MODEL THEORY OF TAME VALUED FIELDS AND BEYOND: RECENT DEVELOPMENTS AND OPEN QUESTIONS

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Abstract. We give a survey on recent developments in the model theory of valued fields since the introduction of the notion of "tame valued field", and the modifications and generalizations of this notion.

1. Introduction

The notion of "tame valued field" was first introduced in the author's PhD thesis; for an extended version of it, see [25]. The name was coined in collaboration with the author's very supportive supervisor Peter Roquette. Since then, tame valued fields and their model theory have been generalized, modified or applied in many research articles. One of the applications was to prove local uniformization of Abhyankar places in positive characteristic ([21]), and local uniformization by alteration of arbitrary places in positive characteristic ([22]). However, the developments in the model theory of valued fields have been much more complex, and they are what this survey is devoted to. It is based on a survey talk given at the conference *Recent Applications of Model Theory* held in June of 2025 at the Institute for Mathematical Science of the National University of Singapore. For the slides of the talk, see `https://www.fvkuhlmann.de/TalkSingapore2025Jun20nopause.pdf`, and `https://www.fvkuhlmann.de/AbstractSingapore2025.pdf` for the extended abstract.

Through the course of this survey, the reader will meet tame, separably tame and roughly tame fields, extremal fields, perfectoid fields, deeply ramified and roughly deeply ramified fields. Without any claim for completeness, several main theorems and contributors are presented and 16 open problems are listed. I hope that this will provide a useful basis for young as well as experienced mathematicians eager to attack these problems.

2. Some preliminaries

For a valued field (K, v), we denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$. By $(L|K, v)$ we denote an extension $L|K$ with a valuation v on L, where K is endowed with the restriction of v. In this

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case, there are induced embeddings of vK in vL and of Kv in Lv. The extension $(L|K,v)$ is called **immediate** if these embeddings are onto. A valued field (K,v) is called **algebraically maximal** if it does not admit nontrivial immediate algebraic extensions, and it is called **maximal** if it does not admit any nontrivial immediate extensions.

A valued field (K,v) is called **henselian** if for each algebraic extension $L|K$ the extension of v to L is unique. A finite extension $(L|K,v)$ of a henselian valued field (K,v) is called **defectless** if

$$[L:K] \;=\; (vL:vK)[Lv:Kv]\,.$$

This definition can be extended to valued fields that are not henselian by using the **fundamental inequality** (cf. (17.5) of [12] or Theorem 19 on p. 55 of [41]): given a valued field (K,v) and a finite field extension $L|K$, there are finitely many extensions of v to L and we have

$$(1)\qquad [L:K] \;\geq\; \sum_{i=1}^{\mathrm{g}} (v_iL:vK)[Lv_i:Kv]\,,$$

where $v_1,\ldots,v_{\mathrm{g}}$ are the distinct extensions of v from K to L. We say that (K,v) is **defectless in** L if equality holds in (1). A valued field is called a **defectless field** if it is defectless in each finite extension. A valued field is a defectless field if and only if any (and then every) of its henselizations is ([26, Theorem 8.9]).

The following are examples for defectless fields:

1) every trivially valued and every algebraically closed valued field,
2) every valued field with residue characteristic 0,
3) maximal fields (see the discussion at the beginning of Section 4 in [8]); in particular:
– with its p-adic valuation v_p, the field $\mathbb{Q}_p$ of p-adic numbers,
– with their t-adic valuations v_t, the Laurent series field $k((t))$ over a field k and every power series field $k((t^\Gamma))$ with coefficients in k and exponents in Γ (see [26, Theorem 8.26 and Corollaries 8.27 and 8.28]),
4) $(\mathbb{Q},v_p)$ and its henselization (by [26, Theorem 8.32]),
5) $(k(t),v_t)$ and its henselization (this is a special case of [29, Theorem 1.1]).

3. Tame fields

An algebraic extension $(L|K,v)$ of a henselian valued field (K,v) is called **tame** if every finite subextension $K'|K$ satisfies the following conditions:

(T1) the ramification index $(vK':vK)$ is not divisible by $\mathrm{char}\,Kv$,

(T2) the residue field extension $K'v|Kv$ is separable,

(T3) the extension $(K'|K,v)$ is defectless.

Remark. This notion of "tame extension" does not coincide with the notion of "tamely ramified extension" as defined on page 180 of O. Endler's book [12]. The latter definition requires (T1) and (T2), but not (T3). Our tame extensions are the defectless tamely ramified extensions in the sense of Endler's book. In particular, in our terminology, proper immediate algebraic extensions of henselian fields are not called tame (in fact, they cause a lot of problems in the model theory of valued fields).

A henselian valued field (K,v) is called a **tame field** if the algebraic closure K^{ac} of K with the unique extension of v is a tame extension of (K,v). It follows from conditions (T1)–(T3) that all tame fields are perfect. For a characterization of tame fields, see Theorem 8 below.

The perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$ of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field $\mathbb{F}_p$ with p elements is perfect but not tame, as the extension generated by a root of the Artin-Schreier polynomial $X^p - X - 1/t$ is immediate and therefore does not satisfy (T3).

For details on tame fields, see [31]. The results of that paper are now frequently applied in the model theory of valued fields. In particular:

Theorem 1 (Kuhlmann (2016))**.** *Tame fields (K,v) satisfy model completeness in the language $\mathcal{L}_{\mathrm{val}}$ of valued rings relative to the elementary theories of their value groups vK in the language of ordered groups and their residue fields Kv in the language of rings. If* $\operatorname{char} K = \operatorname{char} Kv$*, then also relative completeness and relative decidability hold.*

However, there are still daunting questions about tame fields that have remained unanswered.

Open problem 1: Do tame fields admit quantifier elimination in a suitable language?

The problem is that we do not know enough about **purely wild extensions**, i.e., algebraic extensions of a henselian valued field that are linearly disjoint from tame extensions. For background, see [34].

