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Manuscripta mathematica

Volume 55 / 1986 / Article

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manuscripta math. 55, 39 - 67 (1986)

manuscripta
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IMMEDIATE AND PURELY WILD EXTENSIONS OF VALUED FIELDS

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Kaplansky's hypothesis A concerning valued fields is put into a Galois theoretic setting. Accordingly, Kaplansky's theorem on maximal immediate extensions can be deduced from the Schur-Zassenhaus theorem about conjugacy of complements in profinite groups. Some generalization of Kaplansky's theory is given, concerning maximal purely wild extensions.

1. Introduction.

Let K be a valued field, with value group vK and residue field $\overline{K}$. Kaplansky [7] has given conditions for vK and $\overline{K}$ (known as hypothesis A), which ensure that all maximal immediate extensions of K are K-isomorphic as valued fields. A valued field extension L of K is called immediate if L has the same value group and the same residue field as K. It is known from the work of Krull that every immediate extension of K is contained in a maximal one. In a sense, such maximal immediate extension behaves somewhat like a completion, provided it is uniquely determined up to K-isomorphisms. In general, uniqueness does not hold if the residue characteristic is p > 0; this gives Kaplansky's theorem its fundamental importance.

In this note we are going to exhibit a Galois theoretic interpretation of Kaplansky's hypothesis (A), in terms of the structure of the Galois group over K and its subgroups defined by Hilbert's ramification theory: decomposition group, inertia group, ramification group. In this way it will be seen that Kaplansky's theorem becomes almost obvious for group theoretical reasons, in view of the theorem of Schur-Zassenhaus about conjugacy of complements in profinite groups.

More precisely, while using Galois theory we have to restrict our consideration to algebraic extensions of K.

From group theoretical arguments we do not obtain the full uniqueness theorem of Kaplansky but only its algebraic part, which asserts uniqueness of the maximal algebraic immediate extension of K, in the presence of hypothesis (A). This algebraic part appears to be the main ingredient of Kaplansky's theorem, for in many respects it seems more appropriate to consider not the full maximal immediate extension in the sense of Krull, but only its algebraic part.

Nevertheless in some instances transcendental extensions seem to be necessary. Therefore we shall complement our Galois theoretic discussion by some further remarks concerning transcendental extensions, thereby leading to a proof of the full theorem of Kaplansky. The crucial embedding property for simple transcendental immediate extensions (section 7) is of the same kind as used for proving model completeness theorems [13], [15]. We do not use the theory of pseudo-Cauchy sequences in the sense of Ostrowski-Kaplansky. Although this notion has played an important role in the past development it now seems to be time to look for more natural, intrinsic concepts and arguments.

In recent years we have observed rising interest in the study of maximal immediate extensions of valued fields, in particular from the model theoretic point of view. This has motivated the publication of the present note. Nearly all its contents are from the unpublished thesis of Matthias Pank [11]. Further investigations into the same direction would be desirable. For instance, one would like to know to which extent hypothesis (A) is necessary for the uniqueness of the maximal immediate extension. A preliminary result is given in the appendix, due to F.-V. Kuhlmann. The problem might perhaps be settled by studying the structure of the Galois group in more detail. Also, one might hope to classify at least in special instances, the various maximal immediate extensions, when uniqueness fails. In view of this general program we shall carry our discussion as far as possible without using hypothesis (A). Our main results are theorem 2.1, theorem 4.3 and theorem 5.3.

In the whole paper the valued field K is always supposed to be Henselian; this is no essential restriction. The Henselian property implies that the valuation of K extends uniquely to the algebraic closure K^a of K. Hence K^a will be regarded as a valued field. Then every subfield $L \subset K^a$ is also a valued field, in a canonical way.

We always suppose that the residue characteristic is a prime number $p > 1$. In case of residue characteristic zero it is well known that the uniqueness theorem for maximal algebraic immediate extensions holds trivially because there are no proper algebraic immediate extensions. However it should be understood that all our results and proofs remain trivially valid (for $p = 1$) in case of residue characteristic zero.

Notational conventions:

- K — is a nontrivially valued field, assumed to be Henselian
- v — the valuation of K, written additively
- vK — the value group of K
- $\overline{K}$ — the residue field
- p — the characteristic of $\overline{K}$, assumed to be > 1
- K^a — the algebraic closure of K, regarded as a valued field extension of K
- G — the Galois group over K, defined as the group of all K-automorphisms of K^a
- K^r — the ramification field over K
- G_r — the ramification group over K, consisting of all automorphisms in G which leave the elements of K^r fixed.

We assume the reader to be familiar with the fundamentals from ramification theory of general valuations. As a reference we mention Endler's book [4], in particular § 19-22. See also [17], Chap. 3.

If L is a valued extension field of K, its valuation is denoted by the same letter v as the valuation of K. vL is a group extension of vK, and $\bar{L}$ is a field extension of $\bar{K}$.

2. Splitting property of the ramification field.

Theorem 2.1. (i) The algebraic closure K^a of K splits over the ramification field K^r. That is, there exists a subfield $L \subset K^a$ such that

$$L \cap K^r = K \quad , \quad L \cdot K^r = K^a \ .$$

Any such field L is called an algebraic K-complement of K^r.

(ii) Every algebraic extension K' of K which is K-linearly disjoint to K^r, is contained in some algebraic K-complement L of K^r.

We will prove theorem 2.1 via Galois theory by verifying the corresponding theorem for the Galois group G over K:

Theorem 2.2. (i) The Galois group G splits over its ramification subgroup G_r. That is, there exists a closed subgroup $H \subset G$ such that

$$H \cdot G_r = G \quad , \quad H \cap G_r = 1 \ .$$

Any such group H is called a group complement of G_r within G.

(ii) Every closed subgroup $G' \subset G$ which satisfies $G' \cdot G_r = G$, contains some group complement H of G_r in G.

If K is perfect then Galois theory yields a 1-1 correspondence between algebraic extensions L of K and closed subgroups H of G. In this correspondence, the properties of L to be an algebraic K-complement of K^r are translated into the properties of H to be a group complement of G_r in G. Hence indeed, theorems 2.2 and 2.1 are equivalent. If K is not perfect then we have to take into account that Galois groups do not distinguish between fields which differ by pure inseparabilities only. The fixed field of any closed subgroup of G is a perfect field. Galois theory yields a 1-1 correspondence between perfect algebraic extensions of K and closed subgroups of G. On the other hand, from $K^r|K$ being separable we infer

1) if $L \cdot K^r = K^a$, then L is perfect
2) if K' is K-linearly disjoint to K^r then the perfect hull of K' too is K-linearly disjoint to K^r.

Hence, to prove theorem 2.1 it suffices to consider <u>perfect</u> algebraic extensions of K. For those, Galois correspondence can be applied, showing that again theorems 2.1 and 2.2 are equivalent. At the same time our discussion yields the following, regardless of whether K is perfect or not:

<u>Lemma 2.3.</u> <u>The algebraic K-complements</u> L <u>of</u> K^r <u>are precisely the fixed fields of the group complements</u> H <u>of</u> G_r <u>in</u> G. <u>In particular, every such field</u> L <u>is perfect</u>.

