Invariance group and invariance valuation ring of a cut*

— preliminary version —

Franz-Viktor Kuhlmann

22. 10. 2011

1 Introduction

In this paper, we consider cuts in ordered abelian groups and in ordered fields. We associate to every cut an (additive) invariance group. This is the maximal convex subgroup of the ordered group (respectively, of the additive group of the ordered field) which can be added to the cut sets without changing them. This notion is a useful tool for the classification of cuts. For example, it helps to determine whether a cut is induced by the upper or lower edge of a convex subgroup, or even of a convex valuation ring. We determine the invariance group of shifted cuts (Lemma 3.10, Lemma 5.1), of the sum of two cuts (Corollary 12), and of the product of two cuts (Corollary 26).

Invariance groups or related notions have been introduced by several authors in various ways (for example, see [M], [W], [T]), and we do not claim that all of our results about cuts in ordered abelian groups (Section 3) are new. We have developed some of these results in [Ku1] and applied them to derive new theorems in the valuation theory of fields, in particular in positive characteristic. These theorems can be found in [Ku2]. The reason for presenting the results in this paper is their application to the classification of cuts in ordered fields (Section 5). Our attention was drawn to this application by a question of J. Madden. During the Special Semester in Real Algebraic Geometry and Ordered Structures, Baton Rouge 1996, he showed us the definition of what we call the invariance valuation ring and asked for the meaning of it. Among other things, this paper presents our answer to his question. It is based on the manuscript [Ku3].

Looking at a cut in an ordered field $(K,<)$, one may ask whether it originates in some way from a cut in the residue field of K with respect to some real place. That is, one would like to know whether the cut can be translated into some "normal position" such that for some convex valuation ring $\mathcal{O}$ of $(K,<)$ with maximal ideal $\mathcal{M}$, it induces a cut

*I would like to thank J. Madden, M. Marshall, S. Kuhlmann and M. Tressl for helpful discussions. In particular, I would like to thank Jim Madden for his support and effort in Baton Rouge.

in the residue field $\mathcal{O}/\mathcal{M}$ via the residue map. If so, one would like to determine how this translation can be done. The invariance valuation ring is a key tool to answer these questions.

A remark by M. Marshall made it clear to us that some of our results of Section 5 are actually a special case of a more general setting. If v is the natural valuation of an ordered field $(K,<)$ and $\mathcal{O}_v$ is its valuation ring, then every convex subgroup of the ordered additive group of $(K,<)$ is an $\mathcal{O}_v$-module. The map

$$M \;\mapsto\; (vK \setminus vM, vM)\;,$$

where vK is the value group of (K,v) and $vM := \{va \mid 0 \neq a \in M\}$, is a bijection between the convex subgroups M of $(K,<)$ and the cuts in the value group vK. This holds more generally for any valuation v of an arbitrary field K and the set of all $\mathcal{O}_v$-modules $M \subseteq K$. We will investigate in Section 4 which information about M can be read off from the invariance group of the cut $(vK \setminus vM, vM)$. We also define the invariance valuation ring of an $\mathcal{O}_v$-module. The invariance valuation ring of a cut is then the invariance valuation ring of the invariance group of the cut.

The invariance valuation ring of a cut has been independently introduced and studied by M. Tressl in [T]. He also showed us the definition and main properties (Theorem 5.24) of the multiplicative invariance group of a cut, which we study in detail in Section 5.5. In Theorem 5.24, we relate the multiplicative invariance group to the additive invariance group and the invariance valuation ring. Tressl proved parts of this theorem in a special case; we choose here a different approach.

2 Preliminaries

2.1 Cuts in ordered sets

Take any ordered set $(S,<)$ (by "ordered", we will always mean "totally ordered"). If S_1, S_2 are nonempty subsets of S and $a \in S$, we will write $a < S_2$ if $a < b$ for all $b \in S_2$, and further, we will write $S_1 < S_2$ if $a < S_2$ for all $a \in S_1$. Similarly, we use the relations $>$, $\leq$ and $\geq$ in place of $<$.

