Hyperstructures in topological categories

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Fachbereich Mathematik

Hyperspaces

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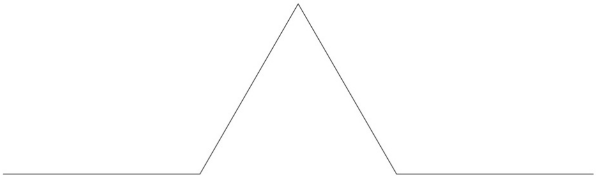
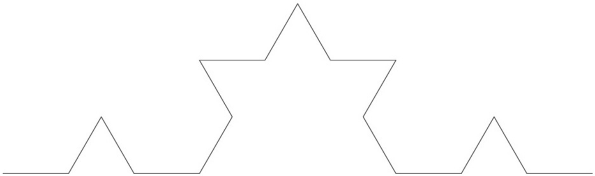
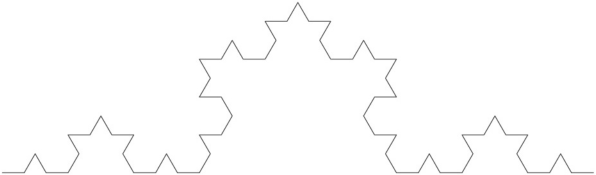
Motivation: a simple fractal

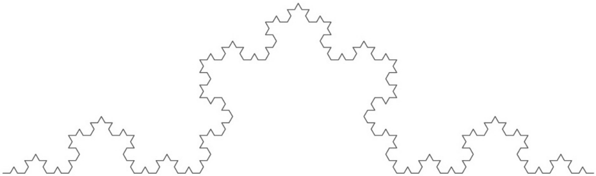
Let $X := \{(x, y) \in R^2 \mid \quad x^2 + y^2 \leq 37\}$,

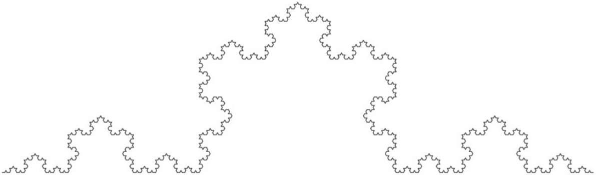
consider $K_0 := [-\frac{3}{2}, \frac{3}{2}] \times \{0\}$

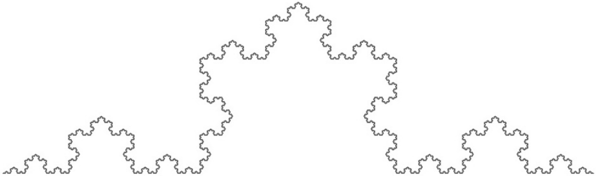












What have we done?

$$f_1 : X \to X : f_1(x) := \tfrac{1}{3}x + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$f_2 : X \to X : f_2(x) := \tfrac{1}{3}\left(\tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} x\right) + \tfrac{1}{4}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

$$f_3 : X \to X : f_3(x) := \tfrac{1}{3}\left(\tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} x\right) + \tfrac{1}{4}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

$$f_4 : X \to X : f_4(x) := \tfrac{1}{3}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$F : K(X) \to K(X) : F(S) := \bigcup_{i=1}^{4} f_i(S)$$

Does a limit object exist?

Yes, if:

1. We have a metric on $K(X)$, and
2. F is contractive on $K(X)$ w.r.t. that metric, and
3. $K(X)$ is complete w.r.t. that metric.

How can we get this?

ad 1. **Hausdorff metric** $d_{\mathcal{H}}$
ad 2. easy calculation because f_1, f_2, f_3, f_4 are contractive

ad 3.
Completeness is not just a metric notion, but a uniform.

We generalize the Hausdorff metric to the **Bourbaki-Uniformity**.

In uniform spaces we have **compact = precompact + complete**

So, if we prove compactness, we get completeness as a gift.

We generalize the Bourbaki-Uniformity to the Vietoris-Topology.

Let (X, τ) be a topological space, $\mathcal{H} \subseteq \mathfrak{P}(X)$ and $M \subseteq X$. We define

$$M^{+\mathcal{H}} := \left\{ H \in \mathcal{H} \middle| \, H \cap M = \emptyset \right\}$$

and

$$M^{-\mathcal{H}} := \left\{ H \in \mathcal{H} \middle| \, H \cap M \neq \emptyset \right\} .$$

(If there is no doubt about $\mathcal{H}$, we omit it in the superscript and write M^{+} resp. M^{-}.)

