Valuation theory, generalized IFS attractors and fractals

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Theorem (T. Craven, 1975)

Every Boolean space is realized as a space of orderings of some formally real field K.

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$$\mathcal{P}(X) \ni S \mapsto F(S) := \bigcup_{i=1}^{n} f_i(S) .$$

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Iterated function systems consisting of weakly contracting functions are studied in e.g. [4, 1, 5].

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Taras Banakh and Magdalena Nowak ([1]) gave a topological analogue of the previous definition:

A compact topological space X is called fractal if there is an iterated function system $F = [f_1, \ldots, f_n]$ consisting of continuous functions $f_i : X \to X$ such that $F(X) = X$ and the following "shrinking condition" is satisfied:

(SC) for every open covering $\mathcal{C}$ of X, there is some $k \in \mathbb{N}$ such that for every sequence $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ there is $U \in \mathcal{C}$ with

$$f_{i_1} \circ \ldots \circ f_{i_k}(X) \subset U .$$

For a detailed continuation of this approach, see [2, 3].

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It is easy to show that each f_s is contracting and that condition (SC) is satisfied.

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Every discrete valuation ring with finite residue field and equipped with the canonical ultrametric is fractal (under both definitions given above).

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In our paper, we give several examples of $*$-IFS attractors and weak $*$-IFS attractors. For instance, we show:

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A locally compact metric space (X, d) is locally fractal if it is the union over a collection of mutually homeomorphic subspaces X_j, $j \in J$, and there is a system $F = [f_1, \ldots, f_n]$ of functions $f_i : X \to X$ such that for every $j \in J$, X_j is fractal w.r.t. the restrictions of the functions f_i to X_j.

A locally compact topological space X is locally fractal if it is the union over a collection of mutually homeomorphic subspaces X_j, $j \in J$, and there is a system $F = [f_1, \ldots, f_n]$ of functions $f_i : X \to X$

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Important discrete valued fields are locally fractal

Proposition

Every discretely valued field with finite residue field is locally fractal under both definitions.

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