Extensions of valuations to rational function fields of arbitrary finite transcendence degree

Franz-Viktor Kuhlmann

AMS Sectional Meeting
Hawaii, March 2019

The extension problem

Given:

The extension problem

Given:

- a valued field (K, v),

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \ldots, x_n)$,

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \dots, x_n)$,

describe:

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \ldots, x_n)$,

describe:

- which value groups and residue fields can appear,

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \ldots, x_n)$,

describe:

- which value groups and residue fields can appear,
- which defects can appear.

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \ldots, x_n)$,

describe:

- which value groups and residue fields can appear,
- which defects can appear.

It is a big mistake to believe that these questions can be answered

The extension problem

Given:

- a valued field (K, v),
- a rational function field $F = K(x_1, \ldots, x_n)$,

describe:

- which value groups and residue fields can appear,
- which defects can appear.

It is a big mistake to believe that these questions can be answered simply by induction on the transcendence degree n.

Extension problem for algebraic function fields

In the case of algebraic function fields,

Extension problem for algebraic function fields

In the case of algebraic function fields, the extension problem breaks down into two steps:

In the case of algebraic function fields, the extension problem breaks down into two steps:

1) first tackle the rational function field generated by a transcendence basis,

Extension problem for algebraic function fields

In the case of algebraic function fields, the extension problem breaks down into two steps:

1) first tackle the rational function field generated by a transcendence basis,

2) then use ramification theory for the remaining algebraic extension.

Extension problem for algebraic function fields

In the case of algebraic function fields, the extension problem breaks down into two steps:

1) first tackle the rational function field generated by a transcendence basis,

2) then use ramification theory for the remaining algebraic extension.

In this presentation, we will address part 1).

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings.

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?

- Which subsets, characterized e.g. by conditions on value group and residue field,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?

- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field $\mathbb{R}(x, y)|\mathbb{R}$,

More advanced problems

Consider the space of all extensions of v from K to F with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?

- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field $\mathbb{R}(x,y)|\mathbb{R}$, see [BM].

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure),

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field.

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

Theorem

Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

Theorem

Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$ can be realized as value group and residue field of a place

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

Theorem

Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$ can be realized as value group and residue field of a place of the rational function field $\mathbb{Q}(x,y)|\mathbb{Q}$

How it all began

Since the work of MacLane and Schilling in the 1930's ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably "simple" algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

Theorem

Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$ can be realized as value group and residue field of a place of the rational function field $\mathbb{Q}(x, y)|\mathbb{Q}$ whose restriction to $\mathbb{Q}$ is the identity.

The defect

We denote the value group of a valued field (F, v) by vF,

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension if the extension of v from F to E is unique

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension if the extension of v from F to E is unique and

$$[E : F] \; > \; (vE : vF)[Ev : Fv] \, .$$

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension if the extension of v from F to E is unique and

$$[E : F] \;>\; (vE : vF)[Ev : Fv] \,.$$

The Lemma of Ostrowski tells us that

$$[E : F] \;=\; p^k (vE : vF)[Ev : Fv]$$

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension if the extension of v from F to E is unique and

$$[E : F] \; > \; (vE : vF)[Ev : Fv]\,.$$

The Lemma of Ostrowski tells us that

$$[E : F] \; = \; p^k (vE : vF)[Ev : Fv]$$

where p is the characteristic of the residue field of K if positive,

The defect

We denote the value group of a valued field (F, v) by vF, and its residue field by Fv.

A finite extension $(E|F, v)$ of valued fields is a defect extension if the extension of v from F to E is unique and

$$[E : F] \; > \; (vE : vF)[Ev : Fv] \, .$$

The Lemma of Ostrowski tells us that

$$[E : F] \; = \; p^k (vE : vF)[Ev : Fv]$$

where p is the characteristic of the residue field of K if positive, and $p = 1$ otherwise.

A second example

As for the value groups and residue fields of rational function fields,

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect.

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

Theorem

Let K be any algebraically closed field of positive characteristic.

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

Theorem

Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

Theorem

Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial,

A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

Theorem

Let K be any algebraically closed field of positive characteristic. Then there exists a valuation v on the rational function field $K(x,y)|K$ whose restriction to K is trivial, such that $(K(x,y),v)$ admits an infinite chain of Galois extensions of degree p and defect p.

