A COMMON GENERALIZATION OF METRIC, ULTRAMETRIC AND TOPOLOGICAL FIXED POINT THEOREMS

— *ALTERNATIVE VERSION* —

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ABSTRACT. We present a general fixed point theorem which can be seen as the quintessence of the principles of proof for Banach's Fixed Point Theorem, ultrametric and certain topological fixed point theorems. It works in a minimal setting, not involving any metrics. We demonstrate its applications to the metric, ultrametric and topological cases, and to ordered abelian groups and fields.

1. INTRODUCTION

What is the common denominator of Banach's Fixed Point Theorem and its ultrametric and topological analogues as developed in [8, 9, 10, 6] and in [13]? Is there a general principle of proof that works for all of these worlds, the (ordinary) metric, ultrametric and topological, and beyond? In this paper, we give an answer to these questions. We draw our inspiration from the notions of "ball" and "spherical completeness" that are used in the ultrametric world.

S. Priess' paper [8] in which she first proved a fixed point theorem for ultrametric spaces initiated an interesting development that led to a better understanding of important theorems in valuation theory and to new results (see, e.g., [KU4]). This was achieved by extracting the underlying principle of the proof of Hensel's Lemma through abstracting from the algebraic operations and only considering the ultrametric induced by the valuation. In this paper we push this development one step further by extracting the underlying principle of various fixed point theorems. In this way, a general

framework is set up that helps understand these theorems in a more conceptual manner and to transfer ideas from one world to the other by analogies (as we will demonstrate, for instance, for the topological fixed point theorem we consider).

The general framework also helps to make the use of fixed point theorems available to situations that are difficult or even impossible to subsume under the above mentioned settings. While investigating spaces of real places, we found that in certain algebraic entities, it may be much easier and more natural to define "balls" than to define the "distance" between two elements. For example, if we are dealing with quotient topologies, like in the case of spaces of real places where the topology is induced by the Harrison topology of spaces of orderings, balls come up naturally as images of certain open or closed sets in the inducing topology. Therefore, we will work with *ball spaces* $(X, \mathcal{B})$, that is, sets X with a nonempty set $\mathcal{B}$ of distinguished nonempty subsets of X, which we call *balls*. We require no further structure on these spaces. We do not even need a topology generated in some way by the balls. But let us mention that the way we formulate our theorems, the balls should be considered closed, rather than open, in such a topology, because singletons appear and are important. One can reformulate everything in an "open ball" approach, but this makes the exposition less elegant.

We found the idea of centering the attention on balls, rather than metrics, in the paper [3]. But there, ball spaces carry much more structure and the conditions for a fixed point theorem are unnecessarily restrictive.

We will now state our most general fixed point theorem for ball spaces. We need two notions. A *nest of balls* in $(X, \mathcal{B})$ is a nonempty collection of balls in $\mathcal{B}$ that is totally ordered by inclusion. If $f : X \to X$ is a function, then a subset $B \subseteq X$ will be called *f-contracting* if it is either a singleton containing a fixed point or satisfies $f(B) \subsetneq B$.

Theorem 1. *Take a function f on a ball space $(X, \mathcal{B})$ which satisfies the following conditions:*

(C1) *there is at least one f-contracting ball,*
(C2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ contains an f-contracting ball,*
(C3) *the intersection of every nest of f-contracting balls contains an f-contracting ball.*

Then f admits a fixed point.

We can obtain uniqueness of the fixed point by strengthening the hypothesis:

Theorem 2. *Take a function f on a ball space $(X, \mathcal{B})$ which satisfies the following conditions:*
(CU1) *X is an f-contracting ball,*

(CU2) *for every f-contracting ball $B \in \mathcal{B}$, the image $f(B)$ is again an f-contracting ball,*
(CU3) *the intersection of every nest of f-contracting balls is again an f-contracting ball.*
Then f has a unique fixed point.

These theorems will be proved in Section 2.

In [13] the authors show that every "J-contraction" on a connected compact Hausdorff space X has a unique fixed point. Using the inspiration from our general framework, we obtain the following strong generalization; note that we do not require the space X to be Hausdorff.

Theorem 3. *Take a compact space X and a closed function $f : X \to X$. If every nonempty closed set B in X with $f(B) \subseteq B$ contains a closed f-contracting subset, then f has a fixed point in X. If every nonempty closed set B in X with $f(B) \subseteq B$ is f-contracting, then f has a unique fixed point in X.*

This theorem and the theorem of [13] are corollaries to Theorems 1 and 2. We will show this in Section 4, where we will also present another version of Theorem 3 that is directly related to Theorem 4 below.

For most applications of these theorems, it is an advantage to have a handy criterion for the existence of the f-contracting balls. From classical fixed point theorems we know the assumption that the function f be strictly contracting. But we have learnt from the ultrametric case that if one does not insist on uniqueness, one can relax the conditions: a function does not need to be strictly contracting on the whole space, but only on the orbits of its elements (and simply contracting otherwise). While this relaxation makes the formulation of the conditions a bit longer, it should be noted that it is important for many applications in which the function under consideration fails for natural reasons to be strictly contracting on the whole space. The following way to present a fixed point theorem may seem unusual, but it turns out to be very close to several applications as it encodes (a weaker form of) the property "strictly contracting on orbits".

Consider a function $f : X \to X$. We will write fx for $f(x)$ and $f^i x$ for the image of x under the i-th iteration of f, that is, $f^0 x = x$, $f^1 x = f(x)$ and $f^{i+1} x = f(f^i x)$. The function f will be called *strongly contracting on orbits* if there is a function

$$X \ni x \mapsto B_x \in \mathcal{B}$$

such that for all $x \in X$, the following conditions hold:

(SC1) $x \in B_x$,
(SC2) $B_{fx} \subseteq B_x$, and if $x \neq fx$, then $B_{f^i x} \subsetneq B_x$ for some $i \geq 1$.

Note that (SC1) and (SC2) imply that $f^i x \in B_x$ for all $i \geq 0$.

We will say that a nest of balls $\mathcal{N}$ is an *f-nest* if $\mathcal{N} = \{B_x \mid x \in S\}$ for some set $S \subseteq X$ that is closed under f. Now we can state our third main theorem:

Theorem 4. *Take a function f on a ball space which is strongly contracting on orbits. If for every f-nest $\mathcal{N}$ in this ball space there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$, then f has a fixed point.*

Theorem 4 does not deal with the question of uniqueness of fixed points; this question is answered in the particular applications by additional arguments that are often very easy.

The condition about the intersection of an f-nest is not needed for Banach's Fixed Point Theorem and may therefore appear alien to readers who are not familiar with the ultrametric case. But there, as in the case of non-archimedean ordered groups and fields, one has to deal with jumps that one could intuitively think of as being a "non-archimedean" or "non-standard" phenomenon. The obstruction is that the intersection of an infinite nest of balls we have constructed may contain more than one element, at which point we have to iterate the construction. The mentioned condition makes this work.

The condition on the intersection of f-nests implies that in particular, they are not empty. This reminds of a similar property of ultrametric spaces, and we take over the corresponding notion. The ball space $(X, \mathcal{B})$ will be called *spherically complete* if every nonempty nest of nonempty balls has a nonempty intersection.

To illustrate the flexibility of the concepts we have introduced and the above explained idea of making fixed point theorems available to totally new settings, we state the following easy but useful result:

Proposition 5. *Take two ball spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ and a function $f : X_1 \to X_2$. Suppose that the preimage of every ball in $\mathcal{B}_2$ is a ball in $\mathcal{B}_1$. If $\mathcal{N}$ is a nest of balls in $(X_2, \mathcal{B}_2)$, then the preimages of the balls in $\mathcal{N}$ form a nest of balls in $(X_1, \mathcal{B}_1)$. If $(X_1, \mathcal{B}_1)$ is spherically complete, then so is $(X_2, \mathcal{B}_2)$.*

In several applications, and in particular in the ultrametric setting, the function under consideration has in a natural way stronger properties than we have used so far. What we have asked for one element in the intersection of an f-nest is often satisfied by every element in the intersection. Therefore, it seems convenient to introduce a notion which reflects this property and in this way to separate it from the condition that the intersections is nonempty. The function f will be called *self-contractive* if in addition to (SC1) and (SC2), it satisfies:

(SC3) if $\mathcal{N}$ is an f-nest and if $z \in \bigcap \mathcal{N}$, then $B_z \subseteq \bigcap \mathcal{N}$.