There are also open problems about the model theory of tame fields (K,v) that have **mixed characteristic**, i.e., $\operatorname{char} K = 0$ while $\operatorname{char} Kv = p > 0$. One question is whether (or under which additional conditions) they satisfy relative completeness and decidability. In the article [6], examples are given of two tame fields (K_1,v_1) and (K_2,v_2) with $v_1K_1 \equiv v_1K_2$ and $K_1v_1 \equiv K_2v_2$ such that $(K_1,v_1) \not\equiv (K_2,v_2)$. The difference to the case of tame fields of positive characteristic is that in this case the restriction of the valuations to the prime fields are trivial while in the mixed characteristic case they are p-adic and hence nontrivial.

Progress on this problem has been made in [37, Theorem 1.2]. Lisinski proves:

Theorem 2 (Lisinski (2021))**.** *Take tame fields (L,v) and (F,w) of mixed characteristic with residue characteristic $p > 0$ such that $vL \equiv wF$ in the language of ordered groups with a constant symbol π interpreted as $v(p)$ and $w(p)$, respectively, and that $Lv \equiv Fw$ in the language of rings. Assume that the relative algebraic closure (K,v) of $\mathbb{Q}$ in (L,v) is algebraically maximal and that vL/vK is torsion free. Assume further that every monic polynomial $f \in \mathbb{Z}[X]$ has a root in $\mathcal{O}_F$ if it has a root in $\mathcal{O}_L$. Then $(L,v) \equiv (F,w)$ in $\mathcal{L}_{\mathrm{val}}$.*

As mentioned already in Theorem 1, relative decidability is proven for tame fields of positive characteristic. As an application, we obtain:

Theorem 3 (Kuhlmann (2016))**.** *Take $q = p^n$ for some prime p and some $n \in \mathbb{N}$, and an ordered abelian group Γ. Assume that Γ is divisible or elementarily equivalent to the p-divisible hull of $\mathbb{Z}$. Then the $\mathcal{L}_{\mathrm{val}}$-elementary theory of the power* *series field* $\mathbb{F}_q((t^\Gamma))$ *with coefficients in the field* $\mathbb{F}_q$ *with* q *elements and exponents in* Γ*, endowed with its canonical valuation* v_t *, is decidable.*

Lisinski improves this result as follows ([37, Theorem 1]):

Theorem 4 (Lisinski (2021))**.** *Take a perfect field* $\mathbb{F}$ *of characteristic* $p > 0$ *whose elementary theory in the language of rings is decidable, and a* p*-divisible group* Γ *whose elementary theory in the language of ordered groups with a constant symbol* 1 *is decidable. Then the* $\mathcal{L}_{\rm val}(t)$*-elementary theory of* $\mathbb{F}((t^\Gamma))$ *is decidable.*

Here, $\mathcal{L}_{\rm val}(t)$ denotes the language $\mathcal{L}_{\rm val}$ with a constant symbol t.

Lisinski also proves a theorem giving a criterion for two tame fields containing $\mathbb{F}_p(t)$ to be equivalent in $\mathcal{L}_{\rm val}(t)$ that is analogous to Theorem 2 (see [37, Theorem 1.1]).

4. The fields $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ and their algebraic extensions

Since in 1965 Ax and Kochen in [5], and independently Ershov in [13], established the decidability of the elementary theory of $\mathbb{Q}_p$, several questions about the decidability of the elementary or the existential theory of local fields and their extensions have been answered, and several others have remained open. We have already seen some results in equal positive charateristic. In contrast, less is known in mixed characteristic, for instance about

- the totally ramified extension $\mathbb{Q}_p(\zeta_{p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining all p^n-th roots of unity, $n \in \mathbb{N}$,
- the totally ramified extension $\mathbb{Q}_p(p^{1/p^\infty})$ obtained from $\mathbb{Q}_p$ by adjoining a compatible system of p^n-th roots of p, $n \in \mathbb{N}$,
- the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$.

These fields with their canonical valuations are studied in [19].

Theorem 5 (Kartas (2024))**.** *The fields* $\mathbb{Q}_p(\zeta_{p^\infty})$ *and* $\mathbb{Q}_p(p^{1/p^\infty})$ *equipped with their unique extensions* v_p *of the* p*-adic valuation admit maximal immediate extensions which have decidable elementary* $\mathcal{L}_{\rm val}$*-theories.*

It was shown by W. Krull in [24] that every valued field (K, v) admits a maximal immediate extension (M, v) (the proof was later simplified by K. A. H. Gravett in [14]). All of these maximal immediate extensions are tame fields. But the fields themselves are not **Kaplansky fields**, i.e., fields satisfying hypothesis (A) in [17], and Kartas shows that there are uncountably many maximal immediate extensions with distinct elementary $\mathcal{L}_{\rm val}$-theories. This implies that uncountably many of them are not decidable.

Kartas proves a "perfectoid transfer theorem" ([19, Theorem A]) which transfers the decidability in certain extensions of $\mathcal{L}_{\rm val}$ of fields in equal positive characteristic to the decidability in $\mathcal{L}_{\rm val}$ of suitable untilts.

By Theorem 4, the perfectoid field $\mathbb{F}_p((t^\Gamma))$, where Γ is the p-divisible hull of $\mathbb{Z}$, is decidable in the language $\mathcal{L}_{\rm val}(t)$. Kartas constructs a suitable untilt K of $\mathbb{F}_p((t^\Gamma))$ which by the perfectoid transfer theorem is decidable in the language $\mathcal{L}_{\rm val}$. As $\mathbb{F}_p((t^\Gamma))$ is a maximal immediate extension of the completion of the perfect hull $\mathbb{F}_p(t^{1/p^\infty})$, which is the tilt of the completion of $\mathbb{Q}_p(p^{1/p^\infty})$, a theorem of Fargues and Fontaine can be used to show that K is a maximal immediate extension of the latter and hence also of $\mathbb{Q}_p(p^{1/p^\infty})$ itself. The case of $\mathbb{Q}_p(\zeta_{p^\infty})$ is similar. This completes the proof of Theorem 5.

