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On places of algebraic function fields

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Introduction and notations

In this paper we study the space of all places of a function field in n variables. We denote by F/k an (algebraic) function field, i.e. F is a finitely generated extension of k of transcendence degree $n \geqq 1$. By a *place* P of F/k we mean a place of F which is trivial on k, i.e. the restriction of P to k yields an isomorphism. The image of $x \in F$ under P will be denoted by xP; correspondingly, FP denotes the set of all images of elements of F. Thus FP consists of the residue field of the valuation v_P associated to P, together with ∞, the value which is assigned to x by P, if x has a "pole at P". Nevertheless, in the following we will simply call FP the residue field of P. The degree of transcendency of FP over k is called the *dimension* of P. It is denoted by $\dim(P)$. In defining $\dim(P)$ we already have identified k with its isomorphic image under P. This will be done frequently in the following sections. The ordered abelian group of values taken by v_P will be denoted by $v_P(F)$. The *rational rank* of P is the dimension of $v_P(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. It will be denoted by $\mathrm{rr}(P)$. The rational rank should not be confused with the number of non-zero convex subgroups of $v_P(F)$ which is sometimes called the *rank* of $v_P(F)$. In this paper we will consider only *non-trivial places* P of F/k, i.e. P is never an isomorphism on F. Consequently, the dimension of P is always at most $n-1$ and the rational rank is at least 1. Moreover, k will always have *characteristic zero*.

The valuation theory used can be found in [2], [5], [7], [19].

For a place P of a function field F/k in n variables, the following inequalities are well known and easily proved:

(i) $0 \leqq \dim(P) \leqq n-1$,

(ii) $1 \leqq \mathrm{rr}(P) \leqq n$,

(iii) $1 \leqq \dim(P) + \mathrm{rr}(P) \leqq n$.

Besides these restrictions, for $n \geqq 2$ almost everything is possible concerning the structure of the residue field FP and the value group $v_P(F)$. In particular, FP need not be finitely generated, hence need not be a function field over k too (see [19], Ch. VI, § 15). In dealing

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*) The results of this paper were presented at the AMS Summer Institut on 'Ordered fields and real algebraic geometry' at Boulder, July 1983.

with places of function fields this is a rather strong drawback. In applications like the determination of certain rings of holomorphy this drawback had to be overcome by the use of "Zariski's Local Uniformization" (see e.g. [9], Lemma 5. 5). In this paper we offer another possibility. The price for that is an application of the Ax-Kochen-Ershov Theorem on the model completeness of certain classes of henselian fields.

In Section 1 we will state and explain the Ax-Kochen-Ershov Theorem. For the convenience of the reader a proof of this important theorem is included as an appendix to this paper.

In Section 2 we will prove that in a very strong sense the places with finitely generated residue fields and finitely generated value groups form a dense subset of the space of all places of F/k. To be more precise, we will prove the following:

Let Q be a place of the function field F/k in n variables. To every finite sequence $x_1, \dots, x_m, x_{m+1}, \dots, x_{m+s} \in F$ there exists a place P of F/k with a finitely generated residue field (over k) and a finitely generated value group such that[1])

$$x_i Q = x_i P \quad \text{for all} \quad 1 \leqq i \leqq m,$$

$$v_Q(x_i) = v_P(x_i) \quad \text{for all} \quad m+1 \leqq i \leqq m+s.$$

Moreover, we can prescribe the dimension and the rational rank of P to be the same as that of Q. But also certain other prescriptions are possible subject only to the above mentioned canonical restrictions (i)—(iii) together with

(iv) $\dim(Q) \leqq \dim(P)$ and $\operatorname{rr}(Q) \leqq \operatorname{rr}(P)$.

This is the content of the Main Theorem.

In Section 3 we show that, under certain assumptions, we can obtain modifications of the Main Theorem with an extended condition (iv). Using these modifications of the Main Theorem we give some applications. One application will be on rings of holomorphy like the real holomorphy ring or the p-adic holomorphy ring of some field[2]). Another application of a certain modification of the Main Theorem (Theorem 3) is contained in the second author's paper [13] where a determination of those real polynomials in $X_1, \dots, X_n$ is given which can be written as a sum of $2m$-th powers of rational functions in $X_1, \dots, X_n$ over $\mathbb{R}$.

1. The Ax-Kochen-Ershov Theorem

There are several ways to state the Ax-Kochen-Ershov Theorem. We prefer to state it in terms of "existentially closed". For other possibilities we refer the reader to [2], [3], [8], [10]. Before stating the theorem we have to explain first some notions from model theory.

In model theory, a substructure $\mathfrak{A}_1$ of a structure $\mathfrak{A}_2$ is called *existentially closed* (e.c.) *in* $\mathfrak{A}_2$, if every existential statement which holds in $\mathfrak{A}_2$ also holds in $\mathfrak{A}_1$. Such an existential statement is expressed in the formal language corresponding to $\mathfrak{A}_1$ (including parameters from $\mathfrak{A}_1$). We will need the notion "existentially closed" only for three types of structures. These are: fields, ordered abelian groups, and valued fields. In each case we will explain separately what "existentially closed" means.

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[1]) Here and in the following sections the residue field FP will always be contained in a fixed extension of the given residue field FQ. Similarly, $v_P(F)$ will be contained in some fixed extension of $v_Q(F)$.

[2]) For the study of these rings we refer the reader to [4] and [14].

A *field* K_1 is e.c. in K_2 if for every finite sequence of polynomials

$$f_1, \dots, f_r \in K_1[X_1, \dots, X_m]$$

we have: whenever $f_1, \dots, f_r$ have a common zero in K_2, they also have a common zero in K_1. As an example we let K_1 be algebraically closed. Then K_1 is e.c. in every extension field K_2. This is just "Hilbert's Nullstellensatz". Another example is provided if K_1 is real closed and K_2 is formally real. Then it follows from the Artin-Lang Homomorphism Theorem that K_1 is e.c. in K_2.

