Elementary properties of power series fields over finite fields *

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Abstract

In spite of the analogies between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ which became evident through the work of Ax and Kochen, an adaptation of the complete recursive axiom system given by them for $\mathbb{Q}_p$ to the case of $\mathbb{F}_p((t))$ does not render a complete axiom system. We show the independence of elementary properties which express the action of additive polynomials as maps on $\mathbb{F}_p((t))$. We formulate an elementary property expressing this action and show that it holds for all maximal valued fields. We also derive an example of a rather simple immediate valued function field over a henselian defectless ground field which is not a henselian rational function field. This example is of special interest in connection with the open problem of local uniformization in positive characteristic.

Contents

1 Elementary properties and additive polynomials

In this paper, we work with valued fields (K, v), denoting the value group by vK, the residue field by Kv and the valuation ring by $\mathcal{O}_v$ or $\mathcal{O}_K$. For elements $a \in K$, the value

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is denoted by va, and the residue by av. We will use the classical additive (Krull) way of writing valuations. That is, the value group is an additively written ordered abelian group, the homomorphism property of v reads as $vab = va + vb$, and the ultrametric triangle law reads as $v(a+b) \geq \min\{va, vb\}$. Further, we have the rule $va = \infty \Leftrightarrow a = 0$. We take $\mathcal{L} = \{+, -, \cdot, 0, 1, \mathcal{O}\}$ to be the language of valued rings, where $\mathcal{O}$ is a binary relation symbol for valuation divisibility. That is, $\mathcal{O}(a, b)$ will say that $va \geq vb$, or equivalently, that a/b is an element of the valuation ring. We will write $\mathcal{O}(X)$ in the place of $\mathcal{O}(X, 1)$ (note that $\mathcal{O}(a, 1)$ says that $va \geq v1 = 0$, i.e., $a \in \mathcal{O}_v$).

Let $\mathbb{F}_p$ denote the field with p elements. The power series field $\mathbb{F}_p((t))$, also called "field of formal Laurent series over $\mathbb{F}_p$", carries a canonical valuation v_t, the t-adic valuation (we write $v_t t = 1$). $(\mathbb{F}_p((t)), v_t)$ is a complete discretely valued field, with value group $v_t\mathbb{F}_p((t)) = \mathbb{Z}$ (that is what "discretely valued" means) and residue field $\mathbb{F}_p((t))v_t = \mathbb{F}_p$. At the first glimpse, such fields may appear to be the best known objects in valuation theory. Nevertheless, the following prominent questions about the elementary theory $\mathrm{Th}(\mathbb{F}_p((t)), v_t)$ are still unanswered:

Is $\mathrm{Th}(\mathbb{F}_p((t)), v_t)$ decidable? Is it model complete? Does $(\mathbb{F}_p((t)), v_t)$ admit quantifier elimination in $\mathcal{L}$ or in a natural extension of $\mathcal{L}$? Does there exist an elementary class of valued fields, containing $(\mathbb{F}_p((t)), v_t)$ and satisfying some Ax–Kochen–Ershov principle?

By an Ax–Kochen–Ershov principle for a class $\mathcal{K}$ of valued fields we mean a principle of the form

$$(K, v), (L, v) \in \mathcal{K} \text{ with } vK \equiv vL\,,\ Kv \equiv Lv \ \text{ implies that } \ (K, v) \equiv (L, v)$$

or a similar version with $\prec$ or $\prec_\exists$ ("existentially closed in") in the place of $\equiv$. Here, vL is the value group of (L, v), and the language is that of ordered groups. Further, Lv is the residue field of (L, v), and the language is that of rings or of fields. For example, the elementary class of henselian fields with residue fields of characteristic 0 satisfies all of these Ax–Kochen–Ershov principles (cf. [AK], [E], [KP], [K2]).

Encouraged by the similarities between $\mathbb{F}_p((t))$ and the field $\mathbb{Q}_p$ of p-adics, one might try to give a complete axiomatization for $\mathrm{Th}(\mathbb{F}_p((t)), v_t)$ by adapting the well known axioms for $\mathrm{Th}(\mathbb{Q}_p, v_p)$. They express that $(\mathbb{Q}_p, v_p)$ is a henselian valued field of characteristic 0 with value group a $\mathbb{Z}$-group (i.e., an ordered abelian group elementarily equivalent to $\mathbb{Z}$), and residue field $\mathbb{F}_p$. They also express that $vp = 1$ (the smallest positive element in the value group). This is not relevant for $\mathbb{F}_p((t))$ since there, $p \cdot 1 = 0$. Nevertheless, we may add a constant name t to $\mathcal{L}$ so that one can express by an elementary sentence that $vt = 1$.

A naive adaptation would just replace "characteristic 0" by "characteristic p" and p by t. But there is an elementary property of valued fields that is satisfied by all valued fields of residue characteristic 0 and all formally p-adic fields, but not by all valued fields in general. It is the property of being **defectless**. A valued field (K, v) is called defectless if the **fundamental equality**

$$n = \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i$$

holds for every finite extension $L|K$, where $n = [L : K]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from K to L, $\mathrm{e}_i = (v_i L : vK)$ are the respective ramification indices, and $\mathrm{f}_i = [Lv_i : Kv]$ are the respective inertia degrees. (Note that $\mathrm{g} = 1$ if (K, v) is henselian.) There is a simple example, probably already due to F. K. Schmidt, which shows that there are henselian discretely valued fields of positive characteristic which are not defectless.

However, each power series field with its canonical valuation is henselian and defectless. In particular, $(\mathbb{F}_p((t)), v_t)$ is defectless. For a less naive adaptation of the axiom system of $\mathbb{Q}_p$, we will thus add "defectless". We obtain the following axiom system in the language $\mathcal{L}(t) = \mathcal{L} \cup \{t\}$:

$$\left.\begin{array}{l} (K, v) \text{ is a henselian defectless valued field} \\ K \text{ is of characteristic } p \\ vK \text{ is a } \mathbb{Z}\text{-group} \\ Kv = \mathbb{F}_p \\ vt \text{ is the smallest positive element in } vK\ . \end{array}\right\} \qquad (1)$$

Let us note that also $(\mathbb{F}_p(t), v_t)^h$, the henselization of $(\mathbb{F}_p(t), v_t)$, satisfies these axioms. It is well-known that this is a defectless field, being the henselization of a global field (cf. [K2]). It is also well-known that $(\mathbb{F}_p(t), v_t)^h$ is existentially closed in $(\mathbb{F}_p((t)), v_t)$; for an easy proof see [K2]. But it is not known whether $(\mathbb{F}_p((t)), v_t)$ is an elementary extension of $(\mathbb{F}_p(t), v_t)^h$.

It did not seem unlikely that axiom system (1) could be complete, until we proved in [K1]:

Theorem 1 *The axiom system (1) is not complete.*

We wish to show how this result is obtained and which additional previously unknown elementary properties of $\mathbb{F}_p((t))$ have been discovered.

We start by noting that for $K = \mathbb{F}_p((t))$, the elements $1, t, t^2, \ldots, t^{p-1}$ form a basis of the field extension $K|K^p$. Thus,

$$K \;=\; K^p \oplus tK^p \oplus \ldots \oplus t^{p-1}K^p\ . \qquad (2)$$

It follows that the $\mathcal{L}(t)$-sentence

$$\forall X \exists X_0 \ldots \exists X_{p-1}\;\; X = X_0^p + tX_1^p + \ldots + t^{p-1}X_{p-1}^p \qquad (3)$$

holds in K. In fact, it holds in every model of (1): since vt is the smallest positive element in vK, the elements $1, t, \ldots, t^{p-1}$ are K^p-linearly independent and thus, Lemma 18 below yields that they form a basis of $K|K^p$.

Since the Frobenius $x \mapsto x^p$ is an endomorphism of every field K of characteristic p, it follows that for every i the polynomial $t^i X^p$ is **additive**. A polynomial $f(X) \in K[X]$

is called additive if $f(a+b) = f(a) + f(b)$ for all a, b in any extension field of K. The additive polynomials in $K[X]$ are precisely the polynomials of the form

$$\sum_{i=0}^{m} c_i X^{p^i} \quad \text{with } c_i \in K\,,\ m \in \mathbb{N}$$

(cf. [L], VIII, §11). If K is infinite, then $f(X) \in K[X]$ is additive if and only if $f(a+b) = f(a) + f(b)$ for all $a, b \in K$. For further details about additive polynomials, see [O], [W1], [W2] and [K2].

