The theory of the defect and its application to the problem of local uniformization, VII

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Characteristic blind Taylor expansion

Take a polynomial

$$f(X) \;=\; c_n X^n + c_{n-1} X^{n-1} + \ldots + c_0 \,.$$

Recall that with the i-th Hasse-Schmidt derivative of f defined as

$$\partial_i f(X) \;:=\; \sum_{j=i}^{n} c_j \binom{j}{i} X^{j-i} \;=\; \sum_{j=0}^{n-i} c_{j+i} \binom{j+i}{i} X^j \,, \qquad (1)$$

we obtain the Taylor expansion

$$f(X+Y) = \sum_{i=0}^{n} \partial_i f(X) Y^i \,. \qquad (2)$$

Polynomials under pseudo Cauchy sequences

We will now list several facts that can be found in Kaplansky's paper *Maximal fields with valuations*.

We assume that $x \in F$ is limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in K of transcendental type. Taking $X = a_\nu$ and $Y = x - a_\nu$ in the Taylor expansion (2), we obtain:

$$f(x) = \sum_{i=0}^{n} \partial_i f(a_\nu)(x - a_\nu)^i \,. \tag{3}$$

By our assumption on $(a_\nu)_{\nu<\lambda}$, for all i the values $v\partial_i f(a_\nu)$ are ultimately fixed. Thus ultimately, the values

$$v(\partial_i f(a_\nu)(x - a_\nu)^i) \;=\; v\partial_i f(a_\nu) \;+\; iv(x - a_\nu)$$

will be linear in $v(x - a_\nu)$ with distinct slopes i.

Polynomials under pseudo Cauchy sequences

Hence from some point on, their graphs will not intersect anymore, so for distinct $i, j \in \{0, \ldots, n\}$ we will have that

$$v\partial_i f(a_\nu) + iv(x - a_\nu) \neq v\partial_j f(a_\nu) + jv(x - a_\nu)$$

and there is $i_0 \in \{1, \ldots, n\}$ such that

$$v\partial_{i_0} f(a_\nu) + i_0 v(x - a_\nu) < v\partial_i f(a_\nu) + iv(x - a_\nu) \qquad (4)$$

for all $i \in \{1, \ldots, n\}, i \neq i_0$.

We need more information on i_0 . The following is proved by Kaplansky in his paper:

Proposition

Let i_0 be as in (4). Then i_0 is a power of the characteristic exponent p of the residue field: $i_0 = p^k$ for some $k \in \mathbb{N} \cup \{0\}$.

Galois extensions of degree p of $K(x)^h$

Similarly as in the proof of the GST, let us consider Galois extensions F of degree $p = \mathrm{char}\, Kv > 0$ of a valued field $E = K(x)^h$. This time we assume that the extension $(F|K, v)$ is immediate and that x is limit of a pseudo Cauchy sequence in K of transcendental type. To avoid technicalities, let us only consider the case where (K, v) is assumed to be a tame field. In particular, K is then a perfect field.
As before, in the equal characteristic case $F|E$ is an Artin–Schreier extension generated by an element ϑ such that $\vartheta^p - \vartheta = a \in E$. In the mixed characteristic case, we assume that K contains all p-th roots of unity, so that $F|E$ is a Kummer extension generated by an element η such that $\eta^p = a \in E$.

As before, we are looking for suitable normal forms for the element a.

Choice of a as a polynomial in x

In what follows, let us assume that $(K(x), v)$ has rank 1. Consequently, $(K(x)^h, v)$ lies in the completion of $(K(x), v)$, and we have already shown that $K[x]$ lies dense in $(K(x), v)$.

First we consider the case of an Artin–Schreier extension. By what we have just shown, there is $f(x) \in K[x]$ such that $a - f(x) \in \mathcal{M}_E \subset \wp(E)$. Thus we can replace a by $f(x)$ without changing the extension.

Choice of a as a polynomial in x

Now we consider the case of a Kummer extension. Since $vK(x)^h = vK$ is p-divisible, there is some $c_1 \in K$ such that $vc_1^p a = 0$. Further, since $K(x)^h v = Kv$ is perfect, there is some $c_2 \in K$ such that $(c_1^p c_2^p a)v = 1$. Since we can replace a by any element in $a(E^\times)^p$ without changing the extension, we can thus assume from the start that a is a 1-unit in $K(x)^h$. Let us write $a = 1 + a'$ with $va' > 0$. Using the density of $K[x]$ in $K(x)^h$, we can find $f(x) \in K[x]$ such that

$$v(f(x) - a') \;>\; \frac{p}{p-1} vp\,.$$

Then by part a) of the lemma on 1-units from lecture VI we can replace a by $1 + a' + f(x) - a' = 1 + f(x)$ without changing the extension.