An *ordered abelian group* Γ_1 is e.c. in Γ_2, if for all sequences of linear forms

$$l_1, \dots, l_r, l'_1, \dots, l'_s \in \mathbb{Z}[X_1, \dots, X_m]$$

and sequences $\gamma_1, \dots, \gamma_r, \gamma'_1, \dots, \gamma'_s \in \Gamma_1$ we have: whenever there are $x_1, \dots, x_m \in \Gamma_2$ such that

$$l_i(\mathbf{x}) = \gamma_i \quad \text{and} \quad l'_j(\mathbf{x}) > \gamma'_j \qquad (1 \leqq i \leqq r,\ 1 \leqq j \leqq s),$$

then there are $x_1, \dots, x_m \in \Gamma_1$ with the same properties. As an example, let Γ_1 be divisible and Γ_2 arbitrary. Then Γ_1 is e.c. in Γ_2. Another example is obtained if Γ_1 is archimedean and dense. It then follows that Γ_1 is e.c. in Γ_2 if Γ_2/Γ_1 is torsion free. (This fact was communicated to us by V. Weispfenning.)

Finally we consider *valued fields*. Let the valuation be given by a valuation ring $\mathcal{O}$. Then, a valued field $(K_1, \mathcal{O}_1)$ is e.c. in the valued field extension[3] $(K_2, \mathcal{O}_2)$ if for all sequences of polynomials $f_1, \dots, f_r, g_1, \dots, g_s, h_1, \dots, h_t \in K[X_1, \dots, X_m]$ we have: whenever there exist $x_1, \dots, x_m \in K_2$ such that for all $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, $1 \leqq l \leqq t$:

$$(*) \qquad f_i(\mathbf{x}) = 0, \quad g_j(\mathbf{x}) \in \mathcal{O}_2, \quad h_l(\mathbf{x}) \notin \mathcal{O}_2,$$

then there exist $x_1, \dots, x_m \in K_1$ having the same properties. Examples are provided by the

Theorem (Ax-Kochen-Ershov). *Let $(K_1, \mathcal{O}_1)$ be a henselian valued field and $(K_2, \mathcal{O}_2)$ a valued field extension of $(K_1, \mathcal{O}_1)$. We denote the corresponding value groups and residue fields by Γ_1, Γ_2 and $\bar{K}_1, \bar{K}_2$ resp. If $\bar{K}_1$ is e.c. in $\bar{K}_2$, the characteristic of $\bar{K}_1$ is zero, and Γ_1 is e.c. in Γ_2, then $(K_1, \mathcal{O}_1)$ is e.c. in $(K_2, \mathcal{O}_2)$.*

In case K_2 is a function field over K_1, the fact that $(K_1, \mathcal{O}_1)$ is e.c. in $(K_2, \mathcal{O}_2)$ can be equivalently expressed by saying that every affine model of K_2 has K_1-rational points, lying dense in the valuation topology of the affine space. A proof of the Ax-Kochen-Ershov Theorem is included as an appendix.

Let us remark that, in no case of the above explained examples, the existential statements made are the most general. However, they suffice in order to test the property of being e.c. The existential statement (for valued fields) which is used in the next section can be easily transformed into one of the above type $(*)$ by introducing new variables. For example, the fact that the non-zero elements x_1 and x_2 have the same value can be expressed by saying that

$$x_1 y_1 = 1, \quad x_2 y_2 = 1, \quad x_1 y_2 \in \mathcal{O}_2, \quad x_2 y_1 \in \mathcal{O}_2$$

for some $y_1, y_2 \in K_2$. Note that the usual formal language for fields does not include inverses x^{-1}, it only includes addition, multiplication and subtraction.

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[3]) We say that $(K_2, \mathcal{O}_2)$ extends $(K_1, \mathcal{O}_1)$ if K_1 is a subfield of K_2 and $\mathcal{O}_1 = K_1 \cap \mathcal{O}_2$.

2. The main theorem

In this section we will state and prove the main theorem of this paper. Before doing so let us introduce one more notation. Let Γ be an ordered abelian group. Any direct product of the type

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \Gamma \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

of Γ with a finite number of copies of $\mathbb{Z}$ is called a *discrete lexicographic extension* of Γ, if this product is ordered lexicographically.

Main theorem. *Let $F|k$ be a function field in n variables with* $\operatorname{char} k = 0$. *Let Q be a place of $F|k$ and $x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+s} \in F$. Then there exists a place P of $F|k$ with a finitely generated value group and a finitely generated residue field (over k) such that*

$$x_i Q = x_i P \quad \text{for all} \quad 1 \leqq i \leqq m,$$
$$v_Q(x_i) = v_P(x_i) \quad \text{for all} \quad m+1 \leqq i \leqq m+s.$$

Moreover, if r_1 and d_1 are natural numbers satisfying

$$\dim(Q) \leqq d_1 \leqq n-1, \quad \operatorname{rr}(Q) \leqq r_1 \leqq n - d_1,$$

then P may be chosen to satisfy in addition:

(1) $\dim(P) = d_1$ *and FP is a subfield of a purely transcendental extension of FQ, finitely generated over k,*

(2) $\operatorname{rr}(P) = r_1$ *and $v_P(F)$ is a finitely generated subgroup of a discrete lexicographic extension of $v_Q(F)$.*

Proof. Let (L, Q) be the henselization of (F, Q), i.e. L is the henselian closure of F with respect to the valuation v_Q on F. Since the extension of v_Q to L is immediate, we use the same letter for the extension of the place Q from F to L. Let $d = \dim(Q)$. We choose $u_1, \ldots, u_d \in F$ such that the residue classes of these elements form a transcendence basis of FQ over k. Let $r = \operatorname{rr}(Q)$. We then choose $z_1, \ldots, z_r \in F$ such that their values under v_Q are independent over $\mathbb{Z}$. By [5], Ch. VI, § 10. 3, Theorem 1, $u_1, \ldots, u_d, z_1, \ldots, z_r$ are algebraically independent over k. Finally we let K' be the relative algebraic closure of $k(u_1, \ldots, u_d, z_1, \ldots, z_r)$ in L and denote by Q' the restriction of Q to K'. We now check that (K', Q') is a henselian field such that

$$K'Q' = LQ \quad \text{and} \quad v_{Q'}(K') = v_Q(L).$$

We first see that (K', Q') is henselian. Using $\operatorname{char} LQ = 0$ we find next that $K'Q' = LQ$. Finally, using the fact that any 1-unit of (L, Q) is a q-th power in L for every prime $q \in \mathbb{N}$, we find $v_{Q'}(K') = v_Q(L)$.[4]) Hence, by the Ax-Kochen-Ershov Theorem, (K', Q') is existentially closed in (L, Q). This fact will be used next.