On $\mathcal{H} \subseteq \mathfrak{P}(X)$ now a topology τ_u is defines by the subbase

$$\left\{ A^{+} \middle| \, A \text{ closed in } X \right\}$$

called **upper Vietoris** topology.

Furthermore, by the subbase

$$\left\{ O^{-} \middle| \, O \in \tau \right\}$$

a topology τ_l is defined, called **lower Vietoris** topology.

$\tau_V := \tau_l \vee \tau_u$ is called **Vietoris topology**.

Let (X, d) be a metric space. On $K(X)$ holds:
The Bourbaki-Uniformity of $\mathcal{U}_d$ coincides with $\mathcal{U}_{d_{\mathcal{H}}}$.

Let $(X, \mathcal{U})$ be an uniform space. On $K(X)$ holds:
The Vietoris topology of $\tau_{\mathcal{U}}$ coincides with the topology generated from the Bourbaki-Uniformity.

$\Longrightarrow$ If we build the Vietoris topology from τ_d we get the same topology as is induced by the Hausdorff metric on $K(X)$.

Theorem

Let (X, τ) be a topological space, $Cl_0(X)$ the family of all nonempty closed subsets of X and τ_V the Vietoris topology for τ on $Cl_0(X)$. Then holds

$$(Cl_0(X), \tau_V) \text{ is compact} \Leftrightarrow (X, \tau) \text{ is compact.}$$

This applies to our problem with the limit object, yielding completeness, as needed.

Topological Categories

A concrete category $\mathcal{C}$ over **Set** is called *topological*, iff

1. For all $X \in |\mathbf{Set}|$ and all families $(f_i, (X_i, \xi_i))_{i \in I}$, indexed by a class I, of $\mathcal{C}$-objects (X_i, ξ_i) and functions $f_i : X \to X_i$ there exists a unique initial $\mathcal{C}$-Object (X, ξ) on the set X, i.e.

$$\forall (Y, \eta) \in |\mathcal{C}|, g : Y \to X :$$
$$g \in [(Y, \eta), (X, \xi)]_{\mathcal{C}} \quad \Leftrightarrow \quad \forall i \in I : f_i \circ g \in [(Y, \eta), (X_i, \xi_i)]_{\mathcal{C}}$$

2. (Fibre-smallness) For all $X \in |\mathbf{Set}|$, the class of $\mathcal{C}$-objects on X is a set.
3. On sets with at most one element exists exactly one $\mathcal{C}$-structure.

A category $\mathcal{C}$ is called **cartesian closed**, iff

4.
   4.1 For every pair (A, B) of $\mathcal{C}$-objects exists a product $A \times B$ in $\mathcal{C}$ and
   4.2 For every pair (A, B) of $\mathcal{C}$-objects exists a $\mathcal{C}$-object B^A and a $\mathcal{C}$-morphism $e : A \times B^A \to B$, s.t. for every $\mathcal{C}$-Object C and every $\mathcal{C}$-morphism $f : A \times C \to B$ there exists a unique $\mathcal{C}$-morphism $\overline{f} : C \to B^A$ with $f = e \circ (\mathbf{1}_A \times \overline{f})$.

A topological category $\mathcal{C}$ is said to be **extensional**, iff for every $\mathbf{Y} \in |\mathcal{C}|$ with underlying set Y, there exists a $\mathcal{C}$-object $\mathbf{Y}^*$ with underlying set $Y^* := Y \cup \{\infty_Y\}$, $\infty_Y \notin Y$, s.t. for every $\mathbf{X} \in \mathcal{C}$ with underlying set X, every $Z \subseteq X$ and every $f : Z \to Y$, where f is a $\mathcal{C}$-morphism w.r.t. the subobject $\mathbf{Z}$ of $\mathbf{X}$ on Z, the map $f^* : X \to Y^*$, defined by

$$f^*(x) := \begin{cases} f(x) & ; \quad x \in Z \\ \infty_Y & ; \quad x \notin Z \end{cases}$$

is a $\mathcal{C}$-morphism.