We will now present the normal forms that can be derived for a under the assumption that (K,v) is a tame field (they also hold if (K,v) is a separably tame field). In the case of char $K = p > 0$, we obtain:

Proposition

The extension $F|E$ admits a generator ϑ such that $\vartheta^p - \vartheta = g(z)$ and

$$\left.\begin{array}{l} g(z) = a_n z^n + \ldots + a_1 z + a_0 \in K[z]\,,\ \textit{where} \\ z = (x-c)/d\,,\ \textit{with } vz = 0\,,\ c \in K \textit{ and } 0 \neq d \in K, \\ \textit{if } j > 0 \textit{ and } p|j, \textit{ then } a_j = 0, \\ \textit{the values } va_j \textit{ of all nonzero } a_j \textit{ are distinct.} \end{array}\right\} \quad (5)$$

Normal forms for a, mixed characteristic case

In the case of char $K = 0$, char $Kv = p > 0$, we obtain:

Proposition

The extension $F|E$ admits a generator η such that $\eta^p = 1 + g(z)$ and

$$\left.\begin{array}{l} g(z) = a_n z^n + \ldots + a_1 z + a_0 \in \mathcal{M}_K[z] \textit{ where} \\ z = (x - c)/d\,,\ \textit{with } vz = 0\,,\ c \in K \textit{ and } 0 \neq d \in K, \\ \textit{there is } j \in \{1, \ldots, n\} \textit{ with } p \nmid j \textit{ such that} \\ a_j \textit{ is the unique coefficient of least value among } a_1, \ldots, a_n\,. \end{array}\right\} \quad (6)$$

Proof in the equal characteristic case

We had already shown that we may assume that $\vartheta^p - \vartheta = f(x) \in K[x]$. Set $\deg f = n$. We consider the following Taylor expansion with variables X and X_0:

$$f(X) = \sum_{i=0}^{n} \partial_i f(X_0)(X - X_0)^i \,. \tag{7}$$

For any i which is divisible by p, say $i = pj$, the summand $\partial_i f(X_0)(X - X_0)^i$ in $f(X)$ is equivalent to

$$\partial_i f(X_0)^{1/p}(X - X_0)^j$$

modulo $\wp(K_1[X, X_0])$, where

$$K_1 \ = \ K\left(\partial_i f(X_0)^{1/p}\right) \,. \tag{8}$$

Proof in the equal characteristic case

By a repeated application of this procedure we find that modulo $\wp(K'[X,X_0])$, with $K'|K(X_0)$ a finite purely inseparable extension, $f(X)$ is equivalent to

$$f(X_0)+\sum_j{}' \left(\partial_j f(X_0)+\sum_k{}^{(j)}\partial_{jp^k} f(X_0)^{1/p^k}\right)(X-X_0)^j\,, \qquad (9)$$

where:

- $\sum_j{}'$ denotes the sum over all $j\leq n$ with $p\nmid j$,
- $\sum_k{}^{(j)}$ denotes the sum over all $k\geq 1$ with $jp^k\leq n$.

Proof in the equal characteristic case

For large enough $\ell \in \mathbb{N}$, the power

$$\left(\partial_j f(X_0) + \sum_k{}^{(j)}\, \partial_{jp^k} f(X_0)^{1/p^k} \right)^{p^\ell} \tag{10}$$

is a polynomial in $K[X_0]$. Since x is limit of a pseudo Cauchy sequence in K of transcendental type, we may choose

$$\alpha_0 \in v(x - K)$$

such that for all $c \in K$ with $v(x-c) \geq \alpha_0$ the value of $f(c)$ as well as the values of (10) for $X_0 = c$ are fixed, for all $j \leq n$ with $p \nmid j$.

Proof in the equal characteristic case

For those c we set

$$\beta_j \;:=\; v\left(\partial_j f(c) + \sum_k{}^{(j)}\, \partial_{jp^k} f(c)^{1/p^k}\right)\,, \tag{11}$$

which is an element of $vK \cup \{\infty\}$ (as vK is p-divisible). Since the set $v(x-K)$ has no greatest element, we may choose $c \in K$ with $v(x-c) \geq \alpha_0$ such that all values

$$\beta_j + j\cdot v(x-c)\,, \qquad \text{for } j \leq n \text{ with } p \nmid j \text{ and } \beta_j \neq \infty, \tag{12}$$

are distinct, nonzero, and not equal to $vf(c)$.