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[4]) For more details see Step 1 of the Appendix and the proof of Theorem 2 in Section 3.

Let $K'F = K'(t_1, \ldots, t_{n'}, y)$ where $n' = n - (d+r)$ and $t_1, \ldots, t_{n'}$ are algebraically independent over K'. Furthermore, let $f \in K'[T_1, \ldots, T_{n'}, Y]$ be irreducible and monic in Y such that $f(\mathbf{t}, y) = 0$. We choose $x'_1, \ldots, x'_{m+s}$ in K' such that

$$x'_i Q = x_i Q \quad \text{for all} \quad 1 \leqq i \leqq m,$$
$$v_Q(x'_i) = v_Q(x_i) \quad \text{for all} \quad m+1 \leqq i \leqq m+s.$$

The originally given elements x_i will be represented as

$$x_i = \frac{g_i(\mathbf{t}, y)}{h_i(\mathbf{t})} \quad (1 \leqq i \leqq m+s),$$

where g_i and h_i are polynomials over K'. Since (K', Q') is e.c. in (L, Q), we can find elements $t'_1, \ldots, t'_{n'}, y'$ in K' such that

(i) $f(\mathbf{t}', y') = 0$ and $\frac{\partial f}{\partial Y}(\mathbf{t}', y') \neq 0$,

(ii) $h_i(\mathbf{t}') \neq 0 \quad (1 \leqq i \leqq m+s)$,

(iii) $\frac{g_i(\mathbf{t}', y')}{h_i(\mathbf{t}')} Q = x'_i Q \quad (1 \leqq i \leqq m)$,

(iv) $v_Q\left(\frac{g_i(\mathbf{t}', y')}{h_i(\mathbf{t}')}\right) = v_Q(x'_i) \quad (m+1 \leqq i \leqq m+s)$.

Indeed, the elements $\mathbf{t}$ and y from L satisfy (i) to (iv) resp. Hence the formula[5]) claiming the existence of such elements holds in (L, Q) and thus must also hold in (K', Q').

Now we let K_1 be the subfield of K' which is generated over k by

— $u_1, \ldots, u_d, z_1, \ldots, z_r$,

— $x'_1, \ldots, x'_{m+s}, t'_1, \ldots, t'_{n'}, y'$,

— the coefficients of f, g_i and h_i $(1 \leqq i \leqq m+s)$.

Clearly, K_1 is a finite algebraic extension of $k(u_1, \ldots, u_d, z_1, \ldots, z_r)$. Thus, we find from [5], Ch. VI, § 10. 3, Corollary 1, that $v_Q(K_1)$ is a finitely generated subgroup of $v_Q(F)$ of rational rank r and that $K_1 Q$ is a subfield of FQ of dimension d, finitely generated over k. Let us denote by P_1 the restriction of Q to K_1.

At this point we may forget about the field L and its place Q. Starting from (K_1, P_1) we will construct some henselian extension (K, P) of (K_1, P_1) which will contain an isomorphic copy of F. The construction of (K, P) will be in such a way that the restriction of P to the embedded copy of F will satisfy the assertions of the theorem.

[5]) As already indicated in Section 1, it is easy to formalize (i) to (iv) in the language of valued fields using a unary predicate for the valuation ring corresponding to Q.

First we adjoin $d_1 - d$ elements, algebraically independent over K_1, and extend P_1 such that the value group does not change and the residue field extends purely transcendental of degree $d_1 - d$. This can be done by an iterated application of [5], Ch. VI, § 10. 1, Proposition 2. The result of this step is denoted by (K_2, P_2).

Next we adjoin $r_1 - r$ elements, algebraically independent over K_2, and extend P_2 such that the residue field does not change and the value group extends to

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus v_{P_1}(K_1) \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z},$$

a discrete lexicographic extension of $v_{P_1}(K_1) = v_{P_2}(K_2)$ by $r_1 - r$ copies of $\mathbb{Z}$. The position of $v_{P_1}(K_1)$ can be chosen arbitrarily. This extension exists by an iterated application of [5], Ch. VI, § 10. 1, Proposition 1. The result of this step is denoted by (K_3, P_3).

Finally we adjoin $n - (d_1 + r_1)$ elements, algebraically independent over K_3, and let v_{P_3} be an immediate extension to this field. This is possible since the completion $\widehat{(K_3, P_3)}$ of the valued field (K_3, P_3) carries an immediate extension $\hat{P}_3$ of P_3 and has an infinite transcendence degree over the function field K_3[6]). The result of this step is denoted by (K_4, P_4).

The henselian field (K, P) mentioned above now is the henselian closure of (K_4, P_4). It remains to show that F can be embedded into K over k. Then P induces a place on F which will satisfy the assertions of the theorem. Actually, we find an embedding of $K_1 F$ over K_1 into K as follows.

We choose elements $t_1^*, \ldots, t_{n'}^* \in K$, algebraically independent over K_1, so close to $t_1', \ldots, t_{n'}'$ that by the Implicit Function Theorem (which holds in any henselian field, cf. [15], Theorem 7. 4) we can find $y^* \in K$ satisfying $f(\mathbf{t}^*, y^*) = 0$ and being so close to y' that, in addition, (ii), (iii), and (iv) hold for $\mathbf{t}^*, y^*$ instead of $\mathbf{t}', y'$. Since $\mathbf{t}', y'$ satisfy (ii)—(iv) and these conditions define an open set in the valuation topology, such elements $t_1^*, \ldots, t_{n'}^*, y^*$ can be found in K. The fact that $t_1^*, \ldots, t_{n'}^*$ can be even chosen to be algebraically independent over K_1 follows from the choice of the transcendence degree of K over K_1 (which is $n' = (d_1 - d) + (r_1 - r) + n - (d_1 + r_1)$), and the easily proved observation that, for any intermediate field $K_1 \subset K^* \subsetneqq K$ which is relatively algebraically closed in K, the elements of $K \setminus K^*$ lie dense in K. Applying this observation inductively yields the result.

