

# FIXED POINT THEOREMS FOR SPACES WITH A TRANSITIVE RELATION

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ABSTRACT. We present general fixed point theorems for spaces that are equipped with a transitive relation. We apply them to prove corresponding theorems for ultrametric spaces, topological spaces, complete lattices, and ordered abelian groups and fields.

## 1. INTRODUCTION

In our papers [1] and [3] we have developed a general framework for fixed point theorems that work with functions which are in some way contracting, or have other properties that allow an application of Zorn's Lemma. Throughout this paper we will work in (ZFC) as we want to have Zorn's Lemma at hand. We will not be concerned with the question whether some of our theorems can be proved in (ZF) plus a weaker form of the axiom of choice; this is left to a later investigation. (Unfortunately, several instances we found in the literature which claimed to prove that certain fixed point theorems imply the axiom of choice were riddled with gaps or obscurity.)

We consider **ball spaces**  $(X, \mathcal{B})$ , which are given by nonempty sets  $X$  with a nonempty set  $\mathcal{B}$  of distinguished nonempty subsets  $B$  of  $X$ . The completeness property we need for our fixed point theorems is inspired by the spherical completeness of ultrametric spaces. A **nest of balls** in  $(X, \mathcal{B})$  is a nonempty totally ordered subset of  $(\mathcal{B}, \subseteq)$ . A ball space  $(X, \mathcal{B})$  is called **spherically complete** if every nest of balls has a nonempty intersection.

Note that  $\mathcal{B}$ , a subset of the power set  $\mathcal{P}(X)$ , is a partially ordered set under reverse inclusion. However, spherical completeness in its simplest form, which we have defined here, does not mean that  $\mathcal{B}$  is **inductively ordered**, i.e., that every increasing chain has an upper bound. But this would be true if for instance every singleton, or every intersection over a nonempty descending chain of balls, is a ball.

See [1] for a collection of fixed point theorems we have proved for ball spaces, their applications, as well as back ground information and references.

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*Date:* June 16, 2015.

*2010 Mathematics Subject Classification.* Primary 03E04; Secondary 54H25, 12J20.

*Key words and phrases.* fixed point theorem, fixed points on graphs, ultrametric spaces, topological spaces, complete lattices, ordered abelian groups and fields.

In this note we add a transitive relation  $R$  to the set  $X$ . We do not require that the relation has any other properties, such as reflexivity, symmetry or antisymmetry. We write  $xRy$  if  $x, y \in X$  and  $x$  is in relation with  $y$ .

If  $X$  is the set of vertices of a graph (directed or not), and if we define  $xRy$  to mean that there is a path of finite length from  $x$  to  $y$ , then  $R$  is a transitive relation. The same is true when we also allow paths of infinite length. Fixed point theorems for spaces with additional graph structure have been abundant in the more recent literature, so it seems to be worthwhile to adapt our above mentioned unified approach to this additional structure.

**For the remainder of this introduction, we assume that  $(X, \mathcal{B})$  is a ball space,  $R$  is a transitive relation on  $X$ , and  $f : X \rightarrow X$  is a function.** We will write  $fx$  in place of  $f(x)$ .

A set  $S \subseteq X$  is called  **$f$ -closed** if  $f(S) \subseteq S$ .

**Theorem 1.** *Assume that the following conditions hold:*

(A<sub>1</sub>) *For every non-singleton  $f$ -closed ball  $B \in \mathcal{B}$  and every  $x \in B$ , there is an  $f$ -closed ball  $B' \subsetneq B$  and some  $x' \in B'$  such that  $xRx'$ .*

(B<sub>1</sub>) *If  $\kappa$  is a regular cardinal,  $(B_\nu)_{\nu < \kappa}$  is a nest of  $f$ -closed balls, and  $x_\nu \in B_\nu$  are elements such that  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \kappa$ , then  $\bigcap_{\nu < \kappa} B_\nu$  contains an  $f$ -closed ball  $B'$  and some  $x' \in B'$  such that  $x_\nu Rx'$  whenever  $\nu < \kappa$ .*

*Then for every  $x$  in any  $f$ -closed ball  $B \in \mathcal{B}$  there is a fixed point  $z \in B$  of  $f$  such that  $xRz$ .*

The intersection over a collection of  $f$ -closed sets is again  $f$ -closed. We can use this fact to simplify condition (B<sub>1</sub>) for an important class of ball spaces. A ball space is called **intersection closed** if the intersection of every collection of balls is again a ball, provided it is nonempty. For example, in every topological space the nonempty closed subsets form an intersection closed ball space.