To prove theorem 2.2 we consider the <u>inertia field</u> over K. This can be characterized as the maximal unramified subextension of K^r and hence will be denoted by K^u. Let G_u be the Galois group over K^u.

<u>Proposition 2.4.</u> (i) G_u <u>is normal in</u> G <u>and</u> $G \supset G_u \supset G_r$.

(ii) <u>The factor group</u> G/G_u <u>is canonically isomorphic to the Galois group</u> $\overline{G}$ <u>over the residue field</u> $\overline{K}$ <u>of</u> K. <u>In particular it follows</u>

$$[G : G_u] = |\overline{G}| = [\overline{K}^s : \overline{K}]$$

<u>where</u> $\overline{K}^s$ denotes the separable algebraic closure of $\overline{K}$.

(iii) <u>The factor group</u> G_u/G_r <u>is abelian and its order</u> $[G_u : G_r]$ <u>is relatively prime to</u> p.

(iv) G_r <u>is a</u> p-<u>group.</u>

For the proof we refer to [4], in particular to the table on page 171. Note that we are dealing with profinite groups; the group indices appearing in this proposition may not be finite. As usual, these group indices and the degrees of infinite algebraic field extensions are to be regarded as <u>supernatural numbers</u>. We refer to Serre's lecture notes [16], in particular page I-3.

Corollary 2.5. The p-Sylow subgroups of G/G_r are free as pro-p-groups.

Proof. By prop. 2.4 (iii), the p-Sylow subgroups of G/G_r are isomorphic to those of G/G_u. Those in turn are isomorphic to the p-Sylow subgroups of $\overline{G}$, in view of prop. 2.4 (ii). Now over any field $\overline{K}$ of characteristic p its Galois group $\overline{G}$ has pro-p-free p-Sylow groups: this is a well known consequence of Artin-Schreier theory. See [16], page II-5, cor. 1.

Proof of theorem 2.2.

(i) Since the p-Sylow subgroups of G/G_r are pro-p-free it follows that G/G_r is of cohomological p-dimension ≤ 1; see [16], page I-25, cor. Hence every profinite extension of G/G_r with pro-p-kernel splits; [16], page I-24, prop. 16 (iii bis). We conclude that, indeed, G splits as an extension of G/G_r with kernel G_r.

(ii) Let $G' \subset G$ be a closed subgroup and suppose $G' \cdot G_r = G$. Then $G'/G' \cap G_r \approx G/G_r$. Hence the p-Sylow subgroups of $G'/G' \cap G_r$ are pro-p-free. As in (i) it follows that G' splits over $G' \cap G_r$. Let $H \subset G'$ be a complement of $G' \cap G_r$ within G'. Then

$$H \cdot (G' \cap G_r) = G' \ , \ H \cap (G' \cap G_r) = 1 \ .$$

It follows

$$H \cdot G_r = G' \cdot G_r = G \ , \ H \cap G_r = 1 \ .$$

Hence H is also a complement of G_r within G.

3. Applying the Schur-Zassenhaus theorem.

Let L be an algebraic K-complement of K^r, according to theorem 2.1. In general there will be many other algebraic K-complements of K^r. Under which condition are any two algebraic K-complements L, L' of K^r isomorphic over K? Note that since K is Henselian, every field theoretic K-isomorphism $\sigma : L \to L'$ is compatible with the valuations on L and L', i.e. σ is an isomorphism of valued fields.

We know from lemma 2.3 that L, L' are the fixed fields of group complements H, H' of G_r in G. By Galois theory, L and L' are K-isomorphic if and only if H and H' are conjugate within G. Hence our above question can be reformulated in group theoretical terms as follows: <u>Under which conditions are any two group complements</u> H, H' <u>of</u> G_r <u>conjugate within</u> G? At this point we invoke:

<u>Theorem 3.1.</u> <u>(Schur-Zassenhaus)</u> <u>Let</u> G <u>be a profinite group and</u> N <u>a closed normal subgroup. Suppose that the index</u> [G : N] <u>is relatively prime to the order</u> |N|. <u>Then</u> G <u>splits over</u> N. <u>Moreover, all group complements of</u> N <u>are conjugate within</u> G.

Usually this theorem is formulated in the context of <u>finite</u> groups; see [6], page 128, Satz 18.3. Its generalization to profinite groups is straightforward, using standard compactness arguments for profinite groups. One could also use nonstandard arguments in the sense of A. Robinson who pointed out that every profinite group G can be represented as a homomorphic image of a suitable starfinite group of an enlargement *G of G. See [14], section 3. We leave the details to the reader.

We shall use theorem 3.1 for the profinite Galois group G over K and its normal ramification subgroup G_r. Prop. 2.4 (iv) shows that $|G_r|$ is a power of p. Therefore we can apply theorem 3.1 under the condition that $[G : G_r]$ is relatively prime to p. By prop. 2.4 again, this is equivalent to $[G : G_u] = [\overline{K}^s : \overline{K}]$ being relatively prime to p. We obtain:

<u>Proposition 3.2.</u> <u>Suppose that</u> $[\overline{K}^s : \overline{K}]$ <u>is relatively prime to</u> p. <u>Then</u> $[K^r : K]$ <u>is relatively prime to</u> p, <u>and all algebraic</u> K-<u>complements of</u> K^r <u>are mutually</u> K-<u>isomorphic.</u>

4. Valuational properties of algebraic K-complements.

Definition. A finite algebraic field extension $E|K$ is called tame if the following three conditions are satisfied:

(a) The ramification index $[vE : vK]$ is relatively prime to p.

(b) The residue field extension $\overline{E}|\overline{K}$ is separable.

(c) The degree formula holds:

$$[E : K] = [vE : vK]\cdot[\overline{E} : \overline{K}] .$$

An infinite algebraic extension of K is called tame if every finite subextension is tame.

This notion of tame extension is different from the notion of "tamely ramified" extension as defined in [4], page 180. The latter definition requires (a) and (b) but not necessarily (c). Condition (c) implies, in the terminology of [4], that the extension is "defectless". Thus the tame extensions in our sense coincide with the "tamely ramified and defectless" extensions in the terminology of [4]. Note that in our terminology, proper immediate extensions are not tame - which seems reasonable in view of the rather exotic behavior of immediate extensions.

It follows from the definition that every subextension of a tame extension is tame. The compositum of finitely or infinitely many tame extensions is tame again. Hence there exists a unique maximal tame extension of K. It is well known that this coincides with the ramification field:

Proposition 4.1. The ramification field K^r can be characterized as the unique maximal tame extension of K.

See [4], page 182, theorem (22.7).

A valued field extension $L|K$ is called wild if it is not tame.

Definition. $L|K$ is called purely wild if the following two conditions are satisfied:

(a') The value group index $[vL : vK]$ is a p-power.

(b') The residue field extension $\overline{L}|\overline{K}$ is purely inseparable.

(If $L|K$ is transcendental, conditions (a') and (b') should be read such as to imply that $vL|vK$ is a torsion group and that $\overline{L}|\overline{K}$ is algebraic.)

In case of purely wild extensions, the behavior of value group and residue field belongs to the other extreme as in the case of tame extensions. We shall use this definition to characterize the algebraic K-complements of the maximal tame extension K^r of K.

