The theory of the defect and its application to the problem of local uniformization, VIII

Franz-Viktor Kuhlmann

Padova, April 23, 2021

The basic Zariski space of a function field

We will consider algebraic function fields $F|K$ of arbitrary characteristic. Recall that by a place of $F|K$ we mean a place of F whose restriction to K is the identity. We denote by $S(F|K)$ the set of all such places. This is called the Zariski space (or Zariski–Riemann manifold) of $F|K$.

A more general Zariski space and the Zariski topology

Let $\wp$ be a fixed place on K. The set of all places of F which extend $\wp$ will be denoted by $S(F|K\,;\,\wp)$. Hence, $S(F|K) = S(F|K\,;\,\mathrm{id}_K)$. Every set $S(F|K\,;\,\wp)$ carries the Zariski-topology, for which the basic open sets are the sets of the form

$$\{P \in S(F|K\,;\,\wp) \mid a_1\,,\ldots,\,a_k \in \mathcal{O}_P\}\,, \tag{1}$$

where $k \in \mathbb{N} \cup \{0\}$ and $a_1,\ldots,a_k \in F$. With this topology, $S(F|K\,;\,\wp)$ is a spectral space (as introduced by Hochster); in particular, it is quasi-compact.

The patch topology

The associated patch topology (or constructible topology) is the finer topology whose basic open sets are of the form

$$\{P \in S(F|K\,;\,\wp) \mid a_1\,,\ldots,\,a_k \in \mathcal{O}_P\,;\, b_1\,,\ldots,\,b_\ell \in \mathcal{M}_P\}\,, \qquad (2)$$

where $k,\ell \in \mathbb{N} \cup \{0\}$ and $a_1,\ldots,a_k,b_1,\ldots,b_\ell \in F$. With the patch topology, $S(F|K\,;\,\wp)$ is a totally disconnected compact Hausdorff space.

Zariski's approach to Resolution of Singularities

Zriski's idea was to use the compactness of the Zariski space under the Zariski topology to deduce Resolution of Singularities from Local Uniformization. Local Uniformization is a Zariski-open property, meaning that Local Uniformization with respect to one place simultaneously holds for all places in a Zariski-open neighborhood. Thus, once Local Uniformization is proven for all places of the function field $F|K$, the entire space is covered by such open neighborhoods, and by compactness, it is already covered by a finite number of them. Now a global resolution should be obtained by patching the finitely many local solutions together. Using this approach, Zariski proved Resolution of Singularities for varieties of dimension 2 over fields of characteristic 0.

Zariski's approach to Resolution of Singularities

In his proof of Resolution of Singularities in all dimensions over fields of characteristic 0, Hironaka did *not* use Zariski's approach. However, Cossart and Piltant use it in their proof of Resolution of Singularities in dimension 3 over fields of arbitrary characteristic.

While it is a common belief that Local Uniformization holds in all dimensions and all characteristics, it appears that patching may not be possible beyond dimension 3.

Bad places

As soon as the transcendence degree of $F|K$ is bigger than 1, the Zariski space $S(F|K)$ will contain "bad places". For some place Q of $F|K$, the residue field FQ may then not be finitely generated over K (even though F is finitely generated over K). This can be a serious obstruction in situations as in the proof of the p-adic Nullstellensatz by M. Jarden and P. Roquette, where a K-rational specialization P of the place Q is needed; we call a place P of an arbitrary field L K-rational if K is a subfield of L, P is trivial on K, and $LP = K$. A specialization of Q is a composition of Q with a place Q' on its residue field FQ; for this specialization to be K-rational, Q' must be K-rational. However, if FQ is not finitely generated over K, chances to find a K-rational place on FQ are slim.

Bad places

Similarly, the value group $v_Q F$ of Q may not be finitely generated. For places with value groups or residue fields that are not finitely generated, local uniformization is much more difficult than for Abhyankar places. We have already shown that Abhyankar places P of $F|K$ admit Local Uniformization in arbitrary characteristic, provided that $FP|K$ is separable. To the other extreme, Local Uniformization can also be proven for K-rational discrete places; we call a place discrete if its value group is isomorphic to $\mathbb{Z}$.