Self-contractive functions will appear in the hypothesis of Theorem 7, in Theorem 11, in Theorem 12, and in the proof of Banch's Fixed Point Theorem. The following fixed point theorem is an easy corollary to Theorem 4:

Theorem 6. *Every self-contractive function on a spherically complete ball space has a fixed point.*

For the proof of Theorems 1, 2 and 4, see Section 2. In Section 3, we state two general attractor theorems. Section 4 is devoted to topological fixed point theorems. In Section 5, we show how to derive ultrametric fixed point theorems, and in Section 6, we discuss ultrametric attractor theorems. In Section 7 we then discuss valued fields that are complete by stages, a notion introduced by P. Ribenboim in [11]. We use Theorem 4 for a quick proof of a fixed point theorem that works in such fields (Theorem 16). This theorem can be used to show that such fields are henselian. Its application to the proof of Hensel's Lemma provides an example for a case where one does not have in any natural way a function that is strictly contracting on all of the space. Note also that the particularly weak form that we have chosen for (SC2) comes in very handy for the formulation of Theorem 16.

In Section 8 we discuss how to derive Banach's Fixed Point Theorem. Our aim is not to provide a new proof of this theorem; in contrast to our other applications, the existing proofs in this case are much shorter. Our aim here is to show how to convert the problem from metric to ball space and to pave the way for one of our fixed point theorems (Theorem 21) for ordered abelian groups and fields. Associated with them are two natural ball spaces:

• the order ball space, where the balls are closed bounded intervals, and
• the ultrametric ball space, where the balls are the ultrametric balls derived from the natural valuation.

We discuss these ball spaces and the corresponding fixed point theorems in Section 9. The flexibility of our notion of ball space is demonstrated in the concept of hybrid ball spaces, in which we use order balls and ultrametric balls simultaneously. One of such hybrid ball spaces is used for a simple characterization of those ordered fields which are power series fields with residue field $\mathbb{R}$ (Theorem 27).

2. Proof of the fixed point theorems for ball spaces

Proof of Theorem 1:
The set of all nests consisting of f-contracting balls is partially ordered by inclusion. There is at least one such nest since by condition (C1), there is at least one f-contracting ball. Further, the union over an ascending chain of nests consisting of f-contracting balls is again such a nest (observe that the cardinality of this union is bounded by the cardinality of the power set of X, so the union is a set). Hence by Zorn's Lemma, there is a maximal

nest $\mathcal{N}$ consisting of f-contracting balls. By condition (C3), $\bigcap \mathcal{N}$ contains an f-contracting ball B. Suppose this ball is not a singleton. But then by condition (C2), $f(B)$ contains an f-contracting ball B'. Since $B' \subseteq f(B) \subsetneq B$, $\mathcal{N} \cup \{B'\}$ is then a nest that properly contains $\mathcal{N}$, which contradicts the maximality. We find that B must be a singleton consisting of a fixed point. $\qquad\square$

Proof of Theorem 2:
Using conditions (CU1), (CU2), (CU3) and transfinite induction, we build a nest $\mathcal{N}$ consisting of f-contracting balls as follows. We set $\mathcal{N}_0 := \{X\}$. Having constructed $\mathcal{N}_\nu$ for some ordinal ν with smallest f-contracting ball $B_\nu \in \mathcal{N}_\nu$, we set $B_{\nu+1} := f(B_\nu)$ and $\mathcal{N}_{\nu+1} := \mathcal{N}_\nu \cup \{B_{\nu+1}\}$. If λ is a limit ordinal and we have constructed $\mathcal{N}_\nu$ for all $\nu < \lambda$, we observe that the union over all $\mathcal{N}_\nu$ is a nest $\mathcal{N}'_\lambda$. We set $B_\lambda := \bigcap \mathcal{N}'_\lambda$ and $\mathcal{N}_\lambda := \mathcal{N}'_\lambda \cup \{B_\lambda\}$.

If B_ν is not a singleton, then $B_{\nu+1} \subsetneq B_\nu$. Hence there must be an ordinal ν of cardinality at most that of X such that $B_{\nu+1} = B_\nu$. But this only happens if B_ν is a singleton consisting of a fixed point x. If $x \neq y \in X$, then $y \notin B_\nu$ which means that there is some $\mu < \nu$ such that $y \in B_\mu$, but $y \notin B_{\mu+1} = f(B_\mu)$. This shows that y cannot be a fixed point of f. Therefore, x is the unique fixed point of f. $\qquad\square$

Theorem 4 can be derived from Theorem 1. But as it takes essentially the same effort, we will give a proof along the lines of the proof of Theorem 1.

Proof of Theorem 4:
Take a function f on the ball space $(X, \mathcal{B})$ which is contractive on orbits. For every $x \in X$, the set $\{B_{f^i x} \mid i \geq 0\}$ is an f-nest. The set of all f-nests is partially ordered by inclusion. The union over an ascending chain of f-nests is again an f-nest. Hence by Zorn's Lemma, there is a maximal f-nest $\mathcal{N}$. By the assumption of Theorem 4, there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$. We wish to show that z is a fixed point of f. If we would have that $z \neq fz$, then by (SC2), $B_{f^i z} \subsetneq B_z \subseteq \bigcap \mathcal{N}$ for some $i \geq 1$, and the f-nest $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\}$ would properly contain $\mathcal{N}$. But this would contradict the maximality of $\mathcal{N}$. Hence, z is a fixed point of f. $\qquad\square$

3. General attractor theorems

Let us derive from Theorem 6 an attractor theorem which is modeled after the ultrametric attractor theorem in [5]. We consider two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ and a function $\varphi : X \to X'$. Take an element $z' \in X'$. If there is a function $f : X \to X$ which is strongly contracting on orbits, and a function

$$X \ni x \mapsto B'_x \in \mathcal{B}'$$

such that for all $x \in X$, the following conditions hold:
(AT1) $z' \in B'_x$ and $\varphi(B_x) \subseteq B'_x$,
(AT2) if $\varphi(x) \neq z'$, then $B'_{f^i x} \subsetneq B'_x$ for some $i \in \mathbb{N}$,

then z' will be called a *weak f-attractor for φ*. If in addition f is self-contractive, then z' will be called an *attractor for φ*.

Theorem 7. (Attractor Theorem 1)
Take a function $\varphi : X \to X'$ and an attractor $z' \in X'$ for φ. If $(X, \mathcal{B})$ is spherically complete, then $z' \in \varphi(X)$.

Indeed, by Theorem 6, f has a fixed point z. But by condition (AT2), $fz = z$ implies that $\varphi(z) = z'$. The following version of the Attractor Theorem follows in a similar way from Theorem 4:

Theorem 8. (Attractor Theorem 2)
Take a function $\varphi : X \to X'$ and a weak f-attractor $z' \in X'$ for φ. If for every f-nest $\mathcal{N}$ in $(X, \mathcal{B})$ there is some $z \in \bigcap \mathcal{N}$ such that $B_z \subseteq \bigcap \mathcal{N}$, then $z' \in \varphi(X)$.

4. FIXED POINT THEOREMS FOR TOPOLOGICAL SPACES

In this section, we consider compact topological spaces X with functions $f : X \to X$. We note:

Lemma 9. *Every compact space X together with any family of nonempty closed subsets is a spherically complete ball space.*

Proof: [1, Proposition 2, p. 57] states that every "centered system" of nonempty closed subsets of a compact space X has a nonempty intersection. Here, a "centered system" means a family of subsets that is linearly ordered under inclusion. This is exactly what we call a "nest". Therefore, the cited proposition proves our lemma. $\square$

In view of this lemma, we will take $\mathcal{B}$ to be the set of all nonempty closed subsets of X. We show how to deduce Theorem 3 from Theorem 1.