Here is what MacLane and Schilling ([MS])

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved.

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field.

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ,

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ.

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$.

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation whose restriction to K is trivial,

MacLane and Schilling, Zariski and Samuel

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation whose restriction to K is trivial, whose value group is Γ

Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

Theorem

Let K be any field. Take Γ to be any non-trivial ordered abelian group of finite rational rank ρ, and K' to be any countably generated extension of K of finite transcendence degree τ. Choose any integer $n > \rho + \tau$. Then the rational function field in n variables over K admits a valuation whose restriction to K is trivial, whose value group is Γ and whose residue field is K'.

A new approach, making essential use of Krasner's Lemma,

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1].

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$,

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of v from K to $K(x)$.

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of v from K to $K(x)$. In fact, the relative algebraic closure of K in $K(x)^h$

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of v from K to $K(x)$. In fact, the relative algebraic closure of K in $K(x)^h$ can be much larger than the henselization K^h contained in it.

The implicit function field

A new approach, making essential use of Krasner's Lemma, was presented in [Ku1]. An important observation we use is that while K is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of v from K to $K(x)$. In fact, the relative algebraic closure of K in $K(x)^h$ can be much larger than the henselization K^h contained in it.

Definition

We call the relative algebraic closure of K in $K(x)^h$ the implicit constant field of $(K(x)|K, v)$ and denote it by $\mathrm{IC}\,(K(x)|K, v)$.

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields.

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{\mathrm{ac}},v)$,

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{\mathrm{ac}},v)$,

$$L^h \;=\; \mathrm{IC}\,(K(x)|K,v)\;.$$

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{\mathrm{ac}},v)$,

$$L^h \;=\; \mathrm{IC}\,(K(x)|K,v)\;.$$

This theorem can be used efficiently to prove the following comprehensive result

Extensions with given implicit constant fields

The following result is proven in [Ku1]:

Theorem

Let $(L|K,v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of v from L to the algebraic closure $L(x)^{\mathrm{ac}} = K(x)^{\mathrm{ac}}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{\mathrm{ac}},v)$,

$$L^h \;=\; \mathrm{IC}\,(K(x)|K,v)\;.$$

This theorem can be used efficiently to prove the following comprehensive result for the extensions of v from K to $K(x)$.

The case of transcendence degree 1

Theorem

Take any valued field (K, v)*,*

The case of transcendence degree 1

Theorem

Take any valued field (K, v)*, an ordered abelian group extension* Γ_0 *of* vK

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0 / vK is a torsion group,

The case of transcendence degree 1

Theorem

Take any valued field (K, v), *an ordered abelian group extension* Γ_0 *of* vK *such that* Γ_0/vK *is a torsion group, and an algebraic extension* k_0 *of* Kv.

The case of transcendence degree 1

Theorem

Take any valued field (K,v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$

The case of transcendence degree 1

Theorem

Take any valued field (K, v)*, an ordered abelian group extension* Γ_0 *of* vK *such that* Γ_0/vK *is a torsion group, and an algebraic extension* k_0 *of* Kv*. Further, take* Γ *to be the abelian group* $\Gamma_0 \oplus \mathbb{Z}$ *endowed with any extension of the ordering of* Γ_0 *.*

The case of transcendence degree 1

Theorem

Take any valued field (K, v)*, an ordered abelian group extension* Γ_0 *of* vK *such that* Γ_0/vK *is a torsion group, and an algebraic extension* k_0 *of* Kv*. Further, take* Γ *to be the abelian group* $\Gamma_0 \oplus \mathbb{Z}$ *endowed with any extension of the ordering of* Γ_0 *.*
Assume first that Γ_0/vK *and* $k_0|Kv$ *are finite.*

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0.
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple.

Theorem

Take any valued field (K, v)*, an ordered abelian group extension* Γ_0 *of* vK *such that* Γ_0/vK *is a torsion group, and an algebraic extension* k_0 *of* Kv*. Further, take* Γ *to be the abelian group* $\Gamma_0 \oplus \mathbb{Z}$ *endowed with any extension of the ordering of* Γ_0 *.*
Assume first that Γ_0/vK *and* $k_0|Kv$ *are finite. If* v *is trivial on* K*, then assume in addition that* $k_0|Kv$ *is simple. Then there is an extension of* v *from* K *to the rational function field* $K(x)$

Theorem

Take any valued field (K,v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0 .
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0.

