Ball Spaces – Generic Fixed Point Theorems for Contracting Functions

Franz-Viktor Kuhlmann
joint work with Katarzyna Kuhlmann

Dresden, 18. Januar 2018

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For most FPTs some sort of "completeness" property of X is needed.

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Note: In metric spaces, the existence of fixed points is usually proved by means of Cauchy sequences, not by means of metric balls.

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Example: $\mathbb{Q}$ together with the p-adic metric is an ultrametric space. More generally, every (Krull) valuation induces an ultrametric.

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An ultrametric space (X,u) is called spherically complete if the intersection of every nest of balls is nonempty.

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- *If for every ball $B \in \mathcal{B}$, $f(B)$ contains an f-contracting ball, then f has a fixed point in every ball.*
- *If $X \in \mathcal{B}$ and for every ball $B \in \mathcal{B}$, $f(B)$* **is** *an f-contracting ball, then f has a* **unique** *fixed point.*

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Theorem (B_x-Theorem)

Every self-contractive function on a spherically complete ball space has a fixed point.

Proof of the ultrametric FPT

Theorem (S.Prieß-Crampe, P. Ribenboim)

Take a function f on a spherically complete ultrametric space (X,u) such that for all $x,y \in X$:
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Then $x \in B_x$, so (SC1) holds,
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so the above theorem follows from the B_x-Theorem.

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The Caristi-Kirk FPT

A function φ from a metric space (X,d) to $\mathbb{R}$ is called lower semicontinuous if for every $y \in X$,

$$\liminf_{x\to y} \varphi(x) \;\geq\; \varphi(y)\,.$$

Theorem (Caristi-Kirk)

Take a complete metric space (X,d) and a lower semicontinuous function $\varphi : X \to \mathbb{R}$ which is bounded from below. If a function $f : X \to X$ satisfies the Caristi condition

(CC) $\quad d(x,fx) \;\leq\; \varphi(x) - \varphi(fx),$

then f has a fixed point on X.

Caristi-Kirk balls

We set

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$$\mathcal{B}_\varphi \;:=\; \{B_x \mid x \in X\}\,.$$

Lemma

Take any function $\varphi : X \to \mathbb{R}$ and a function $f : X \to X$ that satisfies condition (CC). Then f is self-contractive in the ball space $(X, \mathcal{B}_\varphi)$.

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Let (X, d) be a metric space. Then the following statements are equivalent:

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Let (X, d) be a metric space. Then the following statements are equivalent:

(i) *The metric space (X, d) is complete.*

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Theorem

Let (X, d) be a metric space. Then the following statements are equivalent:

(i) *The metric space (X, d) is complete.*

(ii) *Every Caristi-Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete.*

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(iii) *For every continuous function $\varphi : X \to \mathbb{R}$ bounded from below, the Caristi-Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete.*

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(i) *The metric space (X, d) is complete.*

(ii) *Every Caristi-Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete.*

(iii) *For every continuous function $\varphi \colon X \to \mathbb{R}$ bounded from below, the Caristi-Kirk ball space $(X, \mathcal{B}_\varphi)$ is spherically complete.*

Together with the previous lemma and the B_x-Theorem, this proves the Caristi-Kirk FPT.

Ordered sets, abelian groups and fields

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Does the spherical completeness of $(X, \mathcal{B})$ imply cut completeness? Then it would not be interesting for ordered abelian groups and fields since all cut complete ordered abelian groups and fields are isomorphic to $\mathbb{R}$.

- Take a nest $\mathcal{N}$ of closed intervals $[a_\nu, b_\nu]$ indexed by ordinals $\nu < \lambda$ such that if $\mu > \nu$, then $a_\nu \leqslant a_\mu \leqslant b_\mu \leqslant b_\nu$.

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- Then $\bigcap \mathcal{N} \neq \emptyset$ if and only if there is $x \in X$ such that $a_\nu \leq x \leq b_\nu$ for every ν.
- If $\bigcap \mathcal{N} = \emptyset$, then $\mathcal{N}$ determines a Dedekind cut which is not filled in X.