Now it is a natural question to ask what might happen if we replace the polynomials $t^i X^p$ in (3) by other additive polynomials. Apart from the additive polynomials cX^{p^n}, the most important is the **Artin-Schreier polynomial** $\wp(X) := X^p - X$. Lou van den Dries observed that if k is a field of characteristic p such that $\wp(k) := \{\wp(x) \mid x \in k\} = k$, then the $\mathcal{L}(t)$-sentence

$$\forall X \exists X_0 \ldots \exists X_{p-1}\ \ X = X_0^p - X_0 + tX_1^p + \ldots + t^{p-1} X_{p-1}^p \tag{4}$$

holds in $k((t))$. However, he was not able to deduce this assertion from axiom system (1) where "$Kv = \mathbb{F}_p$" is replaced by "Kv is perfect and $\wp(Kv) = Kv$". Observe that $\wp(\mathbb{F}_p) = \{0\} \neq \mathbb{F}_p$. To get an assertion valid in $\mathbb{F}_p((t))$, we have to introduce a corrective summand Y:

$$\forall X \exists Y \exists X_0 \ldots \exists X_{p-1}\ \ X = Y + X_0^p - X_0 + tX_1^p + \ldots + t^{p-1} X_{p-1}^p \ \wedge\ \mathcal{O}(Y) \tag{5}$$

Lemma 2 *The $\mathcal{L}(t)$-sentence (5) holds for every intermediate field (K, v) between the fields $(\mathbb{F}_p(t), v_t)$ and $(\mathbb{F}_p((t)), v_t)$.*

Proof: Take $x \in K$. If $vx \geq 0$, then we set $y = x$ and $x_i = 0$ to obtain that $x = y = y + x_0^p - x_0 + tx_1^p + \ldots + t^{p-1} x_{p-1}^p$ with $vy \geq 0$. For $vx < 0$, we can proceed by induction on $-vx$ since $vK = \mathbb{Z}$. Suppose that $m \in \mathbb{N}$ and that we have shown the assertion to hold for every x of value $vx > -m$. Take $x \in K$ such that $vx = -m$. There is $\ell \in \{0, \ldots, p-1\}$ such that $vx \equiv \ell$ modulo $p\mathbb{Z} = pvK$. Choose some $z \in K$ such that $vx = \ell + pvz = vt^\ell z^p$. Then $v(x/t^\ell z^p) = 0$, and the residue of $x/t^\ell z^p$ is some element $j \in \mathbb{F}_p$. It follows that $v(x/t^\ell (jz)^p - 1) = v(j^{-1} x / t^\ell z^p - 1) > 0$. Hence, $v(x - t^\ell (jz)^p) > vt^\ell (jz)^p = vx$. If $\ell > 0$, then we set $x' := x - t^\ell (jz)^p$, so that $vx' > vx$. If $\ell = 0$, then we set $x' := x - (jz)^p + jz$; since $vjz < 0$, we have that $vx = v(jz)^p < vjz$ and thus again, $vx' \geq \min\{v(x - (jz)^p), vjz\} > vx$. So by induction hypothesis, there are $y, x'_0 \ldots x'_{p-1}$ such that $vy \geq 0$ and $x' = y + (x'_0)^p - x'_0 + t(x'_1)^p + \ldots + t^{p-1}(x'_{p-1})^p$. We set $x_\ell = x'_\ell + jz$ and $x_i = x'_i$ for $i \neq \ell$, to obtain by additivity that $x = y + x_0^p - x_0 + tx_1^p + \ldots + t^{p-1} x_{p-1}^p$. $\square$

This lemma shows that in analogy to (2), every intermediate field (K, v) between $(\mathbb{F}_p(t), v_t)$ and $(\mathbb{F}_p((t)), v_t)$ satisfies:

$$K = \mathcal{O}_K + \wp(K) + tK^p + \ldots + t^{p-1} K^p\ . \tag{6}$$

If in addition (K,v) is henselian, then we can improve this representation to

$$K = \mathbb{F}_p + \wp(K) + tK^p + \ldots + t^{p-1}K^p \; . \tag{7}$$

This is seen as follows. Using Hensel's Lemma, one proves that the valuation ideal $\mathcal{M}_K$ of any henselian field (K,v) is contained in $\wp(K)$. On the other hand, $Kv = \mathbb{F}_p$ implies that $\mathcal{O}_K = \mathbb{F}_p + \mathcal{M}_K$. Consequently, $\mathbb{F}_p + \wp(K) = \mathcal{O}_K + \wp(K)$.

Theorem 1 is proved by constructing a valued field (L,v) which satisfies axiom system (1) but not sentence (5):

Theorem 3 *Take (K,v) to be $(\mathbb{F}_p(t),v_t)^h$ or $(\mathbb{F}_p((t)),v_t)$. Then there exists an extension (L,v) of (K,v) such that:*
a) $L|K$ is a regular extension of transcendence degree 1,
b) (L,v) satisfies (1),
c) sentence (5) does not hold in (L,v).

The construction will be given in Section 4. We construct (L,v) also over $(\mathbb{F}_p(t),v_t)^h$ because this leads to a valued field having only transcendence degree 2 over its prime field. This allows us to derive an interesting example in connection with the problem of local uniformization in positive characteristic (cf. [K3], [K5]). Note that a field extension $L|K$ is said to be **regular** if it is linearly disjoint from the algebraic closure of K, that is, if it is separable and K is relatively algebraically closed in L.

Since (L,v) as in Theorem 3 does not satisfy (5), it cannot be an elementary extension of $(\mathbb{F}_p((t)),v_t)$. This contrasts with the fact that, according to another theorem proved in [K1], $(\mathbb{F}_p((t)),v_t)$ is existentially closed in (L,v).

Having seen that the sentence (5) is independent of the axioms in (1), we now pursue two main questions. The first of them is:

A) Are there further assertions similar to (5) and independent of (1)? What happens if we replace the additive polynomials $\wp(X)$, tX^p , ... , $t^{p-1}X^p$ appearing in (5) by other additive polynomials? Which corrective summands are then needed? Can we find a form that asserts essentially the same but dispenses with the use of the corrective summands Y, $\mathcal{O}$, $\mathbb{F}_p$ in (5), (6) and (7)?

Before we formulate the second question, let us give some background. In the model theory of valued fields, the **maximal fields** play a crucial role. These are valued fields not admitting any proper immediate extensions. (An extension of valued fields is called **immediate** if it leaves value group and residue field unchanged.) It was shown by Krull [KL] that every valued field has at least one maximal immediate extension; this must be a maximal field. (Later, Gravett [G] gave a beautiful short proof replacing Krull's complicated argument.) As it is the case for power series fields (which in fact are maximal), also all maximal fields are henselian defectless.

Because of the special role of maximal fields, it would be important to know whether all maximal fields satisfy assertions similar to (5). But for an arbitrary maximal field

(M,v), also the degree $[M:M^p]$ is arbitrary and thus, the basis $1,t,\ldots,t^{p-1}$ has to be replaced adequately. So we ask:

B) Do all maximal fields satisfy assertions similar to (5)? Is there a way to formulate these assertions simultaneously for all maximal fields M, not involving the degree $[M:M^p]$?

In order to formulate our answer to these questions, we have to introduce some notation. Take a valued field (K,v) of characteristic $p>0$ and additive polynomials $f_0,\ldots,f_n\in K[X]$. We define an $\mathcal{L}$-formula

$$\mathrm{pd}(z_0,\ldots,z_n,z_0',\ldots,z_n') \;\;:\Leftrightarrow\;\; v\left(\sum_{i=0}^n z_i - \sum_{i=0}^n z_i'\right) > v\sum_{i=0}^n z_i \,\wedge\, v\sum_{i=0}^n z_i' = \min_i vz_i'$$

and an $\mathcal{L}$-sentence

$$\mathrm{PD}(f_0,\ldots,f_n) \;\;:\Leftrightarrow\;\; \forall X_0,\ldots,X_n\exists Y_0,\ldots,Y_n\; \mathrm{pd}(f_0(X_0),\ldots,f_n(X_n),f_0(Y_0),\ldots,f_n(Y_n))$$

with the coefficients of the polynomials as parameters. To understand the meaning of PD observe that $v\sum_{i=0}^n z_i \geq \min_i vz_i$ by the ultrametric triangle law, but that equality need not hold in general. In this situation, we would like to replace the z_i's by z_i''s such that $\sum_{i=0}^n z_i = \sum_{i=0}^n z_i'$ and $v\sum_{i=0}^n z_i' = \min_i vz_i'$. If one restricts the choice of the z_i''s to certain sets (e.g., the images of the f_i's), then this might not always be possible. Asking for the equality of the sums is quite strong; for our purposes, a weaker condition will suffice. We replace the equality by the expression $v(\sum_{i=0}^n z_i - \sum_{i=0}^n z_i') > v\sum_{i=0}^n z_i$. This means that the new sum "approximates" the old, in a certain sense. Note that this implies that $v\sum_{i=0}^n z_i = v\sum_{i=0}^n z_i'$.

At this point, observe that the images $f_i(K)$ of K under f_i are subgroups of the additive group of K because the f_i's are additive. Now if we have subgroups $G_0,\ldots,G_n$ then we call their sum **direct (as valued groups)** if $v\sum_{i=0}^n z_i = \min_i vz_i$ for every choice of $z_i\in G_i$. In fact, $K=\mathbb{F}_p((t))$ is the direct sum of the subgroups $K^p,tK^p,\ldots,t^{p-1}K^p$ not only in the ordinary sense, but also as valued groups (see Lemma 17 in Section 3). On the other hand, the sum of the subgroups $\wp(K),tK^p,\ldots,t^{p-1}K^p$ is not direct since $t^i\mathcal{O}_K\subset\mathcal{M}_K\subset\wp(K)$ for all $i\geq 1$. Therefore, we introduce the notion **pseudo direct**: we call the sum of the G_i pseudo direct if for every choice of $z_i\in G_i$ there are $z_i'\in G_i$ such that $\mathrm{pd}(z_0,\ldots,z_n,z_0',\ldots,z_n')$ holds.