**Remark.** Assume that  $(X, \mathcal{B})$  is intersection closed and the following condition holds:

(B'<sub>1</sub>) *If  $\kappa$  is a regular cardinal,  $(B_\nu)_{\nu < \kappa}$  is a nest of  $f$ -closed balls, and  $x_\nu \in B_\nu$  are elements such that  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \kappa$ , then  $\bigcap_{\nu < \kappa} B_\nu$  contains an element  $x'$  such that  $x_\nu Rx'$  whenever  $\nu < \kappa$ .*

Then also condition (B<sub>1</sub>) holds.

We will use Theorem 1 to prove the next theorem, which in its hypothesis encodes the notion of a contracting function in the ball space setting. Given a function  $f : X \rightarrow X$ , a ball  $B \in \mathcal{B}$  will be called  **$f$ -contracting** if it is either a singleton containing a fixed point, or  $f(B) \subsetneq B$  holds.

**Theorem 2.** *Assume that the following conditions hold:*

(A<sub>2</sub>) For every  $f$ -contracting ball  $B \in \mathcal{B}$  and every  $x \in B$ , the image  $f(B)$  contains an  $f$ -contracting ball  $B'$  and an element  $x' \in B'$  such that  $xRx'$ .

(B<sub>2</sub>) If  $\kappa$  is a regular cardinal,  $(B_\nu)_{\nu < \kappa}$  is a nest of  $f$ -contracting balls, and  $x_\nu \in B_\nu$  are elements such that  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \kappa$ , then  $\bigcap_{\nu < \kappa} B_\nu$  contains an  $f$ -contracting ball  $B'$  and some  $x' \in B'$  such that  $x_\nu Rx'$  whenever  $\nu < \kappa$ .

Then for every  $x$  in any  $f$ -contracting ball there is a fixed point  $z$  of  $f$  such that  $xRz$ .

The following theorem gives conditions for the existence of a unique fixed point.

**Theorem 3.** Assume that the following conditions hold:

(A<sub>3</sub>) For every  $f$ -contracting ball  $B \in \mathcal{B}$  and every  $x \in B$ , the image  $f(B)$  is again an  $f$ -contracting ball and contains an element  $x' \in B'$  such that  $xRx'$ .

(B<sub>3</sub>) If  $\kappa$  is a regular cardinal,  $(B_\nu)_{\nu < \kappa}$  is a nest of  $f$ -contracting balls, and  $x_\nu \in B_\nu$  are elements such that  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \kappa$ , then  $\bigcap_{\nu < \kappa} B_\nu$  is an  $f$ -contracting ball and contains an element  $x'$  such that  $x_\nu Rx'$  whenever  $\nu < \kappa$ .

Then every  $f$ -contracting ball  $B$  contains a unique fixed point  $z$  of  $f$ , and we have that  $xRz$  for all  $x \in B$ .

A function  $f : X \mapsto X$  will be called  **$R$ -compatible** if  $xRfx$  for all  $x \in X$ . Condition (A<sub>3</sub>) can be replaced by the condition

(A'<sub>3</sub>) The function  $f$  is  $R$ -compatible and the image  $f(B)$  of every  $f$ -contracting ball  $B \in \mathcal{B}$  is again an  $f$ -contracting ball.

Finally, we present a fourth theorem that generalizes Theorem 4 of [1]. It is helpful for applications in ordered abelian groups and fields.

A function  $f$  on a ball space  $(X, \mathcal{B})$  will be called **contracting on orbits** if there is a function

$$X \ni x \mapsto B_x \in \mathcal{B}$$

such that for all  $x \in X$ , the following conditions hold:

(SC1)  $x \in B_x$ ,

(SC2)  $B_{fx} \subseteq B_x$ , and if  $x \neq fx$ , then  $B_{f^i x} \subsetneq B_x$  for some  $i \geq 1$ .

Note that (SC1) and (SC2) imply that  $f^i x \in B_x$  for all  $i \geq 0$ .

**Theorem 4.** Assume that the following conditions hold:

(A<sub>4</sub>) The function  $f$  is  $R$ -compatible and contracting on orbits.