Now $t_i \mapsto t_i^*$ $(1 \leqq i \leqq n')$ and $y \mapsto y^*$ defines an embedding of $K_1 F$ into K. Let us identify $K_1 F$ with its image in K. By the construction we see that $K_4(\mathbf{t}^*, y^*)$ is a finite algebraic extension of F, having a purely transcendental extension of $K_1 P_1$ of degree $d_1 - d$ as its residue field and a discrete lexicographic extension of $v_{P_1}(K_1)$ by $r_1 - r$ copies of $\mathbb{Z}$ as its value group. Thus FP and $v_P(F)$ satisfy the condition (1) and (2) of the theorem.

[6]) Since $v_{P_3}(K_3)$ is finitely generated, $\hat{P}_3$ contains an isomorphic copy of the field $k((X))$ of formal Laurent series which has infinite transcendence degree over k.

Lastly, we must check the conditions on $x_1, \dots, x_{m+s}$. After identification of $K_1 F$ with its image, we have

$$x_i = \frac{g_i(\mathbf{t}^*, y^*)}{h_i(\mathbf{t}^*)}.$$

Now the result follows from (iii) and (iv) for $\mathbf{t}^*, y^*$ together with $x'_i P = x'_i Q = x_i Q$ for $1 \leqq i \leqq m$ and $v_P(x'_i) = v_Q(x'_i) = v_Q(x_i)$ for $m+1 \leqq i \leqq m+s$. q.e.d.

Remark. If in the Main Theorem we have $n > d + r$, then we actually obtain the existence of infinitely many places P of F/k satisfying the asserted conditions. Indeed, let us first fix a sequence $(t_1^{(j)})_{j \in \mathbb{N}}$ in K_1 such that all $t_1^{(j)}$ are so close to t'_1 that we can find $y^{(j)}$ in the henselian closure of (K_1, P_1) such that $t_1^{(j)}, t'_2, \dots, t'_{n'}, y^{(j)}$ satisfy (i)—(iv) instead of $t'_1, \dots, t'_{n'}, y'$, and such that for all $j \in \mathbb{N}$

$$v_{P_1}(t_1^{(j+1)} - t'_1) > v_{P_1}(t_1^{(j)} - t'_1).$$

Next we choose $t_1^{(j)*}, t_2^*, \dots, t_{n'}^*, y^{(j)*}$, as in the proof of the Main Theorem, corresponding to $t_1^{(j)}, t'_2, \dots, t'_n, y^{(j)}$ such that, in addition,

$$v_{P_4}(t_1^{(j)*} - t_1^{(j)}) > v_{P_1}(t_1^{(j)} - t'_1).$$

It then follows that

$$v_{P_4}(t_1^{(j)*} - t'_1) = v_{P_1}(t_1^{(j)} - t'_1).$$

Thus the embeddings of $K_1 F$ corresponding to different choices of j induce different (equivalence classes of) places P on $K_1 F$. Since $K_1 F$ is a finite algebraic extension of F, we still get infinitely many (equivalence classes of) places P on F satisfying the assertions of the Main Theorem.

3. Modifications and applications

In the preceding section we have been concerned with the problem of approximating a given place Q of the function field F/k as close as possible by some other place P having a finitely generated value group and a finitely generated residue field. The approximation becomes better the more x_i's retain their residue classes or their values. In applications, however, the number of x_i's is very often fixed (e.g. there is just one x). In those cases we can put more conditions on the structure of the value group or on the residue field. In particular, one would be interested in the value group to be $\mathbb{Z}$ or the residue field to be k. Actually, it is possible to generalize the Main Theorem such as to cover all these cases. We rather prefer to discuss some interesting special cases. From this discussion the reader can see how to formulate a global generalization.

The procedure for obtaining generalizations will be always the same. In the construction of the place P in the Main Theorem we make the following *modification* (using the notation of the Main Theorem and its proof):

We choose $u_1, \dots, u_{d'}$ in F such that their residue classes form (over k) a transcendence base of the field generated by the residue classes of $x_1, \dots, x_m$. Thus for example, if all $x_1, \dots, x_m$ have their residue classes already lying in k, then the set $u_1, \dots, u_{d'}$ is empty. Next we choose $z_1, \dots, z_{r'} \in F$ such that the values of these elements form (over $\mathbb{Z}$) a maximal independent subset of the group generated by the values of $x_{m+1}, \dots, x_{m+s}$.

In case all elements $x_{m+1}, \ldots, x_{m+s}$ have value zero (or in case $s=0$), we let $r'=1$ and choose $z_1 \in F$ such that $v_Q(z_1) \neq 0$. Now the main point of the modification is the replacement of the henselian closure of (F, Q) by a suitable algebraic extension (L, Q) of the henselian closure. The choice of the henselian extension (L, Q) of (F, Q) is such that for the relative algebraic closure K' of $k(u_1, \ldots, u_{d'}, z_1, \ldots, z_{r'})$ in L we have

(a) $K'Q$ is e.c. in LQ,

(b) $v_Q(K')$ is e.c. in $v_Q(L)$.

Clearly, the choice of L depends on the situation we are interested in. After having made such a choice, we conclude from the Ax-Kochen-Ershov Theorem that (K', Q') is e.c. in (L, Q) where Q' is the restriction of Q to K'. Following the proof of the Main Theorem we now end up with a *modified version* where the conditions on $d_1 = \dim(P)$ and $r_1 = \mathrm{rr}(P)$ are replaced by

$$d' \leqq d_1 \leqq n-1, \quad r' \leqq r_1 \leqq n-d_1,$$

and, in the properties (1) and (2), the residue field FQ is replaced by K_1Q and the value group $v_Q(F)$ is replaced by $v_Q(K_1)$. Thus, particular properties of FP and $v_P(F)$ are reflected through $K'Q$ and $v_Q(K')$ which depend on the choice of L. (Recall that K_1 is a certain subfield of K'.) As in the Main Theorem we may even obtain infinitely many places P satisfying these conditions in case we know that $n > d' + r'$.

We will now consider three different choices of L and, at the same time, give three applications of our method.

