APPROXIMATION OF ELEMENTS IN HENSELIZATIONS

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Abstract. For valued fields K of rank higher than 1, we describe how elements in the henselization K^h of K can be approximated from within K; our result is a handy generalization of the well-known fact that in rank 1, all of these elements lie in the completion of K. We apply the result to show that if an element z algebraic over K can be approximated from within K in the same way as an element in K^h, then $K(z)$ is not linearly disjoint from K^h over K.

1. Introduction

Complete valued fields of rank 1 are henselian, but for valuations v of arbitrary rank, this does not hold in general. However, there is a connection between Hensel's Lemma and completions, but these completions have to be taken for residue fields of suitable coarsenings of v. This connection was worked out by Ribenboim [R] who used **distinguished pseudo Cauchy sequences** to characterize the so called **stepwise complete** fields; it had been shown by Krull that these fields are henselian. We want to give a more precise description of this connection.

Take any extension $(L|K,v)$ of valued fields, that is, an extension $L|K$ of fields and a valuation v on L. By vL and vK we denote the value groups of v on L and on K, and by Lv and Kv the residue fields of v on L and on K, respectively. Similarly, vz and zv denote the value and the residue of an element z under v. For $z \in L$, we define

$$v(z-K) \;:=\; \{v(z-c) \mid c \in K\} \;\subseteq\; vL \cup \{\infty\}\;.$$

We call z **weakly distinguished over** K if there is a non-trivial convex subgroup Δ of vK and some $\alpha \in vK$ such that the coset $\alpha + \Delta$ is cofinal in $v(z-K)$, that is, $\alpha + \Delta \subseteq v(z-K)$ and for all $\beta \in v(z-K)$ there is $\gamma \in \alpha + \Delta$ such that $\beta \leq \gamma$. If this holds with $\alpha = 0$, that is, if some non-trivial convex subgroup of vK is cofinal in $v(z-K)$, then we call z **distinguished over** K. This name is chosen since distinguished elements induce distinguished pseudo Cauchy sequences in the sense of Ribenboim [R], p. 105.

The extension $(L|K,v)$ is **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto.

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Now take an arbitrary valued field (K,v) and extend its valuation v to its algebraic closure $\tilde{K}$. Then $\tilde{K}$ contains a unique henselization K^h with respect to this extension. We will prove:

Theorem 1.1. *Every element $a \in K^h \setminus K$ is weakly distinguished over K. In particular, the henselization is an immediate extension of (K,v).*

Note that if (K,v) is of rank 1, that is, has archimedean ordered value group, then its henselization lies in its completion and every element $a \notin K$ of the henselization K^h is distinguished over K (with $\Delta = vK$).

We will give two proofs for Theorem 1.1. The first one is an adaptation of the proof found in [Z–S] for the fact that the henselization of a valued field is an immediate extension. The second proof uses the fact that the henselization can be constructed as a union of finite extensions generated by roots of polynomials that satisfy the conditions of Hensel's Lemma.

By "$\alpha > v(a-K)$" we mean $\alpha > v(a-c)$ for all $c \in K$. We use Theorem 1.1 to prove the following result:

Theorem 1.2. *Take $z \in \tilde{K} \setminus K$ such that*

$$v(a-z) \;>\; v(a-K)$$

for some $a \in K^h$. Then K^h and $K(z)$ are not linearly disjoint over K, that is,

$$[K^h(z) : K^h] < [K(z) : K]$$

and in particular, $K(z)|K$ is not purely inseparable.

Theorem 1.1 answers a question from Bernard Teissier. Theorem 1.2 has a crucial application in [Ku3] to the classification of Artin-Schreier extensions with non-trivial defect. This classification was originally obtained in [Ku1] under the additional assumption that the fields in question are henselian. With the help of Theorem 1.2 this assumption can be dropped, and so the classification becomes available for valued function fields.

Theorems 1.1 and 1.2 were also proved in [Ku1], but the proofs given in Sections 5 and 6 are improved versions of the original proofs, using much less technical machinery, and the proof of Theorem 1.1 given in Section 4 is new.

2. Some preliminaries

We will assume the reader to be familiar with the basic facts of valuation theory, and we will often use them without further references. We recommend [End], [Eng–P], [R], [W], [Z–S] and [Ku2] for the general valuation theoretical background.

If v and w are two valuations on a field K and $\mathcal{O}_v$ and $\mathcal{O}_w$ are their valuation rings, then w is called a **coarsening** of v if $\mathcal{O}_v \subseteq \mathcal{O}_w$. If this holds, then $vc \geq vd$ implies $wc \geq wd$ and in particular, $vc \geq 0$ implies $wc \geq 0$ and $wd > 0$ implies $vd > 0$.