Proof of Theorem 3: Take a compact space X and a closed function $f : X \to X$. Assume first that every closed set B in X with $f(B) \subseteq B$ contains a closed f-contracting subset. Since X is closed, this implies that condition (C1) is satisfied. If B is an f-contracting closed set in X, then $f(B)$ is closed since f is a closed function. Also, we have that $f(B) \subseteq B$, which yields that $f(f(B)) \subseteq f(B)$. Hence by assumption, $f(B)$ contains an f-contracting closed set, so condition (C2) is satisfied. The intersection $\bigcap \mathcal{N}$ of a nest $\mathcal{N} = \{B_i \mid i \in I\}$ of f-contracting closed sets B_i is closed; since $f(B_i) \subseteq B_i$ for all $i \in I$, we also have that $f(\bigcap \mathcal{N}) \subseteq \bigcap \mathcal{N}$. Hence by assumption, $\bigcap \mathcal{N}$ contains an f-contracting closed set. So condition (C3) is satisfied, and Theorem 1 shows that f has a fixed point.

Now assume that every closed set B in X with $f(B) \subseteq B$ is f-contracting. Then X is f-contracting, and for every f-contracting closed set B also $f(B)$ is closed with $f(f(B)) \subseteq f(B)$ and hence f-contracting. Further, the intersection $\bigcap \mathcal{N}$ of a nest of f-contracting closed sets is closed with

$f(\bigcap \mathcal{N}) \subseteq \bigcap \mathcal{N}$ and hence f-contracting. Therefore, Theorem 2 shows that f has a unique fixed point. $\qquad\square$

We will now show how to deduce a fixed point theorem of [13] from Theorem 3. For this theorem, we assume that X is compact, Hausdorff and connected. An open cover $\mathcal{U}$ of X is said to be *J-contractive for f* if for every $U \in \mathcal{U}$ there is $U' \in \mathcal{U}$ such that $f(\mathrm{cl}\, U) \subseteq U'$, where $\mathrm{cl}\, U$ denotes the closure of U. The function $f : X \to X$ is called *J-contraction* if every open cover $\mathcal{U}$ has a finite J-contractive open refinement $\mathcal{V}$ for f. For every non-connected compact Hausdorff space X there is a J-contraction of X which has no fixed points (cf. [13, Proposition 3, p. 553]); therefore, the approach using J-contractions only works for connected compact Hausdorff spaces. We cite two important facts about a J-contraction f on a connected compact Hausdorff space X:

(J1) If B is a closed subset of X with $f(B) \subseteq B$, then the restriction of f to B is also a J-contraction ([13, Proposition 1, p. 552]);

(J2) If f is onto, then $|X| = 1$ ([13, Proposition 4, p. 554]).

The following is Theorem 4 of [13]:

Theorem 10. *Take a connected compact Hausdorff space X and a continuous J-contraction $f : X \to X$. Then f has a unique fixed point.*

Proof: We claim that by (J1) and (J2), every closed subset B of X with $f(B) \subseteq B$ is f-contracting. Indeed, since by (J1), $f|_B$ is a J-contraction on B, (J2) shows that either $f|_B$ is not onto, or B is a singleton $\{x\}$ and since $fx \in f(B) \subseteq B$, we have that $fx = x$. Now Theorem 10 follows from Theorem 3. $\qquad\square$

The next theorem shows how to apply Theorem 4 to topological spaces.

Theorem 11. *Take a compact space X and a closed function $f : X \to X$. Assume that for every $x \in X$ with $fx \neq x$ there is a closed subset B of X such that $x \in B$ and $x \notin f(B) \subseteq B$. Then f has a fixed point in B. Moreover, f is self-contractive, and for every $x \in X$ with $fx \neq x$ there is a smallest closed subset B of X such that $x \in B$ and $x \notin f(B) \subseteq B$.*

Proof: For every $x \in X$ we consider the following family of balls:

$$\mathfrak{B}_x := \{B \mid B \text{ closed subset of } X, \, x \in B \text{ and } f(B) \subseteq B\}.$$

Note that $\mathfrak{B}_x$ is nonempty because it contains X. We define

$$B_x := \bigcap \mathfrak{B}_x \,.$$

We see that $x \in B_x$ and that $f(B_x) \subseteq B_x$. Further, B_x is closed, being the intersection of closed sets. This shows that B_x is the smallest member of $\mathfrak{B}_x$.

For every $B \in \mathfrak{B}_x$ we have that $fx \in B$ and therefore, $B \in \mathfrak{B}_{fx}$. Hence we find that $B_{fx} \subseteq B_x$.

Assume that $fx \neq x$. Then by hypothesis, there is a closed set B in X such that $x \in B$ and $x \notin f(B) \subseteq B$. Since f is a closed function, $f(B)$ is closed. Moreover, $f(f(B)) \subseteq f(B)$ and $fx \in f(B)$, so $f(B) \in \mathfrak{B}_{fx}$. Since $x \notin f(B)$, we conclude that $x \notin B_{fx}$, whence $B_{fx} \subsetneq B_x$. We have now proved that f is strongly contracting on orbits. Further, $B \in \mathfrak{B}_x$, whence $B_x \subset B$, $f(B_x) \subset f(B)$ and therefore, $x \notin f(B_x)$. This shows that B_x is the smallest of all closed sets B in X for which $x \in B$ and $x \notin f(B) \subseteq B$.

Take an f-nest $\mathcal{N}$. Lemma 9 shows that $\bigcap \mathcal{N}$ is nonempty. Take any $z \in \bigcap \mathcal{N}$. Choose an arbitrary $B_x \in \mathcal{N}$. Then $z \in B_x$ and thus, $B_x \in \mathfrak{B}_z$. So we have that $B_z \subseteq B_x$. Therefore, $B_z \subseteq \bigcap \mathcal{N}$. We have proved that f is self-contractive.

Theorem 11 now follows from Theorem 4. $\qquad \square$

5. Ultrametric fixed point theorems

Let (X, d) be an ultrametric space. That is, d is a function from $X \times X$ to a partially ordered set Γ with smallest element 0, satisfying that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $d(x, y) = 0$ if and only if $x = y$,
(U2) if $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma$, then $d(x, z) \leq \gamma$,
(U3) $d(x, y) = d(y, x)$ (symmetry).

(U2) is the ultrametric triangle law; if Γ is totally ordered, it can be replaced by

(UT) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

We obtain the *ultrametric ball space* $(X, \mathcal{B}_u)$ from (X, d) by taking $\mathcal{B}_u$ to be the set of all

$$B(x, y) := \{z \in X \mid d(x, z) \leq d(x, y)\} .$$

It follows from the ultrametric triangle law that $B(x, y) = B(y, x)$ and that

(1) $\quad B(t, z) \subseteq B(x, y)$ if and only if $t \in B(x, y)$ and $d(t, z) \leq d(x, y)$.

In particular,

$$B(t, z) \subseteq B(x, y) \quad \text{if } t, z \in B(x, y) .$$

Two elements γ and δ of Γ are *comparable* if $\gamma \leq \delta$ or $\gamma \geq \delta$. Hence if $d(x, y)$ and $d(y, z)$ are comparable, then $B(x, y) \subseteq B(y, z)$ or $B(y, z) \subseteq B(x, y)$. If $d(y, z) < d(x, y)$, then in addition, $x \notin B(y, z)$ and thus, $B(y, z) \subsetneq B(x, y)$. We note:

(2) $\qquad d(y, z) < d(x, y) \implies B(y, z) \subsetneq B(x, y) .$

If Γ is totally ordered and B_1 and B_2 are any two balls with nonempty intersection, then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

The ultrametric space (X, d) is called *spherically complete* if the corresponding ball space is spherically complete. The following theorem (with $i = 1$ in (3)) appeared in [10]:

Theorem 12. (Strong Ultrametric Fixed Point Theorem)
Take a spherically complete ultrametric space (X, d) and a function $f : X \to X$. Assume that f satisfies, for all $x, z \in X$:

$$(3) \qquad x \neq fx \implies \exists i \geq 1 : d(f^i x, f^{i+1} x) < d(x, fx) ,$$

$$(4) \qquad d(z, fx) \leq d(fx, f^2 x) \implies d(z, fz) \leq d(x, fx) .$$

Then f has a fixed point.

