The theory of the defect and its application to the problem of local uniformization, V

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Reduction to rank 1

Let us continue with our reduction steps in the proof of the GST.

Every valuation v of rank n can be written as a composition $v = v_1 \circ \ldots \circ v_n$ of rank 1 valuations. Here "composition" does not mean the usual composition of functions, but the place associated with v is indeed the usual composition of the places associated with the v_i (well, with a little extra care for the value ∞). Then we have that v is defectless if and only if all v_i are; more precisely:

Proposition

A valued field $(E, v_1 \circ \ldots \circ v_n)$ is a defectless field if and only if (E, v_1) and $(Ev_1 \ldots v_k, v_{k+1})$, $1 \leq k \leq n-1$, are.

Reduction to rank 1

Note that if $v = v_1 \circ v_2$, then an element t that is value-transcendental with respect to v may be residue-transcendental with respect to the coarsening v_1 of v.

After these reduction steps, we have arrived at the task of showing:
If $(K(t)|K, v)$ is a valued rational function field of rank 1 over an algebraically closed field K with a valuation-transcendental generator t, then $(K(t)^h, v)$ is a defectless field.

Lifting extensions to the absolute ramification field

Take a henselian field (K, v) and a finite extension $(L|K, v)$. Our goal is to investigate the defect of $(L|K, v)$. As before, we denote the absolute ramification field of (K, v) by (K^r, v).

We recall the following proposition from the second lecture:

Proposition

Take a henselian field (K, v) and a tame extension (N, v) of (K, v). Then for any finite extension $(L|K, v)$,

$$\mathrm{d}(L|K, v) \;=\; \mathrm{d}(L.N|N, v)\,.$$

Further, we recall that an extension $(N|K, v)$ is tame if and only if it is a subextension of $(K^r|K, v)$.

Consequently, we have that

$$d(L|K, v) \;=\; d(L.K^r|K^r, v)\,.$$

If char $K = p > 0$, then the extension $L|K$ may not be separable, in which case L is larger than the maximal separable subextension L_s of $L|K$. Likewise, $L_s.K^r|K^r$ is the maximal separable subextension of $L.K^r|K^r$. The purely inseparable extension $L.K^r|L_s.K^r$ can be presented as a tower of extensions of degree p.

Now we wish to analyze the structure of the maximal separable subextension. For simplicity we assume that $L|K$ is separable. We do not assume that K has positive characteristic, but we do assume that char $Kv = p > 0$.

In a theorem in the first lecture it was stated that for every normal extension the ramification group is a p-group. Applying this to the normal extension $K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K, we see that the absolute Galois group of K^r, the Galois group of $K^{\mathrm{sep}}|K^r$, is a (pro-)p-group, i.e., $K^{\mathrm{sep}}|K^r$ is a p-extension. We take N to be the normal hull of the extension $L.K^r|K^r$. Then the Galois group G of $N|K^r$ is a quotient of a pro-p-group and is thus a finite p-group.

A theorem about p-groups

The Frattini subgroup of an arbitrary finite group G is defined to be the intersection of all maximal proper subgroups of G and is denoted by $\Phi(G)$. A group G is called elementary-abelian if it is of the form $\mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_n\mathbb{Z}$ for (not necessarily distinct) prime numbers $p_1, \ldots, p_n$. Consequently, an elementary-abelian p-group is a finite product of copies of $\mathbb{Z}/p\mathbb{Z}$, that is, a finite dimensional $\mathbb{F}_p$-vector space.

Theorem

Let G be any finite p-group.
a) If H is a maximal proper subgroup of G, then $H \lhd G$ and $(G : H) = p$. Consequently, $G/\Phi(G)$ is elementary-abelian.
b) For every subgroup $H \subset G$ there exists a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ such that $H_{i-1} \lhd H_i$ and $(H_i : H_{i-1}) = p$ for $i = 1, \ldots, n$. In particular, every finite p-group is solvable.