As we are working with valued fields (K,v) of higher rank (that is, with non-archimedean ordered value groups vK), we will use convex subgroups Δ of vK and the corresponding coarsenings of v. The ordering of vK induces an ordering on vK/Δ: the set of positive elements in the latter group is just the image under the canonical epimorphism of the set vK^+ of positive elements in vK. Hence, $\alpha \geq \beta$ implies $\alpha + \Delta \geq \beta + \Delta$. More precisely, $\alpha + \Delta \geq \beta + \Delta$ holds if and only if there is some $\gamma \in \Delta$ such that $\alpha + \gamma \geq \beta$. The coarsening v_Δ of v is the valuation whose valuation ring is $\{c \in K \mid vc \in vK^+ \cup \Delta\}$; this contains the valuation ring $\mathcal{O}_v$ of v. The value group of v_Δ on K is canonically isomorphic to vK/Δ. Note that

$$v_\Delta c \,>\, 0 \iff vc \,>\, \Delta\,. \tag{2.1}$$

The valuation v also induces a valuation $\overline{v}_\Delta$ on the residue field Kv_Δ such that v is (equivalent to) the composition $v_\Delta \circ \overline{v}_\Delta$ (in this paper, we will identify equivalent valuations). If $\mathcal{O}_{v_\Delta}$ and $\mathcal{M}_{v_\Delta}$ denote the valuation ring and valuation ideal of v_Δ, then the valuation ring of $\overline{v}_\Delta$ is the image of $\mathcal{O}_v$ under the canonical epimorphism $\mathcal{O}_{v_\Delta} \rightarrow \mathcal{O}_{v_\Delta}/\mathcal{M}_{v_\Delta} = Kv_\Delta$. The value group of $\overline{v}_\Delta$ on Kv_Δ is canonically isomorphic to Δ via

$$\overline{v}_\Delta(a + \mathcal{M}_{v_\Delta}) \,\mapsto\, va \quad \text{for } a \notin \mathcal{M}_{v_\Delta}\,. \tag{2.2}$$

If $(L|K,v)$ is an arbitrary extension of valued fields, then the convex hull Γ of Δ in vL is a convex subgroup of vL, and v_Γ is an extension of v_Δ from K to L. If vL/vK is a torsion group (which is the case if $L|K$ is algebraic), then taking convex hulls induces a bijective inclusion preserving mapping from the chain of convex subgroups of vK to the chain of convex subgroups of vL, and v_Γ is the unique coarsening of v on L which extends v_Δ.

We will need some facts from ramification theory.

Lemma 2.1. *Let $(N|K,v)$ be an arbitrary normal algebraic extension and w a coarsening of v on N. Then*

$$(N|K)^{d(w)} \subseteq (N|K)^{d(v)} \subseteq (N|K)^{i(v)} \subseteq (N|K)^{i(w)}\,, \tag{2.3}$$

where $(N|K)^{d(v)}$ and $(N|K)^{d(w)}$ denote the decomposition fields of $(N|K,v)$ with respect to v and w, respectively, and $(N|K)^{i(v)}$ and $(N|K)^{i(w)}$ denote the inertia fields of $(N|K,v)$ with respect to v and w, respectively.

Proof. For $\sigma \in \mathrm{Gal}\,(N|K)$, $v \circ \sigma = v$ implies $w \circ \sigma = w$. Hence the decomposition group with respect to v is contained in the decomposition group with respect to w. This proves the first inclusion. The second inclusion is well known from ramification theory (cf. [Eng-P], p. 124). For $\sigma \in \mathrm{Gal}\,(N|K)$, if $w(x - \sigma x) > 0$ for all x such that $wx \geq 0$, then $v(x - \sigma x) > 0$ for all x such that $vx \geq 0$. Hence the inertia group with respect to w is contained in the inertia group with respect to v. This proves the third inclusion. □

Lemma 2.2. *Let $(N|K,w)$ be a finite normal extension of valued fields with decomposition field Z and inertia field T. If $z \in T$ then there is $c \in Z$ such that*

$$w(z-c) \;=\; \max w(z-Z) \;\in wZ\ .$$

Proof. From ramification theory we know that $n := [T:Z] = [Tw:Zw]$. We choose $b_1 = 1, \ldots, b_n \in T$ such that $wb_1 = \ldots = wb_n = 0$ and $b_1w, \ldots, b_nw$ is a basis of $Tw|Zw$. Then $b_1, \ldots, b_n$ are Z-linearly independent and thus form a basis of $T|Z$. Since $b_1w, \ldots, b_nw$ are Zw-linearly independent, we have that $w(c_1b_1 + \ldots + c_nb_n) = \min_{1\leq i\leq n} w(c_ib_i) \leq \min_{2\leq i\leq n} w(c_ib_i) = \min_{2\leq i\leq n} w(c_i) \in wZ$. Hence if $z = c_1b_1 + \ldots + c_nb_n$ and we set $c = c_1b_1 = c_1 \in Z$, then $w(z-c) = \min_{2\leq i\leq n} w(c_ib_i) = \max w(z-Z)$. □

3. Properties of weakly distinguished elements

Throughout this section, let $(L|K,v)$ be an extension of valued fields. General valuation theory tells us that the extension is immediate if and only if for every $z \in L \setminus K$ and every $c \in K$ there is $c' \in K$ such that $v(z-c') > v(z-c)$. This holds if z is weakly distinguished over K since then, $v(z-K)$ has no maximal element (as a non-trivial convex subgroup of vK has no maximal element). This proves:

Lemma 3.1. *If every $z \in L\setminus K$ is distinguished over K, then $(L|K,v)$ is immediate.*

A subset S of an ordered set T is a **final segment** of T if $S \ni \beta < \gamma \in T$ implies $\gamma \in S$, and an **initial segment** of T if $S \ni \beta > \gamma \in T$ implies $\gamma \in S$.