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0.
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0.
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0.

The case of transcendence degree 1

Theorem

Take any valued field (K,v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0 .
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0 .
Now assume that v is non-trivial on K and that Γ_0/vK and $k_0|Kv$ are countably generated.

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0.
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0.
Now assume that v is non-trivial on K and that Γ_0/vK and $k_0|Kv$ are countably generated. Suppose that at least one of them is infinite or that (K, v) admits an immediate transcendental extension.

The case of transcendence degree 1

Theorem

Take any valued field (K, v), an ordered abelian group extension Γ_0 of vK such that Γ_0/vK is a torsion group, and an algebraic extension k_0 of Kv. Further, take Γ to be the abelian group $\Gamma_0 \oplus \mathbb{Z}$ endowed with any extension of the ordering of Γ_0.
Assume first that Γ_0/vK and $k_0|Kv$ are finite. If v is trivial on K, then assume in addition that $k_0|Kv$ is simple. Then there is an extension of v from K to the rational function field $K(x)$ which has value group Γ and residue field k_0. If v is non-trivial on K, then there is also an extension which has value group Γ_0 and as residue field a rational function field in one variable over k_0.
Now assume that v is non-trivial on K and that Γ_0/vK and $k_0|Kv$ are countably generated. Suppose that at least one of them is infinite or that (K, v) admits an immediate transcendental extension. Then there is an extension of v from K to $K(x)$ which has value group Γ_0 and residue field k_0.

The converse

Theorem

Let $(K(x)|K, v)$ be a valued rational function field.

The converse

Theorem

Let $(K(x)|K,v)$ be a valued rational function field. Then one and only one of the following three cases holds:

The converse

Theorem

Let $(K(x)|K, v)$ be a valued rational function field. Then one and only one of the following three cases holds:

1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$*, where* $\Gamma_0|vK$ *is a finite extension of ordered abelian groups, and* $K(x)v|Kv$ *is finite;*

The converse

Theorem

Let $(K(x)|K, v)$ be a valued rational function field. Then one and only one of the following three cases holds:

1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$, where $\Gamma_0|vK$ is a finite extension of ordered abelian groups, and $K(x)v|Kv$ is finite;

2) $vK(x)/vK$ is finite, and $K(x)v$ is a rational function field in one variable over a finite extension of Kv;

The converse

Theorem

Let $(K(x)|K,v)$ be a valued rational function field. Then one and only one of the following three cases holds:

1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$, *where* $\Gamma_0|vK$ *is a finite extension of ordered abelian groups, and* $K(x)v|Kv$ *is finite;*

2) $vK(x)/vK$ *is finite, and* $K(x)v$ *is a rational function field in one variable over a finite extension of* Kv*;*

3) $vK(x)/vK$ *is a torsion group and* $K(x)v|Kv$ *is algebraic.*

The converse

Theorem

Let $(K(x)|K, v)$ be a valued rational function field. Then one and only one of the following three cases holds:

1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$, where $\Gamma_0|vK$ is a finite extension of ordered abelian groups, and $K(x)v|Kv$ is finite;

2) $vK(x)/vK$ is finite, and $K(x)v$ is a rational function field in one variable over a finite extension of Kv;

3) $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic.

In all cases, $vK(x)/vK$ is countable and $K(x)v|Kv$ is countably generated.

Theorem

Let $(K(x)|K,v)$ be a valued rational function field. Then one and only one of the following three cases holds:

1) $vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}$, where $\Gamma_0|vK$ is a finite extension of ordered abelian groups, and $K(x)v|Kv$ is finite;

2) $vK(x)/vK$ is finite, and $K(x)v$ is a rational function field in one variable over a finite extension of Kv;

3) $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic.

In all cases, $vK(x)/vK$ is countable and $K(x)v|Kv$ is countably generated.

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$,

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K, v) is a maximal field,

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K, v) is a maximal field, i.e., it does not admit any nontrivial immediate extension,

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K, v) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions $\Gamma | vK$ and $k | Kv$ such that

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K, v) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions $\Gamma|vK$ and $k|Kv$ such that

$$\operatorname{rr} \Gamma/vK \;+\; \operatorname{trdeg} k|Kv \;\leq\; 1\,.$$

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1,\ldots,x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K,v) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions $\Gamma|vK$ and $k|Kv$ such that

$$\mathrm{rr}\,\Gamma/vK \;+\; \mathrm{trdeg}\,k|Kv \;\leq\; 1\,.$$

Is it possible to find an extension of v to $K(x,y)$

The case of higher transcendence degree

When asking which value group and residue field extensions may appear when v is extended from K to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on n.