A topological category $\mathcal{C}$ is called a **topological universe**, iff it is cartesian closed and extensional.

Problem: How to define „natural“ Hyperstructures for arbitrary topological categories?

Selections

If X is a set and $\mathfrak{P}_0(X)$ the set of all nonempty subsets of X, let

$$\mathcal{A}(X) := \{f \in X^{\mathfrak{P}_0(X)} \mid \quad \forall A \in \mathfrak{P}_0(X) : f(A) \in A\}$$

the family of all *selections* on $\mathfrak{P}_0(X)$.

One can show, for instance:

Proposition

Let (X, τ) *be a locally compact topological space,* $P := \{p \in X \mid \exists f \in \mathcal{A}(X) : f(\widehat{\varphi}) \xrightarrow{\tau} p\}$ *and let* $\widehat{\varphi}$ *be an ultrafilter on* $\mathfrak{P}_0(X)$ *with* $\widehat{\varphi} \xrightarrow{\tau_l} A \in \mathfrak{P}(X)$. *Then* $A \subseteq \overline{P}$ *holds.*

By $\mathfrak{F}(M)$ we denote the set of all filters on a set M and by $\mathfrak{F}_0(M)$ the set of all ultrafilters on M.

Proposition

Let (X, τ) be a nested neighbourhood space, let $\widehat{\varphi}$ be an ultrafilter on $\mathfrak{P}_0(X)$ with $\widehat{\varphi} \xrightarrow{\tau_l} A \in \mathfrak{P}(X)$ and let
$P := \{p \in X \mid \exists \mathcal{F} \in \mathfrak{F}(\mathcal{A}(X)) : \mathcal{F}(\widehat{\varphi}) \xrightarrow{\tau} p\}$.
Then $A \subseteq P$ holds.

For uniform spaces we get even a quite nice characterization:

Theorem

Let $(X, \mathcal{U})$ be a uniform space, $\tau_{\mathcal{U}}$ the induced topology on X, and $\mathfrak{X}$ of compact subsets of X and $\hat{\mathcal{U}}$ the induced Bourbaki uniformity on $\mathfrak{X}$. For $\underline{\varphi} \in \mathfrak{F}(\mathfrak{X})$ are equivalent

1. $\underline{\varphi} \xrightarrow{\tau_{\hat{\mathcal{U}}}} A \in \mathfrak{X}$,
2. 2.1 $\forall f \in \mathcal{A}(X), \underline{\psi} \in \mathfrak{F}_0(\underline{\varphi}) : \exists a \in A : f(\underline{\psi}) \xrightarrow{\tau_{\mathcal{U}}} a$ *and*
   2.2 $\forall a \in A : \exists f \in \mathcal{A}(X) : f(\underline{\varphi}) \xrightarrow{\tau_{\mathcal{U}}} a$.

Nevertheless definitions by selections needs precise analyse of the concrete structure (topology, uniformity ...).
Moreover, it can lead rapidly to some hard set theoretical difficulties.

For a filter φ on a set X and a function $f: X \to Y$ we mean by the *image of* φ *under* f the filter $f(\varphi) := \{B \subseteq Y \mid \exists P \in \varphi : f[P] \subseteq B\}$.

We say, a filter Φ has *Property (A) w.r.t. X* iff Φ is a filter on $\mathfrak{P}_0(X)$ and fullfills

$$\forall f \in \mathcal{A}(X) : \exists x_f \in X : f(\Phi) = \overset{\bullet}{x}_f \tag{A}$$

(Here $\overset{\bullet}{x}_f$ is the singleton filter generated by x_f.)

Question: If Φ has property (A) w.r.t. X, must Φ itself be a singleton filter on $\mathfrak{P}_0(X)$?

Proposition

If a filter Φ *has property (A) w.r.t. a set* X*, then it is an ultrafilter on* $\mathfrak{P}_0(X)$*.*

Lemma

If Φ *has property (A) w.r.t. a set* X*, then it is countably complete.*

Corollary

If Φ *has property (A) w.r.t. a* **countable** *set* X*, then it is a singleton filter on* $\mathfrak{P}_0(X)$*.*

1. Countably complete **free** ultrafilter exist, iff ω-measurable cardinals exist.
2. ω-measurable cardinals exist, iff measurable cardinals exist.
3. Every measurable cardinal is inaccessible.