Lemma 3.2. *Take $z \in L$. If $v(z-K)$ has no maximal element, then it is an initial segment of vK.*

Proof. By our assumption, for every $c \in K$ there is $c' \in K$ such that $v(z-c) < v(z-c')$, whence $v(z-c) = \min\{v(z-c), v(z-c')\} = v(c'-c) \in vK$. This proves that $v(z-K) \subseteq vK$. If $v(z-c) > \gamma \in vK$, then take $d \in K$ such that $vd = \gamma$ to obtain that $\gamma = vd = \min\{v(z-c), vd\} = v(z-(c+d)) \in v(z-K)$. This proves that $v(z-K)$ is an initial segment of vK. □

If $z \in L$ is distinguished over K with the convex subgroup Δ of vK cofinal in $v(z-K)$, and if Γ is the convex hull of Δ in vL, then for all $c \in K$, $v(z-c) \geq 0$ implies $v_\Gamma(z-c) \geq 0$ (but the converse is not true). On the other hand, $v_\Gamma(z-c) > 0$ is impossible since by (2.1) this would imply that $v(z-c) > \Gamma$, whence $v(z-c) > \Delta$, a contradiction. Therefore,

$$v(z-c) \;\geq\; 0 \;\Longrightarrow\; v_\Gamma(z-c) \;=\; 0\ . \tag{3.1}$$

We will denote by $(Kv_\Delta)^{c(\overline{v}_\Delta)}$ the completion of Kv_Δ with respect to $\overline{v}_\Delta$.

Lemma 3.3. *Take $z \in L$ and suppose that Δ is a non-trivial convex subgroup of vK and $\alpha \in vK$ such that $\alpha + \Delta$ is cofinal in $v(z-K)$ (so that z is weakly distinguished over K). Then $z \notin K$, $v(z-K) \subseteq vK$, and $\alpha + \Delta$ is a final segment of $v(z-K)$.*

If in addition $\alpha = 0$ *(so that* z *is distinguished over* K*),* Γ *is the convex hull of* Δ *in* vL*, and* $v_\Gamma z = 0$*, then*

$$zv_\Gamma \in (Kv_\Delta)^{c(\overline{v}_\Delta)} \setminus Kv_\Delta \,. \tag{3.2}$$

Conversely, if there exists a decomposition $v = v_\Gamma \circ \overline{v}_\Gamma$ *on* L *such that (3.2) holds, then* z *is distinguished over* K *with* $\Delta = \Gamma \cap vK$ *cofinal in* $v(z-K)$*.*

Proof. Suppose that Δ is a non-trivial convex subgroup of vK and $\alpha \in vK$ such that $\alpha + \Delta$ is cofinal in $v(z-K)$. Then $v(z-K)$ has no maximal element, and Lemma 3.2 shows that $v(z-K) \subseteq vK$. In particular, $\infty \notin v(z-K)$, which shows that $z \notin K$. Since Δ and hence also $\alpha + \Delta$ is convex in vK, the assumption that $\alpha + \Delta$ is cofinal in $v(z-K) \subseteq vK$ implies that it is a final segment of $v(z-K)$.

Now suppose that Δ is cofinal in (and hence a final segment of) $v(z-K)$, Γ is the convex hull of Δ in vL, and $v_\Gamma z = 0$. Then $0, \infty \neq zv_\Gamma \in Kv_\Gamma$. Via the isomorphism (2.2), let us identify the value group of $\overline{v}_\Delta$ on Kv_Δ with Δ and the value group of $\overline{v}_\Gamma$ on Kv_Γ with Γ. Take any $\delta \in \Delta$. Then we can choose $d \in K$ such that $v(z-d) > \delta$. This implies that $v(z-d) \in \Delta$, whence $v_\Gamma(z-d) = 0$ so that $(z-d)v_\Gamma \in Lv_\Gamma$. Now $\overline{v}_\Gamma((z-d)v_\Gamma) = v(z-d) > \delta$. This yields that $\overline{v}_\Gamma(zv_\Gamma - dv_\Delta) > \delta$. Since $\delta \in \Delta$ was arbitrary, we see that $zv_\Gamma \in (Kv_\Delta)^{c(\overline{v}_\Delta)}$. On the other hand, if zv_Γ would lie in Kv_Δ and thus would equal dv_Δ for some $d \in K$, then we would have that $v_\Gamma(z-d) > 0$ and hence $v(z-d) > \Delta$, a contradiction.

For the converse, let Δ be any convex subgroup of vK and Γ its convex hull in vL, and assume that (3.2) holds. Then for every $\delta \in \Delta$ there is $d \in K$ such that $\overline{v}_\Gamma(zv_\Gamma - dv_\Gamma) = \delta$, that is, $v(z-d) = \delta$. This shows that $\Delta \subseteq v(z-K)$. But there is no $d \in K$ such that $v(z-d) > \Delta$ since otherwise, $zv_\Gamma - dv_\Gamma = (z-d)v_\Gamma = 0$, which would mean that $zv_\Gamma \in Kv_\Delta$. □

Lemma 3.4. *Take any coarsening* w *of* v *on* L*. If* $z \in L$ *is weakly distinguished over* K *with respect to* w*, then also with respect to* v*.*

Proof. We denote by Γ_w the convex subgroup of vL associated with the coarsening w. Via the canonical isomorphism, we identify wL with vL/Γ_w. Further, $\Delta_w = \Gamma_w \cap vK$ is the convex subgroup of vK associated with the restriction of w to K, and wK is canonically isomorphic to vK/Δ_w. We denote by $\overline{\Delta}$ the convex subgroup and by $\overline{\alpha}$ the element of wK such that $\overline{\alpha} + \overline{\Delta}$ is cofinal in $w(z-K)$. We choose $\alpha \in vK$ such that $\alpha + \Delta_w = \overline{\alpha}$. We set $\Delta = \{\delta \in vK \mid \delta + \Delta_w \in \overline{\Delta}$; this is a convex subgroup of vK.