Finite subextensions of p-extensions

Via Galois correspondence, this theorem implies:

Corollary

Every finite subextension of a p-extension is a tower of Galois extensions of degree p.

So we find that the extension $L.K^r|K^r$ (if separable) is a tower of Galois extensions of degree p.

What we aim to prove

Recall what we wish to prove:

If $(K(t)|K,v)$ is a valued rational function field of rank 1 over an algebraically closed field K with a valuation-transcendental generator t, then $(K(t)^h,v)$ is a defectless field.

We take a finite extension $(L|K(t)^h,v)$ and aim to show that it is defectless. As we have already shown that $(K(t)^h,v)$ is an inseparably defectless field, and the same holds for the maximal separable subextension (L_s,v) of $(L|K(t)^h,v)$, we know that the purely inseparable extension $(L|L_s,v)$ is defectless. It remains to show that the separable extension $(L_s|K(t)^h,v)$ is defectless, so we may assume from the start that $L|K(t)^h$ is separable.

Reduction to Galois extensions of degree p

By what we have shown, $L.K(t)^r|K(t)^r$ is a tower of Galois extensions of degree p, so the idea is to show by induction on the extensions in the tower that each of them is defectless. However, the field $K(t)^r$ is too large for us to handle. While it is still of rank 1, it is far from being the henselization of a valued rational function field $K(t)$ with valuation-transcendental generator t.

Reduction to Galois extensions of degree p

Let $K(t)^r(a)|K(t)^r$ be the first Galois extension of degree p in the tower. Again, we employ the tool of "field of definition": there is a finite extension E of $K(t)^h$ in $K(t)^r$ such that $E(a)|E$ is a Galois extension of degree p. Since $(K(t)^r|E,v)$ is tame, we have that

$$\mathrm{d}(E(a)|E,v) = \mathrm{d}(K(t)^r(a)|K(t)^r,v)\,.$$

Hence it suffices to show that $(E(a)|E,v)$ is defectless. However, what is the structure of E?

Since $(E|K(t)^h,v)$ is a finite extension, the Fundamental Inequality shows that $vE/vK(t)$ and $Ev|K(t)v$ are finite.

Assume first that t is a value-transcendental generator. Since $vK(t) = vK \oplus \mathbb{Z}vt$, vK is divisible and $vE|vK(t)$ is finite, we have that

$$vE \;=\; vK \oplus \mathbb{Z}\frac{vt}{n}$$

for some $n \geq 1$. In the value-transcendental case, $K(t)v = Kv$, and as Kv is algebraically closed and $Ev|K(t)v$ is finite, we have that $Ev = Kv = K(t)v$. Since $(E|K(t)^h, v)$ is a tame extension, it follows that $[E : K(t)^h] = (vE : vK(t)) = n$ and that n is prime to char Kv. Therefore, Hensel's Lemma can be used to find an element s in the henselian field E such that $s^n = ct$ for some $c \in F$ of value 0 (exercise left to the audience). Then $vs = \frac{vt}{n}$, $K(t)^h \subset K(s)^h$, and

$$[E : K(t)^h] \;\geq\; [K(s)^h : K(t)^h] \;\geq\; (vK(s) : vK(t)) \;\geq\; n \;=\; [E : K(t)^h] .$$

Hence, equality holds everywhere, so $E = K(s)^h$. As vs is rationally independent over vK, this is a henselized rational function field with value-transcendental generator s.

The residue-transcendental case

Now assume that t is a residue-transcendental generator. Then $vK(t) = vK$, which is divisible, and since $vE/vK(t)$ is finite, we have that $vE = vK = vK(t)$. Hence E lies in the absolute inertia field of $K(t)^h$. In this case we call E a henselized inertially generated function field. Since $Ev|K(t)v$ is a finite separable extension, we can choose a generator ζ of this extension and use Hensel's Lemma to find an element $a \in E$ with $av = \zeta$ and $E = K(t)^h(a)$. In general, E will not be henselized rational, as the residual function field $Kv(tv, av)|Kv$ may not be rational. So we will have to deal with Galois extensions of degree p of henselized inertially generated function fields of the above form.