Kartas notes that all tilts of the undecidable maximal immediate extensions of $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ are maximal immediate extensions of $\mathbb{F}_p((t))^{1/p^\infty}$. Being tame fields, they are decidable in the language $\mathcal{L}_{\mathrm{val}}$. But they are not decidable in the language $\mathcal{L}_{\mathrm{val}}(t)$.

Open problem 2: What is the structure of these extensions (apart from the fact that they are infinite)? What are the indications in their structure that distinguish the decidable from the undecidable extensions?

Kartas also notes that $\mathbb{Q}_p^{ab}$, being a Kaplansky field, admits a unique maximal immediate extension, and that it follows from the model theory of algebraically maximal Kaplansky fields that this extension is decidable in $\mathcal{L}_{\mathrm{val}}$.

Open problem 3: Are $\mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p^{ab}$ decidable in $\mathcal{L}_{\mathrm{val}}$? Are $\mathbb{F}_p((t))^{1/p^\infty}$ and $\mathbb{F}_p^{\mathrm{ac}}((t))^{1/p^\infty}$ decidable in $\mathcal{L}_{\mathrm{val}}$ or even $\mathcal{L}_{\mathrm{val}}(t)$?

Here $\mathbb{F}_p^{\mathrm{ac}}$ denotes the algebraic closure of $\mathbb{F}_p$.

We do not know the answers, but there are some conditional results connecting decidability of the mixed characteristic fields with those of the positive characteristic fields. Although these fields are not perfectoid, Kartas succeeds to deduce the following from the perfectoid transfer theorem.

Theorem 6 (Kartas (2024))**.**
(a) If $\mathbb{F}_p((t))^{1/p^\infty}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}(t)$-theory, then $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ have decidable elementary or existential $\mathcal{L}_{\mathrm{val}}$-theories, respectively.

(b) If $\mathbb{F}_p^{\mathrm{ac}}((t))^{1/p^\infty}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}(t)$-theory, then $\mathbb{Q}_p^{ab}$ has a decidable elementary or existential $\mathcal{L}_{\mathrm{val}}$-theory, respectively.

Open problem 4: What about the reverse direction?

In fact, if $\mathbb{Q}_p(\zeta_{p^\infty})$ or $\mathbb{Q}_p(p^{1/p^\infty})$ has a decidable $\mathcal{L}_{\mathrm{val}}$-theory, then $\mathbb{F}_p((t))^{1/p^\infty}$ has a decidable $\mathcal{L}_{\mathrm{val}}$-theory, and if $\mathbb{Q}_p^{ab}$ has a decidable $\mathcal{L}_{\mathrm{val}}$-theory, then so does $\mathbb{F}_p^{\mathrm{ac}}((t))^{1/p^\infty}$. This essentially follows from [16, Corollary 1.7.6], which says that if a perfectoid field K has a decidable $\mathcal{L}_{\mathrm{val}}$-theory, then so does its tilt. Since problem 4 is about non-complete valued fields, one also needs to use that the involved fields have the same elementary theory as their respective completions. So a more precise version of problem 4 is whether this argument can be improved to get decidability in $\mathcal{L}_{\mathrm{val}}(t)$ on the positive characteristic side. (For a related result, compare [16, Proposition 7.2.3].)

Let us point out that $\mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p^{ab}$ with their unique extensions of the p-adic valuation are not tame extensions of $\mathbb{Q}_p$. This brings us to the question: what can be said about infinite tame extensions of $\mathbb{Q}_p$? In [19], Kartas works with a precise formulation of resolution of singularities, which he calls "Log-Resolution". By the work of Hironaka, it is known that Log-Resolution holds in characteristic 0. Under the assumption that it also holds in positive characteristic, Kartas proves an existential Ax-Kochen/Ershov principle ([18, Theorem A]), from which the following decidability results for infinite tame extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ can be deduced:

- for primes $\ell \neq p$, $\mathbb{Q}_p(p^{1/\ell^\infty})$ has a decidable existential theory in the language of rings,
- for primes $\ell \neq p$, $\mathbb{F}_p((t))(t^{1/\ell^\infty})$ has a decidable existential theory in the language of rings enriched by a constant symbol t,
- the maximal tame extensions (also known as absolute ramification fields, see [30, Section 4]) of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ have decidable existential theories in the respective languages.

5. Separably tame fields

A henselian field is called a **separably tame field** if every separable-algebraic extension is a tame extension. We let $\mathcal{L}_{\rm val,Q}$ denote the language $\mathcal{L}_{\rm val}$ enriched by m-ary predicates Q_m, $m \in \mathbb{N}$, for p-independence. That is, in a field K of characteristic $p > 0$, Q_m is interpreted by

$$Q_m(x_1,\ldots,x_m) \Leftrightarrow \left\{ \begin{array}{l} \text{the monomials of exponents} < p \text{ in the } x_i\text{'s} \\ \text{are linearly independent over the subfield } K^p \\ \text{of } p\text{-th powers.} \end{array} \right.$$

Field extensions $L|K$ as $\mathcal{L}_{\rm val,Q}$-structures are separable, i.e., linearly disjoint from the perfect hull of K.

The model theory of separably tame fields is studied in the article [33].

Theorem 7 (Kuhlmann – Pal (2016)). *Separably tame fields (K, v) of positive characteristic and finite degree of inseparability satisfy completeness and decidability in $\mathcal{L}_{\rm val}$ relative to the elementary theories of their value groups vK in the language of ordered groups and of their residue fields Kv in the language of rings.*

In the language $\mathcal{L}_{\rm val,Q}$, they also satisfy relative model completeness.