In the first case we choose L to be the algebraic closure of the function field F. Then K' is also algebraically closed. Hence $K'Q$ is algebraically closed and $v_Q(K')$ is divisible. Thus (as we explained in Section 1) the above conditions (a) and (b) are satisfied. Applying this situation leads us to the following strengthening of a well known theorem (cf. [18], Part. I, § 4).[8])

Theorem 1. *Let A be an affine domain over k and $\mathfrak{p}$ a prime ideal of dimension d' in A. Then there exist places P of* Quot$(A)/k$ *which contain A and are centered at $\mathfrak{p}$ such that*

(1) *the residue field of P is finitely generated over k of dimension d_1,*

(2) *the value group of P is the r_1-fold product $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, ordered lexicographically,*

where d_1 and r_1 may be chosen freely, subject only to

$$d' \leqq d_1 \leqq n-1 \quad \textit{and} \quad 1 \leqq r_1 \leqq n-d_1$$

with n being the Krull dimension of A.

Proof. By Chevalley's Place Existence Theorem we extend the canonical homomorphism $A \to A/\mathfrak{p}$ to some place Q of Quot$(A)/k$ into the algebraic closure of $A/\mathfrak{p}$. Let $x_1, \ldots, x_m$ be generators of A. Thus, in the above explained modification of the Main Theorem, we may take $d' = \dim(Q) = \dim \mathfrak{p}$ and $r'=1$. The place P obtained by this modification satisfies the conditions (1) and (2) of Theorem 1. Note that, in the above construction, $v_Q(K_1)$ is generated by one element, hence is isomorphic to $\mathbb{Z}$.

q.e.d.

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[7]) Actually, we treat here only the case of char $k=0$. However, char $k=p$ can be treated similarly using A. Robinson's result on the model completeness of the theory of algebraically closed fields with a fixed valuation ring.

In the second application we consider a class $\mathscr{K}$ of field extensions of k. We assume that $\mathscr{K}$ is closed under subfields and purely transcendental extensions. Moreover, we assume that k is e.c. in k_1 for every $k_1 \in \mathscr{K}$. Examples of such classes are

— the class of all extensions of an algebraically closed field k,

— the class of all formally real fields, extending a real closed field k,

— the class of all formally p-adic fields of a fixed p-rank, extending a p-adically closed field k of the same p-rank (cf. [14]),

— the class of all extensions k_1 of a field k in which k is e.c.,

— the class of all totally real extensions of a maximal PRC-field k (see [12]),

— the class of all totally real and regular extensions of a PRC-field k (see [11]).

A place P of a function field F/k in n variables will be called a $\mathscr{K}$-place if $FP \in \mathscr{K}$. We define the *$\mathscr{K}$-holomorphy ring* of F/k to be

$$H(\mathscr{K}, F) = \bigcap \mathcal{O}_P$$

where the intersection runs over all $\mathscr{K}$-places P and $\mathcal{O}_P$ denotes the valuation ring of P.

Theorem 2. *Let $\mathscr{K}$ and $H(\mathscr{K}, F)$ be as introduced above. Then, for every $0 \leqq d_1 \leqq n-1$,*

$$H(\mathscr{K}, F) = \bigcap_{\substack{\dim(P)=d_1 \\ v_P(F)=\mathbb{Z}}} \mathcal{O}_P .$$

The intersection is taken over $\mathscr{K}$-places only. Moreover, we may restrict the intersection to $\mathscr{K}$-places P such that FP is a finitely generated subfield of a purely transcendental extension of k.

Proof. Clearly, it suffices to prove that, if for some $\mathscr{K}$-place Q and some non-zero $x \in F$ we have $xQ = 0$, we can find some $\mathscr{K}$-place P, admitted for the above intersection, such that $xP = 0$. We will obtain such a place in two steps. In the first step we deal with the residue field, in the second step we deal with the value group.

By an application of the Main Theorem, we may already assume that the $\mathscr{K}$-place Q has a finitely generated residue field FQ. By the assumption on $\mathscr{K}$, k is e.c. in FQ. By the lemma below, the function field FQ/k admits a rational place $\bar{P}$. Thus the composition of Q with $\bar{P}$ yields a rational place of F/k sending x to 0. Hence we may assume right from the beginning that Q is rational, i.e. $FQ = k$.

Now, let Q be a place of F/k such that $FQ = k$ and $xQ = 0$. In the above explained modification of the Main Theorem, we let $m = 0$, $s = 1$ and $x_{m+1} = x$. Hence $d' = 0$ and $r' = 1$. We choose L such that $LQ = FQ$ and $v_Q(L)$ is divisible. In particular, $LQ = K'Q$, which yields condition (a). Moreover, using $LQ = K'Q$, we will show that $v_Q(L)/v_Q(K')$ is torsion free. Indeed, let $\gamma \in v_Q(L)$ such that $q\gamma \in v_Q(K')$ for some prime $q \in \mathbb{N}$. Let $b \in L$ have value γ and $a \in K'$ have value $q\gamma$. Then $b^q a^{-1}$ is a unit in L. Since $LQ = K'Q$, there is some $e \in K'$ such that $b^q \cdot a^{-1} Q = eQ$. Hence $b^q \cdot a^{-1} \cdot e^{-1}$ is a 1-unit in L. Since $\operatorname{char} k = 0$, every 1-unit of the henselian field (L, Q) is a q-th power in L. Thus $a \cdot e$ is a q-th power in L, and hence in K'. Consequently, γ belongs to $v_Q(K')$. From $v_Q(L)/v_Q(K')$ being torsion free we see that $v_Q(K')$ is divisible too. Thus condition (b) also holds. Therefore, we can find a place P satisfying (1) and (2) of the modified version such that $xP = 0$. Hence we may have $v_P(F) = \mathbb{Z}$, $\dim(P) = d_1$, and FP being a subfield of a purely transcendental extension of $K'Q = k$. q.e.d.