Now the problem:

4. In ZFC+„there exists an inaccessible cardinal“ the consistency of ZFC can be proved.
5. If ZFC is consistent, then ZFC+„there exists *no* inaccessible cardinal“ is consistent, too.

Question: If Φ is a filter on $\mathfrak{P}_0(X)$ such that for every $f \in \mathcal{A}(X)$ the image $f(\Phi)$ is an ultrafilter on X. Must Φ itself be an ultrafilter on $\mathfrak{P}_0(X)$?

Hyperspaces and function spaces I

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In function spaces one is mainly concerned with continuous functions.
What to do, if a context leads to other functions like

$$f : R \to R : \; f(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & : \quad x \neq 0; \\ 0 & : \quad x = 0 \end{cases} \quad ?$$

[Naimpally, 1966] introduced a topology for such "almost continuous" functions:

Definition

Let (X, τ), (Y, σ) be topological spaces, let $X \times Y$ be equipped with product topology. For any open set $O \subseteq X \times Y$ define

$$\hat{O} := \{f \in Y^X | \; f \subseteq O\}$$

The topology Γ generated from the base consisting of all $\hat{O}$, O open in $X \times Y$, is the **graph topology** w.r.t. τ, σ.

Remark: The set of "almost continuous" functions is just the closure of $C(X, Y)$ in Y^X w.r.t. Γ.

Theorem [Naimpally, 1966]

1. If X is T_1, then the graph topology on Y^X contains the pointwise topology.
2. If X is T_2, then the graph topology on Y^X contains the compact-open topology.
3. If X is T_2, Y not trivial, and the graph topology on Y^X coincides with the compact-open, then X is compact.

[Poppe, 1967] remarked, that the graph topology is just the restriction of the (upper) *Vietoris topology* from $\mathfrak{P}_0(X \times Y)$ to Y^X and generalized the approach using other suitable seeming hypertopologies for $X \times Y$.

Theorem [Poppe, 1967]

1. If X is compact and T_2, then the graph topology on $C(X, Y)$ coincides with the compact-open topology.
2. If X is completely regular and the graph topology on $C(X, R)$ coincides with the compact-open, then X is compact.

Theorem [Naimpally, 1966]

Let X, Y be uniform spaces and $UC(X, Y)$ the set of uniformly continuous functions.

1. The graph topology on $UC(X, Y)$ contains the uniform topology.
2. If X is compact T_2, then the graph topology on $C(X, Y)$ coincides with the uniform topology.

Theorem [Poppe, 1967]

Let X be a topological and Y an uniform space.

1. The graph topology on $C(X, Y)$ contains the uniform topology.
2. If the graph topology on $C(X)$ coincides with the uniform topology, then X is countably compact.

For sets X we define a relation $\preceq$ between elements of $\mathfrak{P}_0(\mathfrak{P}_0(X))$:

$$\alpha_1 \preceq \alpha_2 :\Leftrightarrow \forall A_1 \in \alpha_1 : \exists A_2 \in \alpha_2 : A_1 \subseteq A_2 .$$

For subsets $\Sigma_1, \Sigma_2 \subseteq \mathfrak{P}_0(\mathfrak{P}_0(X))$:

$$\Sigma_1 \preceq \Sigma_2 \ :\Leftrightarrow \ \forall \alpha_2 \in \Sigma_2 : \exists \alpha_1 \in \Sigma_1 : \alpha_1 \preceq \alpha_2 .$$

$\preceq$ is reflexive and transitive, but not symmetric, not antisymmetric and not asymmetric in general.

Definition multifilter

Let X be a set. A family $\Sigma \subseteq \mathfrak{P}_0(\mathfrak{P}_0(X))$ is called a **multifilter** on X, iff

1. $\sigma_1 \in \Sigma \wedge \sigma_1 \preceq \sigma_2 \implies \sigma_2 \in \Sigma$ and
2. $\sigma_1, \sigma_2 \in \Sigma \implies \exists \sigma_3 \in \Sigma : \sigma_3 \preceq \sigma_1$ and $\sigma_3 \preceq \sigma_2$

hold. The set of all multifilters on a set X we denote by $\widehat{\mathfrak{F}}(X)$.