The following lemma will be proved in Section 3:

Lemma 4 *Assume that (K,v) is a valued field of characteristic $p>0$ with $t\in K$ such that vt is the smallest positive element in the value group vK. Then the sum of the groups $\wp(K),tK^p,\ldots,t^{p-1}K^p$ is pseudo direct. That is,* $\mathrm{PD}(\wp(X),tX^p,\ldots,t^{p-1}X^p)$ *holds in (K,v).*

We need one further notion, which will play a key role in our results. A subset S of a valued field (K,v) will be called an **optimal approximation subset in** (K,v) if for every $z \in K$ there is some $y \in S$ such that $v(z-y) = \max\{v(z-x) \mid x \in S\}$, i.e., if the following holds in (K,v):

$$\forall Z\, \exists Y \in S\, \forall X \in S\;\; \mathcal{O}(Z-Y\,,\, Z-X)\,. \tag{8}$$

Note that if S is $\mathcal{L}$-definable with parameters, then (8) is an $\mathcal{L}$-sentence with the same parameters. For additive polynomials $f_0,\ldots,f_n \in K[X]$, we define:

$$\begin{array}{rcl}\mathrm{OA}(f_0,\ldots,f_n) & :\Leftrightarrow & \text{the sum of the images of } f_0,\ldots,f_n \\ & & \text{is an optimal approximation subset.}\end{array}$$

Since the subgroup $f_0(K)+\ldots+f_n(K)$ of $(K,+)$ is $\mathcal{L}$-definable with the coefficients of the polynomials as parameters, $\mathrm{OA}(f_0,\ldots,f_n)$ is an $\mathcal{L}$-sentence with the coefficients of the polynomials as parameters.

Parameters from K can be avoided by quantifying over the coefficients of the polynomials $f_0,\ldots,f_n$. The elementary $\mathcal{L}$-sentence

$$\forall Z_{0,0}\ldots\forall Z_{n,n}\;\mathrm{PD}\left(\sum_{j=0}^{n} Z_{0,j}X^{p^j},\ldots,\sum_{j=0}^{n} Z_{n,j}X^{p^j}\right)\;\Rightarrow\;\mathrm{OA}\left(\sum_{j=0}^{n} Z_{0,j}X^{p^j},\ldots,\sum_{j=0}^{n} Z_{n,j}X^{p^j}\right)$$

talks about at most $n+1$ additive polynomials of degrees at most p^n. Letting n run through all natural numbers, we obtain a recursive $\mathcal{L}$-axiom scheme expressing the following elementary property:

(PDOA) for every $n \in \mathbb{N}$ and every choice of additive polynomials $f_0,\ldots,f_n$,
$\mathrm{PD}(f_0,\ldots,f_n) \;\Rightarrow\; \mathrm{OA}(f_0,\ldots,f_n)\,.$

One of our main results is:

Theorem 5 *(PDOA) holds in every maximal field.*

We will give a proof in Section 2 below. (For maximal fields of characteristic 0, the theorem is trivial because then the only additive polynomials are of the form cX.)

Keeping some faith in our original sentence (5), let us observe:

Lemma 6 *If (K,v) satisfies axiom system (1) and (PDOA), then (5) holds in (K,v).*

The proof will be given in Section 3. Theorem 5 and Lemma 6 yield:

Corollary 7 *If (K,v) is a maximal field which satisfies axiom system (1), then (5) holds in (K,v).*

Let us take advantage of the fact that (PDOA) is already formalized in $\mathcal{L}$, without needing the constant t. So far, we have kept secret the fact that we are much more interested in the $\mathcal{L}$-axiom system

$$\left.\begin{array}{l}(K,v) \text{ is a henselian defectless valued field}\\ K \text{ is of characteristic } p\\ vK \text{ is a } \mathbb{Z}\text{-group}\\ Kv = \mathbb{F}_p\end{array}\right\} \tag{9}$$

rather than in the $\mathcal{L}(t)$-axiom system (1). We only formulated (1) to show what the sentence (5) tells us about it. But now, we can derive:

Theorem 8 *The axiom system (9) is not complete.*

Indeed, Theorem 5 shows that the model $(\mathbb{F}_p((t)), v_t)$ satisfies (PDOA), whereas Lemma 6 shows that the model (L, v) given in Theorem 3 cannot satisfy (PDOA).

Now our main open question is:

Is the axiom system (9) + (PDOA) complete?

If this is the case, then it will also follow that $\mathrm{Th}(\mathbb{F}_p((t)), v_t)$ is decidable. We do not know an answer to this question. But we know that (PDOA) plays an important role in the structure theory of valued function fields. In fact, it allows us to derive structure theorems of the same sort as we employed to prove the Ax–Kochen–Ershov principles for the elementary class of tame fields (cf. [K1], [K2], [K7]). Also, we can show that valued fields (K, v) satisfying (9) + (PDOA) will satisfy the Ax–Kochen–Ershov principle with $\prec_\exists$ for arbitrary extensions (L, v), provided that the extension $L|K$ is of transcendence degree 1. However, this needs an abundance of valuation theoretical machinery. The reason is that (PDOA) does not have as nice properties as "henselian" (or "tame"). Let us present one of the problems. It is a well known fact that a relatively algebraically closed subfield of a henselian field is again henselian. (The same holds for "tame" in the place of "henselian" if the extension is immediate.) But now consider an arbitrary maximal immediate extension (M, v) of the field (L, v) which is given in Theorem 3. By Theorem 5, (PDOA) holds in (M, v). But it does not hold in (L, v). On the other hand, the fact that (L, v) is henselian defectless yields that (L, v) is **algebraically maximal**, that is, it admits no proper immediate algebraic extension. Therefore, it is relatively algebraically closed in M. Hence:

Theorem 9 *There is an immediate extension $(L, v) \subset (M, v)$ of henselian defectless fields such that L is relatively algebraically closed in M and (PDOA) holds in (M, v), but not in (L, v).*

Another important property of "henselian" is: if (K, v) is henselian, then so is each of its algebraic extensions. Also, "defectless" carries over to every finite extension (but not to every algebraic extension in general). So the following yet unanswered questions arise:

Does (PDOA) carry over to finite extensions or even to any algebraic extensions, provided they are henselian defectless fields? What are the "algebraic properties" of (PDOA)?

For the conclusion of this section, let us think about generalizations of Theorem 5:

1) A polynomial $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ is called **additive** if it induces an additive map on L^n for every extension field L of K. With $f_i(X_i) = f(0, \ldots, 0, X_i, 0 \ldots, 0)$, it follows by additivity that the f_i's are additive and $f(a_1, \ldots, a_n) = f_1(a_1) + \ldots + f_n(a_n)$ for all $(a_1, \ldots, a_n) \in L^n$. Hence, if in a given maximal field $\mathrm{OA}(f_0, \ldots, f_n)$ holds for *all* additive polynomials $f_0, \ldots, f_n$, then the image of *every* additive polynomial in this field

is an optimal approximation subset. In a subsequent paper [DK] we will consider the question for which maximal fields this generalization holds.

2) Perhaps, the image of *every* polynomial in several variables on a maximal field is an optimal approximation subset. This would be an amazing generalization of Theorem 5 and of Lemma 13 of the next section. Our hope is that one could derive such a result from generalization 1) by approximating arbitrary polynomials by suitably chosen additive polynomials. For polynomials in one variable, this is implicit in Kaplansky's work [KA].

Having seen the crucial role played by subgroups defined by additive polynomials, we also would like to know:

Which are the definable subgroups in valued fields of positive characteristic? Are they optimal approximation subgroups? Do they carry other valuation theoretical properties which are independent of axiom systems like (9) + (PDOA)?