For the convenience of the reader we add the following well known

Lemma. *Let K/k be a function field such that k is existentially closed in K. Then K/k admits a rational place.*

Proof. Let $K=k(x_1,\dots,x_d,y)$ where $x_1,\dots,x_d\in K$ are algebraically independent over k and $f\in k[x_1,\dots,x_d,Y]$ is the irreducible polynomial of y over $k(x_1,\dots,x_d)$. Since $x_1,\dots,x_d,y$ satisfy

$$f(\mathbf{x},y)=0 \quad\text{and}\quad \frac{\partial f}{\partial Y}(\mathbf{x},y)\neq 0$$

in K, we infer from k being e.c. in K that there are $a_1,\dots,a_d,b\in k$ satisfying

$$f(\mathbf{a},b)=0 \quad\text{and}\quad \frac{\partial f}{\partial Y}(\mathbf{a},b)\neq 0.$$

We consider the field $L=k((X_1))\cdots((X_d))$ of iterated Laurent series. The iterated composition of the canonical places corresponding to the variables X_i yields a place P of L such that $LP=k$ and $v_P(L)=\mathbb{Z}\times\cdots\times\mathbb{Z}$, d-times. Moreover, (L,P) is a henselian valued field. Since $X_1+a_1,\dots,X_d+a_d$ are algebraically independent over k, the map $x_i\mapsto X_i+a_i$ yields an embedding of $k(x_1,\dots,x_d)$ into L. Since the polynomial

$$f(X_1+a_1,\dots,X_d+a_d,Y)$$

has b as a simple zero in the residue field of (L,P), it also has a zero in L. Thus K embeds into L. Therefore P induces a rational place on K/k. q.e.d.

In the proof of Theorem 2 we have used the assumption that k is e.c. in every $k_1\in\mathcal{K}$. In case k fails to do so (i.e. $\mathcal{K}$ is just closed under subfields and purely transcendental extensions), we can still conclude that

$$(**)\qquad H(\mathcal{K},F)=\bigcap_{P\in\mathrm{Div}}\mathcal{O}_P$$

where Div denotes the class of $\mathcal{K}$-places of codimension 1, i.e. $\dim(P)=n-1$. This follows from the second part of the above proof, arranging it such that $LQ=K'Q=FQ$. However, we can no longer guarantee that FP is a subfield of a purely transcendental extension of k.

For the class $\mathcal{K}$ of formally real extensions of k, $(**)$ was proved by E. Becker in [4]. For k being real closed this was first proved by H. Schülting in [16]. In view of Schülting's result in [17], it should be pointed out that, in Theorem 2 for $n\geqq 2$, a finite number of places may be always omitted in the intersection yielding $H(\mathcal{K},F)$. Indeed, in the last part of the proof of Theorem 2 we apply the modified version to the case $d'=0$ and $r'=1$. Hence $r'+d'<n$ and thus we obtain infinitely many places P of the desired form.

For the class $\mathcal{K}$ of formally real extensions of a real closed field k, C. Andradas proved a theorem similar to Theorem 1 above ([1] Theorem 4. 6). It also results from our more general approach.

In the third application we will again consider the situation where $xQ=0$ for some non-zero element $x\in F$. This time however, we do not care about the residue field. We rather like to preserve as much information about the value $v_Q(x)$ as possible.

Theorem 3. *Let Q be a place of F/k and let $x \in F$ be non-zero. For every fixed prime $q \in \mathbb{N}$, there exists a place P of F/k with $v_P(F) = \mathbb{Z}$ and FP being a subfield of FQ, finitely generated over k, such that for every integer l: if l is prime to q and does not divide $v_Q(x)$ in $v_Q(F)$, then l does also not divide $v_P(x)$.*

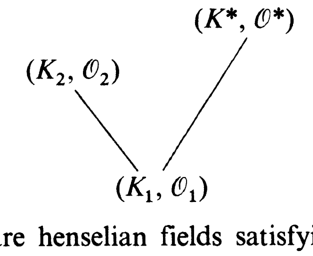
Proof. We let $m = \dim(Q)$, $x_1 = u_1, \ldots, x_m = u_m$, $s = 1$ and $x_{m+1} = x$. In the above modification of the Main Theorem we choose L such that $LQ = FQ$ and $v_Q(L)$ is the q-divisible hull of $v_Q(F)$. Thus $K'Q = LQ$ and $v_Q(L)/v_Q(K')$ is torsionfree (cf. the proof of Theorem 2). In particular, $v_Q(K')$ is archimedean and densely ordered. Since $v_Q(L)/v_Q(K')$ is torsionfree, we conclude that $v_Q(K')$ is e.c. in $v_Q(L)$[8]). Applying now the modified version to $d_1 = \dim Q$ and $r_1 = 1$, we obtain a place P of F/k such that FP is a subfield of FQ, finitely generated over k, and $v_P(F) = \mathbb{Z}$. Clearly, if l divides $v_P(x)$ in $\mathbb{Z}$, it also divides $v_Q(x) = v_P(x)$ in $v_Q(L)$. Since q is prime to l, this divisibility remains true in $v_Q(F)$. q.e.d.

Using the same arguments as in the proof of Theorem 2, one can generalize Theorem 3 similarly. If, for example, k is e.c. in FQ, we may have in addition that $FP = k$. This is used in [13].

Appendix

We will now give a proof of the Ax-Kochen-Ershov Theorem stated in Section 1. Using two well known facts from general model theory, we will first reduce the theorem to a specific assertion on embeddings of function fields into saturated fields. These general facts from model theory will be used without explanation. For more information the reader is refered to [6]. After having obtained the reduction, the rest of the proof — which actually is the main part — is algebraic in nature, using only twice the very definition of saturatedness.

By the Existence Theorem on elementary saturated extensions (cf. [6], Lemma 5. 1. 4) we let $(K^*, \mathcal{O}^*)$ be a $|K_2|^+$-saturated extension of $(K_1, \mathcal{O}_1)$. Since this extension is elementary, $(K^*, \mathcal{O}^*)$ is a henselian valued field too. Moreover, the value group Γ^* is an elementary and $|K_2|^+$-saturated extension of Γ_1. Since Γ_1 is e.c. in Γ_2, we can embed Γ_2 into Γ^* over Γ_1 ([5], Lemma 5. 2. 1). By the same argument we find that the residue field $\bar{K}_2$ can be embedded into the residue field $\bar{K}^*$ of $(K^*, \mathcal{O}^*)$ over $\bar{K}_1$. We identify Γ_2 and $\bar{K}_2$ with their images resp. Thus we have obtained the following situation:



where $(K_1, \mathcal{O}_1)$ and $(K^*, \mathcal{O}^*)$ are henselian fields satisfying:

(i) $(K^*, \mathcal{O}^*)$ is $|K_2|^+$-saturated,

(ii) $\bar{K}_1 \subset \bar{K}_2 \subset \bar{K}^*$, and $\bar{K}_1$ is algebraically closed in $\bar{K}_2$,

(iii) $\Gamma_1 \subset \Gamma_2 \subset \Gamma^*$, and Γ_2/Γ_1 is torsionfree.