A subset S' of S is called **convex in** $(S,<)$ if for every two elements $a,b \in S'$ and every $c \in S$ such that $a \leq c \leq b$, it follows that $c \in S'$. A subset S_1 of S is an **initial segment of** S if for every $a \in S_1$ and every $c \in S$ with $c \leq a$, it follows that $c \in S_1$. Symmetrically, S_2 is a **final segment of** S if for every $a \in S_2$ and every $c \in S$ with $c \geq a$, it follows that $c \in S_2$. Note that S_1 is an initial segment of S if and only if S_1 is convex and $S_1 < S \setminus S_1$. Note also that $\emptyset < S$ and $S < \emptyset$ by definition; so $\emptyset$ is an initial segment as well as a final segment of S.

If $S_1 \subseteq S$ and $S_2 \subseteq S$ are such that $S_1 \leq S_2$ and $S = S_1 \cup S_2$, then we will call (S_1, S_2) a **quasi-cut** in S. Then S_1 is an initial segment of S, S_2 is a final segment of S, and the intersection of S_1 and S_2 consists of at most one element. If this intersection is empty, then (S_1, S_2) will be called a **cut** in S. In this case, we will write $\Lambda^L = S_1$, $\Lambda^R = S_2$ and

$$\Lambda \;=\; (\Lambda^L, \Lambda^R) \;.$$

A cut (Λ^L, Λ^R) with $\Lambda^L \neq \emptyset$ and $\Lambda^R \neq \emptyset$ is called a **Dedekind cut**. If $(T, <)$ is an extension of $(S, <)$ and $a \in T$ is such that $\Lambda^L \leq a \leq \Lambda^R$, then we will say that a **realizes** (Λ^L, Λ^R) (in $(T, <)$).

For any subset $M \subseteq S$, we let M^+ denote the cut

$$M^+ \;=\; (\{s \in S \mid \exists m \in M : s \leq m\} \,,\, \{s \in S \mid s > M\}) \;.$$

That is, if $M^+ = (\Lambda^L, \Lambda^R)$ then Λ^L is the least initial segment of S which contains M, and Λ^R is the largest final segment having empty intersection with M. If $M = \emptyset$ then $\Lambda^L = \emptyset$ and $\Lambda^R = M$, and if $M = S$, then $\Lambda^L = M$ and $\Lambda^R = \emptyset$. Symmetrically, we set

$$M^- \;=\; (\{s \in S \mid s < M\} \,,\, \{s \in S \mid \exists m \in M : s \geq m\}) \;.$$

That is, if $M^- = (\Lambda^L, \Lambda^R)$ then Λ^L is the largest initial segment having empty intersection with M, and Λ^R is the least final segment of S which contains M. If $M = \emptyset$ then $\Lambda^L = M$ and $\Lambda^R = \emptyset$, and if $M = S$, then $\Lambda^L = \emptyset$ and $\Lambda^R = M$.

If $M = \{a\}$, we will write a^+ instead of $\{a\}^+$ and a^- instead of $\{a\}^-$. These two cuts are called **principal**. Hence if M has a largest element a, then $M^+ = a^+$ is principal, and if M has a smallest element a, then $M^- = a^-$ is principal. The cut (Λ^L, Λ^R) is non-principal if and only if Λ^L has no largest element and Λ^R has no smallest element.

If $a, b, c \in S$ such that $a < c < b$, then we will say that c is **strictly between** a and b. We will say that $(S, <)$ is **dense** if for every two distinct elements of S there is a third element of S strictly between them. This holds if and only if there are no cuts (Λ^L, Λ^R) in S for which Λ^L has a last element *and* Λ^R has a first element. If $a < b$ and there is no element strictly between a and b, then b is called the **immediate successor** of a, and a is called the **immediate predeccessor** of b. Further, $(S, <)$ is called **discretely ordered** if for every $a \in S$ there is an immediate successor if a is not the last element in S, and an immediate predecessor if a is not the first element in S. The properties "dense" and "discretely ordered" are mutually exclusive (if $S \neq \emptyset$).