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A symmetric argument works when the cofinality of D is larger than the coinitiality of E.

The cut (D, E) is called asymmetric if the cofinality of D is not equal to the coinitiality of E. By what we have seen, nests of intervals $[a_\nu, b_\nu]$ over asymmetric cuts will always have nonempty intersection.

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Theorem

$(X, \mathcal{B})$ is spherically complete if and only if X is symmetrically complete.

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- (2013) In joint work with S. Shelah we extended his result to ordered sets and abelian groups, characterized all symmetrically complete ordered abelian groups and fields, and proved an analogue of the Banach FPT. (Israel J. Math. **208** (2015))

What are the balls in topological spaces?

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Theorem

The ball space $(X, \mathcal{B})$ is spherically complete if and only if X is compact.

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Observe that in both 1) and 2) the ball spaces have a stronger property than just spherical completeness: intersections of nests contain balls or are themselves balls. We will now introduce a classification of ball spaces according to these stronger properties. But first we will give an overview of the various instances of spherical completeness that we have talked about so far.

What we got so far

spaces	balls	completeness property
ultrametric spaces	all closed ultrametric balls	spherically complete
metric spaces	metric balls with radii in suitable sets of positive real numbers Caristi-Kirk balls	complete
linearly ordered sets, ordered abelian groups and fields	all intervals $[a,b]$ with $a \leq b$	symmetrically complete
topological spaces	all nonempty closed sets	compact
posets	all intervals $[a,\infty)$	inductively ordered

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Moreover, observe that in a topological space arbitrary intersections of closed sets are again closed. What about ball spaces in which all (nonempty) intersections of nests, directed systems, or centered systems, are again balls?

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We will also write $\mathbf{S}^*$ for $\mathbf{S}_4^c$ because this turns out to be the "star" (the strongest) among the ball spaces:

Hierarchy of spherical completeness

$$\begin{array}{ccccccc}
\mathbf{S}_1 & \Leftarrow & \mathbf{S}_1^d & \Leftarrow & \mathbf{S}_1^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_2 & \Leftarrow & \mathbf{S}_2^d & \Leftarrow & \mathbf{S}_2^c & & \\
\Uparrow & & \Uparrow & & \Uparrow & & \\
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\Uparrow & & \Uparrow & & \Uparrow & & \\
\mathbf{S}_4 & \Leftarrow & \mathbf{S}_4^d & \Leftarrow & \mathbf{S}_4^c & =: & \mathbf{S}^*
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Theorem

1) If the ball space $(X, \mathcal{B})$ is finitely intersection closed, then $\mathbf{S}_i^d$ is equivalent to $\mathbf{S}_i^c$, for $i = 1, 2, 3, 4$.

Intersection closed ball spaces

A ball space $(X, \mathcal{B})$ will be called finitely intersection closed if $\mathcal{B}$ is closed under nonempty intersections of any finite collection of balls, and it will be called intersection closed if $\mathcal{B}$ is closed under nonempty intersections of arbitrary collections of balls.

Theorem

1) If the ball space $(X, \mathcal{B})$ is finitely intersection closed, then $\mathbf{S}_i^d$ is equivalent to $\mathbf{S}_i^c$, for $i = 1, 2, 3, 4$.
2) If the ball space $(X, \mathcal{B})$ is intersection closed, then all properties in the hierarchy are equivalent.

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- The ball space of a lattice with top and bottom, consisting of all intervals of the form $[a, b]$, is finitely intersection closed, and it is intersection closed if and only if the lattice is complete.

Summary of $\mathbf{S}^*$ ball spaces

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- A poset is a complete lattice if and only if it has a bottom and a top element and the ball space defined by its nonempty closed intervals $[a, b]$ is S^* .

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$\mathrm{scl}_{\mathcal{B}}(S) \in \mathcal{B}$ *is the smallest ball containing* S.

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Proposition

The associated ball space of $\mathbb{R}$ is $\mathbf{S}^*$. For all other densely ordered abelian groups or fields the associated ball space can at best be $\mathbf{S}_1$ or $\mathbf{S}_2$.