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[8]) See Section 1

Claim. *Under these conditions,* $(K_2, \mathcal{O}_2)$ *embeds into* $(K^*, \mathcal{O}^*)$.

Once we have proved this claim, we are done. Indeed, in the above construction, $(K^*, \mathcal{O}^*)$ is an elementary extension of $(K_1, \mathcal{O}_1)$. Thus, an existential statement which holds in $(K_2, \mathcal{O}_2)$ clearly must hold in $(K_1, \mathcal{O}_1)$. Note that, for the proof of the claim, we do not require that $(K^*, \mathcal{O}^*)$ elementarily extends $(K_1, \mathcal{O}_1)$.

Proof. The proof of the claim proceeds in three steps. Let us first assume w.l.o.g. that $(K_2, \mathcal{O}_2)$ is also henselian.

Step 1. In this step we extends the embedding of $(K_1, \mathcal{O}_1)$ into $(K^*, \mathcal{O}^*)$ (which is the identity) to a subfield of $(K_2, \mathcal{O}_2)$ which has residue field $\bar{K}_2$. Assume that $(K, \mathcal{O})$ is a maximal subfield of $(K_2, \mathcal{O}_2)$ having value group $\Gamma = \Gamma_1$, such that $(K, \mathcal{O})$ can be embedded into $(K^*, \mathcal{O}^*)$. We identify $(K, \mathcal{O})$ with its image. If $\bar{K} \subsetneqq \bar{K}_2$, we let $\bar{x} \in \bar{K}_2 \setminus \bar{K}$ and consider two cases, both leading to a contradiction to the maximality of $(K, \mathcal{O})$.

Case 1. $\bar{x}$ is algebraic over $\bar{K}$. Let $f \in \mathcal{O}[X]$ be monic such that $\bar{f}$ is the minimal polynomial of $\bar{x}$ over $\bar{K}$. Then f is irreducible over K. By Hensel's Lemma it has zeros $x \in K_2$ and $x^* \in K^*$ both having residue class $\bar{x}$. Now the assignment $x \mapsto x^*$ defines an embedding of $K(x)$ into K^* which is value-preserving. Therefore $(K, \mathcal{O})$ would not be maximal.

Case 2. $\bar{x}$ is transcendental over $\bar{K}$. Let $x \in K_2$ and $x^* \in K^*$ be preimages of $\bar{x}$ resp. Clearly, both are transcendental over K. Thus $K(x)$ is isomorphic to $K(x^*)$ via $x \mapsto x^*$. This isomorphism is value-preserving, since there is a unique valuation on $K(x)$ and $K(x^*)$ resp., extending $\mathcal{O}$ and assigning the residue class $\bar{x}$ to x and x^* resp. (cf. [5], Ch. VI, § 10.1, Proposition 2). Also in this case, $(K, \mathcal{O})$ would not be maximal.

Step 2. In this step we extend the above embedding further to a subfield of $(K_2, \mathcal{O}_2)$ which has value group Γ_2. Let now $(K, \mathcal{O})$ be a maximal subfield of $(K_2, \mathcal{O}_2)$ which embeds into $(K^*, \mathcal{O}^*)$ such that $\bar{K} = \bar{K}_2$ and Γ_2/Γ is torsionfree. (As above, Γ denotes the value group of $(K, \mathcal{O})$.) Such a subfield exists by Zorn's Lemma. We identify $(K, \mathcal{O})$ with its image. Assume that $\Gamma_2 \setminus \Gamma$ contains an element γ. By the assumption on Γ, we have $\Gamma \cap \mathbb{Z}\gamma = \{0\}$. Let $x \in K_2$ have value γ. Assigning γ to x defines a unique extension of $\mathcal{O}$ to the rational function field $K(x)$ (cf. [5], Ch. VI, § 10.1, Proposition 1). From $K(x)$ we now pass to an algebraic extension K' inside K_2 such that Γ_2/Γ' is torsionfree where Γ' denotes the value group corresponding to $\mathcal{O}' = \mathcal{O}_2 \cap K'$. This can be done in the following manner. If some $\delta \in \Gamma_2 \setminus (\Gamma + \mathbb{Z}\gamma)$ satisfies $q\delta \in \Gamma + \mathbb{Z}\gamma$ for some prime $q \in \mathbb{N}$, we choose $y \in K_2$ having value δ and $a \in K(x)$ having value $q\delta$. Then $y^q \cdot a^{-1}$ is a unit in $\mathcal{O}_2$. Since $\bar{K}_2 = \overline{K(x)}$ we find a unit e in $K(x)$ such that $y^q \cdot a^{-1} \cdot e^{-1}$ is a 1-unit in $\mathcal{O}_2$. Since $\operatorname{char} \bar{K}_2 = 0$, every 1-unit of $\mathcal{O}_2$ is a q-th power in K_2. Thus ae is a q-th power in K_2, say $ae = z^q$ for some $z \in K_2$. Thus the value group of $K(x, z)$ contains δ. Transfinite repetitions of this procedure (or simply an application of Zorn's Lemma) yield an algebraic extension K' of $K(x)$ of the desired nature. It remains to find an embedding of $(K', \mathcal{O}')$ into $(K^*, \mathcal{O}^*)$.

At this place we use the fact that $(K^*, \mathcal{O}^*)$ is $|K_2|^+$-saturated. Because of this assumption it suffices to find an embedding into $(K^*, \mathcal{O}^*)$ for every subfield of $(K', \mathcal{O}')$, finitely generated over K. It then follows that $(K', \mathcal{O}')$ admits an embedding into $(K^*, \mathcal{O}^*)$. Therefore we may assume that $(K', \mathcal{O}')$ itself is a finitely generated extension of K.