But consider the following case: (K, v) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions $\Gamma|vK$ and $k|Kv$ such that

$$\mathrm{rr}\,\Gamma/vK \;+\; \mathrm{trdeg}\,k|Kv \;\leq\; 1\,.$$

Is it possible to find an extension of v to $K(x, y)$ such that $vK(x, y) = \Gamma$ and $K(x, y) = k$?

The immediate extension problem

From the extension theorem for the transcendence degree 1 case

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of v to $K(x, y)$.

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of v to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of v to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee that (L, v) admits a maximal immediate extension of infinite transcendence degree.

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of v to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee that (L, v) admits a maximal immediate extension of infinite transcendence degree. Through a far reaching generalization of a construction method introduced by MacLane and Schilling in [MS],

The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of v to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of v to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee that (L, v) admits a maximal immediate extension of infinite transcendence degree. Through a far reaching generalization of a construction method introduced by MacLane and Schilling in [MS], such criteria were given in [BK2].

Transcendence degree of immediate extensions

Theorem

Take an extension $(L|K,v)$ of finite transcendence degree ≥ 0, with v nontrivial on L. Assume that one of the following four cases holds:
<u>valuation-transcendental case</u>: vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;
<u>value-algebraic case</u>: vL/vK contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq vL$ containing vK such that Γ/vK is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of Kv;
<u>residue-algebraic case</u>: Lv contains elements of arbitrarily high degree over Kv;
<u>separable-algebraic case</u>: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ is infinite.
Then each maximal immediate extension of (L,v) has infinite transcendence degree over L.

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series.

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem.

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead,

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski.

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky's theorems says that if $(K(x)|K,v)$ is immediate,

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky's theorems says that if $(K(x)|K, v)$ is immediate, then x is the limit of a pseudo Cauchy sequence

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky's theorems says that if $(K(x)|K,v)$ is immediate, then x is the limit of a pseudo Cauchy sequence that has no limit in K.

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky's theorems says that if $(K(x)|K,v)$ is immediate, then x is the limit of a pseudo Cauchy sequence that has no limit in K. Further, Kaplansky distinguishes pseudo Cauchy sequence of algebraic type

Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky's theorems says that if $(K(x)|K, v)$ is immediate, then x is the limit of a pseudo Cauchy sequence that has no limit in K. Further, Kaplansky distinguishes pseudo Cauchy sequence of algebraic type and those of transcendental type.

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic),

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K,

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}}, v)$ becomes immediate

Valuation algebraic extensions

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}},v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Valuation algebraic extensions

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}},v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties,

Valuation algebraic extensions

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}},v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}}, v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

This observation is extensively used in our main theorem

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}}, v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

This observation is extensively used in our main theorem on the extensions of a valuation from K to $K(x_1, \ldots, x_n)$ (see [Ku1, BK1]).

Valuation algebraic extensions

It should be noted that if $(K(x)|K,v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}},v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

This observation is extensively used in our main theorem on the extensions of a valuation from K to $K(x_1,\ldots,x_n)$ (see [Ku1, BK1]). (Due to the failure of simple induction,

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{\mathrm{ac}}(x)|K^{\mathrm{ac}}, v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

This observation is extensively used in our main theorem on the extensions of a valuation from K to $K(x_1, \ldots, x_n)$ (see [Ku1, BK1]). (Due to the failure of simple induction, this theorem needs several case distinctions

Valuation algebraic extensions

It should be noted that if $(K(x)|K, v)$ is an extension such that $vK(x)/vK$ is a torsion group and $K(x)v|Kv$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure K^{ac} of K, the extension $(K^{ac}(x)|K^{ac}, v)$ becomes immediate and x will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in K^{ac}.

This observation is extensively used in our main theorem on the extensions of a valuation from K to $K(x_1, \ldots, x_n)$ (see [Ku1, BK1]). (Due to the failure of simple induction, this theorem needs several case distinctions and is too long to be put on one slide.)