We show that $\alpha + \Delta$ is cofinal in $v(z-K)$. Take any $c \in K$. By assumption there is $\overline{\delta} \in \overline{\Delta}$ such that $\overline{\alpha} + \overline{\delta} > w(z-c) = v(z-c) + \Gamma_w$. Take $\delta \in \Delta$ such that $\delta + \Delta = \overline{\delta}$; then $\alpha + \delta > v(z-c)$. On the other hand, for every $\delta \in \Delta$ we can take $\beta \in \Delta$ and some $c' \in K$ such that $\overline{\alpha} + \overline{\delta} < \overline{\alpha} + \overline{\beta} \leq w(z-c')$. This implies that $\alpha + \delta < v(z-c')$. This completes our proof. □

We leave the easy proof of the following lemma to the reader.

Lemma 3.5. *Take $z \in L$ and $b, c \in K$, $b \neq 0$. Then*

$$v(bz + c - K) \;=\; vb + v(z - K)\;.$$

Consequently,

1) $bz + c$ is weakly distinguished over K if and only if z is,

2) if z is distinguished over K, then $bz + c$ is weakly distinguished over K,

3) if z is weakly distinguished over K, then there is some $d \in K$ such that dz is distinguished over K.

Lemma 3.6. *Let $(L|K, v)$ and $(L(z)|L, v)$ be arbitrary extensions of valued fields. Assume that every element $x \in L \setminus K$ is weakly distinguished over K. If z is weakly distinguished over L, then also over K.*

Proof. From Lemma 3.1 we know that $vL = vK$. We have that

$$v(z - K) \;\subseteq\; v(z - L)\;.$$

If "$=$" holds, we are done. So we assume that "$\neq$" holds. Then there exists an element $x \in L$ such that $v(z - c) < v(z - x)$ for every $c \in K$, whence $v(z - c) = v(x - c)$. This shows that

$$v(z - K) \;=\; v(x - K)\;.$$

Since x is weakly distinguished over K by hypothesis, this shows that also z is weakly distinguished over K. □

4. DISTINGUISHED ELEMENTS IN HENSELIZATIONS

Our goal in this section is to show that every element in the henselization K^h is weakly distinguished over K. For valuations of rank 1, this is a direct consequence of the well known fact that the completion of a valued field of rank 1 contains its henselization. Indeed, all elements in this completion and hence also all elements in the henselization that do not lie in K are distinguished over K.

If (K, v) is of rank > 1, then the distinct extensions of v to a given algebraic extension field L may not be independent. In this case, the Strong Approximation Theorem may fail. As a substitute, for the proof that the henselization is an immediate extension, Ribenboim [R] gives a generalized version of the Strong Approximation Theorem where the independence condition is replaced by conditions on the given data that have to be satisfied by the requested element. But in our context, the method of Zariski and Samuel [ZA–SA2] is more natural: it proceeds by induction on the number of extensions of the valuation v and treats dependent extensions by an investigation of suitable coarsenings of v. We adapt this method to prove the more informative Theorem 1.1.

Lemma 4.1. *Take a normal separable-algebraic extension $(N|K, v)$ of valued fields and assume that the distinct extensions of v from K to N are independent. Further, assume that $a \in N$ has the property that $v \neq v \circ \sigma$ on N for every $\sigma \in \mathrm{Gal}\,(N|K)$ such that $\sigma a \neq a$. Then a lies in the completion of (K, v).*

Proof. Given any $\alpha \in vK$, we have to show that there exists $c \in K$ such that $v(a-c) \geq \alpha$. All extensions of v from K to N are conjugate, that is, of the form $v \circ \sigma$ with $\sigma \in \mathrm{Gal}\,(N|K)$ (cf. [Eng–P], Theorem 3.2.15). As we assume that all of them are independent, the same will be true for the finitely many extensions of v from K to the normal hull $N_a \subseteq N$ of $K(a)|K$.

We show that $\sigma a \neq a$ implies that $v \neq v \circ \sigma$ already holds on N_a. Indeed, if the latter is false, then v and $v \circ \sigma$ are both extensions of $v = v \circ \sigma$ from N_a to N. Hence there is $\tau \in \mathrm{Gal}\,(N|N_a)$ such that $v = v \circ \sigma \circ \tau$ on N. The assumption of our lemma then yields that $\sigma\tau a = a$. As $\sigma\tau a = \sigma a$, we obtain that $\sigma a = a$.