Every subextension of a purely wild extension is purely wild. Immediate extensions are purely wild. Every purely inseparable extension is purely wild. The only extension $L|K$ which is purely wild and tame is the trivial one: $L = K$.

Lemma 4.2. An algebraic extension $L|K$ is purely wild if and only if L is K-linearly disjoint to K^r.

Proof. If L is purely wild over K then so is its subextension $L \cap K^r$. On the other hand $L \cap K^r \subset K^r$ and hence $L \cap K^r$ is tame over K, by prop. 4.1. It follows $L \cap K^r = K$. Thus L is K-linearly disjoint to K^r.

Conversely suppose $L \cap K^r = K$. We have to show that conditions (a'), (b') are satisfied. Let us start with (b'). Let $\alpha \in \overline{L}$ be separable over $\overline{K}$; we claim that $\alpha \in \overline{K}$. Let $\overline{f}(X) \in \overline{K}[X]$ be the monic irreducible polynomial for α over $\overline{K}$, and let $f(X) \in K[X]$ be a monic foreimage of $\overline{f}(X)$. Since α is a simple root of $\overline{f}(X)$ we conclude from Hensel's lemma that there is a unique $a \in L$ such that $f(a) = 0$ and $\overline{a} = \alpha$. Note that L is Henselian because L is an algebraic extension of the Henselian field K. Consider $E = K(a) \subset L$. If n denotes the degree of $f(X)$ and $\overline{f}(X)$ then clearly

$$[E : K] \leqq n = [\overline{K}(\alpha) : \overline{K}] \leqq [\overline{E} : \overline{K}] \leqq [\overline{E} : \overline{K}]\cdot[vE : vK].$$

From general valuation theory we know that

$$[vE : vK]\cdot[\overline{E} : \overline{K}] \leqq [E : K] ,$$

see [4] , page 99, cor. (13.10). We conclude

$$[E : K] = n = [\overline{E} : \overline{K}] \ , \ \overline{E} = \overline{K}(\alpha) \ , \ [vE : vK] = 1 \ .$$

These relations show that $E|K$ is tame. By prop. 4.1, $E \subset K^r$. Hence $E \subset L \cap K^r = K$. In particular $a \in K$ and therefore $\overline{a} = \alpha \in \overline{K}$ as contended.

We are now going to prove condition (a'). Let $\alpha \in vL$ be such that its order modulo vK, say n, is relatively prime to p; we claim that $\alpha \in vK$. It suffices to show that $p^h\alpha \in vK$ for some suitable integer $h \geqq 0$. Let us choose a, b, c such that

$$a \in L \ , \ v(a) = \alpha$$
$$b \in K \ , \ v(b) = n\alpha$$
$$c = \frac{a^n}{b} \in L \ .$$

Then $0 \neq \overline{c} \in \overline{L}$. We know already that $\overline{L}|\overline{K}$ is purely inseparable, hence $\overline{c}^{p^h} \in \overline{K}$ for some p-power p^h. As said above, we may replace α by $p^h\alpha$ and, accordingly, we replace a, b, c by a^{p^h}, b^{p^h}, c^{p^h}. Thus we may assume from now on that $\overline{c} \in \overline{K}$. Let $d \in K$ be such that $\overline{c} = \overline{d}$. Then $v(d) = 0$. We replace b by $bd \in K$; this will not affect the value $v(b) = n\alpha$. After changing notation we now have $\overline{c} = 1$. This implies that c is an n-th power in L. For let $f(X) = X^n - c \in L[X]$; then $\overline{f}(X) = X^n - 1$. Since n is relatively prime to p it follows that $\overline{f}(X)$ admits 1 as a <u>simple</u> root. We conclude from Hensel's lemma that there exists a unique $w \in L$ such that $f(w) = 0$ and $\overline{w} = 1$. Thus

$$\frac{a^n}{b} = c = w^n .$$

We replace a by $aw^{-1} \in L$; this will not affect the value $v(a) = \alpha$. After changing notation again we obtain $c = 1$,

$$a^n = b.$$

Note that $b \in K$ by construction. Hence a is an n-th root of $b \in K$. Let $E = K(a) \subset L$. Then $\alpha \in vE$ and

$$[E : K] \leqq n \leqq [vE : vK] \leqq [vE : vK]\cdot[\overline{E} : \overline{K}] \leqq [E : K] \ .$$

We conclude

$$[E : K] = n = [vE : vK], \ [\bar{E} : \bar{K}] = 1.$$

These relations show that $E|K$ is tame. Therefore from prop. 4.1: $E \subset L \cap K^r = K$. In particular $a \in K$ and $v(a) = \alpha \in vK$ as contended.

Lemma 4.2 is proved. Comparing it with theorem 2.1 we obtain:

<u>Theorem 4.3. The algebraic</u> K-<u>complements of</u> K^r <u>can be characterized as the maximal algebraic purely wild extensions of</u> K.

To prove the next proposition we will need:

<u>Lemma 4.4. Every valued field</u> L <u>with nontrivial valuation</u> v <u>and</u> $char(L) = p > 0$, <u>which is closed under purely wild extensions by roots of polynomials of the form</u> $X^p - X - c$, $c \in L$, $v(c) < 0$ <u>has</u> p-<u>divisible value group and perfect residue field.</u>

<u>Proof.</u> Any root a of the polynomial $X^p - X - c$, $v(c) < 0$, will have value $v(a) = v(c)/p$. Now if $v(c)/p \notin vL$, by the same arguments as in the proof of lemma 4.2 we may conclude that $[L(a) : L] = p = [v(L(a)) : vL]$, $[\overline{L(a)} : \bar{L}] = 1$. Hence $L(a)|L$ would be a proper purely wild extension, contrary to the hypotheses. Thus vL is p-divisible.

Now let $c \in L$ have value $v(c) = 0$. We choose $d \in L$ with $v(d) < 0$. Any root a of $X^p - X - d^pc$ will have value $v(a) = v(d)$ and will satisfy $\overline{a/d} = \bar{c}^{1/p}$. If $\bar{c}^{1/p} \notin \bar{L}$, it follows as above that $L(a)|L$ would be a proper purely wild extension, contrary to the hypothesis. Hence $\bar{L}$ is perfect.

Lemma 4.4 implies that every Artin-Schreier-closed valued field of characteristic p has p-divisible value group and perfect residue field. Note that this is also true for every perfect valued field of characteristic p, since it is closed under p-th roots.

Proposition 4.5. (i) Let L be an algebraic K-complement of K^r or, what is the same by theorem 4.3, L should be a maximal algebraic purely wild extension of K. Then vL is the p-divisible hull of vK, and $\overline{L}$ is the perfect hull of $\overline{K}$.

(ii) More generally, the same holds for any subextension L' of L which is closed under purely wild extensions by roots of polynomials of the form $X^p - X - c$, $c \in L'$, $v(c) < 0$. Consequently L is an immediate extension of L'.

(iii) In particular, L is an immediate extension of L_s, the maximal subfield of L which is separable over K. L_s is closed under separable algebraic purely wild extensions.

Proof: L_s has no proper separable algebraic purely wild extension L_1 because otherwise the perfect hull of L_1 would be a proper algebraic purely wild extension of L.