Proof: Our theorem follows from Theorem 6 once we have shown that f is self-contractive on the ball space $(X, \mathcal{B}_u)$. We define

$$(5) \qquad\qquad B_x := B(x, fx)$$

and observe that $x \in B_x$. Taking $z = fx$ in (4), we find that $d(fx, f^2 x) \leq d(x, fx)$. Hence by (1), $B_{fx} = B(fx, f^2 x) \subseteq B(x, fx) = B_x$. By induction on i it follows that $f^i x \in B_x$. By (3), $d(f^i x, f^{i+1} x) < d(x, fx)$ for some $i \geq 1$. Then by (2), we have that $B_{f^i x} = B(f^i x, f^{i+1} x) \subsetneq B(x, fx) = B_x$. So we have proved that f satisfies (SC1) and (SC2).

To show that also (SC3) holds, we take an f-nest $\mathcal{N}$ and any $z \in \bigcap \mathcal{N}$. We have to show that $B_z \subseteq \bigcap \mathcal{N}$, that is, $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. Since $z \in \bigcap \mathcal{N} \subseteq B_{fx} = B(fx, f^2 x)$, we have that $d(z, fx) \leq d(fx, f^2 x)$. By (4), this implies that $d(z, fz) \leq d(x, fx)$. Since we know that $z \in B_x$, (1) now shows that $B_z = B(z, fz) \subseteq B(x, fx) = B_x$. $\square$

A function $f : X \to X$ is called *contracting* if $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$. It is shown in [10] that the following theorem (in the case of $i = 1$ in (3)) follows from Theorem 12:

Theorem 13. (Ultrametric Fixed Point Theorem)
Every contracting function on a spherically complete ultrametric space which satisfies (3) has a fixed point.

This theorem follows directly from Theorem 6 by way of the following result:

Lemma 14. *Take a contracting function f on an ultrametric space (X, d) and define the balls B_x as in (5). Then f satisfies (SC3), and $f(B_x) \subseteq B_x$ and $B_{fx} \subseteq B_x$ for all $x \in X$. If f also satisfies (3), then it is self-contractive.*

Proof: We claim that for contracting f we have, as in the topological case,

$$(6) \qquad\qquad B_z \subseteq B_x \quad \text{for all } z \in B_x .$$

Indeed, $z \in B_x$ means that $d(x, z) \leq d(x, fx)$. Since f is contracting, we then have that $d(fx, fz) \leq d(x, z) \leq d(x, fx)$. Together with the trivial inequality $d(x, fx) \leq d(x, fx)$ and the ultrametric triangle law, this yields

that $d(x, fz) \le d(x, fx)$. Together with $d(z, x) = d(x, z) \le d(x, fx)$ and the ultrametric triangle law, this yields that $d(z, fz) \le d(x, fx)$. Now (1) shows that $B_z = B(z, fz) \subseteq B(x, fx) = B_x$, which proves (6). We also obtain that $fz \in B_x$, and as $z \in B_x$ was arbitrary, this shows that $f(B_x) \subseteq B_x$.

Taking $z = fx \in B_x$ in (6), we find that $B_{fx} \subseteq B_x$.

If $\mathcal{N}$ is an f-nest and $z \in \bigcap \mathcal{N}$, then for every $B_x \in \mathcal{N}$ we have that $z \in B_x$ and by (6), $B_z \subseteq B_x$. This implies that $B_z \subseteq \bigcap \mathcal{N}$, which proves (SC3).

The last assertion of the lemma is clear. □

6. Ultrametric attractor theorems

In this section, we present a generalization of the attractor theorem of [5] to ultrametric spaces with partially ordered value sets, and show how to derive it from Theorem 7.

Take ultrametric spaces (X, d) and (X', d') and a function $\varphi : X \to X'$. An element $z' \in X'$ is called *attractor for* φ if for every $x \in X$ such that $z' \ne \varphi x$, there is an element $y \in X$ which satisfies:

(UAT1) $d'(\varphi y, z') < d'(\varphi x, z')$,
(UAT2) $\varphi(B(x, y)) \subseteq B(\varphi x, z')$,
(UAT3) if $t \in X$ such that $d'(\varphi x, z') < d'(\varphi t, z')$ and $\varphi(B(t, x)) \subseteq B(\varphi t, z')$, then $d(t, x)$ and $d(x, y)$ are comparable.

Condition (UAT1) says that the approximation φx of z' from within the image of φ can be improved, and condition (UAT2) says that this can be done in a somewhat continuous way. Condition (UAT3) is always satisfied when the value set of (X, d) is totally ordered, which implies that any two balls with nonempty intersection are comparable by inclusion. For this reason, it does not appear as a condition in the attractor theorem of [5]. But if the value set of (X, d) is not totally ordered, then it can happen that several "parallel universes" exist around a point; (UAT3) then guarantees that we can keep our approximations to remain in the same universe.

Theorem 15. *Assume that $z' \in X'$ is an attractor for $\varphi : X \to X'$ and that (X, d) is spherically complete. Then $z' \in \varphi(X)$.*

Proof: For $x \in X$, we define $B'_x := B'(\varphi x, z')$. Then we define a function $f : X \to X$ as follows. If $\varphi x = z'$, we set $fx = x$. If $\varphi x \ne z'$, then we choose some $y \in X$ that satisfies (UAT1), (UAT2), (UAT3) and set $fx := y$. In both cases, we set $B_x := B(x, fx)$.

We have that $z' \in B'_x$ by definition. If $\varphi x = z'$, then $B_x = \{x\}$ and $\varphi(B_x) = \varphi(\{x\}) = \{\varphi x\} = B'_x$. If $\varphi x \ne z'$, then $\varphi(B_x) = \varphi(B(x, fx)) \subseteq B(\varphi x, z') = B'_x$ holds by (UAT2). Hence, (AT1) is satisfied.

In order to prove that (AT2) is satisfied, we assume that $\varphi x \ne z'$. By (UAT1), we have that $d'(\varphi fx, z') < d'(\varphi x, z')$, which by (2) implies that

$B'_{fx} = B(\varphi fx, z') \subsetneq B(\varphi x, z') = B'_x$. Thus, (AT2) holds. We also see that $\varphi x \notin B(\varphi fx, z')$.

Now we show that f is strongly contracting on orbits. (SC1) holds by definition, and (SC2) holds trivially when $x = fx$. So we assume that $x \neq fx$. We run the above construction again with fx in place of x to obtain f^2x. If $\varphi fx = z'$, then $f^2x = fx$ and $B_{fx} = \{fx\} \subsetneq B_x$ since $x \neq fx$ by assumption. Now assume that $\varphi fx \neq z'$. Then by what we have already shown, $\varphi(B_{fx}) \subseteq B'_{fx}$, so $x \notin B_{fx}$. From (UAT3), where we replace t, x, y by x, fx, f^2x, we infer that $d(x, fx)$ and $d(fx, f^2x)$ are comparable. Therefore, $B_x \subseteq B_{fx}$ or $B_{fx} \subseteq B_x$. Since $x \notin B_{fx}$, it follows that $B_{fx} \subsetneq B_x$, which proves that f is strongly contracting on orbits.