In [1, Corollary 1.6], Anscombe removes the condition of finite degree of inseparability from the relative decidability result in $\mathcal{L}_{\rm val}$, and in [1, Theorem 1.5] from the other assertions of Theorem 7 in a language $\mathcal{L}_{{\rm val},\lambda}$ which is $\mathcal{L}_{\rm val,Q}$ with the predicates Q_m replaced by function symbols for Lambda functions (see [1, Definition 2.5]). To this end, Anscombe proves that the $\mathcal{L}_{{\rm val},\lambda}$-theory of equal characteristic separably tame valued fields has the **Lambda Relative Embedding Property** (see [1, Definition 4.10]). This is done by adapting the proofs of [31, Theorem 7.1], which shows that the elementary class of tame fields has the **Relative Embedding Property** (see [31, Section 6]), and of [33, Teorem 5.1], which shows that the elementary class of separably tame fields of finite degree of inseparability has the **Separable Relative Embedding Property** (see [33, Section 4]).

6. Perfectoid and deeply ramified fields

In [39, Definition 1.2] Peter Scholze defines a **perfectoid field** to be a complete nondiscrete rank 1 valued field of residue characteristic $p > 0$ such that $\mathcal{O}_K/p\mathcal{O}_K$ is **semiperfect**, that is, the Frobenius is surjective on $\mathcal{O}_K/p\mathcal{O}_K$. This implies that the value group is p-divisible. A valued field has **rank 1** if its value group is embeddable in the ordered additive group $\mathbb{R}$.

Neither "complete" nor "rank 1" are elementary properties. A suitable elementary class of valued fields containing the perfectoid fields is that of deeply ramified fields, studied in the article [35]. A nontrivially valued field (K, v) is a **deeply ramified field** if and only if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group vK, then Γ_2/Γ_1 is not isomorphic to $\mathbb{Z}$ (that is, no archimedean component of vK is discrete),

(DRvr) if $\mathrm{char}\, Kv = p > 0$, then $\mathcal{O}_K/p\mathcal{O}_K$ is semiperfect if $\mathrm{char}\, K = 0$, and $\mathcal{O}_{\widehat{K}}/p\mathcal{O}_{\widehat{K}}$ is semiperfect if $\mathrm{char}\, K = p$, where $\mathcal{O}_{\widehat{K}}$ is the valuation ring of the completion $\widehat{K}$ of (K, v).

If (K, v) has rank 1, then (DRvg) just means that (K, v) is not discrete. If (K, v) is complete, then (DRvr) means that $\mathcal{O}_K/p\mathcal{O}_K$ is semiperfect. Hence every perfectoid field is deeply ramified. Every perfect valued field of positive characteristic p (in particular, $\mathbb{F}_p((t))^{1/p^\infty}$) and every tame field is deeply ramified. The former as well as all tame fields of residue characteristic $p > 0$ have p-divisible value group, but this does not necessarily hold for deeply ramified fields of characteristic 0 with residue characteristic $p > 0$. In [35], we define (K, v) to be a **semitame field** if it is a deeply ramified field whose value group is p-divisible if $\mathrm{char}\, Kv = p > 0$. Semitame fields form a smaller elementary class which still contains all perfectoid fields.

7. Roughly deeply ramified and roughly tame fields

Inspired by the notion "roughly p-divisible value group" introduced by Will Johnson, we call (K, v) a **roughly deeply ramified field**, or in short an **rdr field**, if it satisfies axiom (DRvr) together with:

(DRvp) if $\mathrm{char}\, Kv = p > 0$, then $v(p)$ is not the smallest positive element in the value group vK.

The two axioms (DRvp) and (DRvr) together imply that the smallest convex subgroup of vK containing $v(p)$ (or equivalently, the interval $[-v(p), v(p)]$) is p-divisible.

For the definition of roughly tame fields, we need the following characterization of tame fields as preparation. The following is part of [31, Theorem 3.2]:

Theorem 8 (Kuhlmann (2016))**.** *A henselian field (K, v) is a tame field if and only if the following conditions hold:*

(TF1) if $\mathrm{char}\, Kv = p > 0$, then vK is p-divisible,

(TF2) Kv is perfect,

(TF3) (K, v) is algebraically maximal.

Replacing (TF1) by

(TF1r) if $\mathrm{char}\, Kv = p > 0$, then $[-v(p), v(p)]$ is p-divisible,

we obtain the definition of a **roughly tame field**.

In the article [38], the following is proven:

Theorem 9 (Rzepka – Szewczyk (2023))**.** *A henselian field is roughly tame if and only if all of its algebraic extension fields are defectless fields.*

We know that being defectless is an important property in the model theory of valued fields. For instance, if a valued field is existentially closed in its maximal immediate extensions, then it is henselian and defectless (see [31, Lemma 5.5]). The converse is not true, see Proposition 12.

The property of being a defectless field is preserved under finite algebraic extensions, but in general not under infinite algebraic extensions. For instance, Example 3.12 of [30] constructs infinite algebraic extensions of $(\mathbb{F}_p(t), v_t)$ and $(\mathbb{F}_p((t)), v_t)$ which are not defectless fields, and Example 3.20 of [30] constructs infinite algebraic extensions of $(\mathbb{Q}, v_p)$ and $(\mathbb{Q}_p, v_p)$ which are not defectless fields.

8. Taming perfectoid fields

In the article [16], Jahnke and Kartas generalize the model theoretic results about tame fields to the elementary class of roughly tame fields. To this end, they prove the Relative Embedding Property for the elementary class of roughly tame fields ([16, Fact 3.3.12]). From this it follows that the assertions of Theorem 1 also hold for the elementary class of roughly tame fields. They put this generalization to work in their approach of "taming perfectoid fields". They work with an elementary class $\mathcal{C}$ of henselian fields (K, v) of residue characteristic $p > 0$ with distinguished element $\pi \in K \setminus \{0\}$, $v\pi > 0$, such that:

($\mathcal{C}$1) the valuation ring $\mathcal{O}_K$ is semitame, and
($\mathcal{C}$2) with the coarsening w of v associated with the valuation ring $\mathcal{O}_v[\pi^{-1}]$, (K, w) is algebraically maximal (which implies that it is roughly tame).