Proof in the equal characteristic case

Having chosen c, we pick $d \in K$ such that $vd = v(x - c)$ and put

$$z = \frac{x - c}{d},$$

so that $vz = 0$. In (7), (8) and (9) we now set $X := x$ and $X_0 := c$. Then K' becomes a finite purely inseparable extension of K, hence equal to K since K is perfect. From (9) we obtain a polynomial that can be written as a polynomial in z as

$$g(z) := f(c) + \sum_j{}' a_j z^j \tag{13}$$

with coefficients

$$a_j = d^j \cdot \left(\partial_j f(c) + \sum_k{}^{(j)} \partial_{jp^k} f(c)^{1/p^k} \right),$$

all of which have nonzero value.

Proof in the equal characteristic case

By construction, $g(z)$ and $f(x)$ are equivalent modulo $\wp(K[x])$. Hence we can replace $f(x)$ by $g(z)$ without changing the extension $F|E$. By definition of the summation $\sum_j{}'$ in (13), it runs only over those $j \leq n$ that are not divisible by p. We set $a_0 := f(c)$ and $a_j := 0$ whenever p divides j. Then the assertions of (6) are satisfied.

The proof of the normal form in the mixed characteristic case is far more complicated, and I will not describe it here. The original proof in my thesis contained a gap. It was found by Yuri Ershov. In

Ershov, Yu. L.: *On Henselian Rationality of Extensions*, Doklady Math. **78** (2008), 724–728,

Ershov proposed a procedure to fill the gap. An improved version of this procedure was then used for the proof in

Kuhlmann, F.-V.: *Elimination of Ramification II: Henselian Rationality*, Israel J. Math. **234** (2019), 927–958.

In order to put to work the normal forms we have found, we will need the following application of Hensel's Lemma:

Lemma

Take an extension $(K(z)|K,v)$ of valued fields of residue characteristic p and assume that $vz = 0$. Further, take a polynomial $h(X) = b_nX^n + \ldots + b_1X \in K[X]$ for which there is an index $j \in \{1,\ldots,n\}$ with $p \nmid j$ such that b_j is the unique coefficient of least value among $b_1,\ldots,b_n$. Then

$$K(z)^h \;=\; K(h(z))^h\,.$$

An application of Hensel's Lemma

Proof.

Consider the polynomial

$$\begin{aligned} H(X) &:= b_j^{-1}[h(X) - h(z)] \\ &= X^j + \sum_{i\neq j} b_j^{-1} b_i X^i - b_j^{-1} h(z) \in K(h(z))[X] . \end{aligned}$$

Since $v(b_j^{-1} b_i) > 0$ for $i \neq j$ and $vz = 0$, we find that $b_j^{-1} h(z) v = zv$ and that the reduction of H modulo v is $X^j - (zv)^j$. Since $p \nmid j$, this polynomial admits zv as a simple root. Hence by Hensel's Lemma, there is a unique root of H in $K(h(z))^h$ whose residue is zv. As $H(z) = 0$ and zv is a simple root of $X^j - (zv)^j$, this root must be z, which shows that $z \in K(h(z))^h$. On the other hand, $h(z) \in K(z)^h$, so $K(z)^h = K(h(z))^h$. □

Use of the normal forms

Let us return to the normal form that we have found in the equal characteristic case. We observe that

$$g(z) \;=\; \vartheta^p - \vartheta \;\in\; K(\vartheta)\,.$$

If we can show that

$$K(z)^h = K(g(z))^h\,,$$

then we will obtain that $K(z)^h \subseteq K(\vartheta)^h$, whence

$$F \;=\; K(z)^h(\vartheta) \;=\; K(\vartheta)^h\,,$$

showing that F is again of the same form as $E = K(z)^h$.

Use of the normal forms

We apply the lemma we have just proved before to $h(X) := g(X) - a_0$. We find that

$$K(z)^h \;=\; K(h(z))^h \;=\; K(h(z)+a_0)^h \;=\; K(g(z))^h\,,$$

as desired.

Similar arguments work in the mixed characteristic case to show that F is of the same form as E.