Now it suffices to prove (ii). Since $L'|K$ is purely wild, $[vL' : vK]$ is a p-power and $\overline{L'}$ is purely inseparable over $\overline{K}$. On the other hand, lemma 4.4 shows that vL' is p-divisible and $\overline{L'}$ is perfect.

Proposition 4.5 may be compared with the following well known proposition giving an explicit description of value group and residue field of the ramification field. Let us define the p'-divisible hull of vK to be the smallest totally ordered group extension of vK which is divisible by all prime numbers $\neq p$.

Proposition 4.6. The value group vK^r is the p'-divisible hull of vK. The residue field $\overline{K^r}$ is the separable algebraic closure of $\overline{K}$.

For a proof see [4] , pages 151, 166. There one can also find a proof of the following proposition which we shall have to use in section 6.

Proposition 4.7. Let $F|K$ be a subextension of K^r. Suppose that F has the same value group and the same residue field as K^r:

$$vF = vK^r \quad , \quad \overline{F} = \overline{K^r} \ .$$

Then $F = K^r$. In other words: The ramification field K^r is minimal with the properties as stated in prop. 4.6.

5. Introducing Kaplansky's hypothesis (A).

Kaplansky's hypothesis (A) consists of two parts, the first giving a condition for the value group:

(A1) *The value group vK is p-divisible.*

As for the second part concerning the residue field, we split it into two parts again, one for purely inseparable extensions and the other for separable extensions:

(A2) *The residue field $\overline{K}$ is perfect, i.e. $\overline{K}$ does not admit any proper purely inseparable extension.*

(A3) $[\overline{K}^s : \overline{K}]$ *is relatively prime to p, i.e. $\overline{K}$ does not admit any finite separable algebraic extension whose degree is divisible by p.* *)

The conjunction (A2) & (A3) means that $\overline{K}$ does not admit any algebraic extension, separable or inseparable, whose degree is divisible by p. Or, equivalently:

(A2/3) $[\overline{K}^a : \overline{K}]$ *is relatively prime to* p.

Actually, in Kaplansky's original paper [7] condition (A2/3) appears in a somewhat disguised form which Kaplansky himself calls "rather unusual". Namely, he requires that for every additive polynomial $\overline{f}(X) \in \overline{K}[X]$ and every $\overline{a} \in \overline{K}$ the equation $\overline{f}(\overline{x}) = \overline{a} \in \overline{K}$ admits a solution $\overline{x} \in \overline{K}$. It has been shown by Whaples [19] that this condition is indeed equivalent to (A2/3); Whaples' result has recently been rediscovered by Delon [3] with a simple proof.

Condition (A3) appears in proposition 3.2 already. It implies that all algebraic K-complements of K^r are mutually K-isomorphic. Combining this with theorem 4.3 we obtain:

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*) Recall that degrees of infinite algebraic extensions are to be understood as supernatural numbers

Theorem 5.1. Suppose that K *satisfies condition (A3). Then all maximal algebraic purely wild extensions of* K *are mutually* K-*isomorphic, as valued fields.*

Lemma 5.2. If K *satisfies conditions (A1) and (A2), then every purely wild extension of* K *is immediate, and conversely.*

This follows directly from the definition of purely wild extension in section 4. For the converse see prop. 4.5.

Combining theorem 5.1 with lemma 5.2 we obtain:

Theorem 5.3. Suppose that K *satisfies Kaplansky's condition (A) = (A1) & (A2) & (A3). Then all maximal algebraic immediate extensions of* K *are mutually* K-*isomorphic, as valued fields.*

6. Transcendental extensions.

We are now going to study transcendental immediate extensions with the aim of proving Kaplansky's theorem:

Theorem 6.1. Suppose that K *satisfies Kaplansky's condition (A). Then all maximal immediate extensions of* K *are* K-*isomorphic, as valued fields.*

We shall prove the following version for purely wild extensions which implies theorem 6.1 by means of lemma 5.2.

Theorem 6.2. Suppose that K satisfies condition (A3). Then all maximal purely wild extensions of K *are* K-*isomorphic, as valued fields.*

In order to make this statement meaningful one has to verify that maximal purely wild extensions exist, and that every purely wild extension L of K is contained in a maximal one. In fact, as in the case of immediate extensions, it can be shown that the cardinality $|L|$ of every purely wild extension is bounded, the bound depending on the cardinalities $|vK|$ and $|\overline{K}|$ only. To see this, one has to verify that the algebraic closure L^a as a valued extension of K^a is immediate. Consequently, by the known result (due to Krull [8]) for immediate extensions, we conclude that $|L^a|$ is bounded, with a bound depending on $|vK^a|$ and $|\overline{K}^a|$ only. The same bound then applies to $|L|$.

Our proof of theorem 6.2 will be based on theorem 5.1 and some additional information concerning transcendental purely wild extensions.

Lemma 6.3. *Let L be a valued extension of K and suppose that $L|K$ is purely wild. Then L is linearly disjoint to K^r. If L is Henselian then the ramification field L^r is the disjoint compositum of L with K^r:*

$$L^r = L \cdot K^r .$$

Recall that the base field K is always assumed to be Henselian. But this does not imply that L is Henselian, because L may be transcendental over K. Hence in lemma 6.3 the Henselian condition for L is not superfluous.

Proof of Lemma 6.3. The linear disjointness is proved precisely as in the proof of lemma 4.2.

In order to prove that $L^r = L \cdot K^r$ we shall show firstly:

(1) $L \cdot K^r$ *is tame over* L.

Hence $L \cdot K^r \subset L^r$ by prop. 4.1. Secondly we shall show:

(2) $v(L \cdot K^r) = v(L^r)$, $\overline{L \cdot K^r} = \overline{L^r}$.

This yields $L \cdot K^r = L^r$ by prop. 4.7.

Proof of (1). K^r is the union of tame extensions $E|K$ of finite degree. Therefore it suffices to show for each E:

$$L \cdot E \text{ is tame over } L .$$

$v(LE)$ contains $vL + vE$. We know that $[vL : vK]$ is a p-power (because $L|K$ is purely wild) and that $[vE : vK]$ is relatively prime to p (because $E|K$ is tame). Hence

$$[vE : vK] = [vL + vE : vL] \leqq [v(LE) : vL] .$$

Similarly for the residue fields: $\overline{LE}$ contains the compositum $\overline{L} \cdot \overline{E}$. We know that $\overline{L}|\overline{K}$ is purely inseparable while

$\overline{E}|\overline{K}$ is separable. Hence

$$[\overline{E} : \overline{K}] = [\overline{L}\cdot\overline{E} : \overline{L}] \leqq [\overline{LE} : \overline{L}] \ .$$

Using general valuation theory and the degree formula (section 4 (c)) for the tame extension $E|K$ we obtain:

$$[E : K] = [LE : L] \geqq [v(LE) : vL]\cdot[\overline{LE} : \overline{L}] \geqq$$

$$\geqq [vE : vK]\cdot[\overline{E} : \overline{K}] = [E : K] \ .$$

Thus equality signs hold. This yields $v(LE) = vL + vE$ and $\overline{LE} = \overline{L}\cdot\overline{E}$. We conclude

(a) $[v(LE) : vL] = [vE : vK]$, relatively prime to p ,

(b) $\overline{LE} = \overline{L}\cdot\overline{E}$, separable over $\overline{L}$

(c) $[v(LE) : vL]\cdot[\overline{LE} : \overline{L}] = [LE : L]$.