2 Spherical completeness and optimal approximation

In the following, we will give the proof of Theorem 5. We need some further definitions. They can be given already in the context of ultrametric spaces, but here we will give them for subsets S of valued fields (K,v). A **closed ball in** S is a set of the form $B_\gamma(a,S) = \{x \in S \mid v(a-x) \geq \gamma\}$ for $a \in S$ and $\gamma \in v(S-S) := \{v(s-s') \mid s,s' \in S\}$. A **nest of (closed) balls B** is a nonempty collection of closed balls such that each two balls in **B** have a nonempty intersection. By the ultrametric triangle law it follows that the balls in **B** are linearly ordered by inclusion. Now (S,v) is called **spherically complete** if every nest of balls **B** in S has a nonempty intersection: $\bigcap_{B\in\mathbf{B}} B \neq \emptyset$. It is easy to prove that (S,v) is spherically complete if and only if every pseudo Cauchy sequence in (S,v) has a pseudo limit in S (see [KA] or [K2] for these notions; note that Kaplansky uses "pseudo-convergent" instead of "pseudo Cauchy"). Therefore, the following characterization of maximal fields is a direct consequence of Theorem 4 of [KA]:

Theorem 10 *A valued field (K,v) is maximal if and only if it is spherically complete.*

On the other hand, we have:

Lemma 11 *Take any subset S of the additive group of a valued field (K,v). If (S,v) is spherically complete, then S is an optimal approximation subset in (K,v).*

Proof: Assume that S is not an optimal approximation subset in (K,v). Then there is an element $z \in K$ such that for every $y \in S$ there is some $x \in S$ satisfying that $v(z-x) > v(z-y)$. Note that by the ultrametric triangle law, the latter implies that $v(z-y) = v(x-y) \in v(S-S)$. From this and the fact that $S \cap B_{v(z-y)}(z,K) = B_{v(z-y)}(y,S)$, it follows that

$$\{\, B_{v(z-y)}(y,S) \;\mid\; y \in S \,\}$$

is a nest of balls in (S,v). Take any $a\in S$ and choose $b\in S$ such that $v(z-b)>v(z-a)$. Then $a\notin B_{v(z-b)}(z,K)$. Hence the nest has an empty intersection, showing that (S,v) is not spherically complete. $\square$

Now a natural question is: if (K,v) is spherically complete and f is an additive polynomial, does it follow that $(f(K),v)$ is spherically complete? In fact, this is true for *every* polynomial. For the proof, we need the following preparation:

Lemma 12 *Let (K,v) be any valued field, $f\in K[X]$, and $(a_\nu)_{\nu<\lambda}$ a sequence in K such that $(f(a_\nu))_{\nu<\lambda}$ is a pseudo Cauchy sequence. Then there is a pseudo Cauchy subsequence $(a_{\nu_\mu})_{\mu<\lambda'}$ of $(a_\nu)_{\nu<\lambda}$ with $(\nu_\mu)_{\mu<\lambda'}$ cofinal in λ.*

Proof: We work in some maximal algebraically closed extension field (K',v) of (K,v). There, we find a pseudo limit b of $(f(a_\nu))_{\nu<\lambda}$ and a decomposition $f(X)-b=c\prod_i(X-c_i)$. Then

$$v(f(a_\nu)-b) \;=\; vc+\sum_i v(a_\nu-c_i)\;.$$

Since b is a pseudo limit of $(f(a_\nu))_{\nu<\lambda}$, the sequence $(v(f(a_\nu)-b))_{\nu<\lambda}$ is stricly increasing; thus, there is some i and a sequence $(\nu_\mu)_{\mu<\lambda'}$ cofinal in λ for which $v(a_{\nu_\mu}-c_i)$ is stricly increasing. This means that $(a_{\nu_\mu})_{\mu<\lambda'}$ is a pseudo Cauchy sequence, with pseudo limit c_i. $\square$

Lemma 13 *If (K,v) is spherically complete, then for every $f\in K[X]$, $(f(K),v)$ is spherically complete and therefore, $f(K)$ is an optimal approximation subset of (K,v).*

Proof: Take $a_\nu\in K$ such that $(f(a_\nu))_{\nu<\lambda}$ is a pseudo Cauchy sequence. Then by Lemma 12, we find a pseudo Cauchy subsequence $(a_{\nu_\mu})_{\mu<\lambda'}$ of $(a_\nu)_{\nu<\lambda}$ with $(\nu_\mu)_{\mu<\lambda'}$ cofinal in λ. By hypothesis, it has a pseudo limit a in (K,v). Now it follows from the general theory of pseudo Cauchy sequences that $f(a)$ is a pseudo limit of $(f(a_{\nu_\mu}))_{\mu<\lambda'}$ and hence also of $(f(a_\nu))_{\nu<\lambda}$. $\square$

Lemma 14 *If (K,v) is algebraically maximal, then for every $f\in K[X]$, $f(K)$ is an optimal approximation subset of (K,v).*

Proof: Suppose that $f(K)$ is not an optimal approximation subset of (K,v). Then there is some $b\in K$ and a pseudo Cauchy sequence $(f(a_\nu))_{\nu<\lambda}$ having pseudo limit b but no pseudo limit in $f(K)$. By Lemma 12, we find a pseudo Cauchy subsequence $(a_{\nu_\mu})_{\mu<\lambda'}$ of $(a_\nu)_{\nu<\lambda}$ with $(\nu_\mu)_{\mu<\lambda'}$ cofinal in λ. Since $f(X)-b\in K[X]$ and the sequence $(v(f(a_{\nu_\mu})-b))_{\mu<\lambda'}$ is increasing, $(a_{\nu_\mu})_{\mu<\lambda'}$ is of algebraic type. Since (K,v) is algebraically maximal, there is a limit a of $(a_{\nu_\mu})_{\mu<\lambda'}$ in (K,v) (since otherwise, Theorem 3 of [KA] would show that (K,v) admits a proper immediate algebraic extension). As in the foregoing proof, it follows that $f(a) \in f(K)$ is a pseudo limit of $(f(a_\nu))_{\nu<\lambda}$. This contradiction proves our assertion. $\square$

The last lemma exhibits an intriguing fact: if we have additive polynomials $f_0,\ldots,f_n$ on K and (K,v) is henselian defectless, then the $f_i(K)$ are optimal approximation subgroups, but their sum is not necessarily an optimal approximation subgroup, even if it is pseudo direct. By virtue of Lemma 16 below, the field (L,v) of Theorem 3 with the additive polynomials $\wp(X), tX^p,\ldots,t^{p-1}X^p$ is an example for this. The situation changes when the subgroups are spherically complete:

Theorem 15 *Let $G_0,\ldots,G_n$ be spherically complete subgroups of an arbitrary valued abelian group (G,v). If their sum is pseudo direct, then it is spherically complete and hence an optimal approximation subset of (G,v).*

The proof is given in [K4], using a theorem about maps on spherically complete ultrametric spaces. (The theorem does not work without a condition on the sum of the G_i's. But we do not know of any counterexample where the G_i's are images of additive polynomials.)

Now Theorem 5 follows from Theorem 10, Lemma 13 and Theorem 15.

3 Valuation independence and pseudo direct sums

Take any valued field extension $(K|K',v)$. The elements $c_0,\ldots,c_m \in K\setminus\{0\}$ will be called **K'-valuation independent** if for every choice of elements $d_0,\ldots,d_m\in K'$, the following holds:

$$v(c_0d_0+\ldots+c_md_m) \;=\; \min_{0\le i\le m} vc_id_i\;.$$

In particular, if $d_k\neq 0$ for at least one k, then $c_kd_k\neq 0$ and thus, $v(c_0d_0+\ldots+c_md_m)\le vc_kd_k<\infty$ which shows that $c_0d_0+\ldots+c_md_m\neq 0$. Hence if $c_0,\ldots,c_m$ are K'-valuation independent, then they are K'-linearly independent. The notion of valuation independent elements was introduced by Walter Baur under the name “separated sequence” ([B1], [B2]).

Lemma 16 *Take a valued field (K,v) of characteristic $p>0$ with K^p-valuation independent elements $c_0,\ldots,c_m$, where $c_0=1$. Then* $\mathrm{PD}(\wp(X),c_1X^p,\ldots,c_mX^p)$ *holds in (K,v).*

Proof: We set $f_0(X)=\wp(X)$ and $f_i(X)=c_iX^p$ for $1\le i\le m$. For $x_0,\ldots,x_m\in K$,

$$f_0(x_0)+\ldots+f_m(x_m) \;=\; -x_0\;+\;c_0x_0^p+\ldots+c_mx_m^p\;.$$

If $vx_0>v(c_0x_0^p+\ldots+c_mx_m^p)$, then

$$\begin{aligned} v(f_0(x_0)+\ldots+f_m(x_m)) &= \min\{vx_0, v(c_0x_0^p+\ldots+c_mx_m^p)\}\\ &= v(c_0x_0^p+\ldots+c_mx_m^p) \;=\; \min_{0\le i\le m} vc_ix_i^p \;=\; \min_{0\le i\le m} vf_i(x_i)\;, \end{aligned}$$

where the last equality holds since if the minimum is $vc_0x_0^p$, then $vx_0 > vc_0x_0^p$ by our assumption on vx_0, which implies that $vc_0x_0^p = vf_0(x_0)$. Hence in this case, choosing $y_i = x_i$ causes

$$\mathrm{pd}(f_0(x_0),\ldots,f_m(x_m),f_0(y_0),\ldots,f_m(y_m)) \tag{10}$$

to hold.