2.2 Convex and symmetric sets in ordered abelian groups

Throughout, let $(G, +, <)$ be an ordered abelian group. For all subsets $S \subseteq G$ and $S' \subseteq G$ and all elements $g \in G$, we define

$$\begin{aligned}
-S &:= \{-a \mid a \in S\} \,, \\
S + g &:= g + S \;:=\; \{a + g \mid a \in S\} \,, \\
S + S' &:= \{a + a' \mid a \in S \,, a' \in S'\} \,, \\
S - S' &:= \{a - a' \mid a \in S \,, a' \in S'\} \,,
\end{aligned}$$

and similarly $S - g = S + (-g)$.

A subset S of G is called **symmetric** if $-S = S$.

The map $a \mapsto a + g$ is an order preserving bijection from G onto G. From this fact, it is easy to deduce the following assertions; the easy proofs are left to the reader.

Lemma 2.1 *Pick any subsets $S \subseteq G$ and $S' \subseteq G$, and an element $g \in G$.*
1) If S is convex, or an initial segment of G, or a final segment of G, then $S + g$ has the same property.
2) If both S and S' are convex, or symmetric, or initial segments of G, or final segments of G, then the same holds for $S + S'$.

Lemma 2.2 *The following assertions are equivalent for every convex symmetric subset S of G:*
a) S is not a convex subgroup of G,
b) there is some element $g \in S$ and some $n \in \mathbb{N}$ such that $ng > S$ if $g > 0$ and $ng < S$ if $g < 0$,
c) there is a positive element $g \in S$ such that $2g > S$.

Proof: The implications c)⇒b) and b)⇒a) are clear. We show "¬c)⇒ ¬a)". Since S is convex and symmetric, and since $-|a| - |b| \leq a + b \leq |a| + |b|$, it suffices to show that $a + b \in S$ for all positive $a, b \in S$. We may assume that $a \leq b$. Then $a \leq a + b \leq 2b$. Since $2b > S$ is not true, there is $c \in S$ such that $2b \leq c$. Hence $a \leq a + b \leq c$ and therefore $a + b \in S$ by convexity. □

For every $S \subseteq G$, we define

$$\mathcal{G}(S) \;:=\; \{g \in G \mid S + g = S\} \,.$$

Lemma 2.3 *Take any non-empty subset $S \subseteq G$. Then we have:*
1) $\mathcal{G}(S)$ is a subgroup of G, and

$$\mathcal{G}(S) \;=\; \{g \in G \mid S + |g| = S\} \;=\; \{g \in G \mid S - |g| = S\} \,. \tag{1}$$

2) Further,

$$\mathcal{G}(S) \;=\; \{g \in G \mid S + g \subseteq S \text{ and } S - g \subseteq S\} \,. \tag{2}$$

3) If $0 \in S$, then $\mathcal{G}(S) \subseteq S$.
4) If S is convex, then $\mathcal{G}(S)$ is a convex subgroup of G.
5) If S is symmetric, then

$$\mathcal{G}(S) \;=\; \{g \in G \mid S + g \subseteq S\} \,. \tag{3}$$

6) If S is convex symmetric, then

$$\mathcal{G}(S) \;=\; \{g \in G \mid S + g \subseteq S\} \;\subseteq\; S \,. \tag{4}$$

Proof: 1): Take $a, b \in \mathcal{G}(S)$. Then $S + a + b = S + b = S$, whence $a + b \in \mathcal{G}(S)$. Further, $S - a = S + a - a = S$, whence $-a \in \mathcal{G}(S)$. This proves that $\mathcal{G}(S)$ is a group. It follows that $g \in \mathcal{G}(S)$ is equivalent to $|g| \in \mathcal{G}(S)$ and to $-|g| \in \mathcal{G}(S)$; this proves (1).

2): The inclusion "$\subseteq$" in (2) follows from (1). If $S + g \subseteq S$ and $S - g \subseteq S$, then $S = S + g - g \subseteq S - g \subseteq S$, whence $S + g = S$. This proves the inclusion "$\supseteq$" in (2).