Induction on the number of extensions in the tower

Assume that we have proved the following result:

(*) *Take an extension $(E|K,v)$ of rank 1 of an algebraically closed field K such that*

a) $E = K(t)^h$ where t is value-transcendental over K, or

b) $E = K(t)^h(a)$ where t is residue-transcendental over K and $K(tv,av)|K(tv)$ is a separable extension of degree equal to $[K(t)^h(a) : K(t)^h]$.

Then every Galois extension $(F|E,v)$ of degree p is defectless.

In order to proceed by induction on the number of extensions in the tower, we need to show that F is again of the same form as E.

Induction on the number of extensions in the tower

Once we have proved **(*)**, there is a straightforward but important consequence:

Corollary

Take $(E|K,v)$ as in the assumptions of **(*)**. *Then (E,v) does not admit any non-trivial immediate algebraic extension.*

Indeed, suppose that $(L|E,v)$ is a non-trivial immediate algebraic extension. Then it also has a finite non-trivial immediate subextension, so we may assume that $L|E$ is finite. It is unibranched, as (E,v) is henselian. Thus its defect $\mathrm{d}(L|E,v)$ is equal to its degree. However,

$$\mathrm{d}(L.E^r|E^r,v) \;=\; \mathrm{d}(L|E,v) \;=\; [L:E] \;\geq\; [L.E^r:E^r]\,,$$

which implies that also $(L.E^r|E^r,v)$ is immediate. This contradicts the fact that by **(*)** the first extension in the tower is defectless.

Induction on the number of extensions in the tower

From this corollary, we obtain:

Lemma

Every henselized function field (E, v) of rank 1 and of transcendence degree 1 without transcendence defect over an algebraically closed ground field K is of the form as in the assumptions of **(*)**.

Induction on the number of extensions in the tower

Let us describe the proof of the lemma in the value-transcendental case.

Assume that E contains a value-transcendental element t_0. Since $K(t_0)v = Kv$ is algebraically closed and $Ev|K(t_0)v$ is finite, we have that $Ev = Kv = K(t_0)v$. Further, vE is a finite extension of $vK(t_0) = vK \oplus \mathbb{Z}vt_0$. Since vK is divisible, we have that $vE = vK \oplus \mathbb{Z}\alpha$ for some $\alpha \in vE$. Choose $t \in E$ such that $vt = \alpha$. Then $vE = vK \oplus \mathbb{Z}vt = vK(t)$ by the Bourbaki theorem, and $Ev = Kv = K(t)v$. Now the henselian field E contains the henselization $K(t)^h$, and we have just shown that $E|K(t)^h$ is an immediate extension. But by the above corollary, $K(t)^h$ does not admit any non-trivial immediate algebraic extension. This shows that $E = K(t)^h$.

If $(E|K,v)$ is as in the assumptions of **(*)**, and $(F|E,v)$ is a Galois extension of degree p, then (F,v) is also a henselized function field of rank 1 and of transcendence degree 1 without transcendence defect over the algebraically closed ground field K, so it satisfies the assumption of the above lemma. Consequently, F is again of the same form as E. Hence we can apply **(*)** again, and by induction on the number of extensions in the tower, we obtain that every finite separable extension of (E,v) is defectless. By what we have said before, this will complete the proof of the first assertion of the GST.

It remains to prove **(*)**.

The equal characteristic case

We will now assume that $(E|K,v)$ satisfies the assumptions of **(*)** and take a Galois extension $(F|E,v)$ of degree p. We wish to show that it is defectless.