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$.

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v),

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ / vK is torsion,

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ / vK is torsion, and an algebraic extension $k|Kv$.

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ / vK is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ/vK is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

If at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ/vK is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

If at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K, v) admits an immediate extension of transcendence degree n,

Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from K to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field (K, v), an extension Γ of vK such that Γ / vK is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

If at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K, v) admits an immediate extension of transcendence degree n, then there is an extension of v from K to F with $vF = \Gamma$ and $Fv = k$.

The converse

In [BK2] the following is proven:

The converse holds if $\operatorname{char} Kv = 0$*, or if* $\operatorname{char} Kv = p$*,*

The converse

In [BK2] the following is proven:

The converse holds if $\mathrm{char}\, Kv = 0$*, or if* $\mathrm{char}\, Kv = p$*,* vK *is* p*-divisible*

The converse

In [BK2] the following is proven:

The converse holds if $\operatorname{char} Kv = 0$*, or if* $\operatorname{char} Kv = p$*,* vK *is* p*-divisible and* Kv *is perfect.*

In [BK2] the following is proven:

The converse holds if $\operatorname{char} Kv = 0$*, or if* $\operatorname{char} Kv = p$*,* vK *is* p*-divisible and* Kv *is perfect.*

For this class of fields (which includes the tame fields

The converse

In [BK2] the following is proven:

The converse holds if char $Kv = 0$*, or if* char $Kv = p$*,* vK *is* p*-divisible and* Kv *is perfect.*

For this class of fields (which includes the tame fields but also fields that allow nontrivial defect extensions)

The converse

In [BK2] the following is proven:

The converse holds if char $Kv = 0$*, or if* char $Kv = p$*,* vK *is* p*-divisible and* Kv *is perfect.*

For this class of fields (which includes the tame fields but also fields that allow nontrivial defect extensions) quite a bit of results are known (see e.g. [BK3]). Valued fields not in this class pose even harder problems.

The power series problem

Take any field k and a power series y in x

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$.

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$?

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\operatorname{char} k = 0$, the answer is not difficult.

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\operatorname{char} k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field,

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\mathrm{char}\, k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\mathrm{char}\, k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series.

The power series problem

Take any field k and a power series y in x with coefficients in k^{ac} and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y),v)$ where v is an extension of the x-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\operatorname{char} k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series.

The answer is provided by homogeneous sequences.

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma.

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\, K \;\text{ and }\; \tau a \neq \sigma a\}\,.$$

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\}\,.$$

Theorem

Take $K(a)|K$ to be a separable-algebraic extension

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\}\,.$$

Theorem

Take $K(a)|K$ to be a separable-algebraic extension and $(K(a,b),v)$ to be any valued field extension of (K,v) such that

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a-\sigma a)\mid \sigma,\tau\in\mathrm{Gal}\,K \text{ and } \tau a\neq\sigma a\}\,.$$

Theorem

Take $K(a)|K$ to be a separable-algebraic extension and $(K(a,b),v)$ to be any valued field extension of (K,v) such that

$$v(b-a) \;>\; \mathrm{kras}(a,K)\,.$$

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a-\sigma a)\mid \sigma,\tau\in \mathrm{Gal}\,K \;\text{ and }\; \tau a\neq\sigma a\}\,.$$

Theorem

Take $K(a)|K$ to be a separable-algebraic extension and $(K(a,b),v)$ to be any valued field extension of (K,v) such that

$$v(b-a) \;>\; \mathrm{kras}(a,K)\,.$$

Then for every extension of v from $K(a,b)$ to its algebraic closure $K(a,b)=K(b)^{\mathrm{ac}}$,

A variant of Krasner's Lemma

The main idea is to employ the following form of Krasner's Lemma. We define

$$\mathrm{kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\}\,.$$

Theorem

Take $K(a)|K$ to be a separable-algebraic extension and $(K(a,b),v)$ to be any valued field extension of (K,v) such that

$$v(b-a) \;>\; \mathrm{kras}(a,K)\,.$$

Then for every extension of v from $K(a,b)$ to its algebraic closure $\widetilde{K(a,b)} = K(b)^{\mathrm{ac}}$, the element a lies in the henselization of $(K(b),v)$ in $(K(b)^{\mathrm{ac}},v)$.

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v).