Examples: Every uniformity in the covering sense (Tukey) is a multifilter. For $x \in X$ the family $\widehat{x} := \{\sigma \subseteq \mathfrak{P}_0(X) | \ \{\{x\}\} \preceq \sigma\}$ is a multifilter.

Let $x \in X$ and $\alpha \subseteq \mathfrak{P}_0(X)$. Then the *star of α at x* is defined as

$$st(x, \alpha) := \bigcup_{A \in \alpha, x \in A} A \,,$$

and the *weak star set* of α at x is defined as

$$\Diamond(x, \alpha) := \{ \bigcup_{i=1}^{n} A_i \mid n \in N, \forall i = 1, ..., n : x \in A_i \in \alpha \} \,.$$

For a partial cover σ of a set X let
$\sigma^{\Diamond} := \bigcup_{x \in X, \Diamond(x,\sigma) \neq \emptyset} \Diamond(x, \sigma)$,
$\sigma^{*} := \{ st(x, \sigma) \mid x \in X, st(x, \sigma) \neq \emptyset \}$, and for a multifilter Σ on X let
$\Sigma^{\Diamond} := \{ \xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) \mid \exists \sigma \in \Sigma : \sigma^{\Diamond} \preceq \xi \}$,
$\Sigma^{*} := \{ \xi \in \mathfrak{P}_0(\mathfrak{P}_0(X)) \mid \exists \sigma \in \Sigma : \sigma^{*} \preceq \xi \}$.

Definition multifilter-space

For a set X and a set $\mathcal{M}$ of multifilters on X we call the pair $(X, \mathcal{M})$ a **multifilter-space**, iff

1. $\forall x \in X : \widehat{x} \in \mathcal{M}$ and
2. $\Sigma_1 \in \mathcal{M} \wedge \Sigma_2 \preceq \Sigma_1 \Rightarrow \Sigma_2 \in \mathcal{M}$

hold. $\mathcal{M}$ is called the **multifilter-structure** of this space.
If $(X_1, \mathcal{M}_1)$, $(X_2, \mathcal{M}_2)$ are multifilter-spaces and $f : X_1 \to X_2$ is a map, then f is called **fine** (w.r.t. $\mathcal{M}_1, \mathcal{M}_2$), iff

3. $f(\mathcal{M}_1) \subseteq \mathcal{M}_2$.

A multifilter-space $(X, \mathcal{M})$ is called

1. *limited* iff $\forall \Sigma_1, \Sigma_2 \in \mathcal{M} : \Sigma_1 \cap \Sigma_2 \in \mathcal{M}$,
2. *principal* iff $\exists \Sigma_0 \in \mathcal{M} : \forall \Sigma \in \mathcal{M} : \Sigma \preceq \Sigma_0$.
3. *weakly uniform* iff $\forall \Sigma \in \mathcal{M} : \Sigma^{\Diamond} \in \mathcal{M}$,
4. *uniform* iff $\forall \Sigma \in \mathcal{M} : \Sigma^{*} \in \mathcal{M}$.

Lemma

The multifilter-spaces as objects and the fine mappings between them as morphisms form a strong topological universe, denoted by **MFS**. The natural function-space between the multifilter-spaces $\mathbf{X} := (X, \mathcal{M})$ and $\mathbf{Y} := (Y, \mathcal{N})$ is $(\mathbf{Y}^{\mathbf{X}}, \mathcal{M}_{\mathbf{X},\mathbf{Y}})$ with $\mathcal{M}_{\mathbf{X},\mathbf{Y}} := \{\Gamma \in \widehat{\mathfrak{F}}(\mathbf{Y}^{\mathbf{X}}) | \; \forall \Sigma \in \mathcal{M} : \Gamma(\Sigma) \in \mathcal{N}\}$.

The subcategories of limited, principal, weak uniform limited, weak uniform principal, uniform limited and uniform principal multifilter-spaces are denoted by **LimMFS**, **PrMFS**, **WULimMFS**, **PrWULimMFS**, **ULimMFS** and **PrULimMFS**, respectively.

Lemma

1. **LimMFS** is bireflective in **MFS**.
2. **PrMFS**, **ULimMFS**, **WULimMFS**, **PrULimMFS**, **PrWULimMFS** are bireflective in **LimMFS**.

The category **UMer** of uniform covering spaces (in the sense of Tukey) and uniformly continuous maps is concretely isomorphic to **PrULimMFS**.

