we show that there is some $\nu<\mu$ such that $\iota_\nu(D_\nu)$ is cofinal in D or $\iota_\nu(E_\nu)$ is coinitial in E. Suppose that the contrary is true. Then $D_\nu\cap E_\nu\neq\emptyset$ for all $\nu<\mu$ and there is a unique element $a\in J$ whose restriction to index set $\nu+1$ lies in $D_\nu\cap E_\nu$, for all $\nu<\mu$. It follows that a is either the largest element of D or the smallest element in E. But this contradicts our assumption that (D,E) is nonprincipal.

We take ν to be minimal with the property that $\iota_\nu(D_\nu)$ is cofinal in D or $\iota_\nu(E_\nu)$ is coinitial in E. From what we have shown above, it follows that $D_{\nu'}\cap E_{\nu'}\neq\emptyset$ for all $\nu'<\nu$ and there is $(\alpha_\rho)_{\rho<\nu}\in J_{<\nu}$ whose restriction to $\nu'+1$ lies in $D_{\nu'}\cap E_{\nu'}$, for all $\nu'<\nu$. Therefore, there must be elements in both D_ν and E_ν whose restrictions to ν are equal to $(\alpha_\rho)_{\rho<\nu}$. Consequently, with

$$\begin{aligned}\overline{D}_\nu &:= \{\alpha_\nu\in I_\nu((\alpha_\rho)_{\rho<\nu})\mid(\alpha_\rho)_{\rho\leq\nu}\in D_\nu\}\text{ and}\\ \overline{E}_\nu &:= \{\alpha_\nu\in I_\nu((\alpha_\rho)_{\rho<\nu})\mid(\alpha_\rho)_{\rho\leq\nu}\in E_\nu\},\end{aligned}$$

$(\overline{D}_\nu,\overline{E}_\nu)$ is a cut in $I_\nu((\alpha_\rho)_{\rho<\nu})$. But $I_\nu((\alpha_\rho)_{\rho<\nu})$ is cut complete, and so there is some $\alpha_\nu\in I_\nu((\alpha_\rho)_{\rho<\nu})$ such that $a=(\alpha_\rho)_{\rho\leq\nu}$ is either the largest element of D_ν or the smallest element of E_ν. We note that $\alpha_\nu\neq\underline{0}$; otherwise, the element $(\alpha_\rho)_{\rho<\mu}$ with $\alpha_\rho=\underline{0}$ for $\nu\leq\rho<\mu$, which is the unique element in J whose restriction to $\nu+1$ is a, would be the largest element of D or the smallest element of E in contradiction to our assumption on (D,E). Hence by construction, for every

$$\alpha\in I_{\nu+1}((\alpha_\rho)_{\rho\leq\nu}) \;=\; \varphi_{\mathrm{right}}(\kappa_a)^*+\mu+\{\underline{0}\}+\mu^*+\varphi_{\mathrm{left}}(\lambda_a)$$

there is an element $(\alpha_\rho)_{\rho<\mu}$ with $\alpha_{\nu+1}=\alpha$ whose restriction to $\nu+1$ is a.

We assume first that $\iota_\nu(D_\nu)$ is cofinal in D. Since (D,E) is nonprincipal, D and hence also D_ν has no largest element. So a is the smallest element of E_ν. Consequently,

$$\iota_{\nu+1}(\{(\alpha_\rho)_{\rho\leq\nu+1}\mid\alpha_{\nu+1}\in I_{\nu+1}((\alpha_\rho)_{\rho\leq\nu})\})$$

is coinitial in E. We observe that $\kappa_a = \mathrm{cf}(D_\nu) = \mathrm{cf}(D) \neq 1$. By construction, the coinitiality of $I_{\nu+1}((\alpha_\rho)_{\rho\leq\nu})$ is $\varphi_{\mathrm{right}}(\kappa_a)$. This proves that the cofinality of (D,E) is $(\kappa_a, \varphi_{\mathrm{right}}(\kappa_a))$.

If on the other hand, $\iota_\nu(E_\nu)$ is coinitial in E, then a is the largest element of D_ν and one shows along the same lines as above that the cofinality of (D,E) is $(\varphi_{\mathrm{left}}(\lambda_a), \lambda_a)$ with $\lambda_a = \mathrm{ci}(E_\nu) = \mathrm{ci}(E) \neq 1$.

We have to prove that the cardinals κ_a and λ_a that appear in the construction, i.e., in definition (3), are elements of $\{1\}\cup R_{\mathrm{left}}$ and $\{1\}\cup R_{\mathrm{right}}$, respectively. We observe that κ_a and λ_a appear above only if $a = (\alpha_\rho)_{\rho\leq\nu} \in J_\nu$ is such that $\alpha_\rho \neq \underline{0}$ for all $\rho \leq \nu$. We show our assertion by induction on $1 \leq \nu \leq \mu$. We do this for κ_a ; for λ_a the proof is similar. First, we consider the successor case $\nu = \sigma + 1$. We set $\overline{a} = (\alpha_\rho)_{\rho\leq\sigma}$. If $\sigma \geq 1$, then our induction hypothesis states that our assertion is true for $\kappa_{\overline{a}}$ and $\lambda_{\overline{a}}$. We observe that

$$\begin{aligned}\kappa_a &= \mathrm{cf}(\{(\beta_\rho)_{\rho\leq\nu} \in J_\nu \mid \beta_\rho = \alpha_\rho \text{ for } \rho < \nu \text{ and } \beta_\nu < \alpha_\nu\}) \\ &= \mathrm{cf}(\{\beta \in I_\nu(\overline{a}) \mid \beta < \alpha_\nu\})\,.\end{aligned}$$