(See [16, Definition 4.2.1 and Proposition 4.2.2].) The class $\mathcal{C}$ contains all henselian roughly deeply ramified fields of mixed characteristic (see [16, Remark 4.2.4]).

If (K', v') is a suitable ultrapower of a perfectoid field (K, v) with distinguished element $\varpi \in K \setminus \{0\}$, $v\varpi > 0$, and w' is the coarsest coarsening of v' on K' such that $w'\varpi > 0$, then $(K', w') \in \mathcal{C}$ for any $\pi \in K \setminus \{0\}$, $v\pi > 0$, and the residue field $K'w'$ with its valuation induced by the one of the ultrapower is an elementary extension of the tilt of K (see [16, Corollary 4.2.6 and Theorem 6.2.3]). This is used to show that certain model theoretic properties hold for perfectoid fields if and only if they hold for their tilts (see [16, Corollary 5.3.1]).

For the class $\mathcal{C}$, Jahnke and Kartas prove analogues of the model theoretic results for (roughly) tame fields, but with the residue fields Kv replaced by the residue rings $\mathcal{O}_K/\pi\mathcal{O}_K$ (see [16, Theorems 5.1.2 and 5.1.4]). This "mods out the non-tame part" of the valued fields in $\mathcal{C}$. So we are still left with the

Open problem 5: What can we say about the model theory of (roughly) deeply ramified fields (relative to their value groups and residue fields), and in particular of $\mathbb{F}_p((t))^{1/p^\infty}$?

It is well known that the henselization $(\mathbb{F}_p(t)^h, v_t)$ of $(\mathbb{F}_p(t), v_t)$ is existentially closed in $(\mathbb{F}_p((t)), v_t)$. This can be deduced from the following more general result (see [31, Theorem 5.12]):

Theorem 10 (Kuhlmann 2016)**.** *Let k be an arbitrary field. Then $(k(t), v_t)^h$ is existentially closed in $(k((t)), v_t)$.*

However, the following has remained a daunting

Open problem 6: Is $\mathbb{F}_p(t)^h$ an elementary substructure of $\mathbb{F}_p((t))$?

In contrast, Jahnke and Kartas prove that $\mathbb{F}_p(t^{1/p^\infty})^h$ is an elementary substructure of $\mathbb{F}_p((t))^{1/p^\infty}$ ([16, Corollary 1.7.5]). This positive result encourages us to ask:

Open problem 7: Is it possible to prove model theoretic results for henselian perfect valued fields of positive characteristic, analoguous to those for tame fields (but under mild additional conditions)?

9. On the model theory of $\mathbb{F}_p((t))$

While model theoretic results about $\mathbb{Q}_p$ and in particular the decidability of $\mathbb{Q}_p$ are known since the work of Ax–Kochen and Ershov, we are still facing the

Open problem 8: What can we say about the model theory and in particular a complete axiomatization and the decidability of $\mathbb{F}_p((t))$?

In the article [27] the following negative result is proven:

Theorem 11 (Kuhlmann (2001))**.** *The $\mathcal{L}_{\rm val}(t)$-elementary axiom system*

(A_t) "henselian defectless valued field of positive characteristic with value group a $\mathbb{Z}$-group with smallest element $v(t)$ and residue field $\mathbb{F}_p$"

is not complete.

This theorem is proven by constructing an extension (L, v) of $(\mathbb{F}_p((t)), v_t)$ with the following properties:

- (L, v) satisfies axiom system (A_t),
- $L|K$ is of trancendence degree 1 and regular (i.e., $L|K$ is separable and K is relatively algebraically closed in L),
- there is an $\forall\exists$-elementary $\mathcal{L}_{\rm val}(t)$-sentence expressing a property of additive polynomials which holds in (K, v) but not in (L, v),

see [27, Theorem 1.3]. A polynomial $f(X) \in K[X]$ is called **additive** if $f(a+b) = f(a) + f(b)$ for all a, b in any extension field of K. If $\mathrm{char}\, K = p > 0$, then the additive polynomials in $K[X]$ are precisely the polynomials of the form

$$\sum_{i=0}^{m} c_i X^{p^i} \quad \text{with } c_i \in K\,,\ m \in \mathbb{N}$$

(see [37, VIII, §11]). If K is infinite, then $f(X) \in K[X]$ is additive if and only if $f(a+b) = f(a) + f(b)$ for all $a, b \in K$. Additive polynomials in several variables are defined in a similar way; but note that they are just sums of additive polynomials in one variable. The special role of additive polynomials for valued fields of characteristic $p > 0$ had been long known; for example, the **Artin-Schreier polynomial** $X^p - X$ is additive.

Moreover, it is shown that (L, v) is not $\mathcal{L}_{\rm val}$-existentially closed in its maximal immediate extensions (cf. [27, Theorem 1.3]). This proves:

Proposition 12 (Kuhlmann (2001))**.** *There are henselian defectless fields that are not $\mathcal{L}_{\rm val}$-existentially closed in their maximal immediate extensions.*

Open problem 9: Is there a handy additional condition on the immediate extensions that remedies this situation?

In the article [27], also the following is shown:

Theorem 13 (Kuhlmann (2001))**.** *The $\mathcal{L}_{\rm val}$-elementary axiom system*
(A) "henselian defectless valued field of positive characteristic with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$"
is not complete.

Further, a (not really handy) axiom scheme, called (PDOA) and stating properties of additive polynomials, is suggested to be added to axiom systems (A) or (A_t). A much more elegant axiom scheme was found after Yuri Ershov introduced the notion of "extremal field" and claimed that $\mathbb{F}_p((t))$ is extremal. However, his definition and proof were faulty. In the article [7] it is shown that $\mathbb{F}_p((t))$ does not satisfy Ershov's definition; in fact, every valued field satisfying this definition must be algabraically closed. A corrected definition is given, and it is shown that $\mathbb{F}_p((t))$ satisfies this corrected definition, which we present now.

