Deeply ramified fields and their relatives

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joint work with Anna Rzepka (formerly Blaszczok)

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In the following, p will always be the characteristic of the residue field.

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where $\mathcal{O}_K$ is the valuation ring of K, $\mathcal{O}_{K^{\mathrm{sep}}}$ is the valuation ring of the separable-algebraic closure of K, and $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra.

Theorem (Gabber & Ramero)

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Another possible direction of generalization is offered by the class of extremal fields —studied in joint work with Durhan (formerly Azgin), Pop, Anscombe— but this is another story.

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Semitame fields are our best bet when it comes to generalizing the results we have proved in the past for tame fields.

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Good news for model theorists

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Theorem (Rzepka & K)

The classes of semitame, deeply ramified and gdr fields of fixed characteristic and residue characteristic are first order axiomatizable in the language of valued fields.

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Extensions in the absolute ramification field

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Note that if (K, v) is henselian, then the condition on (L, v) just means that it is a tame extension of (K, v).

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For the reduction, we then also use the fact that the absolute Galois group of an absolute ramification field of residue characteristic $p > 0$ is a p-group, which implies that every finite extension of this field is a tower of normal extensions of degree p.

By the way, this argument had already been used by Abhyankar in his work on resolution of singularities in positive characteristic.

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There are many open problems about independent defect fields. For instance, if (K, v) is an independent defect field, does the same hold for its absolute ramification field? (The converse holds.)

Questions of this type are very hard to study when the valued field under consideration does not have a p-divisible value group or does not have a perfect residue field.

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Further results and work in progress

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Further results and work in progress

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