3): If $0 \in S$, then $\mathcal{G}(S) = 0 + \mathcal{G}(S) \subseteq S + \mathcal{G}(S) = S$.

4): By part 1), it suffices to show that $\mathcal{G}(S)$ is convex. Take $a, b \in \mathcal{G}(S)$ and $g \in G$ with $a \leq g \leq b$. Then for all $s \in S$, $s + a \leq s + g \leq s + b$ and $s + a, s + b \in S$. Hence $s + g \in S$ by convexity of S. This proves that $S + g \subseteq S$. In the same way, we also obtain that $S - g \subseteq S$ because $-a, -b \in \mathcal{G}(S)$ (since $\mathcal{G}(S)$ is a group) and $-b \leq -g \leq -a$. By part 2) it follows that $g \in \mathcal{G}(S)$.

5): From (2) we see that $\mathcal{G}(S) \subseteq \{g \in G \mid S + g \subseteq S\}$. If $S + g \subseteq S$, then $S - g = -S - g = -(S = g) = -S = S$ by the symmetry of S. This shows that $\{g \in G \mid S + g \subseteq S\} \subseteq \mathcal{G}(S)$ and thus proves (3).

6): Since S is non-empty, convex and symmetric, we have $0 \in S$. Hence by part 3), $\mathcal{G}(S) \subseteq S$. The equality in (4) follows from part 5). □

For any subset $S \subseteq G$, we call $\mathcal{G}(S)$ the **invariance group** of S.

Lemma 2.4 *1) If S_1, S_2 are arbitrary subsets of G, then $\mathcal{G}(S_1) \cup \mathcal{G}(S_2) \subseteq \mathcal{G}(S_1 + S_2)$.*
2) If both S_1, S_2 are initial segments or final segments of G, then

$$\mathcal{G}(S_1 + S_2) \;=\; \mathcal{G}(S_1) \cup \mathcal{G}(S_2)\;.$$

Proof: 1): Take $g_1 \in \mathcal{G}(S_1)$ and $g_2 \in \mathcal{G}(S_2)$. Then for all $a_1 \in S_1$ and $a_2 \in S_2$, $(a_1 + a_2) + g_1 = (a_1 + g_1) + a_2 \in S_1 + S_2$ and $(a_1 + a_2) + g_2 = a_1 + (a_2 + g_2) \in S_1 + S_2$. Hence, $g_1, g_2 \in \mathcal{G}(S_1 + S_2)$.

2): In view of part 1), it suffices to show that $\mathcal{G}(S_1 + S_2) \subseteq \mathcal{G}(S_1) \cup \mathcal{G}(S_2)$. Let S_1 and S_2 be initial segments of G; the proof for final segments is similar. Take a positive $g \in G$ such that $g \notin \mathcal{G}(S_1) \cup \mathcal{G}(S_2)$. Then there are $a_1 \in S_1$ and $a_2 \in S_2$ such that $a_1 + g \notin S_1$ and $a_2 + g \notin S_2$. Since S_1 and S_2 are initial segments of G, this means that $a_1 + g > S_1$ and $a_2 + g > S_2$. Thus $a_1 + a_2 + 2g > S_1 + S_2$, which means that $2g \notin \mathcal{G}(S_1 + S_2)$ and hence $g \notin \mathcal{G}(S_1 + S_2)$. □

Note that in the ordering given by inclusion on the set of all convex subgroups,

$$H_1 \cup H_2 \;=\; H_1 + H_2 \;=\; \max\{H_1, H_2\}$$

for every two convex subgroups H_1 and H_2.

3 Cuts in ordered abelian groups

Throughout this section, we let $\Lambda = (\Lambda^L, \Lambda^R)$ be a cut in an ordered abelian group $(G,<)$. It will be called a **positive cut** if $0 \in \Lambda^L$; otherwise, it is a **negative cut**.