These properties show that LE is tame over L, as contended.

Proof of (2). In the foregoing proof we have shown that $v(L\cdot E) = vL + vE$ for any tame extension $E|K$ of finite degree. Taking the union over all E we obtain:

$$v(L\cdot K^r) = vL + vK^r.$$

Here vK^r is the p'-divisible hull of vK, by prop. 4.6. Since vL/vK is a torsion group we deduce that $vL + vK^r$ is the p'-divisible hull of vL. Hence, using prop. 4.6 for L^r we obtain:

$$v(L\cdot K^r) = vL^r \ .$$

Similarly for the residue field: for any finite tame extension $E|K$ we have shown that $\overline{L\cdot E} = \overline{L}\cdot\overline{E}$. Hence

$$\overline{L\cdot K^r} = \overline{L}\cdot\overline{K^r} \ .$$

Here $\overline{K^r}$ is the separable algebraic closure of $\overline{K}$, by prop. 4.6. Since $\overline{L}|\overline{K}$ is algebraic we deduce that $\overline{L}\cdot\overline{K^r}$ is the separable algebraic closure of $\overline{L}$. Hence, using prop. 4.6 for L^r we obtain:

$$\overline{L\cdot K^r} = \overline{L^r} \ ,$$

as contended. Lemma 6.3 is proved.

Proposition 6.4. *Let L be a purely wild extension of K. Suppose that L does not permit any proper algebraic purely wild extension. Then $L \cap K^a$ has the same property. Hence $L \cap K^a$ is a maximal algebraic purely wild extension of K. Moreover L is immediate over $L \cap K^a$.*

Proof. After replacing K by $L \cap K^a$ we may assume that K is algebraically closed within L. Our contention is that K does not admit any proper algebraic purely wild extension. By theorem 4.3 this means that K itself is an algebraic K-complement of K^r. Equivalently: $K^r = K^a$. By assumption this is true for L instead of K: $L^r = L^a$. In particular L^a is separable over L, hence L is perfect. Consequently $K = L \cap K^a$ is perfect too. Therefore the relation $K = L \cap K^a$ implies the linear disjointness of L and K^a over K. We conclude: in order to prove $K^r = K^a$ it suffices to prove that $L \cdot K^r = L \cdot K^a$. Using lemma 6.3 we compute: $L \cdot K^r = L^r = L^a \supset L \cdot K^a \supset L \cdot K^r$, hence $L \cdot K^r = L \cdot K^a$, as contended.

By prop. 4.5, $v(L \cap K^a)$ is p-divisible and $\overline{L \cap K^a}$ is perfect. Hence $L \cap K^a$ satisfies conditions (A1) and (A2). Consequently every purely wild extension of $L \cap K^a$ is immediate. Proposition 6.4 is proved.

This being said let us now start with the

Proof of theorem 6.2.

Let L, L' be two maximal purely wild extensions of K. It suffices to show that L can be K-isomorphically embedded into L', as a valued field.

We consider fields F between K and L and K-isomorphic embeddings

$$\sigma : F \to L' ,$$

as valued fields. Using Zorn's lemma we assume that $\sigma : F \to L'$ is a maximal such embedding, not extendable to a proper overfield of F in L. We have to show that F = L.

The fields L, L' are both Henselian. Hence L contains the Henselization F^h of F, and the embedding $\sigma : F \to L'$ extends uniquely to an embedding $\sigma^h : F^h \to L'$. By the maximality property of σ, we conclude $F = F^h$. Thus F is Henselian.

Since $F|K$ is purely wild, the residue field extension $\overline{F}|\overline{K}$ is purely inseparable, hence linearly disjoint to $\overline{K}^s$. This implies $\overline{F}^s = \overline{F}\cdot\overline{K}^s$ and $[\overline{F}^s : \overline{F}] = [\overline{K}^s : \overline{K}]$, relatively prime to p.

Hence F inherits from K the Henselian property and property (A3). We identify F with its isomorphic image $\sigma F \subset L'$ and, changing notation, write again K instead of F. We now have the same situation as before with the additional information that no proper extension of K within L is K-isomorphically embeddable into L'. We have to show that K = L.

By prop. 6.4 $L \cap K^a$ and $L' \cap K^a$ are maximal algebraic purely wild extensions of K. Applying theorem 5.1 we obtain a K-isomorphism $L \cap K^a \to L' \cap K^a$. Hence $L \cap K^a = K$, i.e. K does not admit any proper algebraic purely wild extension. Also, L and L' are immediate over K.

If $K \neq L$ let $x \in L \setminus K$; then K(x) is a simple transcendental immediate extension of K. We prove in the next section that K(x) can be embedded into every maximal immediate extension L' of K. Contradiction. Hence K = L, as contended.

7. The embedding theorem for simple transcendental immediate extensions.

In the foregoing proof we have used the following embedding theorem.

Theorem 7.1. *Suppose that* K *does not admit any proper algebraic purely wild extension. Let* K(x) *be a simple transcendental extension of* K, *valued such that* K(x) *is immediate over* K. *Then* K(x) *can be* K-*isomorphically embedded into every maximal immediate extension* L *of* K.

Here and in the following, all fields considered are valued fields, and isomorphic embeddings are to be understood in the sense of valued fields.

Proof. Suppose K(x) cannot be embedded into L. Then we shall show that K(x) can be embedded into some simple transcendental immediate extension of L; this yields a contradiction since L is supposed to be maximal.

It is well known that the theory of valued fields has the amalgamation property: Any two valued extension fields of K can be K-isomorphically embedded into a common valued overfield. See [2], page 171, ex. § 2 (2). Hence there exists a valued field extension Ω of K which contains K-isomorphic copies of K(x) and of L. Identifying K(x) and L with their images in Ω, we may assume:

$$K(x) \subset \Omega \ , \ L \subset \Omega \ .$$

After enlarging Ω we may assume that Ω is algebraically closed, hence Ω contains K^a and L^a. The valuation of Ω is denoted by v again. In this setting we prove:

(1) Suppose that K(x) cannot be K-isomorphically embedded into L. Then x is transcendental over L and L(x) is immediate over L.

Let $t \in L$. Then there exists a polynomial $f(x) \in K[x]$ such that $vf(x) \neq vf(t)$. For otherwise the substitution $x \mapsto t$ would yield a K-isomorphism $K(x) \approx K(t) \subset L$. Decomposing f(x) into linear factors over K^a we conclude: there exists $c \in K^a$ such that

(2) $$v(x-c) \neq v(t-c) \ .$$

We claim that such c can be found in K already; this is seen as follows.