We use the Strong Approximation Theorem (cf. [Eng–P], Theorem 2.4.1) to find $b \in N_a$ such that $v(a-b) \geq \alpha$ and $v(\sigma b) = (v \circ \sigma)\, b \geq \alpha$ whenever $\sigma a \neq a$. Writing $c = \sum_\sigma \sigma b$ for the trace $\mathrm{Tr}_{N_a|K}(b)$, we find that

$$v(a-c) \;\geq\; \min\{v(a-b), v\sigma b \mid \sigma|_{N_a} \neq \mathrm{id}\} \;\geq\; \alpha\;.$$

□

Lemma 4.2. *The assumption on the element a in Lemma 4.1 is satisfied when a lies in the decomposition field Z of $(N|K,v)$.*

Proof. If $\sigma a \neq a$ then $\sigma \notin \mathrm{Gal}\,(N|Z)$. As the latter is the decomposition group of $(N|K,v)$, this shows that $v \neq v \circ \sigma$ on N. □

Take a valued field (K,v) and extend v to the separable-algebraic closure K^{sep} of K. The henselization K^h of (K,v) is the decomposition field of $(K^{\mathrm{sep}}|K,v)$ (cf. [Eng–P], Theorem 5.2.2). From the two preceding lemmata, we obtain:

Corollary 4.3. *If $(N|K,v)$ is a normal separable-algebraic extension of valued fields and all extensions of v from K to N are independent, then the decomposition field of $(N|K,v)$ is contained in the completion of (K,v). If all extensions of v from K to K^{sep} are independent (which in particular is the case if the rank of (K,v) is 1), then K^h is contained in the completion of (K,v).*

Now we are ready for the

Proof of Theorem 1.1:
Since K^h is the decomposition field of $(K^{\mathrm{sep}}|K,v)$, it follows that for every normal separable-algebraic extension $(N|K,v)$, the decomposition field is $K^h \cap N$ (cf. [End], (15.6) c)). Hence K^h is the union over the decomposition fields of all finite normal separable-algebraic extensions of (K,v). Thus we may assume that a lies in the decomposition field (Z,v) of some finite Galois extension $(N|K,v)$. Let $v_1 = v, v_2, \ldots, v_n$ be all extensions of v from K to N. (Note that $n \geq 2$ because the assumption $a \notin K$ implies that $Z \neq K$.)

If $n \geq 3$, then suppose that the lemma is already proved for the case where the number of extensions of the valuation v from K to N is smaller than n. In view of Corollary 4.3, we only have to treat the case where the extensions $v_1, \ldots, v_n$ are not independent on N. Hence, there are i, j such that v_i and v_j admit a non-trivial common coarsening. The restriction of this coarsening to K is also a non-trivial coarsening of the valuation v on K. (Indeed, as $N|K$ is algebraic, restriction induces an inclusion preserving bijection between the coarsenings of v_i and the coarsenings of v on K which preserves inclusion between the corresponding valuation rings.) Among all the coarsenings of v on K that we find in this way, running through all common coarsenings of all possible pairs v_i and v_j, let w be the finest one. (Its valuation ring is the intersection of the valuation rings of all of these coarsenings.) We write $v = w \circ \overline{w}$. Now w admits an extension (again called w) to N which is a coarsening of at least two of the v_i's.

W.l.o.g., we may assume that w is a coarsening of $v_1 = v$. Indeed, since all extensions of v from K to N are conjugate, we may choose $\sigma \in \mathrm{Gal}\,(N|K)$ such that $v_i \circ \sigma = v_1$, and we obtain that $w \circ \sigma$ is an extension of w from K to N and a coarsening of $v_i \circ \sigma = v_1$ and of $v_j \circ \sigma \neq v_1$.

For the coarsening w of v, we may infer from Lemma 2.1, using the notation of that lemma:

$$(N|K)^{d(w)} \subset (N|K)^{d(v)} \subset (N|K)^{i(v)} \subset (N|K)^{i(w)} .$$

We set $L = (N|K)^{d(w)}$; note that $Z = (N|K)^{d(v)}$.

Every extension of w from K to N may be refined to an extension of v from K to N (just by composing it with any extension of $\overline{w}$ from Kw to Nw). Since the extension w gives already rise to at least two extensions of v from K to N, we see that there cannot be more than $n-1$ extensions of w from K to N. By our induction hypothesis, we find that every element $a \in L \setminus K$ is weakly distinguished over K with respect to w, and by Lemma 3.4, also with respect to v. Note that if there is only one extension of w from K to N, then $L = K$ and the assertion is trivially true. In view of Lemma 3.6, it now suffices to show that every element $z \in Z \setminus L$ is weakly distinguished over L.

Since Z is contained in $(N|K)^{i(w)}$, we may infer from Lemma 2.2 the existence of an element $c \in L$ such that $w(z-c) = \max w(z-L) \in wL$. We choose $b \in L$ such that $wb(z-c) = 0$. By Lemma 3.5, z is weakly distinguished over L if and only if $b(z-c)$ is. Consequently, we may assume $c = 0$, $b = 1$ and

$$0 \;=\; wz \;=\; \max w(z-L)$$

from the start.

After a suitable renumbering, we may assume that precisely the extensions $v_1 = v, v_2, \ldots, v_m$ of v are composite with w, and we write $v_j = w \circ \overline{w}_j$ for $1 \leq j \leq m$. Now (Z, v) is also the decomposition field of $(N|L, v)$ (cf. [End], (15.6) b)). Since L was chosen to be the decomposition field of $(N|K, w)$, the extension of w from L to N is unique. Every $\tau \in \mathrm{Gal}\,(Nw|Lw)$ is induced by some $\sigma \in \mathrm{Gal}\,(N|L)$ (this follows from [Eng-P], Lemma 5.2.6 (1)). If $\tau(zw) \neq zw$, then $\sigma z \neq z$, and by Lemma 4.2, $v \circ \sigma \neq v$ while $w \circ \sigma = w$ on N. This implies that $\overline{w}_1 \circ \tau \neq \overline{w}_1$ on Nw.