Now assume that $v\sum_{i=0}^m f_i(x_i) > \min_i vf_i(x_i)$. Then by what we just have shown,

$$vx_0 \le v(c_0x_0^p + \ldots + c_mx_m^p) \;=\; \min_{0\le i\le m} vc_ix_i^p \;\le\; vc_0x_0^p \;=\; pvx_0\;.$$

But $vx_0 \le pvx_0$ can only hold if $vx_0 \ge 0$, in which case also $vf_0(x_0) \ge 0$. We also find that $0 \le vx_0 \le \min_i vc_ix_i^p \le vc_jx_j^p = vf_j(x_j)$ for all $j \ge 1$. Hence, $\min_i vf_i(x_i) \ge 0$. Now it follows from our assumption that $v\sum_{i=0}^m f_i(x_i) > 0$. We set $y_0 = -\sum_{i=0}^m f_i(x_i)$. Observe that $vy_0 > 0$ implies that $vy_0^p > vy_0$. Hence,

$$v\left(\sum_{i=0}^m f_i(x_i) \;-\; \wp(y_0)\right) \;=\; vy_0^p \;>\; vy_0 \;=\; v\sum_{i=0}^m f_i(x_i)\;.$$

Taking $y_i = 0$ for $i \ge 1$, we obtain that (10) holds. □

If in the situation of this lemma, (K,v) is henselian, then we can even get that $\sum_{i=0}^m f_i(y_i) = \sum_{i=0}^m f_i(x_i)$. Indeed, using that $\mathcal{M} \subset \wp(K)$, in the second part of the proof we just have to choose $y_0 \in K$ such that $\wp(y_0) = \sum_{i=0}^m f_i(x_i)$.

Lemma 17 *Assume that (K,v) is a valued field of characteristic $p > 0$ with $t \in K$ such that vt is the smallest positive element in the value group vK. Then the elements $1,t,t^2,\ldots,t^{p-1}$ are K^p-valuation independent.*

Proof: For every choice of elements $d_0,\ldots,d_{p-1}$ we have that $vt^id_i^p \in ivt + pvK$. As vt is the smallest positive element of vK by assumption, the cosets $pvK, vt+pvK, 2vt+pvK,\ldots,(p-1)vt+pvK$ are all distinct. This shows that $vt^id_i^p \ne vt^jd_j^p$ for $0 \le i < j \le p-1$. Hence, $v(d_0 + td_1 + \ldots + t^{p-1}d_{p-1}) = \min_{0\le i\le p-1} vt^id_i\,.$ □

Now Lemma 4 follows from Lemmas 16 and 17.

A valued field (K,v) is called **inseparably defectless** if the fundamental equality holds for every finite purely inseparable extension. We will need the following characterization of inseparably defectless fields, which was proved by F. Delon [D] (see also [K2]):

Lemma 18 *Take a valued field (K,v) of characteristic $p > 0$ such that $(vK : pvK) < \infty$ and $[Kv : (Kv)^p] < \infty$. Then (K,v) is inseparably defectless if and only if*

$$[K:K^p] \;=\; (vK:pvK)[Kv:(Kv)^p]\;. \tag{11}$$

From this lemma we obtain:

Lemma 19 *Assume that (K,v) is an inseparably defectless valued field of characteristic $p>0$ with $t\in K$ such that vt is the smallest positive element in the value group vK. Assume further that vK is a $\mathbb{Z}$-group and that Kv is perfect. Then $1,t,t^2,\ldots,t^{p-1}$ is a basis of $K|K^p$.*

Proof: By Lemma 17 and our remark in the beginning of this section we know that $1,t,t^2,\ldots,t^{p-1}$ are K^p-linearly independent. By the foregoing lemma, (11) holds. Since vK is a $\mathbb{Z}$-group, we have that $(vK:pvK)=p$. Since Kv is perfect, we have that $[Kv:(Kv)^p]=1$. Thus, $[K:K^p]=p$, which shows that $1,t,t^2,\ldots,t^{p-1}$ is a basis of $K|K^p$. □

Lemma 20 *Let the assumptions be as in Lemma 19. Take $z\in K$ and assume that the set*

$$\{v(z-y)\mid y\in\wp(K)+tK^p+\ldots+t^{p-1}K^p\} \tag{12}$$

admits a maximum. Then this maximum is either 0 or ∞ (the latter meaning that z lies in $\wp(K)+tK^p+\ldots+t^{p-1}K^p$).

Proof: Assume that $y_0\in K$ is such that $v(z-y_0)$ is the maximum of (12). After replacing z by $z-y_0$ we can assume that $y_0=0$.

Suppose that $vz>0$. Then $vz=\infty$ since otherwise, we could set

$$y\;:=\;-z^p+z\;=\;(-z)^p-(-z)+t\cdot 0+\ldots+t^{p-1}\cdot 0\;\in\;\wp(K)+tK^p+\ldots+t^{p-1}K^p$$

to obtain that $v(z-y)=vz^p>vz$, a contradiction.

Now suppose that $vz<0$. We have to deduce a contradiction from this assumption. By Lemma 19, we can write

$$z\;=\;b_0^p+tb_1^p+\ldots+t^{p-1}b_{p-1}^p\quad\text{with } b_0,\ldots,b_{p-1}\in K\;.$$

By Lemma 17 we have that

$$\min_{0\le i\le p-1} vt^ib_i^p\;=\;v(b_0^p+tb_1^p+\ldots+t^{p-1}b_{p-1}^p)\;=\;vz\;<\;0\;.$$

Hence if $vb_0<0$, then

$$vb_0\;>\;pvb_0\;=\;vb_0^p\;\ge\;\min_{0\le i\le p-1} vt^ib_i^p\;=\;vz\;.$$

On the other hand, if $vb_0\ge 0$, then $vb_0>vz$, too. Hence in every case,

$$v(z\;-\;(\wp(b_0)+tb_1^p+\ldots+t^{p-1}b_{p-1}^p))\;=\;vb_0\;>\;vz\;,$$

a contradiction to the maximality of vz. □

Proof of Lemma 6: Assume that (K,v) satisfies axiom system (1) and (PDOA). By Lemma 4, $\mathrm{PD}(\wp(X), tX^p, \ldots, t^{p-1}X^p)$ holds in (K,v). Thus, (PDOA) yields that $\mathrm{OA}(\wp(X), tX^p, \ldots, t^{p-1}X^p)$ holds in (K,v). Take $z \in K$ and suppose that $z \notin \wp(K) + tK^p + \ldots + t^{p-1}K^p$. Then by Lemma 20, there is some $y \in \wp(K) + tK^p + \ldots + t^{p-1}K^p$ such that $v(z-y) = 0$. Since $Kv = \mathbb{F}_p$, there is some $j \in \mathbb{F}_p$ such that $v(z-y-j) > 0$. Again by Lemma 20, we obtain that $z-y-j \in \wp(K) + tK^p + \ldots + t^{p-1}K^p$. This proves that K satisfies (7). □

4 Construction of a crucial example

We need some preparations. The **rank** of (K,v) is the number of proper convex subgroups of the value group vK (if finite); (K,v) has rank 1 if and only if vK is archimedean, i.e., embeddable in the additive group of the reals. If (K,v) has rank n, then v is the composition of n valuations of rank 1.

Here is a well-known fact about pseudo Cauchy sequences.

Lemma 21 *Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in a valued field (L,v) (where λ is a limit ordinal). If b is not a pseudo limit of this sequence, then there is some $\nu_0 < \lambda$ such that for all $\nu \geq \nu_0$, $\nu < \lambda$,*

$$v(b - a_\nu) \;<\; v(a_{\nu_0+1} - a_{\nu_0}) \;.$$

We will need the following characterization of henselian defectless fields; for a proof, see [K1], [K2].

Theorem 22 *Let (K,v) be an inseparably defectless field of characteristic $p > 0$. If in addition (K,v) is algebraically maximal, then (K,v) is henselian defectless.*

Now we are ready for the construction of a basic example, which we will then use to prove Theorem 3. Let K be a field of characteristic $p > 0$. Further, assume that $[K : K^p]$ is finite, and choose a basis $c_0, \ldots, c_m$ of $K|K^p$ with $c_0 = 1$. We work in the power series field $K((s^{\mathbb{Q}}))$ with its canonical (s-adic) valuation v_s. As this is henselian, it contains the henselization of the subfield $K(s^{1/n} \mid n \in \mathbb{N}\,, (p,n) = 1)$ with respect to (the restriction of) v_s. We will denote this henselization by L_1. We have that

$$v_s L_1 \;=\; \sum_{n\in\mathbb{N}\,,\,(p,n)=1} \frac{1}{n}\mathbb{Z}\;. \tag{13}$$

In particular, $1/q \in v_s L_1$ and $s^{1/q} \in L_1$ for every prime number $q \neq p$.

We take a strictly increasing sequence of prime numbers q_j, $j \in \mathbb{N}$, such that

$$p^{j+1} \;<\; q_j \quad \text{for all } j \in \mathbb{N}\;. \tag{14}$$

In particular, $p < q_j$ and thus $s^{1/q_j} \in L_1$ for all $j \in \mathbb{N}$. Now assume that ζ is a pseudo limit of the pseudo Cauchy sequence

$$\left(\sum_{j=1}^{k} s^{-1/q_j}\right)_{k\in\mathbb{N}} \tag{15}$$

in some extension of (L_1, v_s). Using the method employed in Example 16.1 of [K5] one shows by use of Hensel's Lemma that

$$v_s K(s,\zeta) \;=\; \sum_{j\in\mathbb{N}} \frac{1}{q_j}\mathbb{Z}\;.$$

By the fundamental inequality, the fact that $(v_s K(s,\zeta) : v_s K(s)) = (v_s K(s,\zeta) : \mathbb{Z})$ is not finite shows that ζ must be transcendental over $K(s)$, and thus also over its algebraic extension L_1. By virtue of Theorem 3 of [KA], this proves that the pseudo Cauchy sequence $(\sum_{j=1}^{k} s^{-1/q_j})_{k\in\mathbb{N}}$ in (L_1, v_s) cannot be of algebraic type; hence it must be of transcendental type.