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Theorem

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Is there an analogue for ball spaces?
Can it be used to transfer the Knaster-Tarski FPT to other applications?

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Let $\mathrm{Fix}(f)$ be the set of fixed points of f, and set

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An open question for topologists

As we are rather interested in the topology on $\mathrm{Fix}(f)$ induced by the original topology of X, we ask:

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Open question: Give a natural criterion on f which guarantees that

$$\{B \cap \mathrm{Fix}(f) \mid B \in \mathcal{B}^f\} = \{B \cap \mathrm{Fix}(f) \mid B \in \mathcal{B}\}.$$

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Theorem

Take **S** *to be any of the properties in the hierarchy of spherical completeness. The product* $(\prod_{j \in J} X_j, \mathcal{B})$ *has property* **S** *if and only if all ball spaces* $(X_j, \mathcal{B}_j)$, $j \in J$, *have property* **S**.

Smaller products

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As in topology when the open sets of the products are defined, one can ask that $B_j = X_j$ for all but finitely many $j \in J$. This leads to a smaller set $\mathcal{B}'$ of balls.
If $(\prod_{j\in J} X_j, \mathcal{B})$ is spherically complete, then so is $(\prod_{j\in J} X_j, \mathcal{B}')$.
However, the stronger properties of the hierarchy may get lost under this restriction.

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The Tychonoff theorem for topological spaces

In which way does Tychonoff's theorem follow from its analogue for ball spaces? The problem in the case of topological spaces is that the product ball space we have defined, while containing only closed sets of the product, does not contain all of them, as it is not necessarily closed under finite unions and arbitrary intersections. If we close it under these operations, are its spherical completeness properties maintained?

Closure under finite unions

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

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In order to prove this theorem, we need a lemma that is inspired by Alexander's Subbase Theorem:

Lemma

If $\mathcal{S}$ is a maximal centered system of balls in $\mathcal{B}'$ (that is, no subset of $\mathcal{B}'$ properly containg $\mathcal{S}$ is a centered system),

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Closure under unions and intersections

From the previous two theorems we obtain:

Theorem

Take an $\mathbf{S}_1^c$ *ball space* $(X, \mathcal{B})$*. If* $\mathcal{B}'$ *is obtained from* $\mathcal{B}$ *by first closing under finite unions and then under arbitrary nonempty intersections, then:*

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b) $\mathcal{B}'$ is intersection closed,

c) $(X, \mathcal{B}')$ is an $\mathbf{S}^*$ ball space.

The topology associated to a ball space

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Which are the topologies we obtain in this way?

Example: the p-adics

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However, $\mathbb{Q}_p$ is known to be locally compact, but not compact. But this refers to the topology which has the balls

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Example: the p-adics

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as basic open sets. It turns out that this topology is finer than the one we derived from the ball space.

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In joint work with R. Bartsch we are considering the question: what are possible notions of continuity for functions on ball spaces?

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Take two ball spaces $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ and a function $f : X \to X'$.

A notion of continuity for functions on ball spaces?

Theorem

a) If

$$\{f^{-1}(B') \mid B' \in \mathcal{B}'\} \;\subseteq\; \mathcal{B}$$

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and $(X',\mathcal{B}')$ is spherically complete, then so is $(X,\mathcal{B})$.
c) If

$$\mathcal{B} = \{f^{-1}(B') \mid B' \in \mathcal{B}'\}, \tag{1}$$

then $(X,\mathcal{B})$ is spherically complete if and only if $(X',\mathcal{B}')$ is.

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then $(X, \mathcal{B})$ is spherically complete if and only if $(X', \mathcal{B}')$ is.
d) If (1) holds and f is surjective, then the posets $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic.

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and f is finite-to-one, and if $(X', \mathcal{B}')$ is spherically complete, then so is $(X, \mathcal{B})$.

Further research

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• Interaction with notions from category theory.
• Applications to domain theory. (Dream.)

The end

Thank you for your attention — and

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stay tuned for further developments!