$(X, \mathcal{M})$ a multifilter-space:

- a filter φ on X is called Cauchy-filter, iff $\exists \Sigma \in \mathcal{M} : \forall \alpha \in \Sigma : \varphi \cap \alpha \neq \emptyset$. The family of all Cauchy-filters is denoted by $\gamma_{\mathcal{M}}$.
- $P \subseteq X$ is called precompact, iff all ultrafilters containing P are Cauchy. The family of all precompact subsets of a given multifilters-space X is denoted by $\mathcal{PC}(X)$.
- a generalized convergence structure $q_{\gamma_{\mathcal{M}}}$ is defined on X by

$$q_{\gamma_{\mathcal{M}}} := \{(\varphi, x) \in \mathfrak{F}(X) \times X | \; \varphi \cap \overset{\bullet}{x} \in \gamma_{\mathcal{M}}(X)\} .$$

- This convergence on **PrULimMFS** coincides with the usual convergence in uniform spaces.
- $(X, q_{\gamma_{\mathcal{M}}})$ is always a symmetric Kent-convergence space.

$A_1, ..., A_n \subseteq X$, $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$:

$$< A_1, ..., A_n >_{\mathfrak{A}} := \{M \in \mathfrak{A} | \ M \subseteq \bigcup_{i=1}^{n} A_i \wedge \forall i = 1, ..., n : M \cap A_i \neq \emptyset\}$$

For $\alpha \subseteq \mathfrak{P}_0(X)$ we set $\alpha_{V,\mathfrak{A}} := \{< A_1, ..., A_n > | \ n \in N, A_i \in \alpha\}$ and for $\Sigma \in \widehat{\mathfrak{F}}(X)$ we define $\Sigma_{V,\mathfrak{A}} := [\{\alpha_{V,\mathfrak{A}} | \ \alpha \in \Sigma\}]_{\widehat{\mathfrak{F}}(\mathfrak{A})}$.

Definition *finite hyperstructure*

Let $(X, \mathcal{M})$ be a limited multifilter-space. Then we call

$$\mathcal{M}_V := \{\underline{\Sigma} \in \widehat{\mathfrak{F}}(\mathcal{PC}(X)) | \ \exists \Xi \in \mathcal{M} : \underline{\Sigma} \preceq \Xi_{V,\mathcal{PC}(X)}\}$$

the **finite hyperstructure** on $\mathcal{PC}(X)$ w.r.t. $\mathcal{M}$.

If $(X, \mathcal{M})$ is a limited multifilter-space, then $(\mathcal{PC}(X), \mathcal{M}_V)$ is a limited multifilter-space, too.

Theorem

Let $(X, \mathcal{M})$ be a limited multifilter-space. Then $(\mathcal{PC}(X), \mathcal{M}_V)$ is precompact, if and only if $(X, \mathcal{M})$ is precompact.

Lemma

If $(X, \mathcal{M})$ is a limited multifilter-space and $\mathfrak{A} \subseteq \mathcal{PC}(X)$, then $\mathfrak{A}$ is precompact w.r.t. $\mathcal{M}_V$ if and only if $\bigcup_{A \in \mathfrak{A}} A$ is precompact w.r.t. $\mathcal{M}$.

We adopt the concept of Naimpally-Poppe for limited multifilter-spaces:

- for $\mathbf{X} = (X, \mathcal{M}), \mathbf{Y} = (Y, \mathcal{N}) \in$ **LimMFS** build the product $\mathbf{X} \times \mathbf{Y}$
- endow $\mathfrak{P}_0(X \times Y)$ with the finite hyperstructure $(\mathcal{M} \times \mathcal{N})_V$
- restrict $(\mathcal{M} \times \mathcal{N})_V$ to a subset $\mathcal{H} \subseteq Y^X$ and use it as function space structure Γ

Theorem

Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be limited multifilter-spaces with $(X, \mathcal{M})$ locally precompact and $(Y, \mathcal{N})$ being weakly uniform and principal. Let $\mathcal{H} \subseteq Y^X$ be the family of fine maps. Let Γ be the Naimpally-Poppe-Structure on $\mathcal{H}$.

1. Γ is finer than $\mathcal{M}_{\mathbf{X},\mathbf{Y}}$
2. If $(X, \mathcal{M})$ is precompact, then $\Gamma = \mathcal{M}_{\mathbf{X},\mathbf{Y}}$.