3.1 Convex symmetric subset and maximal convex subgroup generated by a cut

We define

$$\mathrm{CS}\,(\Lambda) \;:=\; \{\pm g \in G \mid 0 \leq g \in \Lambda^L \text{ or } 0 \geq g \in \Lambda^R\}\,.$$

This set is a convex symmetric subset of G, and we call it the **center set of** Λ. Note that

Λ is a positive cut $\;\Leftrightarrow\;$ $\mathrm{CS}\,(\Lambda) = \Lambda^L \cap -\Lambda^L$ $\;\Leftrightarrow\;$ $\Lambda = \mathrm{CS}\,(\Lambda)^+$,
Λ is a negative cut $\;\Leftrightarrow\;$ $\mathrm{CS}\,(\Lambda) = \Lambda^R \cap -\Lambda^R$ $\;\Leftrightarrow\;$ $\Lambda = \mathrm{CS}\,(\Lambda)^-$.

Observe that $0 \in \mathrm{CS}\,(\Lambda)$. Since the union over any chain of convex subgroups is again a convex subgroup, there is a maximal convex subgroup contained in $\mathrm{CS}\,(\Lambda)$; we will denote it by $\mathrm{CG}\,(\Lambda)$ and call it the **center group of** Λ. Since the convex hull of a subgroup is again a subgroup, $\mathrm{CG}\,(\Lambda)$ is at the same time the maximal subgroup contained in $\mathrm{CS}\,(\Lambda)$. By Lemma 2.2 it follows that:

Lemma 3.1 *1) For every element $g \in \mathrm{CS}\,(\Lambda) \setminus \mathrm{CG}\,(\Lambda)$ there is some $n \in \mathbb{N}$ such that $ng > \mathrm{CS}\,(\Lambda)$ if $g > 0$ and $ng < \mathrm{CS}\,(\Lambda)$ if $g < 0$.*
2) If $\mathrm{CS}\,(\Lambda) \neq \mathrm{CG}\,(\Lambda)$ then there is a positive element $g \in \mathrm{CS}\,(\Lambda)$ such that $2g > \mathrm{CS}\,(\Lambda)$.

3.2 The invariance group of a cut

Since the map $a \mapsto a + g$ is an order preserving bijection, the **shifted cut**

$$\Lambda + g \;:=\; (\Lambda^L + g\,,\, \Lambda^R + g)$$

is again a cut in $(G,<)$. $\Lambda + g = \Lambda$ if and only if $\Lambda^L + g = \Lambda^L$, if and only if $\Lambda^R + g = \Lambda^R$.

Theorem 3.2 *The set*

$$\mathcal{G}(\Lambda) \;:=\; \{g \in G \mid \Lambda + g = \Lambda\} \;=\; \mathcal{G}(\Lambda^L) \;=\; \mathcal{G}(\Lambda^R) \tag{5}$$

is a convex subgroup of $(G,<)$, and

$$\{0\} \;\subseteq\; \mathcal{G}(\Lambda) \;=\; \mathcal{G}(\mathrm{CS}\,(\Lambda)) \;\subseteq\; \mathrm{CG}\,(\Lambda) \;\subseteq\; \mathrm{CS}\,(\Lambda)\,. \tag{6}$$

Proof: The equalities in (5) follow from the fact that for every $g \in G$,

$$\Lambda + g = \Lambda \iff \Lambda^L + g = \Lambda^L \iff \Lambda^R + g = \Lambda^R .$$

Since Λ^L is an initial segment and hence convex, part 4) of Lemma 2.3 shows that $\mathcal{G}(\Lambda)$ is a convex subgroup of G.

The inclusions $\{0\} \subseteq \mathcal{G}(\Lambda)$ and $\mathrm{CG}(\Lambda) \subseteq \mathrm{CS}(\Lambda)$ in (6) follow directly from the definitions. From parts 3) and 4) of Lemma 2.3 it follows that $\mathcal{G}(\mathrm{CS}(\Lambda))$ is a convex subgroup contained in $\mathrm{CS}(\Lambda)$. This implies that $\mathcal{G}(\mathrm{CS}(\Lambda)) \subseteq \mathrm{CG}(\Lambda)$.