HRT in the case of an algebraically closed ground field

Assume that K is algebraically closed (hence in particular a tame field). Then vK is divisible and Kv is algebraically closed. We consider an immediate function field $(F|K,v)$ of transcendence degree 1. Since K is perfect, we can pick $x \in F$ such that $F|K(x)$ is finite and separable. Since $(K(x)|K,v)$ is immediate, $vK(x) = vK$ is divisible and $K(x)v = Kv$ is algebraically closed. Consequently, $K(x)^r = K(x)^h$, and thus the extension $F.K(x)^h|K(x)^h$ is a finite tower of Galois extensions of degree p. By induction on the extensions in the tower, using what we have just shown before, we can find some $x' \in F.K(x)^h = F^h$ such that $F.K(x)^h = K(x')^h$, so that $F \subset K(x')^h$. However, to prove HRT in this special case, we need to find $x' \in F$.

HRT in the case of an algebraically closed ground field

Assume first that x' lies in the completion K^c of (K, v). Since K is algebraically closed, so is its completion. Thus $K(x')$ and hence also $K(x')^h$ and F can be assumed embedded in K^c. Then it follows from the second special case we discussed in the last lecture that (F, v) is henselized rational.

Now assume that $x' \notin K^c$.

Theorem

Take a valued field (K, v) of rank 1 and an immediate function field $(F|K, v)$ of transcendence degree 1. Assume that there is some $x' \in F^h \setminus K^c$ such that $F^h = K(x')^h$ and x' is limit of a pseudo Cauchy sequence in K of transcendental type. Then there is already some $y \in F$ such that $F^h = K(y)^h$.

HRT in the case of an algebraically closed ground field

This theorem is proven in

Kuhlmann, F.-V. – Vlahu, I.: *The relative approximation degree*, Mathematische Zeitschrift **276** (2014), 203-235.

The idea of the proof is as follows. Since $x' \in F^h$ and (F, v), as an immediate extension of (K, v), has rank 1, it follows that x' lies in the completion of (F, v). On the other hand, as $x' \notin K^c$, we know that there is some $\gamma \in vK$ such that $\gamma > v(x' - K)$. So there exist elements $y \in F$ with $v(x' - y) \geq \gamma$, and it is shown that for every such y we have that $K(y)^h = K(x')^h$.

Separable-algebraically closed ground field

Now assume that (K, v) is separable-algebraically closed (hence in particular a separably tame field), and take an immediate function field $(F|K, v)$ of transcendence degree 1 such that $F|K$ is separable. Since $(F|K, v)$ is immediate and transcendental, v cannot be trivial on K. Therefore, again we have that vK is divisible and Kv is algebraically closed. Since $F|K$ is assumed to be separable, also in this case we can pick $x \in F$ such that $F|K(x)$ is finite and separable. Now the remainder of the proof proceeds as before.

HRT in the case of a tame ground field of rank 1

Having proved the HRT in the case of algebraically closed ground fields of rank 1, we wish to generalize the result to the case of tame ground fields of rank 1. This means that we have to pull down henselian rationality through tame extensions. The following theorem was proved in the above cited article:

Theorem

Let (K,v) be an algebraically maximal field of rank 1, and let $(F|K,v)$ be an immediate function field of transcendence degree 1, with $F \not\subset K^c$. If $(F.L|L,v)$ is a henselized rational function field for some tame extension $(L|K,v)$, then also $(F|K,v)$ is a henselized rational function field.

The degree $[K(x)^h : K(f(x))^h]$

The two theorems we have cited from the above article are based on the compution of the degree

$$[K(x)^h : K(f(x))^h]$$

in the situation where $(K(x)|K,v)$ is an immediate extension with $x \notin K^c$ limit of a pseudo Cauchy sequence in K of transcendental type, and $f \in K[X]$. We have already seen that if $vx = 0$ and there is $j \leq \deg f$ such that j is not divisible by $p = \operatorname{char} Kv$ and the coefficient of x^j is the unique one of minimal value among all coefficients of f, other than the constant term, then

$$[K(x)^h : K(f(x))^h] \;=\; 1\,.$$

The degree $[K(x)^h : K(f(x))^h]$

For arbitrary polynomials f, the approach is to write

$$f(x) \; = f(c) + \sum_{i>0} \partial_i f(c)(x-c)^i \quad \text{with } c \in K\,.$$