Since K does not admit any proper algebraic purely wild extension we have $K^a = K^r$, by theorem 4.3. Hence K^a is the union of finite tame extensions $E|K$. We choose E such as to contain the given element $c \in K^a$ satisfying (2). Now we choose a special basis of $E|K$. Let $u_1,\ldots,u_f \in E$ be elements whose residues $\overline{u}_i$ form a $\overline{K}$-basis of $\overline{E}$, with $f = [\overline{E} : \overline{K}]$. We may suppose that $u_1 = 1$. Similarly let $z_1, \ldots, z_e \in E$ be elements whose values $v(z_j)$ form a system of representatives of vE modulo vK, with $e = [vE : vK]$. We may suppose that $z_1 = 1$. Then the $e \cdot f$ elements $u_i z_j$ are linearly independent over K, hence form a basis of $E|K$ in view of the degree formula $[E : K] = e \cdot f$. Moreover, if $c \in E$ is represented in the form

$$c = \sum_{i,j} c_{ij} u_i z_j$$

with coefficients $c_{ij} \in K$ then

$$v(c) = \min v(c_{ij} u_i z_j).$$

Such basis $u_i z_j$ is called a <u>valuation basis</u> of E over K. See e.g. [4], page 99, (13.9) for a proof of the above statements. Now since $K(x)$ is immediate over K, the $u_i z_j$ remain to be a valuation basis for $E(x)$ over $K(x)$, and similarly for $E(t)$ over $K(t)$. Accordingly let us write

$$x-c = x-c_{11} - \sum_{i,j}{}' c_{ij} u_i z_j$$

where the $c_{ij} \in K$ are as above, and where the prime at the summation sign indicates that $(i,j) \neq (1,1)$. We obtain

$$v(x-c) = \min[v(x-c_{11}), \min_{i,j}{}' v(c_{ij} u_i z_j)].$$

Similarly we obtain:

$$v(t-c) = \min[v(t-c_{11}), \min_{i,j}{}' v(c_{ij} u_i z_j)].$$

Consequently if $c \in E$ satisfies (2) then we deduce

$$v(x-c_{11}) \neq v(t-c_{11}) .$$

Let us write c instead of c_{11}. We have proved:

(3) <u>Let</u> $t \in L$. <u>Then there exists</u> $c \in K$ <u>such that</u> $v(x-c) \neq v(t-c)$.

With the same meaning of c we now claim:

(4) <u>If</u> $a \in K$ satisfies $v(x-a) > v(x-c)$ <u>then</u> $v(x-a) > v(t-a)$.

For we have $v(c-a) = \min[v(x-c), v(x-a)] = v(x-c)$. Therefore, if we write $t-a = (t-c) + (c-a)$ we infer from (3) that both summands have different values, hence $v(t-a) = \min[v(t-c), v(c-a)] \leqq v(c-a) = v(x-c) < v(x-a)$.

At this point it is appropriate to recall that there exist elements $a \in K$ which satisfy the hypothesis of (4):

(5) <u>For every</u> $c \in K$ <u>there exists</u> $a \in K$ <u>such that</u> $v(x-a) > v(x-c)$.

This is a well known consequence of the fact that $K(x)$ is immediate over K, and it is proved as follows: Firstly there is $b \in K$ such that $v(x-c) = v(b)$. Secondly there is $b' \in K$ such that the residue class of $\frac{x-c}{b}$ is $\overline{b'}$; this implies $v(\frac{x-c}{b} - b') > 0$. Hence if we put $a = c + bb'$ then $v(x-a) = v(x-c-bb') > v(b) = v(x-c)$.

In view of (5) we may now reformulate (4) as follows:

(6) <u>Let</u> $t \in L$. <u>If</u> $a \in K$ <u>is sufficiently close to</u> x, <u>we have</u> $v(x-a) > v(t-a)$.

Next we claim that (6) holds not only for $t \in L$ but also for each $t \in L^a$. Applying theorem 4.3 to L we see that

$$L^a = L^r = L \cdot K^r$$

in view of lemma 6.3. Hence every given $t \in L^a$ is contained in $L \cdot E$ for some finite tame extension $E|K$. We use a valuation basis $u_i z_j$ for $E|K$ as explained above; this remains being a valuation basis for $L \cdot E$ over L. We write

$$t = \sum_{i,j} t_{ij} u_i z_j$$

with $t_{ij} \in L$. Then

$$t-a = (t_{11}-a) + \sum_{i,j}{}' t_{ij} u_i z_j ,$$

$$v(t-a) = \min[v(t_{11}-a), \min_{i,j}{}' v(t_{ij} u_i z_j)] ,$$

$$v(t-a) \leqq v(t_{11}-a) .$$

Since $t_{11} \in L$ we infer from (6) that if $a \in K$ is sufficiently close to x then

$$v(x-a) > v(t_{11}-a) \geqq v(t-a) .$$

Thus, indeed, (6) holds also for $t \in L^a$. In particular it follows that $x \notin L^a$, i.e. x is transcendental over L.

Note that if $v(x-a) > v(t-a)$ then

$$v\left(\frac{x-t}{a-t} - 1\right) = v\left(\frac{x-a}{a-t}\right) > 0 . \tag{7}$$

This means that $\frac{x-t}{a-t} \in L(x)$ is a unit with residue 1, briefly: it is a 1-unit.

Now let $f(x) \in L(x)$ be an arbitrary nonzero rational function. We decompose f(x) into linear factors over L^a:

$$f(x) = t \cdot \prod_i (x-t_i)^{e_i} ,$$

where $t \in L$, $t_i \in L^a$, $e_i \in \mathbb{Z}$. We obtain for $a \in K$:

$$\frac{f(x)}{f(a)} = \prod_i \left(\frac{x-t_i}{a-t_i}\right)^{e_i} .$$

If $a \in K$ is sufficiently close to x then we know from (6), (7) that each of the factors $\frac{x-t_i}{a-t_i}$ is a 1-unit. Since the 1-units form a multiplicative group it follows that $\frac{f(x)}{f(a)}$ is a 1-unit, hence

$$v\left(\frac{f(x)}{f(a)} - 1\right) > 0. \tag{8}$$

In view of (4) this holds for all $a \in K$ which satisfy a condition $v(x-a) > v(x-c)$ with $c \in K$ depending on f(x). From (5) we know that there exist such $a \in K$.

Relation (8) implies $vf(x) = vf(a) \in vL$ for each $f(x) \in L(x)$. This shows

$$vL(x) = vL .$$

If $vf(x) = 0$ then (8) implies $\overline{f(x)} = \overline{f(a)} \in \overline{L}$, hence

$$\overline{L(x)} = \overline{L} .$$

Thus L(x) is immediate over L, and statement (1) is proved.

8. Appendix

We refer to the situation of theorem 5.1, using the same notations. If (A3) does not hold, is it possible that the conclusion of theorem 5.1 remains valid? What are the precise conditions for K to possess a unique maximal algebraic purely wild extension (up to K-isomorphisms)? First we observe, rather trivially:

Lemma 8.1. Suppose that K does not admit any proper separable algebraic purely wild extension. Then the perfect hull K^* is the unique algebraic purely wild extension of K and $K^*|K$ is immediate.

For since $K^*|K$ is purely inseparable, it is purely wild. From proposition 4.5 (iii) it follows that $K^*|K$ is immediate.

Is this the only situation where uniqueness holds in absence of condition (A3)? A preliminary answer to this question is the following.

Proposition 8.2. Assume that (A3) does not hold, i.e. $[\overline{K}^s : \overline{K}]$ is divisible by p. Also, we suppose that there is a proper separable-algebraic extension $L|K$ which is purely wild. Then there exists a finite tame extension $E|K$ of degree relatively prime to p such that E admits at least two maximal algebraic purely wild extensions which are not E-isomorphic.