We will now construct a purely inseparable algebraic extension L_2 of L_1 such that $c_0, \ldots, c_m$ is again a basis of $L_2|L_2^p$. We define recursively

$$\xi_1 = s^{-1/p} \;\text{ and }\; \xi_{j+1} = (\xi_j - c_1 s^{-p/q_j})^{1/p}\;. \tag{16}$$

Since v_s is trivial on K, we have that $v_s c_1 = 0$. Using this and (14), one shows by induction on j that

$$v_s\xi_j \;=\; -\frac{1}{p^j} \;<\; -\frac{p}{q_j} \;=\; v_s(c_1 s^{-p/q_j}) \;<\; 0 \quad\text{for all } j \in \mathbb{N}\;. \tag{17}$$

We put

$$L_2 \;:=\; L_1(\xi_j \mid j \in \mathbb{N})\;.$$

To prove that $c_0, \ldots, c_m$ generate the extension $L_2|L_2^p$, we take any $a \in L_2$. Then for a suitable $k \in \mathbb{N}$,

$$a \in L_1(\xi_1, \ldots, \xi_k) = L_1(\xi_k)\;.$$

Now one deduces by induction that $c_\mu \xi_k^\nu$, $0 \le \mu \le m$, $0 \le \nu < p$, is a basis for $L_1(\xi_k)|L_1(\xi_k)^p$ and that

$$\xi_k \;=\; \xi_{k+1}^p + c_1 s^{-p/q_k} \in L_1(\xi_{k+1})^p + c_1 L_1(\xi_{k+1})^p \;\subset\; K.L_2^p\;.$$

This shows that

$$a \in \sum_{\mu,\nu} c_\mu \xi_k^\nu L_1(\xi_k)^p \subset K.L_2^p \;=\; c_0 L_2^p + c_1 L_2^p + \ldots + c_m L_2^p\;.$$

Hence, $L_2 = c_0 L_2^p + c_1 L_2^p + \ldots + c_m L_2^p$.

Since the extension $L_2|L_1$ is purely inseparable, there is a unique extension w of v_s to L_2. (Note that we are now working outside of $K((s^{\mathbb{Q}}))$). For each j we have that $\frac{1}{p^j} = w\xi_j \in wL_1(\xi_j)$ and thus, $(wL_1(\xi_j) : v_sL_1) \geq p^j = [L_1(\xi_j) : L_1]$. By the fundamental inequality, $[L_1(\xi_j) : L_1] \geq (wL_1(\xi_j) : v_sL_1)$. Hence, $[L_1(\xi_j) : L_1] = (wL_1(\xi_j) : v_sL_1)$, and

$$wL_1(\xi_j) \;=\; v_sL_1 \,+\, \frac{1}{p^j}\mathbb{Z}\,. \tag{18}$$

Again by the fundamental inequality it follows that $L_1(\xi_j)w = L_1v_s = K$ and therefore,

$$L_2w \;=\; \bigcup_{j\in\mathbb{N}} L_1(\xi_j)w \;=\; K\;.$$

Hence,

$$[L_2 : L_2^p] \;\geq\; [L_2w : L_2^pw] \;=\; [L_2w : (L_2w)^p] \;=\; [K : K^p] \;=\; m+1\;,$$

which proves that $c_0,\ldots,c_m$ are L_2^p-linearly independent. Hence, $c_0,\ldots,c_m$ form a basis of $L_2|L_2^p$.

By (18) and (13),

$$wL_2 \;=\; \bigcup_{j\in\mathbb{N}} wL_1(\xi_j) \;=\; \bigcup_{j\in\mathbb{N}} \left(v_sL_1 \,+\, \frac{1}{p^j}\mathbb{Z}\right) \;=\; \mathbb{Q}\;. \tag{19}$$

Now we choose (L,w) to be a maximal immediate algebraic extension of (L_2,w) (which exists by Zorn's Lemma since its cardinality is bounded by that of the algebraic closure of L_2). Then

$$\begin{aligned}
[L : L^p] \;&\leq\; [L_2 : L_2^p] \;=\; m+1 \;=\; [K : K^p] \;=\; [L_2w : (L_2w)^p]\\
&=\; [L_2w : (L_2w)^p]\cdot(\mathbb{Q} : p\mathbb{Q}) \;=\; [L_2w : (L_2w)^p]\cdot(wL_2 : pwL_2)\\
&=\; [Lw : (Lw)^p]\cdot(wL : pwL) \;\leq\; [L : L^p]\;.
\end{aligned}$$

Hence, equality holds everywhere. By Lemma 18, the last equality implies that (L,w) is inseparably defectless. Since (L,w) is a maximal immediate algebraic extension and thus algebraically maximal, Theorem 22 shows that (L,w) is a henselian defectless field.

We set

$$x \;:=\; s^{-1}\;.$$

Assume that there is an extension $(L'|L,v)$ such that $L'v = Lv$, and that there exist elements $x_0,x_1,\ldots,x_m,y \in L'$ such that

$$x = y + x_0^p - x_0 + c_1x_1^p + \ldots + c_mx_m^p \quad \text{with } wy \geq 0\;. \tag{20}$$

We wish to show that then x_1 must be a pseudo limit of the pseudo Cauchy sequence (15), which yields that x_1 is transcendental over L. This in turn shows that (20) cannot hold in L.

Suppose that x_1 is not a pseudo limit of (15). Then by Lemma 21, there exists some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$w\left(x_1 - \sum_{j=1}^{k} s^{-1/q_j}\right) \;<\; ws^{-1/q_{k_0+1}} \;=\; -\frac{1}{q_{k_0+1}}\;. \tag{21}$$

We can choose k large enough to also guarantee that $p^k > q_{k_0+1}$, that is,

$$-\frac{1}{q_{k_0+1}} \;<\; -\frac{1}{p^k} \;=\; w\xi_k\;. \tag{22}$$

We set

$$\tilde{x_0} \;:=\; x_0 - \sum_{j=1}^{k} \xi_j \quad \text{and} \quad \tilde{x}_1 \;:=\; x_1 - \sum_{j=1}^{k-1} s^{-1/q_j}\;. \tag{23}$$

According to (21) and (22), we have that

$$pw\tilde{x}_1 \;<\; w\tilde{x}_1 \;<\; w\xi_k \;<\; 0\;. \tag{24}$$

Now we compute

$$\begin{aligned}
\tilde{x_0}^p - \tilde{x_0} &= x_0^p - x_0 + (-\sum_{j=1}^{k} \xi_j)^p + \sum_{j=1}^{k} \xi_j \\
&= x_0^p - x_0 - \xi_1^p - \sum_{j=1}^{k-1} (\xi_{j+1}^p - \xi_j) + \xi_k \\
&= x_0^p - x_0 - x + \sum_{j=1}^{k-1} c_1 s^{-p/q_j} + \xi_k \\
&= \xi_k - y - (c_1\tilde{x}_1^p + c_2 x_2^p + \ldots + c_m x_m^p)\;.
\end{aligned}$$

Since $w\xi_{k+1} < 0$ and $wy \geq 0$, and by virtue of (24), we have that

$$0 \;>\; w(\xi_k - y) \;=\; w\xi_k \;>\; pw\tilde{x}_1 \;=\; w\tilde{x}_1^p \;\geq\; \min\{w\tilde{x}_1^p, wx_2^p, \ldots, wx_m^p\} \;=:\; \alpha\;.$$

We set $\tilde{x}_i := x_i$ for $2 \leq i \leq m$ and take $i_1, \ldots, i_\ell \in \{1, \ldots, m\}$ to be all indices i for which $w\tilde{x}_i^p = \alpha$. Then $w(\tilde{x}_{i_\nu}^p / \tilde{x}_{i_1}^p) = 0$ and $w((\xi_{k+1} - y)/\tilde{x}_{i_1}^p) > 0$. Therefore, and since the elements $1, c_1, \ldots, c_m \in K$ are linearly independent over $K^p = (Lw)^p = (L'w)^p$,

$$\frac{\tilde{x_0}^p - \tilde{x_0}}{\tilde{x}_{i_1}^p} w \;=\; \left(c_{i_1}\frac{\tilde{x}_{i_1}^p}{\tilde{x}_{i_1}^p} + \ldots + c_{i_\ell}\frac{\tilde{x}_{i_\ell}^p}{\tilde{x}_{i_1}^p}\right) w \;=\; c_{i_1} + c_{i_2}\left(\frac{\tilde{x}_{i_2}}{\tilde{x}_{i_1}} w\right)^p + \ldots + c_{i_\ell}\left(\frac{\tilde{x}_{i_\ell}}{\tilde{x}_{i_1}} w\right)^p \notin (L'w)^p\;.$$

In particular, the residue is nonzero, which implies that

$$w(\tilde{x_0}^p - \tilde{x_0}) \;=\; \tilde{x}_{i_1}^p \;=\; \alpha \;<\; 0\;.$$

This yields that $w\tilde{x_0} < 0$. Consequently, $w\tilde{x_0} > w\tilde{x_0}^p$ and thus,

$$\frac{\tilde{x_0}^p - \tilde{x_0}}{\tilde{x}_{i_1}^p} w \;=\; \frac{\tilde{x_0}^p}{\tilde{x}_{i_1}^p} w \;=\; \left(\frac{\tilde{x_0}}{\tilde{x}_{i_1}} w\right)^p \;\in\; (L'w)^p \;.$$

This contradiction proves that x_1 must be a pseudo limit of (15).