Finally, we show that $\mathcal{G}(\Lambda) = \mathcal{G}(\mathrm{CS}(\Lambda))$. Assume that Λ is a positive cut; the case of a negative cut is similar. By this assumption, $\mathrm{CS}(\Lambda)$ is a final segment of Λ^L. Thus for every non-negative element $g \in G$, we have $\Lambda^L + g = \Lambda^L$ if and only if $\mathrm{CS}(\Lambda) + g \subseteq \mathrm{CS}(\Lambda)$. The former is equivalent to $g \in \mathcal{G}(\Lambda)$, and by part 5) of Lemma 2.3, the latter is equivalent to $g \in \mathcal{G}(\mathrm{CS}(\Lambda))$. Since both sets are groups, it follows from what we have just proved that they are equal. □

We call $\mathcal{G}(\Lambda)$ the **(additive) invariance group** of the cut Λ. We have $\mathcal{G}(S) = \mathcal{G}(S^+)$ if S is an initial segment, and $\mathcal{G}(S) = \mathcal{G}(S^-)$ if S is a final segment. Recall that by our notation introduced in Section 2.2,

$$\Lambda^R - \Lambda^L = \{g_2 - g_1 \mid g_1 \in \Lambda^L \text{ and } g_2 \in \Lambda^R\} .$$

Lemma 3.3 *The group $\mathcal{G}(\Lambda)$ is the largest of all convex subgroups H of G satisfying that $(\Lambda^L + H) \cap \Lambda^R = \emptyset$. The latter holds if and only if $\Lambda^L \cap (\Lambda^R + H) = \emptyset$, and this holds if and only if $H \cap (\Lambda^R - \Lambda^L) = \emptyset$. Further,*

$$\Lambda^R - \Lambda^L = G^{>0} \setminus \mathcal{G}(\Lambda) . \tag{7}$$

Proof: Since every convex subgroup H of G contains 0, it satisfies $\Lambda^L \subseteq \Lambda^L + H$. The latter is an initial segment of G since it is the union of initial segments of the form $\Lambda^L + g$, with $g \in H$. Therefore, as G is the disjoint union of Λ^L and Λ^R, we have that $\Lambda^L = \Lambda^L + H$ if and only if $(\Lambda^L + H) \cap \Lambda^R = \emptyset$. This yields the first assertion. For the second and third assertion, just observe that $(\Lambda^L + g) \cap \Lambda^R \neq \emptyset$ if and only if $\Lambda^L \cap (\Lambda^R - g) \neq \emptyset$, if and only if $g \in \Lambda^R - \Lambda^L$. In particular, $\mathcal{G}(\Lambda) \cap (\Lambda^R - \Lambda^L) = \emptyset$. Since $a < b$ whenever $a \in \Lambda^L$ and $b \in \Lambda^R$, it follows that $\Lambda^R - \Lambda^L \subseteq G^{>0} \setminus \mathcal{G}(\Lambda)$. For the converse, assume that $g \in G^{>0} \setminus \mathcal{G}(\Lambda)$. Then there is some $a \in \Lambda^L$ such that $a + g \in \Lambda^R$ and therefore, $g \in \Lambda^R - \Lambda^L$. This proves (7). □

Note that if Λ is realized in G, i.e., Λ^L admits a last or Λ^R admits a first element, then $\mathcal{G}(\Lambda)$ is trivial (i.e., $\mathcal{G}(\Lambda) = \{0\}$). The converse is not true. Indeed, every Dedekind cut in $\mathbb{Q}$ has trivial invariance group.