(B<sub>4</sub>) If  $\lambda$  is a limit ordinal and  $(B_{x_\nu})_{\nu < \lambda}$  is a nest such that  $x_{\nu+1} = fx_\nu$  and  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \lambda$ , then  $\bigcap_{\nu < \lambda} B_{x_\nu}$  contains an element  $x_\lambda$  such that  $B_{x_\lambda} \subseteq \bigcap_{\nu < \lambda} B_{x_\nu}$  and  $x_\nu Rx_\lambda$  whenever  $\nu < \lambda$ .

Then for every  $x \in X$  there is a fixed point  $z$  of  $f$  such that  $xRz$ .

The proofs of these theorems will be given in Section 2. In the remaining sections, we apply our theorems to prove corresponding fixed point theorems for various types of spaces. In Section 3, we apply Theorem 1 to compact topological spaces. In Section 4, we use Theorem 1 to give a fixed point theorem for order preserving functions on complete lattices. In Section 5, we will then derive a fixed point theorem for ultrametric spaces from Theorem 2. We will conclude the paper in Section 6 with a fixed point theorem for ordered abelian groups and fields that we obtain from Theorem 4.

## 2. PROOF OF THE MAIN THEOREMS

Take a ball space  $(X, \mathcal{B})$ , a transitive relation  $R$  on  $X$ , and a function  $f : X \rightarrow X$ . For the proof of our first theorem, we take  $\mathcal{B}^f$  to be the set of all  $f$ -closed balls in  $\mathcal{B}$ . Then we introduce a partial order on the set

$$\mathcal{S} = \{(B, x) \mid B \in \mathcal{B}^f \text{ and } x \in B\}$$

as follows:

$$(B, x) < (B^*, x^*) \Leftrightarrow B^* \subsetneq B \text{ and } xRx^* .$$

**Lemma 5.** *Take a ball  $B \in \mathcal{B}^f$  and an element  $x \in B$ . Set*

$$\mathcal{S}_{(B,x)} = \{(B^*, x^*) \in \mathcal{S} \mid (B, x) \leq (B^*, x^*)\} \subseteq \mathcal{S} .$$

*If condition  $(B_1)$  of Theorem 1 holds, then  $(\mathcal{S}_{(B,x)}, <)$  is inductively ordered. If in addition  $(X, \mathcal{B})$  is intersection closed, then  $(\mathcal{S}_{(B,x)}, <)$  is chain complete (that is, every chain has a supremum).*

*Proof:* We take any chain  $\mathcal{C}$  in  $(\mathcal{S}, <)$ . As the balls in this chain form a strictly descending sequence, the length of  $\mathcal{C}$  is bounded by the cardinality of  $X$ . We take  $\kappa$  to denote the cofinality of  $\mathcal{C}$ . Then  $\kappa$  is a regular cardinal. We may assume that  $\kappa$  is infinite since otherwise,  $\mathcal{C}$  has a last element which then is a supremum of  $\mathcal{C}$ .

We choose a subchain  $((B_\nu, x_\nu))_{\nu < \kappa}$  that is cofinal in  $\mathcal{C}$ . By condition  $(B_1)$  of Theorem 1,  $\bigcap_{\nu < \kappa} B_\nu$  contains an  $f$ -closed ball  $B'$  and some  $x' \in B'$  such that  $x_\nu R x'$  whenever  $\nu < \kappa$ . Then  $(B', x')$  is an upper bound for  $\mathcal{C}$ . This shows that  $(\mathcal{S}_{(B,x)}, <)$  is inductively ordered.

If in addition  $(X, \mathcal{B})$  is intersection closed,  $\bigcap_{\nu < \kappa} B_\nu$  is a ball (as it contains  $x'$  and is thus nonempty). Being an intersection of  $f$ -closed sets, also  $\bigcap_{\nu < \kappa} B_\nu$  is  $f$ -closed. Therefore,  $(\bigcap_{\nu < \kappa} B_\nu, x') \in \mathcal{S}_{(B,x)}$ , and it is a supremum of  $\mathcal{C}$ . This proves our second assertion.  $\square$

In order to prove Theorem 1, we take an  $f$ -closed ball  $B \in \mathcal{B}$  and an element  $x \in B$ . We assume that conditions  $(A_1)$  and  $(B_1)$  hold. Then by Lemma 5,  $(\mathcal{S}_{(B,x)}, <)$  is inductively ordered. By Zorn's Lemma, it admits a maximal element  $(B^*, z)$ . If  $B^*$  is not a singleton, then by condition  $(A_1)$ , there is an  $f$ -closed ball  $B' \subsetneq B$  and an element  $x' \in B'$  such that  $zRx'$ . It follows that  $(B^*, z) < (B', x')$ , which contradicts the maximality of  $(B^*, z)$ .