Furthermore, by our choice of w, it is the finest coarsening of v on K which is induced by a common coarsening of at least two v_i's. Consequently, the $\overline{w}_i$'s must be independent since otherwise, a common non-trivial coarsening of them could be composed with w to obtain a finer valuation, a contradiction. We have thus shown that the extension $(Zw|Lw, \overline{w_1})$ satisfies the hypotheses of Lemma 4.1. We conclude that zw lies in the completion of $(Lw, \overline{w}_1)$. On the other hand, $\max w(z-L) = 0$ shows that there is no element $c \in L$ such that $w(z-c) > 0$. This proves $zw \notin Lw$. Hence by Lemma 3.3, z is distinguished over L.

Now the second assertion of Theorem 1.1 follows from Lemma 3.1. □

5. Building up the henselization

We will give a different approach to the proof of Theorem 1.1. It starts with the following observation:

Lemma 5.1. *Let (K, v) be an arbitrary valued field and $f \in \mathcal{O}_v[X]$ be non-linear, monic and irreducible over K. Assume that $a \in \tilde{K}$ is a root of f such $av \in Kv$ and $vf'(a) = 0$. Then a is distinguished over K.*

Proof. From the Taylor expansion we infer the existence of some $\tilde{h}(X, Z) \in \mathcal{O}_v[X, Z]$ such that

$$f(Z) - f(X) = f'(X)(Z - X) + (Z - X)^2 \tilde{h}(X, Z) \; .$$

Since $av \in Kv$, there is $c \in \mathcal{O}_v$ such that $v(c - a) > 0$. Given any such c, we note that $vf'(c) = vf'(a) = 0$, which follows from the corresponding Taylor expansion since $f' \in \mathcal{O}_v[X]$. We set

$$c' \; := \; c - \frac{f(c)}{f'(c)} \; \in \; \mathcal{O}_v \; . \tag{5.1}$$

Then

$$f(c') - f(c) \; = \; f'(c)(c' - c) + (c' - c)^2 \tilde{h}(c', c) \; = \; -f(c) + \left(\frac{f(c)}{f'(c)}\right)^2 \tilde{h}(c', c)$$

with $\tilde{h}(c', c) \in \mathcal{O}_v$, so that

$$vf(c') \; = \; 2vf(c) + v\tilde{h}(c', c) \; \geq \; 2vf(c) \; . \tag{5.2}$$

On the other hand,

$$f(c) \; = \; f(c) - f(a) \; = \; f'(a)(c - a) + (c - a)^2 \tilde{h}(c, a) \; .$$

Since $vf'(a) = 0$, $v(c - a)^2 > v(c - a)$ and $v\tilde{h}(c, a) \geq 0$, it follows that

$$vf(c) \; = \; v(c - a) \; > \; 0 \; . \tag{5.3}$$

Now (5.1) implies that $v(c' - c) > 0$, whence $v(c' - a) > 0$. Replacing c by c' in the above argument, we obtain that $v(c' - a) = vf(c')$. Then using (5.2) and (5.3), we deduce that

$$v(c'-a) \;=\; vf(c') \;\geq\; 2vf(c) \;=\; 2v(c-a)\,. \tag{5.4}$$

First of all, this yields that $v(a-K)$ has no maximal element (note that $\infty\notin v(a-K)$ as $a\notin K$ by our assumption on f). Hence by Lemma 3.2, we know that the nonempty set of positive elements in $v(a-K)$ is convex in vK. Therefore, (5.4) yields that it is closed under addition and thus the set of positive elements of a convex subgroup. This proves that a is distinguished over K. $\square$

We will consider a very special type of immediate extensions $(K(z),v)|(K,v)$, and build up the henselization by a transfinite repetition of such extensions. We call an element z **strictly distinguished** over K if there exists a coarsening w of v such that the following three conditions hold:

(SD1) $wz=0\,,$

(SD2) $zw\in(Kw)^{c(\overline{w})}\setminus Kw\,,$

(SD3) for all $n\in\mathbb{N}$, if $1,z,\ldots,z^n$ are linearly independent over K, then $1,zw,\ldots,(zw)^n$ are linearly independent over Kw.

Here, $(Kw)^{c(\overline{w})}$ denotes the completion of Kw with respect to $\overline{w}$. The third condition implies that $[K(z):K]=[Kw(zw):Kw]$; in particular, if z is transcendental over K, then zw is transcendental over Kw. Lemma 3.3 shows that if z is strictly distinguished over K, then z is distinguished over K.

The next lemma shows that strictly distinguished elements generate extensions with a nice property. For $f\in\mathcal{O}_K[X]$, we denote by fv the polynomial obtained from f by replacing the coefficients by their v–residues.

Lemma 5.2. *Let z be strictly distinguished over K. Then every element $y\in K(z)\setminus K$ is weakly distinguished over K.*

Proof. Let the decomposition $v=w\circ\overline{w}$ be as in the above definition of strictly distinguished elements. In the first step, we will prove the lemma under the assumption that $y=f(z)$ with $f\in K[X]$ and $\deg f<[K(z):K]$ if the latter is finite. (If z is algebraic over K, then this assumption is no loss of generality.) By Lemma 3.5, for every $b\in K^\times$ and $c\in K$ we have that y is weakly distinguished over K if and only if $by-c$ is; after picking suitable elements b,c and replacing f by $bf-c$ we may thus assume that f has no constant term and that $f\in\mathcal{O}_{(K,w)}[X]\setminus\mathcal{M}_{(K,w)}[X]$. Consequently, $fw\not\equiv 0$, and since $wz=0$, we have $f(z)w=(fw)(zw)$. By our assumption on the degree of f, the elements $1,z,\ldots,z^{\deg f}$ are linearly independent over K, and by condition (SD3), the same holds for the elements $1,zw,\ldots,(zw)^{\deg f}$ over Kw. Hence $(fw)(zw)\notin Kw$. But since zw is an element of the completion of $(Kw,\overline{w})$, the element $f(z)w=(fw)(zw)$ also lies in the completion of $(Kw,\overline{w})$. In view of Lemma 3.3, this shows $f(z)$ to be weakly distinguished over K.