Replacing bad by good places

Therefore, the question arises whether we can "replace bad places Q by good places P". Certainly, in doing so we want to keep a certain amount of information unaltered. For instance, we could fix finitely many elements on which Q is finite and require that also P is finite on them too. This amounts to asking whether the "good" places lie Zariski-dense in the Zariski space.

Now if we would mean by a "good place" just a place with finitely generated value group and residue field finitely generated over K, then the answer would be trivial: the identity is a suitable place, as it lies in every Zariski-open neighborhood. The situation becomes non-trivial when we work with the patch topology instead of the Zariski topology.

In addition, we can even try to keep more information on values or residues, e.g. rational (in)dependence of values or algebraic (in)dependence of residues.

Density results in characteristic 0

In

Kuhlmann, F.-V. – Prestel, A.: *On places of algebraic function fields*, J. reine angew. Math. **353** (1984), 181–195,

such problems are solved, in the case of places that are trivial on a ground field of characteristic 0, by an application of the following Ax–Kochen–Ershov Theorem:

Theorem

Take a henselian field (E, v) of residue characteristic 0 and a valued field extension $(F|E, v)$. If vE is existentially closed in vF and Ev is existentially closed in Fv, then (E, v) is existentially closed in (F, v).

Density results in arbitrary characteristic

The results were generalized in

Kuhlmann, F.-V.: *On places of algebraic function fields in arbitrary characteristic*, Advanves in Math. **188** (2004), 399–424

to the case of arbitrary characteristic and places that extend a fixed place $\wp$ on the ground field K. This uses the following Ax–Kochen–Ershov Theorem for tame fields:

Theorem

Take a tame field (E, v) and a valued field extension $(F|E, v)$. If vE is existentially closed in vF and Ev is existentially closed in Fv, then (E, v) is existentially closed in (F, v).

As in the case of Local Uniformization, in order to work with tame fields, "alterations" have to be taken into account that can enlarge value groups and residue fields.

Dense subsets of the Zariski space

When we say "dense" we will always mean "dense with respect to the patch topology". Throughout, we let $F|K$ be a function field of transcendence degree n, and $\wp$ a place on K. We set $p = \operatorname{char} K$ if this is positive, and $p = 1$ otherwise.

We will now state the key result of the second paper, which is a generalization of the Main Theorem of the first paper. Take any ordered abelian group Γ and $r \in \mathbb{N}$. If the direct product $\Gamma \oplus \bigoplus_r \mathbb{Z}$ of Γ with r copies of $\mathbb{Z}$ is equipped with an arbitrary extension of the ordering of Γ, then it will be called an r-extension of Γ.

For a place $P \in S(F|K\,;\,\wp)$, we set

$$\dim P \;:=\; \operatorname{trdeg} FP|K\wp \quad \text{and} \quad \operatorname{rr} P \;:=\; \operatorname{rr} v_P F/v_\wp K\,.$$

Theorem

Take a place $Q \in S(F|K\,;\,\wp)$ *and*

$$a_1, \ldots, a_m \in F\,.$$

Then there exists a place $P \in S(F|K\,;\,\wp)$ *with value group finitely generated over* $v_\wp K$ *and residue field finitely generated over* $K\wp$*, such that*

$$a_i P = a_i Q \quad \text{and} \quad v_P a_i = v_Q a_i \quad \text{for} \quad 1 \leq i \leq m\,.$$

Moreover, if r_1 *and* d_1 *are natural numbers satisfying*

$$\dim Q \leq d_1 \quad \text{and} \quad \operatorname{rr} Q \leq r_1 \quad \text{with} \quad 1 \leq d_1 + r_1 \leq n\,,$$

then P *may be chosen to satisfy in addition:*

Theorem

(a) $\dim P = d_1$ *and* FP *is a subfield of the rational function field in* $d_1 - \dim Q$ *variables over the perfect hull of* FQ*,*

(b) $\mathrm{rr}\, P = r_1$ *and* $v_P F$ *is a subgroup of an arbitrarily chosen* $(r_1 - \mathrm{rr}\, Q)$*-extension of the* p*-divisible hull of* $v_Q F$*.*

The above remains true even for $d_1 = 0 = r_1$ *, provided that each finite extension of* $(K, \wp)$ *admits an immediate extension of transcendence degree* n*.*

The last condition mentioned in the theorem holds for instance for all $(K, \wp)$ for which the completion is of transcendence degree at least n. Note that the case of $d_1 = 0 = r_1$ only appears when $\wp$ is nontrivial.