We have shown that z' is a weak f-attractor for φ. Our theorem will thus follow from Theorem 7 once we have proved that f also satisfies (SC3). Take an f-nest $\mathcal{N}$ and some $z \in \bigcap \mathcal{N}$. We have to show that $B_z \subseteq \bigcap \mathcal{N}$, that is, $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. If $\varphi x = z'$, then $z \in B_x = \{x\}$, whence $z = x$ and $B_z = B_x$. Hence we will again assume that $\varphi x \neq z'$.

Since $z \in \bigcap \mathcal{N} \subseteq B_{fx}$, we have that

$$\varphi z \in \varphi(B_{fx}) \subseteq B'_{fx} = B(\varphi fx, z').$$

It follows that

$$\varphi(B_z) \subseteq B'_z = B(\varphi z, z') \subseteq B(\varphi fx, z').$$

But we have already shown that $\varphi x \notin B(\varphi fx, z')$, hence $x \notin B_z$. We have that

$$(7) \qquad\qquad d'(\varphi z, z') \leq d'(\varphi fx, z') < d'(\varphi x, z')$$

and

$$(8) \qquad\qquad \varphi(B(x, z)) \subseteq \varphi(B_x) \subseteq B'_x = B(\varphi x, z').$$

From (UAT3), where we replace t, x, y by x, z, fz, we infer that $d(x, z)$ and $d(z, fz)$ are comparable. We find that $B(x, z) \subseteq B(z, fz)$ or $B(z, fz) \subseteq B(x, z)$. Since $x \notin B_z$, we obtain that $B_z \subseteq B(x, z) \subseteq B_x$. $\qquad\square$

Note that our condition (UAT3) is somewhat stronger than condition (8) of [PR] because we do not start with a given function f (which is called g in [PR]), but construct it in our proof. Rewritten in our present notation, condition (8) of [PR] states:

(UAT3′) if $d'(\varphi z, z') < d'(\varphi x, z')$ and $B(z, fz) \cap B(x, fx) \neq \emptyset$, then $d(z, fz) < d(x, fx)$.

If the function f, strongly contracting on orbits, is already given, then condition (UAT3) in Theorem 15 can be replaced by (UAT3′). Let us show how (UAT3′) is used at the end of the proof of Theorem 15 to deduce that (SC3) holds. For $z \in \bigcap \mathcal{N}$, we have to show that $B_z \subseteq B_x$ for all $B_x \in \mathcal{N}$. As in the above proof, one shows that (7) holds. Since $z \in \bigcap \mathcal{N} \subseteq B_x = B(x, fx)$, we have that $B(z, fz) \cap B(x, fx) \neq \emptyset$. Hence by (UAT3′),

$d(z, fz) < d(x, fx)$, which by (1) implies that $B_z \subseteq B_x$. Observe that the condition "$d(z, fz) < d(x, fx)$" in (UAT3′) can be replaced by the weaker condition that $d(z, fz)$ and $d(x, fx)$ are comparable.

7. Completeness by stages

In this section, we consider valued fields (K, v). In order to be compatible with the way we have presented ultrametrics in the previous sections, we write the valuation v multiplicatively, that is, the value group is a multiplicatively written ordered abelian group with neutral element $1 = v(1)$, and we add a smallest element $0 = v(0)$. The axioms for a valuation in this notation are:

(VF1) $v(x) = 0 \Leftrightarrow x = 0$,
(VF2) $v(xy) = v(x)v(y)$,
(VF3) $v(x + y) \leq \max\{v(x), v(y)\}$.
The underlying ultrametric is obtained by setting $d(x, y) := v(x - y)$.

We will work in the valuation ideal $\mathcal{M} = \{x \in K \mid v(x) < 1\}$ of (K, v) since this facilitates the notation, and the typical applications of the fixed point theorem we are going to prove can be made to deal with functions $f : \mathcal{M} \to \mathcal{M}$.

We assume the reader to be familiar with the theory of pseudo Cauchy sequences (see for instance [4]). P. Ribenboim introduced in [11] the notion of *distinguished pseudo Cauchy sequence*. For a pseudo Cauchy sequence $(a_\nu)_{\nu < \lambda}$ in $\mathcal{M}$, indexed by a limit ordinal λ, the original definition of "distinguished" is equivalent to the following: for every $\mu < \lambda$ there is $\nu < \lambda$ such that

$$v(a_{\nu+1} - a_\nu) \leq v(a_{\mu+1} - a_\mu)^2 .$$

The valued field (K, v) is called *complete by stages* if every distinguished pseudo Cauchy sequence in $\mathcal{M}$ has a pseudo limit in K. Ribenboim proves in [11] that every such field is henselian. For this proof, one can use a theorem like the following:

Theorem 16. *Take a valued field (K, v) that is complete by stages and a contracting function $f : \mathcal{M} \to \mathcal{M}$. If for every $x \in K$ there is $j \in \mathbb{N}$ such that*

$$v(f^j x - f^{j+1} x) \leq v(x - fx)^2 ,$$

then f has a fixed point in K.

Note that $v(x - fx) < 1$ since $x, fx \in \mathcal{M}$. In the theorem, $\mathcal{M}$ can be replaced by any ultrametric ball B in K as long as $v(x - fx) < 1$ for all $x \in B$.

In order to deduce this theorem from Theorem 4, we need to show the connection between completeness by stages and nests of balls with certain properties. We will call a nest $\mathcal{N}$ of ultrametric balls in $\mathcal{M}$ *distinguished* if

for all $x, y \in \mathcal{M}$ with $B(x, y) \in \mathcal{N}$ there are $x', y' \in \mathcal{M}$ such that $v(x' - y') \leq v(x - y)^2$ and $B(x', y') \in \mathcal{N}$. Then the following holds:

Lemma 17. *A valued field (K, v) with valuation ideal $\mathcal{M}$ is complete by stages if and only if each distinguished nest of ultrametric balls in $\mathcal{M}$ has a nonempty intersection.*

In this lemma, $\mathcal{M}$ can be replaced by the valuation ring $\mathcal{O}$, and also by the ultrametric balls $a + \mathcal{M}$ and $a + \mathcal{O}$ for all $a \in K$. The proof of the lemma is similar to the proof of the fact that an ultrametric space is spherically complete if and only if every pseudo Cauchy sequence in this space has a pseudo limit. It is based on the following easy observation:

Lemma 18. *Every nest of balls admits, in the ordering given by reverse inclusion, a cofinal well ordered subnest.*

Now the method of proof is to associate such a coinitial well ordered subnest with a pseudo Cauchy sequence such that each pseudo limit will lie in the intersection of the nest — and vice versa. Since every subnest of a distinguished nest is again distinguished, this will give rise to a distinguished pseudo Cauchy sequence. Conversely, if $(a_\nu)_{\nu < \lambda}$ is a distinguished pseudo Cauchy sequence in $\mathcal{M}$, then $\{B(a_\nu, a_{\nu+1}) \mid \nu < \lambda\}$ is a distinguished nest of balls in $\mathcal{M}$.

Proof of Theorem 16: As before, we set $B_x := B(x, fx)$ and note that $x \in B_x$. Since f is contracting, we have that $v(fx - f^2x) \leq v(x - fx)$, which by (1) implies that $B_{fx} \subseteq B_x$. If $fx \neq x$, then by assumption, there is $i \in \mathbb{N}$ such that $v(f^ix - f^{i+1}x) \leq v(x - fx)^2 < v(x - fx)$; this yields that $x \notin B_{f^ix}$ and $B_{f^ix} \subsetneq B_x$. We have now proved that f is strongly contracting on orbits. Since f is contracting, this implies by way of Lemma 14 that f also satisfies (SC3).

Take an f-nest $\mathcal{N}$. Every ball in $\mathcal{N}$ is of the form $B(x, fx)$ and there is some $i \in \mathbb{N}$ such that $v(f^ix - f^{i+1}x) \leq v(x - fx)^2$. By the definition of an f-nest, $B_{f^ix} \in \mathcal{N}$. Thus, $\mathcal{N}$ is distinguished. Since (K, v) is assumed to be complete by stages, Lemma 17 shows that $\mathcal{N}$ has a nonempty intersection. Since f also satisfies (SC3) and is strongly contracting on orbits, this implies that the conditions of Theorem 4 are satisfied, and we obtain the assertion of Theorem 16.