Therefore,  $B^*$  is a singleton. As  $B^*$  is  $f$ -closed, it follows that  $z$  is a fixed point of  $f$ . If  $(B^*, z) \neq (B, x)$ , then the fact that  $(B^*, z) \in \mathcal{S}_{(B,x)}$  implies that  $xRz$ . If  $(B^*, z) = (B, x)$  (or equivalently,  $\mathcal{S}_{(B,x)}$  has only one element), then  $(B, x)$  is a singleton and  $z = x$ ; setting  $\kappa = 1$ ,  $B_0 = B$  and  $x_0 = x$ , we obtain from  $(B_1)$  that  $xRx$ . This completes the proof of Theorem 1.  $\square$

In order to prove Theorem 2, we apply Theorem 1 to the ball space  $(X, \mathcal{B}^{cf})$ , where  $\mathcal{B}^{cf}$  is the collection of all  $f$ -contracting balls. We note that all  $f$ -contracting balls and hence all balls in  $\mathcal{B}^{cf}$  are  $f$ -closed. Take a nonsingleton  $B \in \mathcal{B}^{cf}$  and  $x \in B$ . Then  $f(B) \subsetneq B$ , and by  $(A_2)$  there is an  $f$ -contracting ball  $B' \subseteq f(B) \subsetneq B$  and some  $x' \in B'$  such that  $xRx'$ . This proves that  $(A_1)$  holds.

In a similar way, condition  $(B_2)$  implies condition  $(B_1)$ . Therefore, Theorem 2 follows from Theorem 1.  $\square$

We turn to the proof of Theorem 3. Now we take  $\mathcal{B}^f$  to be the subset of  $\mathcal{B}$  consisting of all  $f$ -contracting balls. Using this new meaning of  $\mathcal{B}^f$ , we define  $\mathcal{S}$  and  $\mathcal{S}_{(B,x)}$  as before.

We assume that conditions  $(A_3)$  and  $(B_3)$  hold, and take an  $f$ -contracting ball  $B \in \mathcal{B}$ . This time, we will replace the use of Zorn's Lemma by that of transfinite induction. We build a chain of elements of  $\mathcal{S}_{(B,x)}$  as follows. We set  $B_1 := B$  and  $x_1 := x$ . Having constructed  $(B_\nu, x_\nu)$  for an ordinal  $\nu$ , we stop the construction if  $B_\nu$  is a singleton; otherwise, making use of condition  $(A_3)$  we set  $B_{\nu+1} := f(B_\nu)$ , which is again an  $f$ -contracting ball and is properly contained in  $B_\nu$ , and choose  $x_{\nu+1}$  in  $f(B_\nu)$  such that  $x_\nu Rx_{\nu+1}$ . We obtain that  $(B_\nu, x_\nu) < (B_{\nu+1}, x_{\nu+1})$ .

If  $\lambda$  is a limit ordinal and we have constructed  $B_\nu$  for all  $\nu < \lambda$ , we proceed as follows. We take  $\kappa$  to be the cofinality of  $\lambda$  and choose a cofinal subsequence  $(B_{\nu_\alpha}, x_{\nu_\alpha})_{\alpha < \kappa}$ . By condition  $(B_3)$ ,  $B_\lambda := \bigcap_{\alpha < \kappa} B_{\nu_\alpha} = \bigcap_{\nu < \lambda} B_\nu$  is an  $f$ -contracting ball, and there is some  $x' \in \bigcap_{\alpha < \kappa} B_{\nu_\alpha} = B_\lambda$  such that  $x_{\nu_\alpha} Rx'$  for all  $\alpha < \kappa$ . As in the proof of the above lemma it follows that  $x_\nu Rx'$  for all  $\nu < \lambda$ . So we can set  $x_\lambda := x'$  to obtain that  $(B_\nu, x_\nu) < (B_\lambda, x_\lambda)$  for all  $\nu < \lambda$ .