Now we take K to be any field of characteristic p containing an element t such that (2) holds (for example, we may take $K = \mathbb{F}_p(t)$, $K = \mathbb{F}_p(t)^h$ or $K = \mathbb{F}_p((t))$). Then we can set $m = p-1$ and $c_i = t^i$ for $0 \leq i \leq m$. We obtain that the existential sentence

$$\exists Y \exists X_0 \ldots \exists X_{p-1}\;\; x = Y + X_0^p - X_0 + tX_1^p + \ldots + t^{p-1}X_{p-1}^p \;\wedge\; \mathcal{O}(Y) \tag{25}$$

does not hold in (L, w). So we have proved:

Theorem 23 *Let K be any field of characteristic $p > 0$ containing an element t such that $K = K^p \oplus tK^p \oplus \ldots \oplus t^{p-1}K^p$. Then there exists a henselian defectless field (L, w), not satisfying property (5), of transcendence degree 1 over its embedded residue field K, having value group $wL = \mathbb{Q}$, and such that*

$$L \;=\; K.L^p \;=\; L^p \oplus tL^p \oplus \ldots \oplus t^{p-1}L^p \;,$$

where "$K.L^p$" denotes the field compositum of K and L^p in L.

Now we can give the

Proof of Theorem 3: We take K to be the field $\mathbb{F}_p(t)^h$ or $\mathbb{F}_p((t))$, with v_t the t-adic valuation on K. We denote by v the composition $w \circ v_t$ of w with v_t on L; this can actually be viewed as an extension of v_t to L. We note that v is finer than w, that is, $\mathcal{O}_v \subset \mathcal{O}_w$. This means that $vy \geq 0$ implies $wy \geq 0$; therefore, since (5) doesn't hold for (L, w), it doesn't hold for (L, v).

We have mentioned already that both $(\mathbb{F}_p(t)^h, v_t)$ and $(\mathbb{F}_p((t)), v_t)$ are defectless fields. On the other hand, we know from our construction that (L, w) is a henselian defectless field. Since the composition of henselian defectless valuations is again henselian defectless (cf. [K2]), it follows that for both choices of K, (L, v) is a henselian defectless field. Since $wL = \mathbb{Q}$ and $v_t(Lw) = v_tK = \mathbb{Z}$, we have that $vt = v_tt$ is the smallest positive element of vL, $\mathbb{Z}vt$ is a convex subgroup of vL, and $vL/\mathbb{Z}vt \simeq \mathbb{Q}$. Hence, vL is a Z-group. Further, $Lv = (Lw)v_t = Kv_t = \mathbb{F}_p$. By construction, $1, t, t^2, \ldots, t^{p-1}$ is a basis of $L|L^p$.

Finally, it remains to show that $L|K$ is regular. Take any finite extension $K'|K$ and an extension of v from L to $K'.L$. Since (K, v_t) is henselian, the restriction of v from $K'.L$ to K' is the unique extension of v_t from K to K'. We set $\mathrm{e} := (vK' : vK)$ and $\mathrm{f} := [K'v : Kv_t]$. Since (K, v_t) is defectless, we have that $[K' : K] = \mathrm{ef}$. As $v_tK = \mathbb{Z}v_tt$, there is $t' \in K'$ such that $\mathrm{e}vt' = vt$; therefore, $t' \in K'.L$ yields that $(vK'.L : vL) \geq \mathrm{e}$. Since $Kv_t = \mathbb{F}_p = Lv$, we also find that $[(K'.L)v : Lv] = [(K'.L)v : \mathbb{F}_p] \geq [K'v : \mathbb{F}_p] = \mathrm{f}$. Thus,

$$[K' : K] \;=\; \mathrm{ef} \;\leq\; (vK'.L : vL)[(K'.L)v : Lv] \;\leq\; [K'.L : L] \;\leq\; [K' : K] \;.$$

Therefore, equality must hold everywhere, showing that $L|K$ is linearly disjoint from $K'|K$. Since $K'|K$ was an arbitrary finite extension, this proves that $L|K$ is regular. $\square$

Remark 24 These examples also show that a field which is relatively algebraically closed in a henselian defectless field that satisfies (5) does itself not necessarily satisfy (5), even if the extension is immediate. Indeed, every maximal immediate extension of our examples (L,v) or (L,w) is a maximal field and thus satisfies (5) according to Theorem 7, and L is relatively algebraically closed in every such extension since (L,v) and (L,w) are henselian defectless and thus algebraically maximal.

From [A] or [RO], we know that in any model of (1) and for each prime $q \neq p$,

$$\mathcal{O}(Y) \;\Leftrightarrow\; \exists Z\; Z^q = 1 + tY^q\;.$$

Hence, the formula (25) is equivalent to an existential formula in the language of rings. By Corollary 7, this formula holds in every maximal immediate extension of (L,v). Since it does not hold in (L,v), we obtain:

Corollary 25 *L is not existentially closed, not even in the language of rings, in any maximal immediate extension of (L,v).*

By a modification of our construction, for every $n \in \mathbb{N}$ we can construct (L,v) in such a way that in addition to the assertions of Theorem 3, the following holds:

If $(L'|L,v)$ is an extension such that $L'v = Lv$ and (PDOA) holds in (L',v), then $\mathrm{trdeg}\, L'|L \geq n$.

If we do not insist in $L|\mathbb{F}_p$ having finite transcendence degree, then we can even get that $\mathrm{trdeg}\, L'|L$ must be infinite.

5 Henselian rationality of immediate function fields

A valued function field $(F|L,v)$ of transcendence degree 1 is called **henselian rational** if there is some $x \in F^h$ such that $F^h = L(x)^h$. In [K1] we have proved that every immediate function field $(F|L,v)$ of transcendence degree 1 over a tame field (L,v) is henselian rational (see also [K2]). This fact has important applications to the model theory of fields ([K7]) as well as to the problem of local uniformization in positive characteristic (cf. [K3], [K5], [K6]). If henselian rationality were true over a larger class of ground fields, this would allow us to improve significantly our results on local uniformization. This being so, it is important to know that there are immediate function fields $(F|L,v)$ of transcendence degree 1 even over a henselian defectless ground field (L,v) which are not henselian rational. With (L,v) as given in Theorem 3 as a ground field, we can construct a rather simple such function field (Theorem 29 below). We will study its structure in detail, as it appears to be closely related to examples studied in singularity theory. Note that we did our construction of (L,v) not only over $\mathbb{F}_p((t))$, but also over the much smaller field $\mathbb{F}_p(t)^h$ because then the function field is of transcendence degree 3 over $\mathbb{F}_p$ and serves to deduce an observation about the role of transcendence bases in local uniformization (cf. [K3] and Section 19 of [K5]). Note also that at present, the only known method for showing that a function field is not henselian rational is the model theoretic method which we will employ below.

For the construction of the valued function field over (L, v), we need two auxiliary lemmas. Note that it is immediate from the definition that a pseudo Cauchy sequence of transcendental type in (k, v) will never have a pseudo limit in k.