8. Banach's Fixed Point Theorem

Banach's Fixed Point Theorem states that every strictly contracting function on a complete metric space (X, d) has a unique fixed point. A function $f : X \to X$ is called *strictly contracting* if there is a positive real number $C < 1$ such that $d(fx, fy) \leq Cd(x, y)$ for all $x, y \in X$. We will show now how Banach's Fixed Point Theorem fits into the setting of Theorems 4 and 6. We work in the ball space $(X, \mathcal{B})$ where $\mathcal{B}$ consists of all balls

$\{y \in X \mid d(x, y) \le r\}$ for $x \in X$ and $r \in \mathbb{R}^{\ge 0}$. This ball space is spherically complete since (X, d) is complete.

We will prove the existence of fixed points under the slightly more general assumption that f is

1) *contracting*, that is, $d(fx, fy) \le d(x, y)$ for all $x, y \in X$, and
2) *strictly contracting on orbits*, that is, there is a positive real number $C < 1$ such that $d(fx, f^2 x) \le C d(x, fx)$ for all $x \in X$.

Take any $x \in X$. Then

$$
\begin{aligned}
d(x, f^i x) &\le d(x, fx) + d(fx, f^2 x) + \ldots + d(f^{i-1} x, f^i x) \\
&\le d(x, fx)(1 + C + C^2 + \ldots + C^{i-1}) \\
&\le d(x, fx) \sum_{i=0}^{\infty} C^i = \frac{d(x, fx)}{1 - C} .
\end{aligned}
$$

Hence if we set

$$
B_x := \left\{ y \in X \mid d(x, y) \le \frac{d(x, fx)}{1 - C} \right\} ,
$$

then $f^i x \in B_x$ for $i \ge 0$. In particular, $x \in B_x$, hence (SC1) holds.

We wish to show that $B_{fx} \subseteq B_x$. Take any $y \in B_{fx}$. Then

$$
\begin{aligned}
d(x, y) &\le d(x, fx) + d(fx, y) \le d(x, fx) + \frac{d(fx, f^2 x)}{1 - C} \\
&\le d(x, fx) + \frac{C}{1 - C} d(x, fx) = \frac{d(x, fx)}{1 - C} .
\end{aligned}
$$

Thus, $y \in B_x$, which proves our assertion.

Since $C < 1$, there is some $i \ge 1$ such that

$$
\frac{C^i}{1 - C} < \frac{1}{2} .
$$

Then

$$
\frac{d(f^i x, f^{i+1} x)}{1 - C} \le \frac{C^i}{1 - C} d(x, fx) < \frac{1}{2} d(x, fx) ,
$$

which implies that x and fx cannot both lie in $B_{f^i x}$. Therefore, $B_{f^i x} \subsetneq B_x$ and we have now proved that (SC2) holds.

Next, we show that also (SC3) holds. Take an f-nest $\mathcal{N}$ and assume that $z \in \bigcap \mathcal{N}$. Pick any $B_x \in \mathcal{N}$ and $i > 0$. Since f is contracting, $d(f^i x, fz) \le d(f^{i-1} x, z) = d(z, f^{i-1} x)$. Using that $z \in \mathcal{N} \subseteq B_{f^i x}$ for all i, we compute:

$$
\begin{aligned}
d(z, fz) &\le d(z, f^i x) + d(f^i x, fz) \le d(z, f^i x) + d(z, f^{i-1} x) \\
&\le \frac{d(f^i x, f^{i+1} x)}{1 - C} + \frac{d(f^{i-1} x, f^i x)}{1 - C} \\
&\le \frac{C^i}{1 - C} d(x, fx) + \frac{C^{i-1}}{1 - C} d(x, fx) = C^{i-1} \frac{C + 1}{1 - C} d(x, fx) .
\end{aligned}
$$

Since $\lim_{i\to\infty} C^i = 0$, we obtain that $fz = z$, so we have found a fixed point. It follows that $B_z \subseteq \bigcap \mathcal{N}$ (in fact, $\bigcap \mathcal{N} = \{z\} = B_z$). This shows that f is self-contractive.

Note that if f is strictly contracting, then the fixed point is unique. Indeed, if there were distinct fixed points x, y, then $d(x, y) = d(fx, fy) < d(x, y)$, a contradiction.

9. The case of ordered abelian groups and fields

In this section we will discuss various forms of fixed point theorems in the case of ordered abelian groups and fields. Here, we always mean that the ordering is total. The ordering induces a natural valuation; we will recall its definition in Section 9.1. This valuation is nontrivial if and only if the ordering is nonarchimedean. Since the valuation induces an ultrametric, our ultrametric fixed point theorems can be translated to the present case. We will do this in Section 9.2.

However, the most natural idea to derive a ball space from the ordering of an ordered abelian group $(G, <)$ is to define the *order balls* in G to be the sets of the form

$$B_o(g; r) := \{z \in G \mid |g - z| \leq r\}$$

for arbitrary $g \in G$ and nonnegative $r \in G$. We set

$$\mathcal{B}_o = \mathcal{B}_o(G, <) := \{B_o(g; r) \mid g \in G,\ 0 \leq r \in G\}\,.$$

Then $(G, \mathcal{B}_o)$ is the *order ball space* associated with $(G, <)$. In Section 9.3 we will state a fixed point theorem corresponding to the order ball space.

Before we continue, we need some preliminaries and general background.

9.1. **Preliminaries on non-archimedean ordered abelian groups and fields.** Take an ordered abelian group $(G, <)$. Two elements $a, b \in G$ are called *archimedean equivalent* if there is some $n \in \mathbb{N}$ such that $n|a| \geq |b|$ and $n|b| \geq |a|$. The ordered group $(G, <)$ is archimedean ordered if all nonzero elements are archimedean equivalent. If $0 \leq a < b$ and $na < b$ for all $n \in \mathbb{N}$, then "a is infinitesimally smaller than b" and we will write $a \ll b$.

We define the *natural valuation* of $(G, <)$ as follows. We denote by va the archimedean equivalence class of a. The set of archimedean equivalence classes is ordered as follows: $va < vb$ if and only if $|a| < |b|$ and a and b are not archimedean equivalent, that is, if $n|a| < |b|$ for all $n \in \mathbb{N}$. We write $0 := v0$; this is the minimal element in the totally ordered set of equivalence classes. The function $a \mapsto va$ is a group valuation on G, i.e., it satisfies $vx = 0 \Leftrightarrow x = 0$ and the ultrametric triangle law

(UT) $v(x - y) \leq \max\{vx, vy\}$.

The natural valuation induces an ultrametric defined by

$$d(x, y) := v(x - y)$$

and hence an ultrametric ball space, with the set $\mathcal{B}_u = \mathcal{B}_u(G, <)$ of balls $B_u(x, y)$ defined as in Section 5. We will call $(G, \mathcal{B}_u)$ the *(natural) ultrametric ball space* of $(G, <)$. Note that all ultrametric balls are cosets of convex subgroups in G.

If $(K, <)$ is an ordered field, then we consider the natural valuation on its ordered additive group and define $va \cdot vb := v(ab)$. This turns the set of archimedean classes into a multiplicatively written ordered abelian group, with neutral element $1 := v1$ and inverses $(va)^{-1} = v(a^{-1})$. In this way, v becomes a field valuation (with multiplicatively written value group). It is the finest valuation on K which is compatible with the ordering. The residue field $Kv := \mathcal{O}/\mathcal{M}$ is archimedean ordered, hence by the version of the Theorem of Hölder for ordered fields, it can be embedded in the ordered field $\mathbb{R}$. Via this embedding, we will always identify it with a subfield of $\mathbb{R}$.