From the maximality condition on $(K, \mathcal{O})$ it follows that $(K, \mathcal{O})$ is henselian. Since $\bar{K} = \bar{K}_2$ and Γ_2/Γ is torsionfree, it follows from $\operatorname{char} \bar{K} = 0$ that K is relatively algebraically closed in K_2. Thus $K' = K$ or K' is a finitely generated extension of K of transcendence degree 1. In the second case, Γ'/Γ must be isomorphic to $\mathbb{Z}$, since it is torsionfree. Thus $\Gamma' = \Gamma + \gamma\mathbb{Z}$ with $\Gamma \cap \gamma\mathbb{Z} = \{0\}$ for some $\gamma \in \Gamma'$. We choose $y \in K'$ with value γ. Then the value group of the transcendental extension $K(y)$ of K is $\Gamma' = \Gamma + \gamma\mathbb{Z}$. The valuation on $K(y)$ is uniquely determined as an extension of $\mathcal{O}$ by the assignment $y \mapsto \gamma$ (see above). Thus, if we choose $y^* \in K^*$ having also value γ, clearly $y \mapsto y^*$ defines a value-preserving isomorphism of $K(y)$ and $K(y^*)$. Since $(K', \mathcal{O}')$ is an immediate extension of $K(y)$, the uniqueness of the henselian closure (together with $\operatorname{char} \bar{K}' = 0$) yields an embedding of $(K', \mathcal{O}')$ into $(K^*, \mathcal{O}^*)$. Thus $(K, \mathcal{O})$ would not be maximal with $\bar{K}_2 = \bar{K}$ and Γ_2/Γ being torsionfree, unless $\Gamma_2 = \Gamma$.

Step 3. Let finally $(K, \mathcal{O})$ be a maximal subfield of $(K_2, \mathcal{O}_2)$ such that $\overline{K_2} = \bar{K}$ and $\Gamma_2 = \Gamma$, and $(K, \mathcal{O})$ embeds into $(K^*, \mathcal{O}^*)$. We identify $(K, \mathcal{O})$ with its image. Clearly, $(K, \mathcal{O})$ is henselian. Using again $\operatorname{char} \bar{K} = 0$, we see that K is relatively algebraically closed in K_2. Thus, each $x \in K_2 \setminus K$ would be transcendental over K. We show that the existence of such an element x would lead to a contradiction, thus proving that $K_2 = K$. This then finishes the proof of the claim.

Let $x \in K_2 \setminus K$. We first prove the existence of some $x^* \in K^*$ satisfying

$$v(x-a) = v^*(x^*-a) \quad \text{for all} \quad a \in K.$$

Here v and v^* denote the valuations corresponding to $\mathcal{O}_2$ and $\mathcal{O}^*$ resp. Since $\Gamma_2 = \Gamma$, we can find some $b_a \in K$ such that $v(x-a) = v(b_a)$. Hence x^* would solve the system

$$v^*(x^*-a) = v^*(b_a) \quad \text{for all} \quad a \in K.$$

This system is easily expressed by a set of formulas over $(K^*, \mathcal{O}^*)$, having cardinality $\leqq |K_2|$. Thus, since $(K^*, \mathcal{O}^*)$ is $|K_2|^+$-saturated, it suffices to solve every finite subsystem of the above system. Let

$$(*) \qquad v^*(x^*-a_i) = v^*(b_i) \quad (1 \leqq i \leqq m)$$

be such a finite subsystem. We then consider the corresponding equations

$$v(x-a_i) = v(b_i) \quad (1 \leqq i \leqq m),$$

satisfied by our fixed element $x \in K_2 \setminus K$. Let $a \in \{a_1, \ldots, a_m\}$ have maximal value $v(x-a) = v(b)$ among the values $v(x-a_i)$ $(1 \leqq i \leqq m)$. Since $v\left(\frac{x-a}{b}\right) = 0$ and $\bar{K}_2 = \bar{K}$, we can find $c \in K$ such that $v\left(\frac{x-a}{b} - c\right) > 0$. Therefore we have

$$v\big(x-(a+bc)\big) > v(b) = v(x-a).$$

Hence putting $y = a + bc$ we see that

$$v(x-y) > v(x-a) \geqq v(x-a_i).$$

Thus we have

$$v(y-a_i) = v\big((x-y)-(x-a_i)\big) = v(x-a_i) = v(b_i).$$

Since $y \in K \subset K^*$, we have found a solution of $(*)$ by taking $x^* = y$.

Now let $x^* \in K^*$ such that $v^*(x^*-a) = v(x-a)$ for all $a \in K$. We observe first that x^* is transcendental over K. Thus $x \mapsto x^*$ defines an isomorphism of $K(x)$ and $K(x^*)$ as fields. We will show that this isomorphism is value-preserving. In fact, we prove by induction on $\deg f$ that

$$v(f(x)) = v^*(f(x^*))$$

for all polynomials $f \in K[x]$.

The case $\deg f = 1$ has just been proved. Assume we already proved this equation for $\deg f = n-1$. Now let $\deg f = n$. By the induction hypothesis, we may assume that f is irreducible. We consider the field $F = K[x]/(f)$ which is isomorphic to

$$V = K + Kx + \cdots + Kx^{n-1},$$

viewed as a K-linear space. The restriction of the valuation v to the subset V of $K[x]$ obviously induces a map (denoted by the same letter)

$$v: F \to \Gamma \cup \{\infty\},$$

satisfying all properties of a valuation on F, except perhaps the multiplicative law. Since $n = \deg f = [F:K] > 1$ and K is a henselian field with $\operatorname{char} \bar{K} = 0$, either this multiplicative law must fail or the residue field of F with respect to v must be a proper extension of $\bar{K}$. The second possibility cannot occur. Thus there must exist polynomials $g, h \in K[x]$ of degree $< n$ such that

$$v(r) \neq v(g) + v(h)$$

where r is the unique polynomial of degree $< n$ such that $g \cdot h = f \cdot s + r$ for some $s \in K[x]$. From the above inequality we find

$$v(f) = -v(s) + \min(v(g) + v(h), v(r)).$$

All polynomials on the RHS have degree $< n$. Thus by the induction hypothesis we conclude that $v(f(x)) = v^*(f(x^*))$. q.e.d.

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