Proof. Let G_p denote a p-Sylow subgroup of G, the Galois group over K. Then $G_p \supset G_r$ by proposition 2.4 (iii).

Let $X(G_p)$ be the group of continuous homomorphisms $\chi : G_p \to \mathbb{Z}/p$. We refer to the elements $\chi \in X(G_p)$ as characters of G_p. A character is called tame if $\chi(G_r) = 0$.

(I) There exists a non-trivial tame character of G_p.

To see this, we remark that the group of tame characters $\chi \in X(G_p)$ coincides, by definition, with the character group $X(G_p/G_r)$. Hence we have to verify that $X(G_p/G_r) \neq 0$.

Since G_p/G_r is a pro-p-group it is sufficient to see that $G_p \neq G_r$. Suppose $G_p = G_r$. Then $[G : G_r]$ would be relatively prime to p. Hence by proposition 2.4, $[\overline{K}^s : \overline{K}]$ would be relatively prime to p, contrary to the hypothesis of the proposition. Hence indeed, $G_p \neq G_r$.

A character $\psi \in X(G_p)$ is called <u>wild</u> if it is not tame.

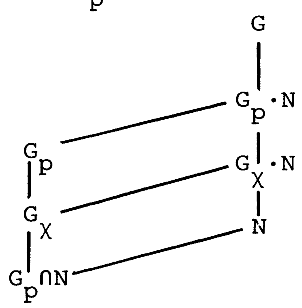
(II) <u>There exists a non-trivial wild character of</u> G_p.

For let $L|K$ be as in the proposition, and let H be the Galois group over L. Then $H \not\supset G_r$ since $L \not\subset K^r$. It follows $H \not\supset G_p$, hence $H \cap G_p$ is a proper subgroup of G_p. Let W be a <u>maximal</u> proper closed subgroup of G_p containing $H \cap G_p$. Since G_p is a pro-p-group, W is normal of index p in G_p. Therefore there exists a character ψ of G_p with kernel W. We have $\psi(H \cap G_p) = 0$, and we claim that ψ is not tame. Note that $H \cdot G_r = G$ since L is K-linearly disjoint to K^r. It follows: $G_r \cdot (H \cap G_p) = (G_r \cdot H) \cap G_p = G \cap G_p = G_p$. Hence if $\psi(G_r) = 0$ then $\psi(G_p) = 0$, contradicting the fact that $\psi \neq 0$ on G_p. Thus $\psi(G_r) \neq 0$ and ψ is not tame.

Let U be an open subgroup of G containing G_p. We say that a character $\chi \in X(G_p)$ can be <u>lifted</u> to U if there exists a character $U \to \mathbb{Z}/p$ which extends χ.

(III) <u>Every character</u> $\chi \in X(G_p)$ <u>can be lifted to some</u> open subgroup $U \supset G_p$.

For let $G_\chi \subset G_p$ be the kernel of χ. We may assume $\chi \neq 0$, $[G_p : G_\chi] = p$. G_χ is closed in G and hence G_χ is the intersection of open subgroups of G. Therefore there exists an open subgroup V of G such that $G_\chi \subset V$ and $G_p \not\subset V$, hence $G_\chi = V \cap G_p$. Let N be a normal open subgroup of G contained in V. Then we have $N \cap G_p \subset G_\chi$. From the diagram we infer:



$$G_p/G_\chi \approx G_p \cdot N/G_\chi \cdot N .$$

By means of this isomorphism the character χ of G_p can be extended to a character of $G_p \cdot N$ which vanishes on $G_\chi \cdot N$. Thus we may put $U = G_p \cdot N$ in order to satisfy (III).

(IV) <u>If</u> $\chi_1, \dots, \chi_n$ <u>are finitely many characters of</u> G_p <u>then there exists an open</u> $U \supset G_p$ <u>such that all</u> χ_i <u>can simultaneously be lifted to</u> U.

Indeed, if χ_i is lifted to U_i then we may take $U = U_1 \cap \dots \cap U_n$.

We now choose a nontrivial tame character $\chi \in X(G_p)$ and a nontrivial wild character $\psi_1 \in X(G_p)$. Let U be an open subgroup of G containing G_p such that both χ and ψ_1 can be lifted to U. Thus χ, ψ_1 now appear as characters in X(U). Let us put

$$\psi_2 = \chi + \psi_1 .$$

Since $\chi(G_r) = 0$, $\psi_1(G_r) \neq 0$, we see that $\psi_2(G_r) \neq 0$. Hence both characters $\psi_1, \psi_2 \in X(U)$ are wild, but their difference $\psi_2 - \psi_1 = \chi$ is tame, and $\chi \neq 0$.

Let $U_1 \subset U$ be the kernel of ψ_1 on U. Since $\psi_1 \neq 0$ we have $[U : U_1] = p$. Since U_1 does not contain G_r it follows $U_1 \cdot G_r = U$. Hence we may apply theorem 2.2 (ii) to U and its subgroup U_1 (instead of G and G'). We conclude: U_1 contains some group complement H_1 of G_r within U. That is, we have:

$$H_1 \cdot G_r = U \quad , \quad H_1 \cap G_r = 1 \quad , \psi_1(H_1) = 0 .$$

Similarly we obtain H_2 such that

$$H_2 \cdot G_r = U \quad , \quad H_2 \cap G_r = 1 \quad , \psi_2(H_2) = 0 .$$

We claim that H_1 and H_2 are <u>not</u> conjugate in U. For if $H_2 = \sigma H_1 \sigma^{-1}$ with $\sigma \in U$, then

$$\psi_1(H_2) = \psi_1(\sigma H_1 \sigma^{-1}) = \psi_1(H_1) = 0 .$$

Hence both ψ_1 and ψ_2 vanish on H_2, and so does $\chi = \psi_2 - \psi_1$. Since $\chi(G_r) = 0$ it follows

$$\chi(U) = \chi(H_2 \cdot G_r) = \chi(H_2) + \chi(G_r) = 0 ,$$

contradicting the fact that $\chi \neq 0$ on U, by construction. Hence indeed, H_1 and H_2 are not conjugate in U.

Let us now interpret this result in terms of field extensions. Let E be the separable fixed field of U. Since U is open, [E : K] is finite. Since $U \supset G_p$, the index $[G : U] = [E : K]$ is not divisible by p. Let L_1, L_2 be the fixed fields of H_1, H_2 in K^a. Applying lemma 2.3 to E instead of K, we see that L_1 and L_2 are algebraic E-complements of $E^r = K^r$. Hence by theorem 4.3, L_1 and L_2 are maximal algebraic purely wild extensions of E. Since H_1 and H_2 are not conjugate in U it follows from Galois theory that L_1 and L_2 are not E-isomorphic. Proposition 8.2 is proved.

If conditions (A1) and (A2) hold then proposition 8.2 deals with <u>immediate</u> extensions (lemma 5.2). But even in the absence of (A1) or (A2) we can modify the above proof such as to remain valid for immediate extensions. We need an auxiliary lemma.