10. Extremal fields

A valued field (K,v) is called **extremal** if for every multi-variable polynomial $f(X_1,\ldots,X_n)$ over K, the set

$$\{v(f(a_1,\ldots,a_n)) \mid a_1,\ldots,a_n \in \mathcal{O}_K\} \subseteq vK \cup \{\infty\}$$

has a maximal element. This is an $\mathcal{L}_{\rm val}$-elementary axiom scheme. Ershov's error was to put "K" in place of "$\mathcal{O}_K$".

Theorem 14 (Azgin – Kuhlmann – Pop (2012))**.** $\mathbb{F}_p((t))$ *is an extremal field.*

Open problem 10: Is (A) + "(K,v) is extremal" a complete axiom system?

In the article [6] an almost complete characterization of extremal valued field is given:

Theorem 15 (Anscombe – Kuhlmann (2016))**.** *Let (K,v) be a nontrivially valued field. If (K,v) is extremal, then it is henselian and defectless, and*

(i) *vK is a $\mathbb{Z}$-group, or*
(ii) *vK is divisible and Kv is large.*

Conversely, if (K,v) is henselian and defectless, and

(i) *$vK \simeq \mathbb{Z}$, or vK is a $\mathbb{Z}$-group and* $\mathrm{char}\, Kv = 0$*, or*
(ii) *vK is divisible and Kv is large and perfect,*

then (K,v) is extremal.

A complete characterization is not known, and there are many more open problems about extremal fields listed in [6]. For instance, in contrast to properties such as "henselian" and "defectless", we do not entirely know how extremality behaves under composition of valuations:

Open problem 11: If $v = w \circ \bar{w}$ with w and $\bar{w}$ extremal and w has divisible value group, does it follow that v is extremal?

Open problem 12: We know that if $v = w \circ \bar{w}$ is extremal, then so is $\bar{w}$ (see [6, Lemma 4.1]). But does it also follow that w is extremal?

On the other hand, it is shown in [6, Theorem 1.13] that in a certain sense, extremal fields are abundant in valuation theory:

Theorem 16 (Anscombe – Kuhlmann (2016)). *Let (K, v) be any $\aleph_1$-saturated valued field. Assume that Γ and Δ are convex subgroups of vK such that $\Delta \subsetneq \Gamma$ and Γ/Δ is archimedean. Let u (respectively w) be the coarsening of v corresponding to Δ (resp. Γ). Denote by $\bar{u}$ the valuation induced on Kw by u. Then $(Kw, \bar{u})$ is maximal, extremal and large, and its value group is isomorphic either to $\mathbb{Z}$ or to $\mathbb{R}$. In the latter case, also $Ku = (Kw)\bar{u}$ is large.*

This shows that extremal fields can be seen as the rank 1 building blocks of valuations, at least of those which are $\aleph_1$-saturated. The properties of the valuations thus built (which can vary wildly) apparently depend on the way the building blocks are "glued together", a process that remains to be investigated further.

Finally, let us mention that in the extension (L, v) of $\mathbb{F}_p((t))$ constructed in [27] not all images of additive polynomials have the optimal approximation property (for its definition, see [6, §3]). It thus follows from [6, Theorem 3.4] that (L, v) is not extremal. This gives rise to the following

Open problem 13: Is every extremal valued field existentially closed in its maximal immediate extensions?

11. The existential theory of $\mathbb{F}_p((t))$ and criteria for large fields to be existentially closed in extensions

Let us return to the model theory of $\mathbb{F}_p((t))$. The following is shown in [4]:

Theorem 17 (Anscombe – Fehm (2016)). *The existential $\mathcal{L}_{\text{val}}$-theory of $\mathbb{F}_p((t))$ is decidable.*

However, as can be seen from our discussion of Kartas' work, we would like to have more. In the article [11], Jan Denef and Hans Schoutens proved in 2003 that the existential $\mathcal{L}_{\text{val}}(t)$-theory of $\mathbb{F}_p((t))$ is decidable, provided that resolution of singularities holds in positive characteristic. In order to discuss more recent improvements of this result, we need some preparations.

In the article [28] the following question is studied: Take a field extension $F|K$ such that F admits a K**-rational place**, or in other words, a valuation with residue field K. Under which additional conditions does it follow that K is existentially closed in F? Here a key role is played by large fields. While they are usually defined in a different way (see [28, Section 1.3]), one can also use the model theoretic approach: A field K is **large** if it is existentially closed in $K((t))$.

Theorem 18 (Kuhlmann (2004)). *Let K be a perfect field. Then the following conditions are equivalent:*
1) K is a large field,
2) K is existentially closed in every power series field $K((t^\Gamma))$,
3) K is existentially closed in every extension field L which admits a K-rational place.

What can we say about fields that are not perfect? For a first answer, we introduce a hypothesis that enables us to generalize the above theorem. **Local uniformization** is a local form, and a consequence, of resolution of singularities. For background, see [21, 22, 40]. In recent decades, doubts have spread in the community of algebraic geometers working on resolution of singularities that it can be proven for all dimensions in positive characteristic. However, there is much more hope for a corresponding general version of local uniformization.

Theorem 19 (Kuhlmann (2004))**.** *If all rational places of arbitrary function fields admit local uniformization, then the three conditions of Theorem 18 are equivalent, for arbitrary fields K.*

In the paper [2] the assumption that implication 1)⇒3) holds for *arbitrary* fields K is called hypothesis **(R4)**. Hence local uniformization implies (R4). The authors prove the following strengthening of Theorem 17:

Theorem 20 (Anscombe – Dittmann – Fehm (2023))**.** *If (R4) holds, then the existential $\mathcal{L}_{\mathrm{val}}(t)$-theory of $\mathbb{F}_p((t))$ is decidable.*

Open problem 14: Does (R4) hold?