For all $c \in K$ with $v(x-c)$ sufficiently large, there will be a single nonzero $j \leq \deg f$ such that $\partial_j f(c)(x-c)^j$ is the unique summand of minimal value among all summands in the sum on the right hand side. By Kaplansky's result that we have cited early on in this lecture, j is always a power of p. In the article cited above, it is shown that

$$[K(x)^h : K(f(x))^h] \; \leq \; j\,.$$

The degree $[K(x)^h : K(y)^h]$

Based on these results, an upper bound can be computed for the degree

$$[K(x)^h : K(y)^h]$$

for elements $y \in K(x)^h$ under the additional condition that (K, v) has rank 1. Because of this assumption, $K[x]$ lies dense in $K(x)^h$, so that y can be approximated by polynomials $f(x) \in K[x]$. For large enough $v(y - f(x))$ it can be shown that the upper bound for $[K(x)^h : K(f(x))^h]$ is also an upper bound for $[K(x)^h : K(y)^h]$.

The precise computation of these degrees is an open problem.

Proof of HRT: final reduction steps

In order to finish the proof of the HRT, it remains to reduce the general assertion to the case of rank 1. As in the proof of the GST, this is done in two steps:

• Reduction from arbitrary rank to finite rank. Again, the idea is to use fields of definition small enough to have finite rank. As can be expected, the lemma on relative algebraic closures in tame fields plays a crucial role.

• Reduction from finite rank to rank 1. This proceeds along similar lines as in the proof of the GST. It should be pointed out that an application of the GST may become necessary; even if $(F|K, v)$ is immediate, it may become residue-transcendental for some coarsening v_1 of v.

We will now describe an application of GST and HRT important for the model theory of valued fields which in turn is used to prove theorems on Zariski spaces of valuations.

Recall: embedding theorem for Abhyankar valuations

Recall the following embedding theorem from Lecture IV:

Theorem

Take a function field $F|K$ and an Abhyankar valuation v of $F|K$ such that (K,v) is a defectless field. Assume that vF/vK is torsion free and $Fv|Kv$ is a separable extension. Assume further that (K^*,v^*) is a henselian extension of (K,v) and there are embeddings

$$\iota_v \,:\; vF \hookrightarrow v^*K^* \quad \text{fixing the elements of } vK$$

and

$$\iota_r \,:\; Fv \hookrightarrow K^*v^* \quad \text{fixing the elements of } Kv\,.$$

Then these embeddings can be lifted to a valuation preserving embedding of (F,v) in (K^*,v^*) over K (i.e., fixing the elements of K).

A characterization of tame fields

We wish to extend this embedding theorem in order to handle extensions of tame fields. We will need the following characterization of tame fields:

Lemma

Take a valued field (K, v) of residue characteristic $p > 0$. Then (K, v) is a tame field if and only if it is algebraically maximal, vK is p-divisible, and Kv is perfect.

Let us take an extension $(L|K,v)$ of valued fields and an elementary extension (K^*,v^*) of (K,v). This means that (K^*,v^*) satisfies the same elementary sentences with parameters from K in the language of valued fields as (K,v).

Further, we assume that (K^*,v^*) is $|L|^+$-saturated. The role of valued function fields becomes obvious from the following lemma:

Lemma

If every finitely generated subextension (F,v) of (L,v) over (K,v) can be embedded over K in (K^*,v^*), then also (L,v) can be embedded over K in (K^*,v^*).

Application: embedding theorem for tame fields

Now let us assume in addition that (K, v) is a tame field and that there are embeddings

$$\iota_v : \; vL \hookrightarrow v^*K^* \quad \text{fixing the elements of } vK$$

and

$$\iota_r : \; Lv \hookrightarrow K^*v^* \quad \text{fixing the elements of } Kv\,.$$

We can construct an algebraic extension $(L'|L, v)$ such that
• vL' is p-divisible and vL'/vL is a p-group,
• $L'v$ is perfect and $L'v|Lv$ is purely inseparable,
• (L', v) is algebraically maximal.
Then by the lemma stated earlier, (L', v) is a tame field.

Since (K^*, v^*) is an elementary extension of (K, v), it is also a tame field. So v^*K^* is p-divisible and K^*v^* is perfect. Consequently, ι_v can be extended to an embedding of vL' in v^*K^*, and ι_r can be extended to an embedding of $L'v$ in K^*v^*. Hence we can assume from the start that (L, v) is a tame field.

Note that we are not assuming that the transcendence degree of $L|K$ is finite.

Maximal subextension without transcendence defect

We wish to choose a set $\mathcal{T}_0$ as in the case of finite transcendence degree. We take

$$\mathcal{T}_0 = \{x_i, y_j \mid i \in I, j \in J\} \subset L$$

such that

• the values vx_i , $i \in I$, form a maximal set of elements in vL rationally independent over vK,

• the residues $y_j v$, $j \in J$, form a transcendence basis of $Lv|Kv$.