Assume that $\mathrm{char}\, E = p$. Then $F|E$ is an Artin–Schreier extension. That is, the extension is of the form

$$F \;=\; E(\vartheta) \quad \text{where} \quad a \;:=\; \vartheta^p - \vartheta \in E\,. \tag{1}$$

A polynomial f over a field E is called additive if $f(b+c) = f(b) + f(c)$ for all b, c in any extension field of E. Over a field of characteristic p, the additive polynomials are exactly the ones of the form

$$\sum_{i=0}^{k} c_i X^{p^i}\,.$$

The equal characteristic case

In particular, the polynomial $\wp(X) = X^p - X$ is additive. Hence for every $d \in E$ we have that $F = E(\vartheta - d)$ and that

$$\wp(\vartheta - d) \;=\; \wp(\vartheta) - \wp(d) \;=\; a - d^p + d \;\in\; a + \wp(E)\;. \tag{2}$$

This shows that we can replace a by any other element of $a + \wp(E)$ without changing the Artin-Schreier extension. Note that by Hensel's Lemma, $X^p - X - a$ has a root in the henselian field E whenever $va > 0$. For the valuation ideal $\mathcal{M}_E$ of E, we thus have:

$$\mathcal{M}_E \;\subset\; \wp(E)\;. \tag{3}$$

The idea is to find a normal form for a that allows us to read off that the extension is defectless.

The mixed characteristic case

Assume now that char $K = 0$ while char $Kv = p$. As E contains the algebraically closed field K, it contains all p-th roots of unity. Hence the Galois extension $F|E$ of degree p is a Kummer extension. That is, the extension is of the form

$$F = E(\eta) \quad \text{where} \quad a := \eta^p \in E\,. \tag{4}$$

For every $d \in E^\times$ we have that

$$F = E(\eta d), \quad (\eta d)^p = ad^p \in a(E^\times)^p\,, \tag{5}$$

showing that we can replace a by any other element of $a(E^\times)^p$ without changing the extension $F|E$. Again, we wish to find a normal form for a that allows us to read off that the extension is defectless.

In what follows, we will sketch parts of the proof for the value-transcendental case. We consider the ring

$$R \;=\; K[t,t^{-1}]$$

which consists of all finite Laurent series

$$r(t) \;=\; \sum_{i\in I} c_i t^i \;,\quad c_i\in K\,,\quad I\subset\mathbb{Z}\text{ finite.} \tag{6}$$

Lemma

Take $E = K(t)^h$ of rank 1 with t value-transcendental over K. Then R is dense in E. If $\operatorname{char} E = p$*, then this implies that $E = R + \wp(E)$.*

The equal characteristic case

We deduce the following normal form, which settles the equal characteristic value-transcendental case:

Proposition

Assume that $\mathrm{char}\, E = p$. *Then*

$$F = E(\vartheta) \quad \textit{where} \quad \vartheta^p - \vartheta = \sum_{i \in I} c_i t^i\,, \quad c_i \in K$$

with finite non-empty $I \subset \mathbb{Z} \setminus p\mathbb{Z}$ *such that*

$$\forall i \in I:\; v c_i t^i < 0\,.$$

The equal characteristic case

Since vt is non-torsion over vK, the summands $c_i t^i$ of

$$a := \sum_{i\in I} c_i t^i$$

have pairwise distinct values. Therefore,

$$va = \min_{i\in I} vc_i + ivt < 0 .$$

Since $I \subset \mathbb{Z} \setminus p\mathbb{Z}$, this value is not p-divisible in $vK(t) = vK \oplus \mathbb{Z}vt$. As $va < 0$, we must have that $v\vartheta < 0$. It follows that $v\vartheta^p = pv\vartheta < v\vartheta$ and thus

$$pv\vartheta = v(\vartheta^p - \vartheta) = va .$$

This shows that $v\vartheta \notin vE$, proving that $(F|E, v)$ is not immediate and must therefore be defectless.

The equal characteristic case

Let us sketch the proof of the proposition.