Hyperspaces and function spaces II

Let X be a set and (Y, σ) a topological space. For $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ we call the topology on Y^X generated by the subbase of all sets

$$(A, O) := \{f \in Y^X \mid \quad f(A) \subseteq O\}$$

with $A \in \mathfrak{A}$ and $O \in \sigma$ the $\mathfrak{A}$-open topology on Y^X (or on $C(X, Y)$, if X has a topology, too, or other subsets of Y^X).

We define a mapping μ_X from Y^X to $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ by

$$\forall M \in \mathfrak{A} : \quad \mu_X(f)(M) := f[M].$$

Lemma

Let $(X,\tau),(Y,\sigma)$ *be topological spaces, let* $\mathfrak{A} \subseteq \mathfrak{P}_0(X)$ *contain the singletons and* $\mathcal{H} \subseteq Y^X$. *Then the map*

$$\mu_X : \mathcal{H} \to \mu_X(\mathcal{H}) := \{\mu_X(f) |\, f \in \mathcal{H}\} \subseteq \mathfrak{P}_0(Y)^{\mathfrak{A}}$$

is open, continuous and bijective, where $\mathcal{H}$ *is equipped with the* $\mathfrak{A}$*-open topology and* $\mathfrak{P}_0(Y)^{\mathfrak{A}}$ *with the pointwise from the Vietoris topology on* $\mathfrak{P}_0(Y)$.

Note:

1. For $\mathfrak{A} = K(X)$ (the family of nonempty compact subsets of X) we get the compact-open topology on $\mathcal{H} := C(X, Y)$.
2. For locally compact (X,τ) the compact-open topology induces the convergence structure of continuous convergence on $C(X, Y)$.
3. The continuous convergence is the „natural“ function space structure in the *topological universe* **PsTop**.

Theorem (F. Schwarz 1989)

For every topological category $\mathcal{C}$ exists a (minimal) topological universe $\mathcal{D}$ s.t. $\mathcal{C}$ is a full subcategory of $\mathcal{D}$.

We have:

$$
\begin{array}{ccccc}
C(X, Y) & \xrightarrow{\mu_X} & K(Y)^{K(X)} & \cong & \prod_{A \in K(X)} K(Y)_A \\
 & & & & \big\downarrow \pi_A \\
 & & & & K(Y)
\end{array}
$$

Now, we can endow $C(X_i, Y)$ with the natural function space structure (as far as available) for every (suitable) (X_i, τ_i), chose $\mathfrak{A}_i := K(X_i)$, for instance, and then define a hyperspace over Y as

the **final topology** on $K(Y)$ (more generally on any subset $\mathfrak{B}$ of $\mathfrak{P}_0(Y)$) w.r.t. all $\left((X_i, \tau_i), \pi_A \circ \mu_{X_i}, A \in K(X_i)\right)_{i \in I}$.

Generally we denote the final structure on $\mathfrak{B} \subseteq \mathfrak{P}_0(Y)$ for certain $\mathfrak{A}_i \subseteq \mathfrak{P}_0(X_i)$ w.r.t. all $\left((X_i, \tau_i), \pi_A \circ \mu_{X_i}, A \in \mathfrak{A}_i\right)_{i \in I}$ by

$$\mathcal{V}\left(\mathfrak{B}, ((X_i, \tau_i), \mathfrak{A}_i)_{i \in I}\right) .$$

Lemma

If (Y, σ) is a locally compact topological space, then

$$\mathcal{V}\left(K(Y), ((X, \tau), K(X))_{(X,\tau) \in |\,\mathbf{lcTop}\,|}\right) \quad = \quad (K(Y), \sigma_V) .$$

- Somashekhar Amrith Naimpally.
  Graph topology for function spaces,
  Transactions of the American Mathematical Society 123 (1966), 267-272.
- Harry Poppe.
  Über Graphentopologien für Abbildungsräume I,
  Bull. Acad. Polon. Sci. XV (1967), no. 2, 71-80
- Harry Poppe.
  Über Graphentopologien für Abbildungsräume II,
  Math. Nachr. 38 (1968), no.1, 89-96
- Gerhard Preuß.
  Foundations of topology,
  Kluwer, 2002