In the second step, it remains to prove the lemma for the case where z is transcendental over K and $y=f(z)/g(z)$ with $f,g\in K[X]$. By a similar argument as above, after multiplication of f and g (and hence of y) with suitable elements from $K^\times$, we may assume that $f,g\in\mathcal{O}_{(K,w)}[X]\setminus\mathcal{M}_{(K,w)}[X]$. To avoid the case where $(f(z)/g(z))w=(f(z)w)/(g(z)w)\in Kw$, we have to do the following. If $m=\deg gw$, then the m-th coefficient of g is not zero; hence there exists an element $d\in K$ such that the m-th coefficient of the polynomial $f-dg$ is 0. Again, after multiplication of $f-dg$ with a suitable element from $K^\times$, we may assume that $f-dg\in\mathcal{O}_{(K,w)}[X]\setminus\mathcal{M}_{(K,w)}[X]$. Then

$$\frac{(f(z)-dg(z))w}{g(z)w}\notin Kw\ ,$$

but this element lies in the completion of $(Kw,\overline{w})$ since the same holds for $(f(z)-dg(z))w$ and $g(z)w$. Since $(f-dg)/g=(f/g)-d$, it follows by Lemma 3.3 and Lemma 3.5 that $y=f(z)/g(z)$ is weakly distinguished over K. □

The next lemma shows how strictly distinguished elements appear in henselizations.

Lemma 5.3. *Let (K,v) be an arbitrary valued field and $f\in\mathcal{O}_v[X]$ be non-linear, monic and irreducible over K. Assume that $a\in\tilde{K}$ is a root of f such that av is an element of Kv and a simple root of fv. Then a is distinguished over K. If in addition, for every coarsening w of v either fw remains irreducible over Kw or admits a root in Kw with $\overline{w}$–residue av, then a is strictly distinguished over K.*

Proof. The first part of the lemma follows directly from Lemma 5.1 via a reformulation of the condition on a. Now let f satisfy the hypothesis of the second part. By the first part of the lemma, a is distinguished over K. By our assumption on f, we have that $va\geq 0$. An application of Lemma 3.3 thus yields $v_\Gamma a=0$ and $av_\Gamma\in(Kv_\Delta)^{c(\overline{v}_\Delta)}\setminus Kv_\Delta$, with Γ and Δ as in that lemma. It remains to show that a also satisfies condition (SD3) for $w=v_\Gamma$. Since av is a simple root of fv, we know that av_Γ is the only root of fv_Γ with $\overline{v}_\Gamma$–residue av. On the other hand, $av_\Gamma\notin Kv_\Delta$, and our hypothesis now yields that fv_Γ is irreducible over Kv_Δ. Since f is monic, we now have $[Kv_\Delta(av_\Gamma):Kv_\Delta]=\deg fv_\Gamma=\deg f=[K(a):K]$ which yields (SD3) for $w=v_\Gamma$. □

The additional condition on the polynomial f that we have introduced in the above lemma is not too restrictive:

Lemma 5.4. *The valued field (K,v) is henselian if and only if it satisfies the following "weaker" version of Hensel's Lemma:*
Let $f\in\mathcal{O}_v[X]$ be monic and $a\in K$ such that fv admits av as a simple zero. Assume in addition that fw admits a root with $\overline{w}$–residue av for every proper coarsening w of v for which fw is reducible. Then f admits a root in K with residue av.

Proof. We have to show the above version implies the original version of Hensel's Lemma (the one without the additional assumption). Assume that (K,v) is not henselian. Then there is some polynomial $g\in\mathcal{O}_v[X]$ having no root in K, and $a\in K$ such that gv admits av as a simple zero. Consider all coarsenings w of v, such that gw admits a factor g_w, irreducible over Kw and of degree > 1, and such that the $\overline{w}$–reduction $g_w\overline{w}$ admits av as a zero. Among these, we choose a coarsening w_0 for which g_{w_0} has least degree. Furthermore, we choose any $f \in \mathcal{O}_v[X]$ with $fw_0 = g_{w_0}$ and $\deg f = \deg g_{w_0}$. Then f satisfies the above condition: fv admits av as a simple zero, and for every coarsening w of v, the polynomial fw is either irreducible or admits a zero whose $\overline{w}$–residue is equal to av. But f does not admit any root in K since its w_0–reduction g_{w_0} is irreducible over Kw_0 and of degree > 1. This shows that (K, v) does not satisfy the above version of Hensel's Lemma. □

The henselization K^h can be generated over K by a transfinitely repeated adjunction of roots x of polynomials which satisfy the hypothesis of Hensel's Lemma. The foregoing lemma shows that this is also true if we replace Hensel's Lemma by the above version. In this case, in every step an element is adjoined which is strictly distinguished over the previous field, according to Lemma 5.3. The next lemma shows why we are choosing this procedure.