Since $K[t,t^{-1}]$ is dense in $E = K(t)^h$, there is $r \in K[t,t^{-1}]$ such that $a - r \in \mathcal{M}_E \subset \wp(E)$. Taking $d \in E$ such that $\wp(d) = a - r$, we obtain that $F = E(\vartheta - d)$ and

$$\wp(\vartheta - d) \;=\; \wp(\vartheta) \;-\; \wp(d) \;=\; a - (a - r) \;=\; r\,.$$

Thus we can replace a by r, so we may assume from the start that a is a finite Laurent series which only contains summands of value ≤ 0. Further, we may replace a summand $c_{jp}t^{jp}$ of a by a summand $c_j' t^j$ with $c_j' = c_{jp}^{1/p} \in K$, since

$$(\vartheta - c_j' t^j)^p \;-\; (\vartheta - c_j' t^j) \;=\; a - c_{jp}t^{jp} + c_j' t^j\,.$$

After a finite repetition of this procedure we arrive at a finite Laurent series $c_0 + \sum_{i\in I} c_i t^i$ with $I \subset \mathbb{Z} \setminus p\mathbb{Z}$. In this procedure, all summands remain of value ≤ 0.

Finally, as K is algebraically closed, $c_0 = \wp(d)$ for some $d \in K$. Then

$$\wp(\vartheta - d) \;=\; \wp(\vartheta) - \wp(d) \;=\; \wp(\vartheta) - c_0\,,$$

and after replacing ϑ by $\vartheta - d$ we may assume that $c_0 = 0$.

We deduce the following normal form, which settles the mixed characteristic value-transcendental case:

Proposition

Assume that char $E = 0$ *while* char $Ev = p$. *Then*

$$F \;=\; E(\eta) \quad \textit{where} \;\; \eta^p = t^m u\,,$$

with $m \in \{0,\ldots,p-1\}$ *and* $u \in K[t,t^{-1}]$ *a* 1*-unit of the form*

$$u \;=\; 1+\sum_{i\in I} c_i t^i\,, \qquad c_i \in K\,,$$

with finite index set $I \subset \mathbb{Z}\setminus\{0\}$ *and* $vc_it^i > vp$ *whenever* $i \in p\mathbb{Z}$. *If* $vc_it^i \geq \frac{1}{p-1}vp$ *for all* $i \in I$, *then it may in addition be assumed that* $I \subset \mathbb{Z}\setminus p\mathbb{Z}$.

We cannot have $m = 0$ and $I = \emptyset$ at the same time since otherwise, $\eta^p = 1$ and the extension $F|E$ would be trivial. If $m \neq 0$, then $v\eta = \frac{m}{p} vt \notin vK \oplus \mathbb{Z} vt$ and as before, the extension is defectless.

Let us now assume that $m = 0$ and $I \neq \emptyset$. Again since vt is non-torsion over vK, there is a unique $i_0 \in I$ such that $c_{i_0} t^{i_0}$ is the summand of smallest value. If i_0 were in $p\mathbb{Z}$, we would have that

$$vc_i t^i \;\geq\; c_{i_0} t^{i_0} \;>\; vp \;\geq\; \frac{1}{p-1} vp$$

for all $i \in I$, whence $I \subset \mathbb{Z} \setminus p\mathbb{Z}$, a contradiction.

The mixed characteristic case

We have now arrived at the case where

$$\eta^p = 1 + c_{i_0} t^{i_0} + \text{ summands of higher value}$$

with $i_0 \notin p\mathbb{Z}$. How do we deduce that the extension is defectless? The answer to this question is closely connected with the answer to the question how we prove the proposition. After all, as $\operatorname{char} E = 0$, we cannot employ additivity; we are dealing with a Kummer extension and not an Artin–Schreier extension. Recall that we can replace a by any other element in

$$a(E^\times)^p$$

without changing the extension. We will transform this multiplicativity into something that looks like additivity, and the Kummer polynomial into a polynomial that looks like an Artin–Schreier polynomial.

p-th roots of 1-units

A well known application of Hensel's Lemma shows that in every henselian field, each 1-unit (i.e., element of the form $1 + b$ with $vb > 0$) is an n-th power for every n not divisible by the residue characteristic p. If p divides n, then one considers the level vb of the 1-unit. For our purposes, we need a sufficiently precise condition for a 1-unit to have a p-th root in a henselian field of mixed characteristic.