Dense subsets of the Zariski space

If $v_Q a_i \geq 0$ for $1 \leq i \leq k$ and if $b_1, \ldots, b_\ell \in F$ such that $v_Q b_j > 0$ for $1 \leq j \leq \ell$, then we can choose P according to the theorem such that also $v_P a_i \geq 0$ for $1 \leq i \leq k$ and $v_P b_j > 0$ for $1 \leq j \leq \ell$. That is, we can find the desired place P in every open neighborhood of Q with respect to the patch topology.

Corollary

The set of all places with value group finitely generated over $v_\wp K$ and residue field finitely generated over $K\wp$ lies dense in $S(F|K\,;\,\wp)$.

Dense subsets of the Zariski space

If $\wp$ is trivial and we choose r_1 and d_1 such that $r_1 + d_1 = n$, then P will be an Abhyankar place. Hence our theorem yields:

Corollary

The set of all Abhyankar places lies dense in $S(F|K)$. Hence, the set of all places that admit Local Uniformization lies dense in $S(F|K)$.

This result shows that "bad places" can be approximated arbitrarily well in the patch topology by places that admit Local Uniformization. Does this imply that also the bad places admit Local Uniformization? If the good places get close enough, will the bad place finally lie in one of those open neighborhoods of a place that admits Local Uniformization in which all other places admit the same Local Uniformization? Unfortunately, we do not know this.

Decreasing the rational rank

In certain cases we would like to obtain value groups of smaller rational rank; for instance, we may want to get discrete places in the case where $\wp$ is trivial.

Theorem

Take a place $Q \in S(F|K;\wp)$ *and* $a_1, \ldots, a_m \in F$. *Choose* r_1 *and* d_1 *such that*

$$\dim Q \le d_1 \ \le \ n-1 \quad \textit{and} \quad 1 \le r_1 \le n - d_1 \,.$$

Then there is a place P *such that*

$$a_i P \ = \ a_i Q \quad \textit{for} \quad 1 \le i \le m$$

and

Theorem

(a) $\dim P = d_1$ *and* FP *is a subfield, finitely generated over* $K\wp$, *of a purely transcendental extension of transcendence degree* $d_1 - \dim Q$ *over the perfect hull of* FQ,

(b) $\mathrm{rr}\, P = r_1$ *and* $v_P F$ *is a subgroup, finitely generated over* $v_\wp K$, *of an arbitrarily chosen* r_1*-extension of the divisible hull of* $v_\wp K$.

The assertions remain true even for $r_1 = 0$, *provided that each finite extension of* $(K, \wp)$ *admits an immediate extension of transcendence degree* n.

Observe that when the rational rank is decreased, it is impossible to preserve the values of arbitrarily chosen elements.

Discrete places

A place $P \in S(F|K)$ of dimension $\operatorname{trdeg} F|K - 1$ is called a prime divisor of $F|K$. Every prime divisor is a discrete Abhyankar place and has a residue field which is finitely generated over K. From the above theorem, applied with $d_1 = n - 1$, we obtain the following result:

Corollary

The prime divisors of $F|K$ lie dense in $S(F|K)$.

Discrete places

If on the other hand we choose $d_1 = \dim Q$, then FP will be contained in a finite purely inseparable extension of FQ. If $\dim Q = 0$, i.e., $FQ|K$ is algebraic, then it follows that FP is a finite extension of K. If in addition K is perfect and Q is K-rational, then also P is K-rational. With $r_1 = 1$, we obtain:

Corollary

If K is perfect, then the discrete K-rational places lie dense in the space of all K-rational places of $F|K$.