We know from [4, Theorem 6] that K can be embedded in the power series field with exponents in the value group and coefficients in the residue field of its natural valuation. (A nontrivial factor set may be needed, unless the positive part of the residue field is closed under radicals, which for instance is the case if K is real closed.) Moreover, the ultrametric ball space of K is spherically complete if and only if the embedding is onto.

9.2. **Ultrametric balls.** Via the natural valuation, the ultrametric fixed point theorems provide fixed point theorems for ordered abelian groups and fields. A valuation is called *spherically complete* if its associated ultrametric ball space is spherically complete, and it is called *complete by stages* if its associated ultrametric ball space is complete by stages.

Take an ordered abelian group G and a function $f : G \to G$. It will be called *o-contracting* if

$$|fx - fy| \leq |x - y|$$

for all $x, y \in G$; note that an o-contracting function is also contracting in the ultrametric sense. The property

(9) $\qquad x \neq fx \implies \exists i \geq 1 : |f^i x - f^{i+1} x| \ll |x - fx|$

implies property (3). Hence, the following theorem is an immediate consequence of Theorem 13. In order to obtain it for an ordered field K, take G to be the additive group of K.

Theorem 19. *Take an ordered abelian group G whose natural valuation is spherically complete. If $f : G \to G$ is an o-contracting function that satisfies (9), then it has a fixed point.*

Take an ordered field K and let v denote its natural valuation. Then the valuation ideal $\mathcal{M}$ of v is the set of all infinitesimals of K, that is, the elements $x \in K$ such that $|x| \ll 1$. The next theorem follows directly from Theorem 16. We suspect that the theorem becomes false if $\mathcal{M}$ is replaced by the valuation ring $\mathcal{O}$.

Theorem 20. *Take an ordered field K with $\mathcal{M}$ the set of its infinitesimals, whose natural valuation is complete by stages. If $f : \mathcal{M} \to \mathcal{M}$ is an o-contracting function and for every $x \in \mathcal{M}$ there is $j \in \mathbb{N}$ such that*

$$|f^j x - f^{j+1} x| \ \leq \ |x - fx|^2 \ ,$$

then f has a fixed point in $\mathcal{M}$.

9.3. **Order balls.** Take an ordered abelian group $(G, <)$. We call a function $f : G \to G$ *strictly o-contracting on orbits* if there is a positive rational number $\frac{m}{n} < 1$ with $m, n \in \mathbb{N}$ such that $n|fx - f^2 x| \leq m|x - fx|$ for all $x \in G$. Note that $n|fx - f^2 x| \leq m|x - fx|$ holds if and only if $|fx - f^2 x| \leq \frac{m}{n}|x - fx|$ in the divisible hull of G with the unique extension of the ordering.

The following theorem is a consequence of Theorem 6:

Theorem 21. *Suppose that the order ball space $(G, \mathcal{B}_o)$ associated with the ordered abelian group $(G, <)$ is spherically complete. Then every o-contracting function on G which is strictly o-contracting on orbits has a fixed point.*

Proof: We compute in the divisible hull of G. We choose $C \in \mathbb{Q}$ such that $\frac{m}{n} < C < 1$. With $d(x, y) := |x - y|$, the computations of Section 8 remain valid, and we may define the balls B_x in exactly the same way. They will then again satisfy (SC1) and (SC2). However, we have to work a little bit harder to show that (SC3) holds. In the present case, we cannot conclude that $d(z, fz) = 0$. But for given $B_x \in \mathcal{N}$ we can conclude that

$$d(z, fz) \ \leq \ C^{i-1} \frac{C+1}{1-C} \, d(x, fx)$$

for all $i > 0$ (which means that $d(z, fz) \ll d(x, fx)$). We choose i so large that

$$C^{i-1} \frac{C+1}{1-C} \ \leq \ C - \frac{m}{n} \ .$$

Then for every $y \in B_z$,

$$d(z, y) \ \leq \ \frac{d(z, fz)}{1-C} \ \leq \ C^{i-1} \frac{C+1}{1-C} \, d(x, fx) \ \leq \ \left(C - \frac{m}{n} \right) d(x, fx) \ .$$

Using that $z \in \bigcap \mathcal{N} \subseteq B_{f^2 x}$, we compute:

$$
\begin{aligned}
d(x, y) \ &\leq \ d(x, fx) + d(fx, f^2 x) + d(f^2 x, z) + d(z, y) \\
&\leq \ \left(1 + \frac{m}{n} \right) d(x, fx) + \frac{C^2}{1-C} \, d(x, fx) + \left(C - \frac{m}{n} \right) d(x, fx) \\
&= \ (1+C) d(x, fx) + \frac{C^2}{1-C} \, d(x, fx) \ = \ \frac{d(x, fx)}{1-C} \ .
\end{aligned}
$$

Hence $y \in B_x$. We have proved that $B_z \subseteq B_x$, as required. Altogether, we have shown that f is self-contractive. Now the assertion of our theorem follows from Theorem 6. $\square$

We leave it to the reader to formulate a corresponding version of Theorem 4.

In the case of an ordered field $(K, <)$, we may actually give a slightly more general definition of "strictly contracting on orbits" by requiring that there is an element $C \in K$ such that $0 < C < 1$, $v(1 - C) = 1$, and $|fx - f^2x| \leq C|x - fx|$ for all $x \in K$, as this condition in fact implies the condition of the original definition; indeed, the condition "$v(1 - C) = 1$" means that C is not infinitesimally close to 1 and therefore, there is some $C' \in \mathbb{Q}$ such that $C \leq C' < 1$. Working in the ordered additive group of the field, one then immediately obtains from Theorem 21 the corresponding version for ordered fields.

At first glance, spherical completeness of the order ball space appears to be a very strong condition. We note:

Lemma 22. *An archimedean ordered abelian group has a spherically complete order ball space if and only if it is isomorphic to the additive group of the integers or of the reals with the canonical ordering.*

Proof: First, observe that every nest $\mathcal{N}$ of order balls in the integers contains a smallest ball, so the integers have a spherically complete order ball space.

Next, take a nest $\mathcal{N}$ of order balls in the reals. Pick a ball B in this nest and consider the nest $\mathcal{N}_0$ of all balls in $\mathcal{N}$ that are contained in B; this nest has the same intersection as $\mathcal{N}$. The order ball B is compact in the order topology of $\mathbb{R}$. Since all balls in $\mathcal{N}_0$ are closed in the order topology, Lemma 9 shows that the intersection of $\mathcal{N}_0$ is nonempty.

From what we have proved it follows that every ordered abelian group which is isomorphic to the additive ordered group of the integers or the reals has a spherically complete order ball space.

For the converse, we use that by the Theorem of Hölder, every archimedean ordered abelian group G can be embedded in the additive ordered group of the reals. We identify G with its image under this embedding and show that if G is spherically complete w.r.t. the order balls, but not isomorphic to $\mathbb{Z}$, then $G = \mathbb{R}$. Since G is not isomorphic to $\mathbb{Z}$, it is dense in $\mathbb{R}$. Hence for every $a \in \mathbb{R}$ there is an increasing sequence $(g_i)_{i \in \mathbb{N}}$ in G converging to a. By density, we can also find a decreasing sequence $(r_i)_{i \in \mathbb{N}}$ in G converging to 0 such that $(g_i + 2r_i)_{i \in \mathbb{N}}$ is a decreasing sequence in G converging to a. Then $\{B_o(g_i + r_i \,; r_i) \mid i \in \mathbb{N}\}$ is a nest of order balls in G. Since G has a spherically complete order ball space, this nest has a nonempty intersection. As this intersection can only contain the element a, we find that $a \in G$. $\quad\square$

One might think that spherical completeness w.r.t. the order balls implies cut completeness, in which case $\mathbb{Z}$ and $\mathbb{R}$ would be the only ordered groups with this property. But fortunately, this is not the case. In [12], Saharon Shelah has shown that every ordered field is contained in one that has a spherically complete order ball space (see also [7] for a power series field

construction of such fields). So there are arbitrarily large ordered fields (and hence also ordered abelian groups) in which our above fixed point theorem holds.