Lemma 26 *Take any henselian field (k, v) and an immediate extension $(k(x)|k, v)$ such that x is the pseudo limit of a pseudo Cauchy sequence of transcendental type in (k, v). Then (k, v) is existentially closed in the henselization $(k(x), v)^h$ of $(k(x), v)$.*

Proof: Let x be the pseudo limit of the pseudo Cauchy sequence $(x_\nu)_{\nu<\lambda}$ of transcendental type in (k, v). We take (k^*, v^*) to be a $|k|^+$-saturated elementary extension of (k, v). Every finite subset of the set $\{\mathcal{O}((X - a_\nu)/(a_{\nu+1} - a_\nu)) \mid \nu < \lambda\}$ of atomic $\mathcal{L}$-sentences with parameters from k is satisfied by a_μ if μ is bigger than all ν appearing in that subset. Thus, there is an element $x^* \in k^*$ which simultaneously satisfies all sentences in this set. Then x^* is a pseudo limit of $(x_\nu)_{\nu<\lambda}$ (we leave the easy proof to the reader). By Theorem 2 of [KA], $x \mapsto x^*$ induces a valuation preserving isomorphism of $k(x)$ onto $k(x^*)$. Since (k^*, v^*) is an elementary extension of (k, v) and "henselian" is an elementary property, (k^*, v^*) is also henselian. Hence by the universal property of henselizations (cf. [R] or [K2]), this isomorphism can be extended to an embedding of $(k(x), v)^h$ in (k^*, v^*). Now every existential $\mathcal{L}$-sentence with parameters from k holding in $(k(x), v)^h$ carries over to (k^*, v^*) through the embedding. Since $(k, v) \prec (k^*, v^*)$, it will therefore also hold in (k, v). $\square$

Lemma 27 *Take a henselian field (k, v), a polynomial $f \in k[X]$ of degree $p = \mathrm{char}\, kv$, and a root a of f. Suppose that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence which does not fix the value of f and has no pseudo limit in (k, v). Then there is an immediate extension of v from k to $k(a)$ such that a is a limit of $(a_\nu)_{\nu<\lambda}$ in $(k(a), v)$.*

Proof: We pick a polynomial $g \in k[X]$ of minimal degree with the property that $(a_\nu)_{\nu<\lambda}$ does not fix the value of g. Take a root b of g. Then by Theorem 3 of [KA] there is an immediate extension of v from k to $k(b)$. Since (k, v) is assumed to be henselian, we have $\mathrm{e} = \mathrm{f} = \mathrm{g} = 1$. By the Lemma of Ostrowski (cf. [R] or [K2]), it follows that $\deg g = [k(b) : k] = [k(b) : k]/\mathrm{efg}$ is a power of p. This proves that f is of minimal degree with the property that $(a_\nu)_{\nu<\lambda}$ does not fix the value of f. Hence, our assertion follows by a second application of Theorem 3 of [KA]. $\square$

Proposition 28 *Let L be the field given by our construction. Then there exists a regular function field F of transcendence degree 1 and generated by two elements over L such that*

L is not existentially closed in F (in the language of rings), but w and v have immediate extensions from L to F.

Proof: We will show that the existential sentence

$$\exists X_0 \ldots \exists X_{p-1} \;\; x = X_0^p - X_0 + tX_1^p + \ldots + t^{p-1} X_{p-1}^p \tag{26}$$

holds already in $L(x_0, x_1)$, where

$$(L(x_0, x_1)|L, w)$$

is a regular immediate function field with $L(x_0, x_1)|L(x_1)$ an Artin-Schreier extension. As $L(x_0, x_1)w = Lw = K$, we can again define $v = w \circ v_t$ on $L(x_0, x_1)$. Then also $(L(x_0, x_1)|L, v)$ is immediate. This will imply the assertion of our lemma.

We take x_1 to be a transcendental element over L. Using Theorem 2 of [KA], we extend w to $L(x_1)$ in such a way that x_1 becomes a pseudo limit of the pseudo Cauchy sequence (15) and $(L(x_1)|L, w)$ is an immediate extension. Then we define a pseudo Cauchy sequence $(a_k)_{k\in\mathbb{N}}$ by setting

$$a_k \;:=\; \sum_{j=1}^{k} \xi_j \;. \tag{27}$$

Now we compute for all $k \in \mathbb{N}$, using (16) and (17):

$$\begin{aligned}
&w(a_k^p - a_k - (x - tx_1^p)) \;=\\
&\;=\; w\left((\sum_{j=1}^{k} \xi_j)^p - \sum_{j=1}^{k} \xi_j - (x - tx_1^p) \right) \;=\; w\left(\xi_1^p + \sum_{j=1}^{k-1} (\xi_{j+1}^p - \xi_j) - \xi_k - (x - tx_1^p) \right)\\
&\;=\; w\left(x - t\sum_{j=1}^{k-1} s^{-p/q_j} - \xi_k - (x - tx_1^p) \right) \;=\; w\left(t(x_1 - \sum_{j=1}^{k-1} s^{-1/q_j})^p - \xi_k \right)\\
&\;=\; \min\left\{ wt(x_1 - \sum_{j=1}^{k-1} s^{-1/q_j})^p, w\xi_k \right\} \;=\; \min\left\{ wts^{-p/q_k}, w\xi_k \right\} \;=\; w\xi_k \;=\; -\frac{1}{p^k}
\end{aligned}$$

(where the first equality of the last line holds since $wts^{-p/q_k} \neq w\xi_k$). This shows that the pseudo Cauchy sequence $(a_k)_{k\in\mathbb{N}}$ does not fix the value of the Artin–Schreier polynomial

$$X^p - X - (x - tx_1^p) \;. \tag{28}$$

Also, we see that a pseudo limit x_0 of $(a_k)_{k\in\mathbb{N}}$ in an arbitrary extension of (L, v) will satisfy

$$w(x_0^p - x_0 - (x - tx_1^p)) \;>\; -\frac{1}{p^k} \quad \text{for all } k \in \mathbb{N} \;,$$

whence

$$w(x_0^p - x_0 - (x - tx_1^p)) \;\geq\; 0 \;.$$

This means that the existential sentence (25) holds in $(L(x_0,x_1),w)$. From Lemma 26 we know that (L,w) is existentially closed in the henselization $(L(x_1),w)^h$ of the immediate rational function field $(L(x_1),w)$. So if x_0 were an element of this henselization, (25) would also hold in (L,w), contrary to what we have already proved. This contradiction shows that the pseudo Cauchy sequence $(a_k)_{k\in\mathbb{N}}$ has no pseudo limit in $(L(x_1),w)^h$. Hence by virtue of Lemma 27, if x_0 is any root of the polynomial (28), then there is an immediate extension of the valuation w from $L(x_1)^h$ to $L(x_1)^h(x_0)$. Its restriction is an immediate extension of w from $L(x_1)$ to the Artin-Schreier extension $L(x_0,x_1)$. Now (26) is satisfied in $L(x_0,x_1)$, as desired.

Using that $(F|L,v)$ is immediate, the regularity of $F|L$ can be shown in a similar way as the regularity of $L|K$ was shown in the proof of Theorem 3. This completes the proof of our lemma. □

Theorem 29 *The immediate function fields $(F|L,w)$ and $(F|L,v)$ of the foregoing proposition are not henselian rational.*

Proof: Suppose that $(F|L,v)$ is henselian rational: $F^h = L(x)^h$ with $x \in F^h$. Since (L,v) is henselian defectless, it is algebraically maximal. Hence in view of Theorem 3 of [KA], any pseudo Cauchy sequence in (L,v) without a pseudo limit in (L,v) must be of transcendental type. Note that $x \notin L$ since otherwise, $F = L$. By Theorem 1 of [KA], x is a pseudo limit of a pseudo Cauchy sequence in (L,v) without a pseudo limit in (L,v), which is consequently of transcendental type. Hence by Lemma 26, (L,v) is existentially closed in (F,v), contradicting the fact that L is not even existentially closed in F. This proves that $(F|L,v)$ is not henselian rational. The same argument holds with w in the place of v. □

The function field F that we have constructed shows the following symmetry between a generating Artin-Schreier extension and a generating purely inseparable extension of degree p. On the one hand, we have the Artin-Schreier extension

$$L(x_0,x_1)|L(x_1)$$

given by

$$x_0^p - x_0 \;=\; x - tx_1^p\;. \tag{29}$$

On the other hand we have the purely inseparable extension

$$L(x_0,x_1)|L(x_0)$$

given by

$$x_1^p \;=\; \frac{1}{t}(-x_0^p + x_0 + x)\;.$$

From equation (29) it is immediately clear that the function field $L(x_0, x_1)$ becomes rational after a constant field extension by $t^{1/p}$; namely

$$F(t^{1/p}) \;=\; L(t^{1/p})(x_0 + t^{1/p}x_1)\;.$$

This shows that the base field L, not being existentially closed in the function field F, becomes existentially closed in the function field after a finite purely inseparable constant extension, although this extension is linearly disjoint from $F|L$.

In our above example there exists also a separable constant extension $L'|L$ of degree p such that $(F.L')^h$ is henselian rational. To show this, we take a constant $d \in L$ and an element a in the algebraic closure of L satisfying

$$t \;=\; a^p - da\;,$$

and we put $L' = L(a)$. If we choose d with a sufficiently high value, then we will have that $vdax_1^p > 0$. From this we deduce by Hensel's Lemma that there is an element $b \in L'(x_1)^h$ such that $b^p - b = -dax_1^p$. If we put $z = x_0 + ax_1 + b \in L'(x_0, x_1)^h$, we get that

$$z^p - z \;=\; x - tx_1^p + a^p x_1^p - ax_1 - dax_1^p \;=\; x - ax_1 + (a^p - da - t)x_1^p \;=\; x - ax_1\;,$$

which shows that

$$x_1 \;\in\; L'(z)\;.$$

This in turn yields that $b \in L'(z)^h$ and consequently,

$$x_0 \;=\; z - ax_1 - b \in L'(z)^h\;.$$

Altogether, we have proved that

$$L'(x_0, x_1)^h \;=\; L'(z)^h$$

is henselian rational.

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