<u>Lemma 8.3</u>. <u>Let</u> $L|K$ <u>be an algebraic extension</u>, K-<u>linearly disjoint to</u> K^r. <u>If</u> $L|K$ <u>is immediate then</u> $L^r|K^r$ is <u>immediate and conversely</u>.

Note that $L^r = L \cdot K^r$ by lemma 6.3.

<u>Proof</u>. Consider first the value group. We have:

$$[vL^r : vK^r][vK^r : vK] = [vL^r : vL][vL : vK] \quad .$$

Here $[vK^r : vK]$ and $[vL^r : vL]$ are not divisible by p, see proposition 4.6. On the other hand, lemma 4.2 shows that $[vL : vK]$ and $[vL^r : vK^r]$ are both p-powers. We conclude

$$[vL : vK] = [vL^r : vK^r] \quad .$$

Hence if one of these indices is 1 then the other is 1 too.

A similar argument applies to the residue field. We have

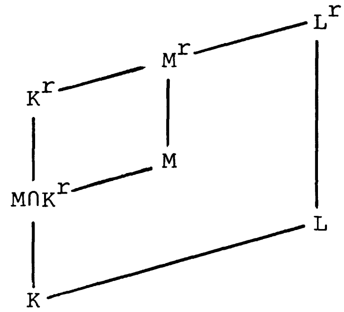
$$[\overline{L^r} : \overline{K^r}][\overline{K^r} : \overline{K}] = [\overline{L^r} : \overline{L}][\overline{L} : \overline{K}] \quad .$$

This time the extensions $\overline{K^r}|\overline{K}$ and $\overline{L^r}|\overline{L}$ are separable while $\overline{L}|\overline{K}$ and $\overline{L^r}|\overline{K^r}$ are purely inseparable. Hence we obtain

$$[\overline{L} : \overline{K}] = [\overline{L^r} : \overline{K^r}] .$$

Again, if one of these degrees is 1 then the other is 1 too.

Corollary 8.4. Let M be any intermediate field between K and L^r. If $L|K$ is immediate then M is immediate over its intersection $M \cap K^r$.



Proof. $L^r|K^r$ is immediate by lemma 8.3. Hence its subextension $M^r = M \cdot K^r$ is immediate over K^r. Applying the lemma to $M|M \cap K^r$ instead of $L|K$ we obtain that $M|M \cap K^r$ is immediate.

Now we can show:

Proposition 8.5. In the same situation as in proposition 8.2 assume in addition that $L|K$ is immediate. Then the field E in proposition 8.2 can be chosen such that, in addition, E admits two maximal algebraic immediate extensions which are not E-isomorphic.

We show that the proof of proposition 8.2, if suitably adapted, leads to proposition 8.5.

A character ψ is called admissible if $\psi(H \cap G_r) = 0$. Every tame character is admissible. The wild character ψ of G_p constructed in the proof of (II) satisfies $\psi(H \cap G_p) = 0$; hence it is admissible since $G_p \supset G_r$. Consequently we may start with admissible characters χ and ψ_1 of G_p, one being tame and the other being wild. Lifting χ and ψ_1 to some open subgroup $U \subset G$ preserves admissibility. Hence χ, ψ_1 now appear as admissible characters of U. Their sum

$\psi_2 = \psi_1 + \chi$ is also admissible.

Let U_1, U_2 be the kernels of ψ_1, ψ_2 in U, and let M_1, M_2 be their separable fixed fields. Each M_i is a normal extension of degree p over E. Since ψ_i is wild, $U_i \not\supset G_r$ and hence $M_i \not\subset K^r$. It follows $M_i \cap K^r = E$. Now $U_i \supset H \cap G_r$ since ψ_i is admissible. Hence $M_i \subset L \cdot K^r = L^r$. Corollary 8.4 shows that $M_i|E$ is immediate (i = 1,2).

Let L_1', L_2' be maximal algebraic immediate extensions of E which contain M_1, M_2 respectively. Let H_1', H_2' be the Galois groups over L_1', L_2'. Then $U_i \supset H_i'$ and therefore $\psi_i(H_i') = 0$ (i = 1,2). With the same arguments as in the proof of proposition 8.2 it follows that H_1' and H_2' cannot be conjugate in U. Hence L_1' and L_2' are not E-isomorphic, as contended.

The question remains open under which conditions 8.5 holds for E = K. The authors have constructed a valued field K which possesses a unique maximal and even normal algebraic purely wild extension, while K satisfies neither (A3) nor the hypothesis of lemma 8.1. The valuation of this field is composite with a valuation whose residue field has characteristic 0, and this fact is essentially used in the proof of the uniqueness.

References

[1] Ax, J. - Kochen, S.: Diophantine problems over local fields, Amer. Journ. Math. 87, 605-648 (1965)

[2] Bourbaki, N.: Eléments de Mathématique, Algèbre commutative, chap. 5/6 (fasc.XXX), Paris (1964)

[3] Delon, F.: Quelques propriétés des corps valués en théorie des modèles. Thèse Paris VII (1981)

[4] Endler, O.: Valuation theory, Berlin-Heidelberg-New York (1972)

[5] Ershov, Ju.L.: Decision problems and constructivizable models. Nauka, Moscow (1980) (Russian)

[6] Huppert, B.: Endliche Gruppen I, Berlin-Heidelberg-New York (1967)

[7] Kaplansky, I.: Maximal fields with valuations, Duke Math. Journ. 9, 303-321 (1942)

[8] Krull, W.: Allgemeine Bewertungstheorie, J. reine angew. Math. 167, 160-196 (1931)

[9] Neukirch, J.: Zur Verzweigungstheorie der allgemeinen Krullschen Bewertungen, Abh. Math. Sem. Univ. Hamburg 32, 207-215 (1968)

[10] Ostrowski, A.: Untersuchungen zur arithmetischen Theorie der Körper, Math. Zeitschr. 39, 269-404 (1935)

[11] Pank, M.: Beiträge zur reinen und angewandten Bewertungstheorie, Dissertation Heidelberg (1976)

[12] Potthoff, K.: Einführung in die Modelltheorie und ihre Anwendungen, Darmstadt (1981)

[13] Prestel, A. - Roquette, P.: Formally p-adic fields, Springer Lecture Notes in Math. 1050 (1984)

[14] Robinson, A.: Nonstandard Arithmetic, Bull. Amer. Math. Soc. 73, 818-843 (1967) = Selected papers vol.2, 132-157 (1979)

[15] Roquette, P.: Some Tendencies in Contemporary Algebra, in: W. Jäger et al. (ed.): Perspectives in Mathematics. Anniversary of Oberwolfach 1984, 393-422, Basel (1984)

[16] Serre, J.P.: Cohomologie Galoisienne, 3rd ed. Springer Lecture Notes in Math. 5 (1965)

[17] Schilling, O.F.G.: The theory of valuations, Math. Surveys A.M.S. (1950)

[18] Steinitz, E.: Algebraische Theorie der Körper, Journ. reine angew. Math. Nr. 137, 167-309 (1910) = reprint Chelsea (1950)

[19] Whaples, G.: Galois cohomology of additive polynomials and n-th power mappings of fields, Duke Math. Journ. 24, 143-150 (1957)

[20] Zassenhaus, H.: The theory of groups, 2nd ed. New York (1958)

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(Received August 16, 1986)