Lemma 5.5. *Let $(M|K, v)$ be an extension of valued fields generated by a set of elements $\{z_\nu \mid \nu < \tau\} \subset M$, where τ is an ordinal number, such that for every $\nu < \tau$, the element z_ν is strictly distinguished over $K_\nu := K(z_\mu|\mu < \nu)$ (where $K_0 := K$). Then every element $z \in M \setminus K$ is weakly distinguished over K.*

Proof. We prove the lemma by transfinite induction on $\rho < \tau$. The assertion holds trivially for the field K. Now assume $\rho \geq 1$ and that the assertion holds for every K_μ with $\mu < \rho$. If ρ is a limit ordinal, then $K_\rho = \bigcup_{\mu<\rho} K_\mu$ showing that the assertion holds for K_ρ too. Now let $\rho = \nu + 1$ be a successor ordinal. Then $K_\rho = K_\nu(z_\nu)$ where z_ν is strictly distinguished over K_ν. Let y be an arbitrary element of $K_\nu(z_\nu) \setminus K_\nu$. By Lemma 5.2, y is weakly distinguished over K_ν. By our induction hypothesis, every element $x \in K_\nu \setminus K$ is weakly distinguished over K. In view of Lemma 3.6, this yields that also y is weakly distinguished over K. Hence, the lemma holds for K_ρ, and the induction step is established. □

This lemma and Lemma 5.3 yield the following corollary, which together with Lemma 3.1 again proves Theorem 1.1:

Corollary 5.6. *Let K be a valued field. The henselization K^h can be generated over K in the way as described in the hypothesis of the foregoing lemma. Thus, every element in $K^h \setminus K$ is weakly distinguished over K.*

6. Proof of Theorem 1.2

We need the following lemma. We assume that the valuation v of a field K is extended to its algebraic closure $\tilde{K}$.

Lemma 6.1. *Assume that $y \in \tilde{K}$ and $v = w \circ \overline{w}$ on $\tilde{K}$ with $wy = 0$ and $yw \in K^h w \setminus Kw$. Then K^h and $K(y)$ are not linearly disjoint over K.*

Proof. Since K^h is henselian for the valuation v, it is also henselian for the coarsening w (because if w would admit two distinct extensions to the algebraic closure of K then we could use them to construct two distinct extensions of v).

Let $f(X) \in K[X]$ be the minimal polynomial of y over K. Our assertion is proved if we are able to show that f is reducible over K^h. At this point, we may assume that all conjugates of y over K have the same value vy since otherwise, the inequality $[K^h(y) : K^h] < [K(y) : K]$ is immediately seen to be true. This assumption together with $wy = 0$ yields that $f \in \mathcal{O}_w[X]$. Because f is monic, its reduction fw is non-trivial. The minimal polynomial $g \in Kw[X]$ for yw over Kw has degree > 1 since $yw \notin Kw$. Furthermore, it must divide fw because $(fw)(yw) = f(y)w = 0$. From Lemma 2.1, we infer that $K^h = (K^{\mathrm{sep}}|K)^{d(v)}$ lies in $L := (K^{\mathrm{sep}}|K)^{i(w)}$. Since $Lw|Kw$ is separable (cf. [Eng-P], Theorem 5.2.7.(1)), we find that yw is a simple root of g. Consequently, g has a second root $\xi \neq yw$ in $\widetilde{Kw} = \tilde{K}w$.

Applying Hensel's Lemma to the henselian field (K^h, w), we conclude that f becomes reducible over K^h; indeed, f factors into two non-trivial polynomials where the roots of the first one all have w–residue yw while there exists at least one root in $\tilde{K}$ of the second polynomial with ξ as its w–residue. This proves our lemma. □

An alternative proof of this lemma reads as follows. We use the fact that $K^h w$ is equal to the henselization of Kw with respect to $\overline{w}$. Hence by the hypothesis of the lemma, $(Kw(yw)|Kw, \overline{w})$ is thus a non-trivial subextension of $(Kw, \overline{w})^h|(Kw, \overline{w})$. By general ramification theory, it admits at least two extensions of the valuation $\overline{w}$ from Kw to $Kw(yw)$. Since these give rise to different extensions of the valuation v from K to $K(y)$, it again follows from general ramification theory that K^h and $K(y)$ are not linearly disjoint over K.

With the help of this lemma and Theorem 1.1, we are now able to give the

Proof of Theorem 1.2:
Assume that (K, v) is any valued field, v is extended to $\tilde{K}$, $z \in \tilde{K} \setminus K$ and $a \in K^h$ such that $v(z - a) > v(a - K)$. Then $a \notin K$ since otherwise, $\infty \in v(a - K)$ and $v(z - a) > \infty$, a contradiction.

Since $a \in K^h \setminus K$, Theorem 1.1 shows that a is weakly distinguished over K. By Lemma 3.5, there is $d \in K^\times$ and a convex subgroup Γ of $v\tilde{K}$ which is cofinal in $v(da - K) = vd + v(a - K)$. By Lemma 3.3, $(da)v_\Gamma \notin Kv_\Gamma$. As $v(z - a) > v(a - K)$ implies that $v(dz - da) > vd + v(a - K)$, we find that $v(dz - da) > \Gamma$. Thus, $(dz)v_\Gamma = (da)v_\Gamma \in K^h v_\Gamma \setminus Kv_\Gamma$. Now Lemma 6.1 shows that K^h and $K(z) = K(dz)$ are not linearly disjoint over K. □

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