The chain of balls thus constructed is strictly descending. Hence there must be an ordinal  $\nu^*$ , bounded by the cardinality of  $X$ , where the construction stops. Then  $B_{\nu^*}$  must be a singleton, that is,  $B_{\nu^*} = \{x_{\nu^*}\}$ . As  $B_{\nu^*}$  is also an  $f$ -contracting ball,  $x_{\nu^*}$  is a fixed point of  $f$ . If  $x_{\nu^*} \neq y \in B$ , then  $y \notin B_{\nu^*}$ , which means that there is some  $\mu < \nu^*$  such that  $y \in B_\mu$ , but  $y \notin B_{\mu+1} = f(B_\mu)$ . This shows that  $y$  cannot be a fixed point of  $f$ . Therefore,  $x_{\nu^*}$  is the unique fixed point of  $f$ . As in the proof of Theorem 1 it is shown that  $xRx_{\nu^*}$ . Since  $x \in B$  was arbitrary, this holds for all  $x \in B$ .  $\square$

Finally, we turn to the proof of Theorem 4. We assume that conditions  $(A_4)$  and  $(B_4)$  hold, and take some  $x \in X$ . We consider the set  $\mathcal{S}_x$  that

consists of all nests of the form  $(B_{x_\nu})_{\nu < \lambda}$ , where  $\lambda$  is any ordinal, such that  $x_{\nu+1} = fx_\nu$  and  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \lambda$ . This set is nonempty, as it contains the nest  $\{B_x\}$ . We introduce a partial order on  $\mathcal{S}_x$  by defining that  $\mathcal{C} \leq \mathcal{C}'$  if and only if the sequence  $\mathcal{C}$  is an initial segment of  $\mathcal{C}'$ .

Take an ascending chain of nests in  $\mathcal{S}_x$  of length  $\kappa$ , where  $\kappa$  is a cardinal. Then it follows from condition (SC2) that there is a strictly descending chain of length  $\kappa$  of balls in  $X$ . This shows that  $\kappa$  is bounded by the cardinality of  $X$ . It follows that the union over any ascending chain of nests in  $\mathcal{S}_x$  is again a nest in  $\mathcal{S}_x$ , that is,  $\mathcal{S}_x$  is chain complete. Hence by Zorn's Lemma,  $\mathcal{S}_x$  admits a maximal nest  $\mathcal{N}$ .

Suppose that  $\mathcal{N}$  is of the form  $(B_{x_\nu})_{\nu < \lambda}$  with  $\lambda$  a limit ordinal. But then condition (B<sub>4</sub>) of Theorem 4 shows the existence of a ball  $B_{x_\lambda}$  such that  $\mathcal{N} \cup \{B_{f^k x_\lambda} \mid k \in \mathbb{N}\}$  properly contains  $\mathcal{N}$ . Since  $f$  is  $R$ -compatible, we have that  $x_\lambda R f^k x_\lambda$  for all  $k \in \mathbb{N}$ , and by the transitivity of  $R$ , we obtain that  $x_\nu R f^k x_\lambda$  for all  $\nu < \lambda$  and all  $k \in \mathbb{N}$ . This shows that  $\mathcal{N} \cup \{B_{x_\lambda}\} \in \mathcal{S}_x$ , contradicting the maximality of  $\mathcal{N}$ . Therefore,  $\mathcal{N}$  must contain a smallest ball  $B_z$ .

We wish to show that  $z$  is a fixed point of  $f$ . If we would have that  $z \neq fz$ , then by (SC2),  $B_{f^i z} \subsetneq B_z \subseteq \bigcap \mathcal{N}$  for some  $i \geq 1$ , and the nest  $\mathcal{N} \cup \{B_{f^k z} \mid k \in \mathbb{N}\}$  would again properly contain  $\mathcal{N}$ . As before, we would obtain a contradiction to the maximality of  $\mathcal{N}$ .

Hence,  $z$  is a fixed point of  $f$ . Since  $\mathcal{N} \in \mathcal{S}_x$ , we also have that  $xRz$ . This completes the proof of Theorem 4.  $\square$

### 3. TOPOLOGICAL SPACES

In this section we will consider topological spaces  $X$ , equipped with a transitive relation  $R$ . A ball space is associated with  $X$  by taking  $\mathcal{B}$  to be the collection of all nonempty closed sets.