Decreasing the dimension

There is also a similar theorem which shows that the dimension can be decreased. This time, the v_Q-values of finitely many elements can be preserved, but not the residues under Q. Using this theorem, on can prove:

Corollary

The zero-dimensional places with finitely generated value group and residue field a finite extension of K lie dense in $S(F|K)$.

By combining previous theorems, one can prove:

Corollary

The discrete zero-dimensional places with residue field a finite extension of K lie dense in $S(F|K)$.

Factoring over the place $\wp$ on the ground field

One aspect had been overlooked in the two papers we have cited so far. This aspect came up in a recent study of real holomorphy rings and the topologies of spaces of real places (these are places with residue fields that admit orderings):

Becker, E. – Kuhlmann, F.-V. – Kuhlmann, K.: *Density of Composite Places in Function Fields and Applications to Real Holomorphy Rings*, to appear in Mathematische Nachrichten.

A place P of F factors over the place $\wp$ of K if there is a place P_0 that is K-rational and such that

$$P = P_0 \circ \wp \, .$$

Factoring over the place $\wp$ on the ground field

Theorem

Take a place $Q \in S(F|K\,;\,\wp)$ *and*

$$a_1, \ldots, a_m \in F^\times .$$

Choose $r \in \mathbb{N}$ *such that* $1 \leq r \leq \operatorname{trdeg} F|K$ *and an arbitrary ordering on* $\mathbb{Z}^r$*; denote by* Γ *the so obtained ordered abelian group. If* $\operatorname{trdeg} F|K > 1$ *and* $\wp$ *is trivial while* Q *is not, then assume in addition that* Γ *is the lexicographic product* $\Gamma' \times \mathbb{Z}$*, where* $\Gamma' = \mathbb{Z}^{r-1}$ *endowed with an arbitrary ordering.*

Then there is a place $P_0 \in S(F|K)$ *and an extension* $\wp'$ *of* $\wp$ *from* K *to* FP_0 *such that, with* $P := P_0 \circ \wp' \in S(F|K\,;\,\wp)$*,*

(a) FP_0 *is a finite extension of* K*,*

(b) $v_{P_0}F \subseteq \Gamma$ *with* $(\Gamma : v_{P_0}F)$ *finite,*

(c) if $a_i \in \mathcal{O}_Q$ *, then* $a_iP_0 \in \mathcal{O}_{\wp'}$ *and* $a_i \in \mathcal{O}_P$ *.*

Application to ordered function fields

Theorem

Assume that K is a real closed field and F is an ordered function field over K. Assume further that $\wp$ is a real place on K and Q is a real place on F. Take elements $a_1, \ldots, a_m \in F$ and let $r \in \mathbb{N}$ and Γ be as in the previous theorem. Then there is a place $P_0 \in S(F|K)$ such that

(a) $FP_0 = K$,

(b) $v_{P_0}F = \Gamma$,

and with $P := P_0 \circ \wp \in S(F|K\,;\,\wp)$,

(c) *if* $a_i \in \mathcal{O}_Q$, *then* $a_iP_0 \in \mathcal{O}_\wp$ *and* $a_i \in \mathcal{O}_P$,

(c′) *if* $a_i > 0$ *and* $a_iQ \neq 0, \infty$, *then* $a_iP > 0$.

The latter implies that if $\infty \neq a_iQ > 0$, *then* $a_iP > 0$.

0-dimensional places of function fields

The following result is an important tool, in particular for the theorem that decreases the dimension of the given place Q, and to find K-rational specializations.

Theorem

Take an arbitary algebraic function field $F|K$ and

$$a_1, \ldots, a_m \in F\,.$$

Choose $r \in \mathbb{N}$ such that $1 \leq r \leq \operatorname{trdeg} F|K$ and an arbitrary ordering on $\mathbb{Z}^r$; denote by Γ the so obtained ordered abelian group. Then there are infinitely many (nonequivalent) places $P \in S(F|K)$ such that $FP|K$ is finite, $v_P F \subseteq \Gamma$ with $(\Gamma : v_P F)$ finite, and $a_i P \neq 0, \infty$ for $1 \leq i \leq m$.