We will study the structure of such ordered fields and abelian groups more closely in a subsequent paper. But we include some basic facts here. The next lemma shows that spherical completeness of the order ball space is stronger than spherical completeness of the ultrametric ball space.

Proposition 23. *If an ordered abelian group $(G, <)$ has a spherically complete order ball space, then it has a spherically complete ultrametric ball space.*

Proof: Assume that the ordered abelian group $(G, <)$ has a spherically complete order ball space and take a nest $\mathcal{N}$ of ultrametric balls in G. If this nest contains a smallest ball, then its intersection is nonempty and there is nothing to show. So we assume that $\mathcal{N}$ does not contain a smallest ball.

In view of Lemma 18, we may assume that the nest is of the form $\{B_u(x_\mu, y_\mu) \mid \mu < \kappa\}$, where κ is a regular cardinal (i.e., equal to its own cofinality), and $B_u(x_\nu, y_\nu) \subsetneq B_u(x_\mu, y_\mu)$ whenever $\mu < \nu < \kappa$. The latter implies that $v(x_\nu - y_\nu) < v(x_\mu - y_\mu)$. For each $\mu < \kappa$, we define an order ball

$$B^\mu := B_o(x_{\mu+1} ; |x_\mu - y_\mu|) \subseteq B_u(x_\mu, y_\mu) .$$

Since $v(x_{\mu+1} - y) \leq v(x_{\mu+1} - y_{\mu+1}) < v(x_\mu - y_\mu)$ implies that $|x_{\mu+1} - y| \ll |x_\mu - y_\mu|$, we find that $B_u(x_{\mu+1}, y_{\mu+1}) \subseteq B^\mu$. It follows that $\{B^\mu \mid \mu < \kappa\}$ is a nest of balls and its intersection is equal to the intersection of $\mathcal{N}$. Since G has a spherically complete order ball space by assumption, this intersection is nonempty. $\square$

Every ordered field contains the field $\mathbb{Q}$ of rational numbers as a subfield. We use this fact in the following lemma:

Lemma 24. *An ordered field is spherically complete w.r.t. the order balls*

$$(10) \qquad\qquad B_o(q ; r) , \quad q, r \in \mathbb{Q} \ \ with \ \ r > 0 ,$$

if and only if its residue field under the natural valuation is $\mathbb{R}$.

Proof: Assume that the ordered field $(K, <)$ is spherically complete w.r.t. the order balls of the form (10). We pick some $a \in \mathbb{R}$ and wish to show that $a \in Kv$. Using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, we take a nest $\{B_o(g_i + r_i ; r_i) \mid i \in \mathbb{N}\}$ of order balls in $G = \mathbb{Q} \subseteq Kv$ as in the proof of Lemma 22. Taken as a nest in Kv, its intersection is either empty or contains only a.

The residue map from $\mathcal{O}$ to Kv induces an isomorphism from the subfield $\mathbb{Q} \subseteq \mathcal{O}$ of K onto the subfield $\mathbb{Q}$ of $\mathbb{R}$. Via this isomorphism, we can see the elements g_i and r_i as elements of K, and taken in K, $\{B_o(g_i ; r_i) \mid i \in \mathbb{N}\}$ is a nest of order balls of the form (10). By assumption, this nest has a nonempty intersection; take $b \in K$ to lie in this intersection. Then its

residue lies in the intersection of the nest $\{B_o(g_i \, ; r_i) \mid i \in \mathbb{N}\}$ in Kv and therefore must be equal to a. We have shown that $a \in Kv$, and as it was an arbitrary element of $\mathbb{R}$, we find that $Kv = \mathbb{R}$.

For the converse, assume that the ordered field $(K, <)$ has residue field $\mathbb{R}$ under its natural valuation. Take a nest $\{B_o(q_i \, ; r_i) \mid i \in I\}$ in K of balls of the form (10). Via the residue map, we can view q_i, r_i as elements of Kv, and we can view $\{B_o(q_i \, ; r_i) \mid i \in I\}$ as a nest of order balls in $Kv = \mathbb{R}$. By Lemma 22, it has a nonempty intersection. Since the residue map is surjective, there is an element $a \in K$ whose residue lies in this intersection. We can take a in the ball $B_o(q_i \, ; r_i)$ in K, for some $i \in I$. Suppose that a does not lie in the intersection of the nest in K. Then there is a smaller ball $B_o(q_j \, ; r_j)$ in which a does not lie. Since the residue of a lies in the intersection of the nest in Kv, there must be some a' in the ball $B_o(q_j \, ; r_j)$ in K which has the same residue as a. We assume without loss of generality that $a > q_j + r_j$; the case of $a < q_j - r_j$ is symmetrical. Then $a' < q_j + r_j < a$, and as a and a' have the same residue, this must also be the residue of $b := q_j + r_j$. Now we show that b lies in the intersection of the nest in K. If this were not true, then there would exist a ball $B_o(q_k \, ; r_k)$ which does not contain b, so $b > q_k + r_k$ or $b < q_k - r_k$. Since $b, q_k, r_k \in \mathbb{Q}$, it would follow that no element in $B_o(q_k \, ; r_k)$ could have the same residue as b and a, which leads to a contradiction. So we find that also the nest in K has a nonempty intersection. This proves that $(K, <)$ is spherically complete w.r.t. the order balls of the form (10). $\qquad\square$

From Proposition 23 and Lemma 24, we obtain the following theorem.

Theorem 25. *Every ordered field with a spherically complete order ball space is isomorphic to a power series field with residue field $\mathbb{R}$.*

The converse is not true. For example, the power series field with value group $\mathbb{Q}$ and residue field $\mathbb{R}$ does not have a spherically complete order ball space. We will give a full characterization of the ordered fields with spherically complete order ball spaces in a subsequent paper, making use of additional conditions on the value group.

9.4. **Hybrid ball spaces.** Using the flexibility of our notion of ball space, we will now give a simple characterization of the ordered fields that are power series fields with residue field $\mathbb{R}$. We just have to enlarge the underlying ultrametric ball space of an ordered field by a suitable set of order balls. We will make use of two very easy principles:

Lemma 26. *1) If the ball space $(X, \mathcal{B})$ is spherically complete and $\mathcal{B}' \subseteq \mathcal{B}$, then also $(X, \mathcal{B}')$ is spherically complete.*

2) If $\mathcal{B}_i$, $1 \le i \le n$, are collections of subsets of X, then the ball space $(X, \bigcup_{i=1}^n \mathcal{B}_i)$ is spherically complete if and only if all of the ball spaces $(X, \mathcal{B}_i)$, $1 \le i \le n$, are spherically complete.

We leave the easy proofs to the reader. Just note that for every nest of balls in $(X, \bigcup_{i=1}^{n} \mathcal{B}_i)$ there must be an i and a cofinal (under reverse inclusion) subnest of balls that all lie in $\mathcal{B}_i$.

We take an ordered field $(K, <)$ and define the *hybrid ball space of* $(K, <)$ to be K together with the union of the set of ultrametric balls with the set of order balls of the form (10), that is,

$$\mathcal{B}_h(K, <) := \mathcal{B}_u(G, <) \cup \{ B_o(q; r) \mid q, r \in \mathbb{Q} \text{ with } r > 0 \} .$$

From Proposition 23 and Lemma 26 it follows that if the order ball space of an ordered field is spherically complete, then so is its hybrid ball space.

From Lemmas 24 and 26, we obtain the following theorem:

Theorem 27. *An ordered field $(K, <)$ is isomorphic to a power series field with residue field $\mathbb{R}$ if and only if its hybrid ball space $(K, \mathcal{B}_h(K, <))$ is spherically complete.*

A different characterization of power series fields with residue field $\mathbb{R}$ was obtained by Ron Brown (cf. [2, Theorem 1.7]), using the notion of "ultracomplete w.r.t. an extended absolute value". This notion appears to be closely related to spherical completeness of suitable ball spaces; details will be worked out in a subsequent paper.

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