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v). We say that a is strongly homogeneous over (K, v)

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v). We say that a is strongly homogeneous over (K, v) if $a \in K^{\mathrm{sep}} \setminus K$,

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v). We say that a is strongly homogeneous over (K, v) if $a \in K^{\mathrm{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique,

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v). We say that a is strongly homogeneous over (K, v) if $a \in K^{\text{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va = \operatorname{kras}(a, K) .$$

Homogeneous approximations

Let (K, v) be any valued field and a, b elements in some valued field extension (L, v) of (K, v). We say that a is strongly homogeneous over (K, v) if $a \in K^{\text{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va = \text{kras}(a, K).$$

We call $a \in L$ a homogeneous approximation of b over K

Let (K,v) be any valued field and a,b elements in some valued field extension (L,v) of (K,v). We say that a is strongly homogeneous over (K,v) if $a \in K^{\mathrm{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va = \mathrm{kras}(a,K) .$$

We call $a \in L$ a homogeneous approximation of b over K if there is some $d \in K$ such that $a - d$ is strongly homogeneous over K

Let (K,v) be any valued field and a,b elements in some valued field extension (L,v) of (K,v). We say that a is strongly homogeneous over (K,v) if $a \in K^{\mathrm{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va \;=\; \mathrm{kras}(a,K)\,.$$

We call $a \in L$ a homogeneous approximation of b over K if there is some $d \in K$ such that $a-d$ is strongly homogeneous over K and

$$v((b-d)-(a-d)) \;=\; v(b-a) \;>\; v(b-d) \;\geq\; vb\,.$$

Let (K,v) be any valued field and a,b elements in some valued field extension (L,v) of (K,v). We say that a is strongly homogeneous over (K,v) if $a \in K^{\mathrm{sep}} \setminus K$, the extension of v from K to $K(a)$ is unique, and

$$va \;=\; \mathrm{kras}(a,K)\,.$$

We call $a \in L$ a homogeneous approximation of b over K if there is some $d \in K$ such that $a-d$ is strongly homogeneous over K and

$$v((b-d)-(a-d)) \;=\; v(b-a) \;>\; v(b-d) \;\geq\; vb\,.$$

It then follows that $va = vb$ and $v(a-d) = v(b-d)$.

Lemma

If $a \in L$ is a homogeneous approximation of b

Applying Krasner's Lemma

Lemma

If $a \in L$ is a homogeneous approximation of b then a lies in the henselization of $K(b)$

Applying Krasner's Lemma

Lemma

If $a \in L$ is a homogeneous approximation of b then a lies in the henselization of $K(b)$ w.r.t. every extension of the valuation v from $K(a,b)$ to $K(b)^{\mathrm{ac}}$.

Homogeneous sequences

Let $(K(x)|K, v)$ be any extension of valued fields.

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\text{ac}}$.

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$.

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1,\ldots,n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i\in S}$$

of elements in K^{ac} will be called a homogeneous sequence for x

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1,\ldots,n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i\in S}$$

of elements in K^{ac} will be called a homogeneous sequence for x if the following condition is satisfied for all $i \in S$:

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0,\ldots,a_{i-1})$

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i \in S}$$

of elements in K^{ac} will be called a homogeneous sequence for x if the following condition is satisfied for all $i \in S$:

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0, \ldots, a_{i-1})$

(where we set $a_0 := 0$).

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1,\ldots,n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i\in S}$$

of elements in K^{ac} will be called a homogeneous sequence for x if the following condition is satisfied for all $i \in S$:

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0,\ldots,a_{i-1})$

(where we set $a_0 := 0$). Then from the definition of "strongly homogeneous" it follows that $a_i \notin K(a_0,\ldots,a_{i-1})^h$.

Homogeneous sequences

Let $(K(x)|K,v)$ be any extension of valued fields. We fix an extension of v to $K(x)^{\mathrm{ac}}$. Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \emptyset$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i \in S}$$

of elements in K^{ac} will be called a homogeneous sequence for x if the following condition is satisfied for all $i \in S$:

(HS) $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over $K(a_0, \ldots, a_{i-1})$

(where we set $a_0 := 0$). Then from the definition of "strongly homogeneous" it follows that $a_i \notin K(a_0, \ldots, a_{i-1})^h$. We set

$$K_{\mathfrak{S}} \;:=\; K(a_i \mid i \in S)\;.$$

Results

Assume that $(a_i)_{i \in S}$ is a homogeneous sequence for x over K.