We will say that  $X$  is  *$R$ -compact* if for every regular cardinal  $\kappa$ , every descending chain  $(B_\nu)_{\nu < \kappa}$  of nonempty closed sets, and any choice of elements  $x_\nu \in B_\nu$  such that  $x_\mu Rx_\nu$  whenever  $\mu < \nu < \kappa$ , the intersection  $\bigcap_{\nu < \kappa} B_\nu$  contains an element  $z$  such that  $x_\nu Rz$  for all  $\nu < \kappa$ . If  $xRy$  for all  $x, y \in X$ , then every  $R$ -compact topological space is compact.

**Theorem 6.** *Take a topological space  $X$  with a transitive relation  $R$  on  $X$  and a function  $f : X \rightarrow X$  such that for every non-singleton closed  $f$ -closed set  $B$  and every  $x \in B$  there is a closed  $f$ -closed set  $B' \subsetneq B$  and some  $x' \in B'$  such that  $xRx'$ . If  $X$  is  $R$ -compact, then for every  $x \in X$  there is a fixed point  $z$  of  $f$  that satisfies  $xRz$ .*

*Proof:* We take  $\mathcal{B}$  to be the set of all nonempty closed sets in  $X$ . Then  $(X, \mathcal{B})$  is intersection closed. By our assumptions, condition (A<sub>1</sub>) of Theorem 1 is satisfied. If  $X$  is  $R$ -compact, then also condition (B<sub>1</sub>) holds.

Hence, our assertion follows from Theorem 1 together with the fact that  $X$  itself is a closed  $f$ -closed set.  $\square$

#### 4. COMPLETE LATTICES

We consider a complete lattice  $(L, <)$ , together with a transitive relation  $R$  on  $L$ . We denote the top element of  $L$  by  $\top$  and the bottom element by  $\perp$ . For any  $a, b \in L$  with  $a \leq b$ , we define the **interval**

$$[a, b] := \{c \in L \mid a \leq c \leq b\}.$$

If we talk of an interval  $[a, b]$ , we will always implicitly assume that it is nonempty. The ball space associated with the lattice is obtained by setting

$$\mathcal{B} := \{[a, b] \mid a, b \in L \text{ with } a \leq b\}.$$

In [3] we prove:

**Proposition 7.** *The ball space associated with a complete lattice is spherically complete and intersection closed.*

A function  $f : L \rightarrow L$  is **order preserving** if  $a < b$  implies  $fa < fb$ . For such a function, an interval  $[a, b]$  is  $f$ -closed if and only if  $fa \geq a$  and  $fb \leq b$ .

From Theorem 1, we derive the following result:

**Theorem 8.** *Take an order preserving  $R$ -compatible function  $f : L \rightarrow L$  and assume that for every regular cardinal  $\kappa$ , every descending chain  $([a_\nu, b_\nu])_{\nu < \kappa}$  of nonempty intervals, and any choice of elements  $x_\nu \in [a_\nu, b_\nu]$  such that  $x_\mu R x_\nu$  whenever  $\mu < \nu < \kappa$ , the intersection  $\bigcap_{\nu < \kappa} [a_\nu, b_\nu]$  contains an element  $z$  such that  $x_\nu R z$  for all  $\nu < \kappa$ . Then for every  $x \in L$  there is a fixed point  $z$  of  $f$  that satisfies  $x R z$ .*

*Proof:* Take a non-singleton  $f$ -closed interval  $[a, b]$  and  $x \in [a, b]$ . Since we assume  $f$  to be order preserving, we find that  $fx \in [fa, fb] \subseteq [a, b]$  and that also  $[fa, fb]$  is  $f$ -closed. Since  $f$  is  $R$ -compatible, we have that  $x R fx$ . If  $[fa, fb] = [a, b]$ , then we replace  $fa$  by  $fx$  if  $f^2x \geq fx$ , and  $fb$  by  $fx$  if  $f^2x \leq fx$ . The so obtained interval is again  $f$ -closed, and it is properly contained in  $[a, b]$  (the proof by case distinction is straightforward). We have proved that condition (A<sub>1</sub>) of Theorem 1 is satisfied.

Since  $(L, \mathcal{B})$  is intersection closed, the assumptions of the theorem yield that also condition (B<sub>1</sub>) holds. Hence, the assertion of our theorem follows from Theorem 1 together with the fact that  $L = [\perp, \top] \in \mathcal{B}$  is  $f$ -closed.  $\square$

#### 5. ULTRAMETRIC SPACES

In this section we consider ultrametric spaces  $(X, d)$ , which are defined as follows. An ultrametric  $d$  on a set  $X$  is a function from  $X \times X$  to a partially

ordered set  $\Gamma$  with smallest element 0, such that for all  $x, y, z \in X$  and all  $\gamma \in \Gamma$ ,

- (U1)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (U2) if  $d(x, y) \leq \gamma$  and  $d(y, z) \leq \gamma$ , then  $d(x, z) \leq \gamma$ ,
- (U3)  $d(x, y) = d(y, x)$  (symmetry).