Consequently, $vL/vK(\mathcal{T}_0)$ is a torsion group, and $Kv|K(\mathcal{T}_0)v$ is algebraic.

It follows that the relative algebraic closure L_0 of $K(\mathcal{T}_0)$ in L is again a tame field and that $(L|L_0,v)$ is immediate. Moreover, it is not hard to show that if $(F|K,v)$ is any finitely generated subextension of $(L_0|K,v)$, then it is an extension of finite transcendence degree without transcendence defect. Hence by the embedding theorem from Lecture IV, the restrictions of ι_v and ι_r can be lifted to a valuation preserving embedding of (F,v) in (K^*,v^*) over K. By the saturation of (K^*,v^*), this implies that ι_v and ι_r can be lifted to a valuation preserving embedding of (L_0,v) in (K^*,v^*) over K.

Now we have to extend the embedding of L_0 to an embedding of L. Again by the saturation of (K^*, v^*), it suffices to show that it can be extended to every finitely generated subextension $(F|L_0, v)$ of the immediate extension $(L|L_0, v)$. We take F' to be the relative algebraic closure of F in L; since $(L|F, v)$ is immediate, (F', v) is a tame field. As in Lecture IV, we can present $(F'|L_0, v)$ as a finite tower of extensions $(F'_{i+1}|F'_i, v)$ of transcendence degree 1 of tame fields. By induction on the extensions in this tower, this reduces our task to showing that an embedding of some (F'_i, v) can be extended to an embedding of (F'_{i+1}, v). Once again, it suffices to embed finitely generated subextensions. These are immediate function fields of transcendence degree 1 over the tame field (F'_i, v). They are contained in $(F'_i(x)^h, v)$ for some x that is limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in F'_i of transcendental type.

Application: embedding theorem for tame fields

In order to construct the embedding of (F_i', v) in (K^*, v^*) to an embedding of $(F_i'(x), v)$, we consider the pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$. As x is a limit, we know that

$$v(x - a_\nu) \;=\; v(a_{\nu+1} - a_\nu) \quad \text{for all } \nu < \lambda\,. \tag{14}$$

Every finite subset of these formulas can be satisfied by some $a \in K$ in place of x. Indeed, let ν_0 be the maximal index appearing in this finite set. Then we can take $a = a_{\nu_0+1}$.
Now the saturation of (K^*, v^*) implies that there is some $x^* \in K^*$ such that (14) holds for x^* in place of x. This means that also x^* is a limit of $(a_\nu)_{\nu<\lambda}$. Theorem 2 of Kaplansky's paper *Maximal fields with valuations* shows that the embedding of F_i' can be extended to a valuation preserving embedding of $(F_i'(x), v)$ by sending x to x^*.
By the universal property of the henselization, the embedding can be extended to $(F_i'(x)^h, v)$ since the tame field (K^*, v^*) is henselian.

Application: embedding theorem for tame fields

We have now proved the following embedding theorem:

Theorem

Assume that (K,v) is a tame field, and that $(L|K,v)$ is an extension of valued fields. Assume further that (K^*,v^*) is an $|L|^+$-saturated elementary extension of (K,v) and there are embeddings

$$\iota_v : \; vL \hookrightarrow v^*K^* \quad \text{over } vK$$

and

$$\iota_r : \; Lv \hookrightarrow K^*v^* \quad \text{over } Kv\,.$$

Then these embeddings can be lifted to a valuation preserving embedding of (L,v) in (K^*,v^*) over K.

This embedding theorem is used to prove the following Ax–Kochen–Ershov Principle for tame fields.

Application to the model theory of tame fields

Theorem

Take a tame field (K,v) and a valued field extension $(L|K,v)$. If vK is existentially closed in vL (in the language of ordered groups) and Kv is existentially closed in Lv (in the language of fields), then (K,v) is existentially closed in (L,v) (in the language of valued fields).

Using model theoretic machinery (in particular, Robinson's Test), one deduces from this theorem the following Ax–Kochen–Ershov Principle:

Theorem

Take an extension $(L|K,v)$ of tame fields. If $vK \subseteq vL$ is an elementary extension (in the language of ordered groups) and $Lv|Kv$ is an elementary extension (in the language of fields), then $(L|K,v)$ is an elementary extension (in the language of valued fields).

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