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$.

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K^h_{\mathfrak{S}} = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x,

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x, $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate,

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x, $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate, and

$$K_{\mathfrak{S}}^h = \mathrm{IC}\,(K(x)|K, v)\,.$$

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x, $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate, and

$$K_{\mathfrak{S}}^h = \mathrm{IC}\left(K(x)|K, v\right).$$

Further, $K_{\mathfrak{S}}v$ is the relative algebraic closure of Kv in $K(x)v$,

Results

Assume that $(a_i)_{i\in S}$ is a homogeneous sequence for x over K. Then the following assertions hold.

- $K_{\mathfrak{S}} \subset K(x)^h$.
- For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)^h$.
- If $S = \{1, \ldots, n\}$, then $K_{\mathfrak{S}}^h = K(a_n)^h$.

Theorem

Assume that $\mathfrak{S} = (a_i)_{i\in S}$ is a homogeneous sequence for x over K with $S = \mathbb{N}$. Then $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}}, v)$ with pseudo limit x, $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}}, v)$ is immediate, and

$$K_{\mathfrak{S}}^h = \mathrm{IC}\,(K(x)|K, v)\,.$$

Further, $K_{\mathfrak{S}}v$ is the relative algebraic closure of Kv in $K(x)v$, and $vK_{\mathfrak{S}}$ is the relative divisible closure of vK in $vK(x)$.

Connection with tame extensions

Proposition

Suppose that (K, v) is henselian.

Connection with tame extensions

Proposition

Suppose that (K, v) is henselian.

1) If $\mathfrak{S}$ is a homogeneous sequence over (K, v),

Connection with tame extensions

Proposition

Suppose that (K, v) is henselian.

1) If $\mathfrak{S}$ is a homogeneous sequence over (K, v), then $K_{\mathfrak{S}}$ is a tame extension of K.

Connection with tame extensions

Proposition

Suppose that (K, v) is henselian.

1) If $\mathfrak{S}$ is a homogeneous sequence over (K, v), then $K_{\mathfrak{S}}$ is a tame extension of K.

2) An element $b \in K^{\mathrm{ac}}$ belongs to a tame extension of (K, v)

Connection with tame extensions

Proposition

Suppose that (K, v) is henselian.

1) If $\mathfrak{S}$ is a homogeneous sequence over (K, v), then $K_{\mathfrak{S}}$ is a tame extension of K.

2) An element $b \in K^{\mathrm{ac}}$ belongs to a tame extension of (K, v) if and only if there is a finite homogeneous sequence $a_1, \ldots, a_k$ for b over (K, v)

Proposition

Suppose that (K, v) is henselian.

1) If $\mathfrak{S}$ is a homogeneous sequence over (K, v), then $K_{\mathfrak{S}}$ is a tame extension of K.

2) An element $b \in K^{\mathrm{ac}}$ belongs to a tame extension of (K, v) if and only if there is a finite homogeneous sequence $a_1, \dots, a_k$ for b over (K, v) such that $b \in K(a_k)$.

References

- Blaszczok, A. – Kuhlmann F.-V.: *Corrections and Notes for the Paper "Value groups, residue fields and bad places of rational function fields"*, Trans. Amer. Math. Soc. **367** (2015), 4505–4515
- Blaszczok, A. - Kuhlmann, F.-V.: *Algebraic independence of elements in completions and maximal immediate extensions of valued fields*, J. Alg. **425** (2015), 179–214
- Blaszczok, A. - Kuhlmann, F.-V.: *On maximal immediate extensions of valued fields*, Mathematische Nachrichten **290** (2017), 7–18
- Brown, R. – Merzel, J.: *The space of real places on* $\mathbb{R}(x, y)$, Ann. Math. Silesianae **32** (2018), 99–131
- Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

References

- Kuhlmann F.-V.: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. **356** (2004), 4559–4600
- Kuhlmann F.-V.: *Defect*, in: Commutative Algebra – Noetherian and non-Noetherian perspectives, Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I. (Eds.), Springer 2011
- MacLane, S. – Schilling, O.F.G.: *Zero-dimensional branches of rank 1 on algebraic varieties*, Annals of Math. **40** (1939), 507–520
- Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin (1960)

Preprints and further information

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html