(U2) is the ultrametric triangle law; if  $\Gamma$  is totally ordered, it can be replaced by

- (UT)  $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ .

A **closed ultrametric ball** is a set  $B_\alpha(x) := \{y \in X \mid d(x, y) \leq \alpha\}$ , where  $x \in X$  and  $\alpha \in \Gamma$ . The problem with general ultrametric spaces is that closed balls  $B_\alpha(x)$  are not necessarily **precise**, that is, there may not be any  $y \in X$  such that  $d(x, y) = \alpha$ . Therefore, we prefer to work only with precise ultrametric balls, which we can write in the form

$$B(x, y) := \{z \in X \mid d(x, z) \leq d(x, y)\},$$

where  $x, y \in X$ . We obtain the **ultrametric ball space**  $(X, \mathcal{B})$  from  $(X, d)$  by taking  $\mathcal{B}$  to be the set of all such balls  $B(x, y)$ .

It follows from symmetry and the ultrametric triangle law that  $B(x, y) = B(y, x)$  and that

- (1)  $B(t, z) \subseteq B(x, y)$  if and only if  $t \in B(x, y)$  and  $d(t, z) \leq d(x, y)$ .

In particular,

$$B(t, z) \subseteq B(x, y) \quad \text{if } t, z \in B(x, y).$$

Two elements  $\gamma$  and  $\delta$  of  $\Gamma$  are **comparable** if  $\gamma \leq \delta$  or  $\gamma \geq \delta$ . Hence if  $d(x, y)$  and  $d(y, z)$  are comparable, then  $B(x, y) \subseteq B(y, z)$  or  $B(y, z) \subseteq B(x, y)$ . If  $d(y, z) < d(x, y)$ , then in addition,  $x \notin B(y, z)$  and thus,  $B(y, z) \subsetneq B(x, y)$ . We note:

- (2)  $d(y, z) < d(x, y) \implies B(y, z) \subsetneq B(x, y)$ .

If  $\Gamma$  is totally ordered and  $B$  and  $B'$  are any two balls with nonempty intersection, then  $B \subseteq B'$  or  $B' \subseteq B$ .

As for ball spaces, we consider nests of (precise) ultrametric balls, and we will represent them in the form  $(B(x_i, y_i))_{i \in I}$  where  $I$  is a totally ordered set and  $B(x_j, y_j) \subsetneq B(x_i, y_i)$  whenever  $i, j \in I$  with  $i < j$ . The ultrametric space  $(X, d)$  with a transitive relation  $R$  on  $X$  will be called  **$R$ -spherically complete** if for every nest of balls  $(B(x_i, y_i))_{i \in I}$  satisfying  $x_i R x_j$  whenever  $i, j \in I$  with  $i < j$  there is  $z \in \bigcap_{i \in I} B(x_i, y_i)$  such that  $x_i R z$  for all  $i \in I$ . Recall that a function  $f : X \rightarrow X$  is said to be  $R$ -compatible if  $x R f x$  for all  $x \in X$ . Further,  $f$  is **non-expanding** if  $d(fx, fy) \leq d(x, y)$  for all  $x, y \in X$ , **contracting** if  $d(fx, fy) < d(x, y)$  for all distinct  $x, y \in X$ , and **contracting on orbits** if  $d(fx, f^2x) < d(x, fx)$  for all  $x \in X$  with  $x \neq fx$ .



**Theorem 9.** *Take an ultrametric space  $(X, d)$ , an element  $x \in X$ , a transitive relation  $R$  on  $X$ , and an  $R$ -compatible function  $f : X \rightarrow X$  which is non-expanding and contracting on orbits.*

*If  $(X, d)$  is  $R$ -spherically complete, then  $f$  admits a fixed point  $z \in B(x, fx)$  that satisfies  $xRz$ . If in addition  $f$  is contracting, then  $f$  admits a unique fixed point  $z$ ; it is independent of the choice of  $x$  and satisfies  $xRz$  for all  $x \in X$ .*

*Proof:* To begin with, we note that for every  $x \in X$  the ball  $B(x, fx)$  is  $f$ -contracting. Indeed, if  $x \neq fx$  then  $d(fx, f^2x) < d(x, fx)$  which by (2) shows that  $B(fx, f^2x) \subsetneq B(x, fx)$ .