In the paper [15] de Jong proved resolution by **alteration**, which means that a finite extension of the function field of the algebraic variety under consideration is taken into the bargain. By valuation theoretical tools, Knaf and Kuhlmann proved local uniformization by alteration in [22].

Open problem 15: Does local uniformization by alteration imply a reasonable (and useful) hypothesis "(R4) by alteration"? Is there a "model theory by alteration"?

You cannot always get what you want – but perhaps after a finite extension?

Indeed, the following is shown in [23]:

Theorem 21 (Kuhlmann – Knaf ?)**.** *Take a large field K and a function field $F|K$ which admits a rational place. Then there is a finite purely inseparable extension $K'|K$ such that K' is existentially closed in the field compositum $F.K'$.*

One possible proof uses Temkin's "inseparable local uniformization" by alteration ([40, Theorem 1.3.2]. However, Arno Fehm has pointed out that the theorem can also more directly be deduced from results in [28].

12. Classification of defects

Let me give some details on the classification of defects which has been introduced in [35]. If $(L|K,v)$ is a finite extension for which the extension of v from K to L is unique, then by the Lemma of Ostrowski ([41, Corollary to Theorem 25, Section G, p. 78]),

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{2}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p}=\operatorname{char} Kv$ if it is positive and $\tilde{p}=1$ otherwise. The factor $d(L|K,v):=\tilde{p}^{\nu}$ is the **defect** of the extension $(L|K,v)$. By our previous definition, if (K,v) is henselian, then $(L|K,v)$ is a defectless extension if $d(L|K,v) = 1$. This always holds if $\operatorname{char} Kv = 0$.

Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p with nontrivial defect; then $p = \operatorname{char} Kv$. For every σ in its Galois group $\operatorname{Gal}(L|K)$, with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma := \left\{ v\left(\frac{\sigma f - f}{f}\right) \,\middle|\, f \in L^\times \right\} . \tag{3}$$

This set is a final segment of vK and independent of the choice of σ (see [35, Theorems 3.4 and 3.5]); we denote it by $\Sigma_{\mathcal{E}}$. We say that $\mathcal{E}$ has **independent defect** if

$$\left\{ \begin{array}{lcl} \Sigma_{\mathcal{E}} & = & \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\} \text{ for some proper convex subgroup } H_{\mathcal{E}} \\ & & \text{of } vK \text{ such that } vK/H_{\mathcal{E}} \text{ has no smallest positive element;} \end{array} \right. \tag{4}$$

otherwise we say that $\mathcal{E}$ has **dependent defect**. If (K,v) has rank 1, then condition (4) just means that $\Sigma_{\mathcal{E}}$ consists of all positive elements in vK.

If (K,v) is of mixed characteristic, then we set $K' := K(\zeta_p)$, where ζ_p is a primitive p-th root of unity; otherwise, we set $K' := K$. Note that every Galois extension L of prime degree of a field K of characteristic 0 containing a primitive p-th root of unity is a **Kummer extension**, i.e., it is generated by an element η with $\eta^p \in K$. Now we call (K,v) an **independent defect field** if for some extension of v to K', all Galois defect extensions of (K',v) of degree p have independent defect. This definition does not depend on the chosen extension of v as all extensions are conjugate.

The following is Theorem 1.5 of [35]:

Theorem 22 (Kuhlmann – Rzepka 2023)**.** *Every algebraic extension of a deeply ramified field is again a deeply ramified field. The same holds for semitame fields and for roughly deeply ramified fields.*

Further, every roughly deeply ramified field is an independent defect field (see [35, Theorem 1.10 (1)]). This proves:

Theorem 23 (Kuhlmann – Rzepka 2023)**.** *Every algebraic extension of a roughly deeply ramified field is an independent defect field.*

In view of Theorem 9, we conjecture that a henselian field is a roughly deeply ramified field if and only if all of its algebraic extensions are independent defect fields. Thus we ask:

Open problem 16: If all algebraic extensions of a henselian field (K,v) are independent defect fields, does it follow that (K,v) is a roughly deeply ramified field?

13. Some definable valuation rings

It is obvious from its definition that an extension $(L|K,v)$ with independent defect gives rise to an $\mathcal{L}_{\mathrm{val}}$-definable coarsening $\mathcal{O}_{\mathcal{E}}$ of the valuation ring $\mathcal{O}_K$, namely

$$\mathcal{O}_{\mathcal{E}} := \{b \in K \mid \exists \alpha \in H_{\mathcal{E}} : \alpha \leq vb\}$$

whose value group is $vK/H_{\mathcal{E}}$. (Note that for this definition it is not needed that (K,v) be henselian, and that in fact, it is applied to deeply ramified fields, which are not required to be henselian.) For the case of henselian (K,v), the above is used in [20, Theorem 4.11] to define corresponding henselian valuations on K that are definable in the language of rings. It follows that a henselian roughly deeply ramified field which is not defectless always has a henselian valuation definable in the language of rings (cf. [20, Corollary 4.14]).

To conclude with, let me present two interesting applications of definable coarsenings of (not necessarily henselian) valuation rings. In [9, Theorem 1.4], the maximal ideal of the valuation ring $\mathcal{O}_{\mathcal{E}}$ is used, under the notation $\mathcal{M}_{v_{\mathcal{E}}}$, in the characterization of extensions with independent defect. In the manuscript [10] the maximal ideal of the valuation ring $\mathcal{O}_{\mathcal{E}}$ that is defined from a ramified Galois extension $\mathcal{E}=(L|K,v)$ of prime degree is used to present the Kähler differential $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ of the extension. These applications are discussed in detail in the manuscript [32].