We take C to be an element in the algebraic closure of $\mathbb{Q}$ such that

$$C^{p-1} = -p\,.$$

Our algebraically closed ground field K of characteristic 0 contains such an element C. Note that

$$C^p = -pC \quad \text{and} \quad vC = \frac{1}{p-1}vp > 0\,. \tag{7}$$

Consider the polynomial

$$X^p - (1+b) \tag{8}$$

with $b \in E$. Performing the transformation

$$X \;=\; CY + 1\,, \tag{9}$$

then dividing by C^p and using that $C^p = -pC$, we obtain the polynomial

$$f(Y) \;=\; Y^p + g(Y) - Y - \frac{b}{C^p} \tag{10}$$

with

$$g(Y) \;=\; \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i \tag{11}$$

a polynomial with coefficients in $\tilde{Q}$ of value greater than 0.

Hence if $v\frac{b}{C^p} > 0$, that is, if

$$vb \;>\; vC^p \;=\; \frac{p}{p-1}\, vp\,,$$

then the reduction of f modulo v is the polynomial $X^p - X$ which splits in Ev. Then by Hensel's Lemma, f has a root $\vartheta \in E$, and the element $\eta = C\vartheta + 1 \in E$ satisfies $\eta^p = 1 + b$, showing that

$$1 + b \;\in\; E^p\,.$$

Using this result, one proves:

Lemma

Take any 1-units $1+b$ and $1+c$ in E. Then the following assertions hold:

a) $1+b \in (1+b+c)\cdot(E^\times)^p$ *if* $vc > \frac{p}{p-1}vp\,.$

b) $1+b \in (1+b+c)\cdot(E^\times)^p$ *if* $1+c \in (E^\times)^p$ *and* $vbc > \frac{p}{p-1}vp\,.$

c) $1+c^p+pc \in (E^\times)^p$ *if* $vc^p > vp\,.$

d) $1+b-pc \in (1+b+c^p)\cdot(E^\times)^p$ *if* $vb \geq \frac{1}{p-1}vp$ *and* $vc^p > vp\,.$

The mixed characteristic case

These facts are used to show that in our proposition the summands $c_i t^i$ can be chosen to satisfy the assertions. For example, part d) is used to replace a summand of the form c^p by $-pc$ when certain conditions are met.

Further, part a) is used to remove every summand of value greater than $\frac{p}{p-1}vp$. Hence in the case where $m = 0$ and $I \neq \emptyset$, we know that the summand $c_{i_0} t^{i_0}$ of minimal value has value less than or equal to $\frac{p}{p-1}vp$. However, as $i_0 \neq 0$ and vt is non-torsion over vK, the value cannot be equal to $\frac{p}{p-1}vp \in vK$.

The mixed characteristic case

Applying the transformation (9) to the polynomial

$$X^p - 1 - \sum_{i\in I} c_i t^i \,,$$

we obtain the polynomial

$$f(Y) \;=\; Y^p + g(Y) + Y - \frac{1}{C^p} \sum_{i\in I} c_i t^i$$

where the constant term has the negative value

$$vc_{i_0} t^{i_0} - \frac{p}{p-1} vp \;\notin\; pvE \,.$$

The mixed characteristic case

Since g has coefficients of value greater than 0, as in the equal characteristic case one finds that the root ϑ of f satisfies

$$p v\vartheta \;=\; v\vartheta^p \;=\; vc_{i_0}t^{i_0} - \frac{p}{p-1}vp \;\notin\; pvE$$

and therefore, $v\vartheta \notin vE$. Hence also in this case the extension $(F|E,v)$ is defectless.

Kuhlmann, F.-V.: *Elimination of Ramification I: The Generalized Stability Theorem*, Trans. Amer. Math. **362** (2010), 5697–5727