If the ball  $B(x, y)$  is  $f$ -contracting, then  $fx \in B(x, y)$ , whence  $B(x, fx) \subseteq B(x, y)$ . By what we have just shown,  $B(x, y)$  properly contains  $B(fx, f^2x)$ . This proves that condition  $(A_2)$  of Theorem 2 holds. Our assumption that  $(X, d)$  is  $R$ -spherically complete implies that also condition  $(B_2)$  holds. So by Theorem 2, for every  $x \in X$  there is a fixed point  $z \in B(x, fx)$  of  $f$ .

If  $f$  is contracting, then it is non-expanding and contracting on orbits. Hence by what we have shown, it admits a fixed point  $z$ . If  $y \neq z$  were another fixed point, then  $d(x, y) = d(fx, fy) < d(x, y)$ , a contradiction. Therefore,  $z$  is the only fixed point of  $f$ , and it follows that  $xRz$  holds for all  $x \in X$ .  $\square$

## 6. ORDERED ABELIAN GROUPS AND FIELDS

For the conclusion of this paper, we consider an ordered abelian group  $(G, <)$  together with a transitive relation  $R$  on  $G$ . Since the underlying additive group of an ordered field is an ordered abelian group  $(G, <)$ , we are implicitly covering also the case of ordered fields.

The associated ball space is given by the collection of all (nonempty) closed bounded intervals in  $(G, <)$ :

$$\mathcal{B} := \{[a, b] \mid a, b \in G \text{ with } a \leq b\}.$$

We call  $(G, <)$  **symmetrically complete** if this ball space is spherically complete. See [4] for more information on this notion and for a characterization of symmetrically complete ordered abelian groups and fields. In particular, we know from this characterization that every symmetrically complete ordered abelian group is divisible.

Take a function  $f : G \rightarrow G$ . We call it **non-expanding** if

$$|fx - fy| \leq |x - y|$$

for all  $x, y \in G$ , and **contracting on orbits** if there is a positive rational number  $\frac{m}{n} < 1$  with  $m, n \in \mathbb{N}$  such that

$$|fx - f^2x| \leq \frac{m}{n}|x - fx|$$

for all  $x \in G$ . Note that the right hand side makes sense since  $G$  is divisible.

From Theorem 4, we derive the following result:

**Theorem 10.** *Take a symmetrically complete ordered abelian group  $(G, <)$ , a transitive relation  $R$  on  $G$ , and an  $R$ -compatible non-expanding function  $f : G \rightarrow G$  which is contracting on orbits. Assume that for every regular cardinal  $\kappa$ , every descending chain  $([a_\nu, b_\nu])_{\nu < \kappa}$  of nonempty intervals, and any choice of elements  $x_\nu \in [a_\nu, b_\nu]$  such that  $x_\mu R x_\nu$  whenever  $\mu < \nu < \kappa$ , the intersection  $\bigcap_{\nu < \kappa} [a_\nu, b_\nu]$  contains an element  $z$  such that  $x_\nu R z$  for all  $\nu < \kappa$ . Then for every  $x \in L$  there is a fixed point  $z$  of  $f$  that satisfies  $x R z$ .*

Proof: We set  $C = \frac{m}{n}$ . As in Section 8 of [1], for every  $x \in G$  we take  $B_x$  to be the closed interval

$$B_x := \left\{ y \in X \mid |x - y| \leq \frac{|x - fx|}{1 - C} \right\}$$

to obtain that  $f^i x \in B_x$  for all  $i \geq 0$ . In particular,  $x \in B_x$ , hence (SC1) holds. It is further shown in Section 8 of [1] that our condition that  $f$  is contracting on orbits implies that also (SC2) holds.

In the proof of Theorem 21 of [1] it is shown that for every element  $z$  in the intersection of a nest as given in condition (B<sub>4</sub>) of Theorem 4, the whole interval  $B_z$  is contained in the intersection. Together with the assumptions of our theorem, this shows that condition (B<sub>4</sub>) holds. Hence, the assertion of our theorem follows from Theorem 4.  $\square$

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