This is the cofinality of a lower cut set of a cut in $I_\sigma(\overline{a})$, which is equal to I_0 if $\sigma = 0$, and to

$$\varphi_{\mathrm{right}}(\kappa_{\overline{a}})^* + \mu + \{\underline{0}\} + \mu^* + \varphi_{\mathrm{left}}(\lambda_{\overline{a}})$$

otherwise. Therefore, if κ_a is infinite, it is an element of $\mathrm{Cofin}(I)\cup\mathrm{Reg}_{<\kappa_0}\cup \mathrm{Reg}_{<\mu} = R_{\mathrm{left}}$ if $\sigma = 0$, and of $\mathrm{Reg}_{<\varphi_{\mathrm{left}}(\lambda_{\overline{a}})} \cup \mathrm{Reg}_{<\mu}$ otherwise. In the latter case, $\lambda_{\overline{a}} \in R_{\mathrm{right}}$ by induction hypothesis, hence $\varphi_{\mathrm{left}}(\lambda_{\overline{a}}) \in R_{\mathrm{left}}$ by (1), which yields that $\mathrm{Reg}_{<\varphi_{\mathrm{left}}(\lambda_{\overline{a}})} \cup \mathrm{Reg}_{<\mu} \subseteq R_{\mathrm{left}}$. Altogether, we have proved that $\kappa_a \in \{1\} \cup R_{\mathrm{left}}$.

Now we consider the case of ν a limit ordinal. Let μ' be its cofinality. Then $\mu' \in \mathrm{Reg}_{<\mu}$. With a similar construction as in the beginning of the proof one shows that the principal cuts generated by elements in $J_{<\nu}$ have cofinalities $(\mu', 1)$ and $(1, \mu')$. This yields that $\kappa_a \in R_{\mathrm{left}}$ and $\lambda_a \in R_{\mathrm{right}}$.

It remains to prove that all cofinalities listed in the assertion of our theorem actually appear as cofinalities of cuts in J. Since for all cardinals $\kappa \in \mathrm{Cofin}(I) \cup \mathrm{Reg}_{<\kappa_0}$, there is a cut in I_0 with cofinality $(\kappa, 1)$, our construction at level $\nu = 1$ shows that $(\kappa, \varphi_{\mathrm{right}}(\kappa))$ appears as the cofinality of a cut in J. Similarly, one shows that $(\varphi_{\mathrm{left}}(\lambda), \lambda)$ appears as the cofinality of a cut in J for every $\lambda \in \mathrm{Coin}(I) \cup \mathrm{Reg}_{<\lambda_0}$.

Now take any regular cardinal $\mu' < \mu$. For an arbitrary $a = (\alpha_0) \in J_0$ we see that there is a cut in $I_1(a)$ with cofinality $(\mu', 1)$. Our construction at level $\nu = 2$ then shows that $(\mu', \varphi_{\mathrm{right}}(\mu'))$ appears as the cofinality of a cut in J. Similarly, one shows that $(\varphi_{\mathrm{left}}(\mu'), \mu')$ appears as the cofinality of a cut in J.

Finally, the cuts $(1,\mu)$ and $(\mu,1)$ appear as the cofinalities of all principal cuts, as shown at the beginning of the proof.

The proof of the last statement of the theorem is again left to the reader. $\square$

The following result is an immediate consequence of Theorem 21, and it proves Theorem 11:

Corollary 22. *Assume in addition to the previous assumptions that*

$$\varphi_{\mathrm{left}}(\kappa) \neq \kappa \neq \varphi_{\mathrm{right}}(\kappa) \quad \textit{for all} \quad \kappa \in R_{\mathrm{left}} \cup R_{\mathrm{right}} \,.$$

Then J is a symmetrically complete extension of I. If in addition μ is uncountable, then J is strongly symmetrically complete.

Remark 23. In both constructions that we have given in this section, every element in the constructed ordered set has, in the terminology of Hausdorff, character (μ,μ).

6. Construction of symmetrically complete ordered extensions

Take any ordered abelian group G. We wish to extend it to an extremely symmetrically complete ordered abelian group. We use the well known fact that G can be embedded in a suitable Hahn product $H_0 = \mathbf{H}_I\,\mathbb{R}$, for some ordered index set I. By Corollary 20, there is an embedding ι of I in an extremely symmetrically complete linearly ordered set J. We set $H = \mathbf{H}_J\,\mathbb{R}$ and note that there is a canonical order preserving embedding φ of $H_0 = \mathbf{H}_I\,\mathbb{R}$ in $H = \mathbf{H}_J\,\mathbb{R}$ which lifts ι by sending an element $(r_\gamma)_{\gamma\in I}$ to the element $(r'_\delta)_{\delta\in J}$ where $r'_\delta = r_\gamma$ if $\delta = \iota(\gamma)$ and $r'_\delta = 0$ if δ is not in the image of ι. By Theorem 6, H is an extremely symmetrically complete ordered abelian group. We have now proved the first part of Theorem 9.

Take any ordered field K. We wish to extend it to an extremely symmetrically complete ordered field. First, we extend K to its real closure K^{rc}. From [Ka] we know that K^{rc} can be embedded in the power series field $\mathbb{R}((G))$ where G is the value group of K^{rc} under the natural valuation. By what we have already shown, G admits an embedding ψ in an extremely symmetrically complete ordered abelian group H. By a definition analogous to the one of φ above, one lifts ψ to an order preserving embedding of the power series field $\mathbb{R}((G))$ in the power series field $\mathbb{R}((H))$. By Theorem 7, $\mathbb{R}((H))$ is an extremely symmetrically complete ordered field. We have thus proved the second part of Theorem 9.

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