References

[1] Sylvy Anscombe: *On Lambda functions in henselian and separably tame valued fields*, arXiv:2505.07518

[2] Sylvy Anscombe – Philip Dittmann – Arno Fehm: *Axiomatizing the existential theory of* $F_q((t))$, Algebra and Number Theory **17** (2023), 2013–2032

[3] Sylvy Anscombe – Philip Dittmann – Franziska Jahnke: *Ax-Kochen-Ershov principles for finitely ramified henselian fields*, Trans. Amer. Math. Soc. **377** (2024), 8963–8988

[4] Sylvy Anscombe – Arno Fehm: *The existential theory of equicharacteristic henselian valued fields*, Algebra and Number Theory **10** (2016), 665–683

[5] James Ax – Simon Kochen: *Diophantine problems over local fields II*, Amer. Journ. Math. **87** (1965), 631–648

[6] Sylvy Anscombe – Franz-Viktor Kuhlmann: *Notes on extremal and tame valued fields*, J. Symb. Logic **81** (2016), 400–416

[7] Salih Azgin – Franz-Viktor Kuhlmann – Florian Pop: *Characterization of extremal valued fields*, Proc. Amer. Math. Soc. **140** (2012), 1535–1547

[8] Pablo Cubides Kovacsics – Franz-Viktor Kuhlmann – Anna Rzepka: *On valuation independence and defectless extensions of valued fields*, J. Algebra **555** (2020), 69–95

[9] Steven Dale Cutkosky – Franz-Viktor Kuhlmann – Anna Rzepka: *On the computation of Kähler differentials and characterizations of Galois extensions with independent defect*, submitted; https://arxiv.org/abs/2305.10022

[10] Steven Dale Cutkosky – Franz-Viktor Kuhlmann: *Kähler differentials of extensions of valuation rings and deeply ramified fields*, submitted; https://arxiv.org/abs/2306.04967

[11] Jan Denef – Hans Schoutens: *On the decidability of the existential theory of* $\mathbb{F}_p[[t]]$, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), 43–60, Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 2003

[12] Otto Endler: Valuation theory. Berlin (1972)

[13] Yuri Ershov: *On elementary theories of local fields*, Algebra i Logika **4** (1965), 5–30

[14] K. A. H. Gravett: *Note on a result of Krull*, Cambridge Philos. Soc. Proc. **52** (1956), 379

[15] Aise Johan de Jong: *Smoothness, semi-stability and alterations*, Publ. Math. IHES **83** (1996), 51–93

[16] Franziska Jahnke – Konstantinos Kartas: *Beyond the Fontaine-Wintenberger theorem*, J. Amer. Math. Soc. **38** (2025), 997–1047

[17] Irving Kaplansky: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

[18] Konstantinos Kartas: *Diophantine problems over tamely ramified fields*, J. Algebra **617** (2023), 127–159

[19] Konstantinos Kartas: *Decidability via the tilting correspondence*, Algebra and Number Theory **18** (2024), 209–24
[20] Margarete Ketelsen – Simone Ramello – Piotr Szewczyk: Definable henselian valuations in positive residue characteristic, J. Symb. Logic, DOI: https://doi.org/10.1017/jsl.2024.55
[21] Hagen Knaf – Franz-Viktor Kuhlmann: *Abhyankar places admit local uniformization in any characteristic*, Ann. Scient. Ec. Norm. Sup. **38** (2005), 833–846
[22] Hagen Knaf – Franz-Viktor Kuhlmann: *Every place admits local uniformization in a finite extension of the function field*, Advances Math. **221** (2009), 428–453
[23] Hagen Knaf – Franz-Viktor Kuhlmann: work in progress
[24] Wolfgang Krull: *Allgemeine Bewertungstheorie*, J. reine angew. Math. **167** (1932), 160–196
[25] Franz-Viktor Kuhlmann: *Henselian function fields and tame fields*, extended version of Ph.D. thesis, Heidelberg (1990)
[26] Franz-Viktor Kuhlmann: *Valuation theory of fields, abelian groups and modules*, Habilitationsschrift, Heidelberg (1994/95), available at `https://fvkuhlmann.de/HAB.pdf`
[27] Franz-Viktor Kuhlmann: *Elementary properties of power series fields over finite fields*, J. Symb. Logic **66** (2001), 771-791
[28] Franz-Viktor Kuhlmann: *On places of algebraic function fields in arbitrary characteristic*, Advances in Math. **188** (2004), 399–424
[29] Franz-Viktor Kuhlmann: *Elimination of Ramification I: The Generalized Stability Theorem*, Trans. Amer. Math. Soc. **362** (2010), 5697–5727
[30] Franz-Viktor Kuhlmann: *The defect*, in: Commutative Algebra – Noetherian and non-Noetherian perspectives. Marco Fontana, Salah-Eddine Kabbaj, Bruce Olberding and Irena Swanson (eds.), 277–318, Springer, New York, 2011
[31] Franz-Viktor Kuhlmann: *The algebra and model theory of tame valued fields*, J. reine angew. Math. **719** (2016), 1–43
[32] Franz-Viktor Kuhlmann: *On definable coarsenings of valuation rings*, work in progress
[33] Franz-Viktor Kuhlmann – Koushik Pal: *The model theory of separably tame fields*, J. Alg. **447** (2016), 74–108
[34] Franz-Viktor Kuhlmann – Matthias Pank – Peter Roquette: *Immediate and purely wild extensions of valued fields*, Manuscripta math. **55** (1986), 39–67
[35] Franz-Viktor Kuhlmann – Anna Rzepka: *The valuation theory of deeply ramified fields and its connection with defect extensions*, Trans. Amer. Math. Soc. **376** (2023), 2693–2738
[36] Serge Lang: Algebra. Addison-Wesley, New York (1965)
[37] Victor Lisinski: *Decidability of positive characteristic tame Hahn fields in* $\mathcal{L}_{\rm val}$, preprint (2021); arXiv:2108.04132v4
[38] Anna Rzepka – Piotr Szewczyk: *Defect extensions and a characterization of tame fields*, J. Algebra **630** (2023), 68–91
[39] Peter Scholze: *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313
[40] Michael Temkin: *Inseparable local uniformization*, J. Algebra **373** (2013), 65–119
[41] Oscar Zariski – Pierre Samuel: *Commutative algebra, vol. II*, New York, 1960

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