If in addition K is existentially closed in F, then these places can be chosen to be K-rational with $v_P F = \Gamma$.

A model theoretic question about rational places

From the above theorem we know that if K is existentially closed in the function field F, then F admits a K-rational point. Is the converse also true? This is in general not the case. We need additional assumptions about K or about the K-rational place.

Large fields

A field K is called a large field if it satisfies one of the following equivalent conditions:

(LF) *For every smooth curve over K the set of K-rational points is infinite if it is non-empty.*

(LF′) *In every smooth, integral variety over K the set of K-rational points is Zariski-dense if it is non-empty.*

(LF″) *For every function field $F|K$ in one variable the set of K-rational places is infinite if it is non-empty.*

For explanations on the equivalence of these conditions, see the second of the above cited articles. Let us use model theory to give a characterization of large fields.

Characterization of large fields

Theorem

The following conditions are equivalent:

a) K is a large field,

b) K is existentially closed in every function field F in one variable over K which admits a K-rational place,

c) K is existentially closed in the henselization $K(t)^h$ of the rational function field $K(t)$ with respect to the t-adic valuation,

d) K is existentially closed in the field $K((t))$ of formal Laurent series,

e) K is existentially closed in every extension field which admits a discrete K-rational place.

In particular, we see: if the converse we have asked for holds for every function field over a given field K, then K must be a large field.

The role of the Implicit Function Theorem

If the field K satisfies an Implicit Function Theorem, then it is easy to show that it satisfies condition (LF), and it is also easy to show that it satisfies assertion c) of the foregoing theorem. Therefore, fields satisfying an Implicit Function Theorem are large fields (but they are not the only ones).

Every field admitting a henselian valuation and every real closed field satisfies an Implicit Function Theorem, so all of these fields are large.

When the converse we asked for holds

The equivalence of a) and b) in the foregoing theorem provides a partial answer to our question: the converse holds if K is a large field and $\operatorname{trdeg} F|K = 1$. What about function fields of arbitrary transcendence degree?

Theorem

Let K be a perfect field. Then the following conditions are equivalent:

1) K is a large field,

2) K is existentially closed in every power series field $K((G))$.

3) K is existentially closed in every extension field L which admits a K-rational place.

The question arises: can the condition that K be perfect be dropped?

Connection with Local Uniformization

Theorem

Let K be a large field and $F|K$ an algebraic function field. If there is a K-rational place of F which admits local uniformization, then K is existentially closed in F.

Consequently, if K is a large field and every K-rational place of an arbitrary function field F over K admits Local Uniformization, then the converse we asked for holds for K.

Connection with Local Uniformization

Corollary

Let K be a large field and $F|K$ an algebraic function field. If there is a K-rational Abhyankar place of F, then K is existentially closed in F. The same holds if there is a place P on F such that (F, P) lies in the completion of a sub-function field (F_0, P) on which P is a K-rational Abhyankar place.

References

- Becker, E. – Kuhlmann, F.-V. – Kuhlmann, K.: *Density of Composite Places in Function Fields and Applications to Real Holomorphy Rings*, to appear in Mathematische Nachrichten
- Cossart, V. – Piltant, O.: *Resolution of singularities of threefolds in positive characteristic. I*, Reduction to local uniformization on Artin-Schreier and purely inseparable coverings. J. Algebra 320 (2008), 1051–1082
- Cossart, V. – Piltant, O.: *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra 321 (2009), 1836–1976
- Hochster, M.: *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60

References

- Jarden, M. – Roquette, P.: *The Nullstellensatz over $\wp$–adically closed fields*, J. Math. Soc. Japan **32** (1980), 425–460
- Kuhlmann, F.-V.: *On places of algebraic function fields in arbitrary characteristic*, Advanves in Math. **188** (2004), 399–424
- Kuhlmann, F.-V.: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43
- Kuhlmann, F.-V. – Prestel, A.: *On places of algebraic function fields*, J. reine angew. Math. **353** (1984), 181–195