Remark 3.4 Recall that an ordered abelian group is discretely ordered if and only if it has a smallest positive element. This already holds if there is at least one element in the group which has an immediate successor. Observe that if Λ is a cut in a discretely ordered abelian group $(G,<)$ with invariance group $\mathcal{G}(\Lambda)=\{0\}$, then Λ^L has a last and Λ^R has a first element. Indeed, if $\mathcal{G}(\Lambda)=\{0\}$ and if g_0 is the smallest positive element of G, then there is some element $g\in\Lambda^L$ such that $g+g_0\in\Lambda^R$. But as $g+g_0$ is the immediate successor of g, we find that g is the last element of Λ^L and $g+g_0$ is the first element of Λ^R.

Remark 3.5 If $0\in\Lambda^L$, then $\mathcal{G}(\Lambda)\subseteq\Lambda^L$ part 3) of Lemma 2.3. Symmetrically, if $0\in\Lambda^R$, then $\mathcal{G}(\Lambda)\subseteq\Lambda^R$. We have that $\mathcal{G}(\Lambda)=G$ if and only if Λ^L or Λ^R is empty.

3.3 Cuts modulo convex subgroups

In what follows, the letter H will always denote an arbitrary convex subgroup of G. Then G/H is again an ordered abelian group, with the ordering induced through the canonical epimorphism $G\to G/H$. That is, this epimorphism preserves the relation $\leq$. It follows that $\Lambda^L/H\leq\Lambda^R/H$ (where for every subset $S\subseteq G$ we denote its image under the canonical epimorphism by S/H). Therefore,

$$\Lambda/H \;:=\; (\Lambda^L/H\,,\,\Lambda^R/H) \tag{8}$$

is a quasicut in G/H.

For the image of an element $g\in G$ under the canonical epimorphism, i.e., the coset $g+H$, we prefer to write g/H since we will also have to deal with subsets of the form $g+H$ in G. With this notation, $S/H=\{g/H\mid g\in S\}$. Recall that by our notation introduced in Section 2.2, $g+H=\{g+h\mid h\in H\}$ and $S+H=\{g+h\mid g\in S\,,\,h\in H\}$.

Theorem 3.6 *The induced quasi-cut (8) in G/H is a cut if and only if $H\subseteq\mathcal{G}(\Lambda)$. If $H\subseteq\mathcal{G}(\Lambda)\neq G$, then Λ/H is a Dedekind cut in G/H.*

Proof: The quasi-cut Λ/H is a cut in G/H if and only if $\Lambda^L/H\cap\Lambda^R/H=\emptyset$. This holds if and only if $\Lambda^L+H\subseteq\Lambda^L$. By definition of $\mathcal{G}(\Lambda)$, this in turn holds if and only if $H\subseteq\mathcal{G}(\Lambda)$.

If $\mathcal{G}(\Lambda)\neq G$ then by Remark 3.5, Λ^L and Λ^R are non-empty. Hence also Λ^L/H and Λ^R/H are non-empty. So Λ/H is a Dedekind cut as soon as it is a cut. □

Corollary 3.7 *If H is a final segment of Λ^L, i.e., if Λ^L is the smallest initial segment of G containing H, then $H=\mathcal{G}(\Lambda)$. The same holds if H is an initial segment of Λ^R, i.e., if Λ^R is the smallest final segment of G containing H. In other words,*

$$\mathcal{G}(H^-)\;=\;\mathcal{G}(H^+)\;=\;H\,.$$

Proof: If H is a final segment of Λ^L, then $0 \in \Lambda^L$ and by Remark 3.5, $\mathcal{G}(\Lambda) \subseteq \Lambda^L$, which implies that $\mathcal{G}(\Lambda) \subseteq H$. On the other hand, it follows that $0/H$ is the last element of Λ^L/H, but is not contained in Λ^R/H since the coset $0+H = H$ has empty intersection with Λ^R. Therefore, Λ/H is a cut in G/H. Hence by the foregoing theorem, $H \subseteq \mathcal{G}(\Lambda)$.

The proof for the case of an initial segment of Λ^R is similar. □

Lemma 3.8 *The invariance group of the cut $\Lambda/\mathcal{G}(\Lambda)$ in $G/\mathcal{G}(\Lambda)$ is trivial. More generally, if $H \subseteq \mathcal{G}(\Lambda)$, then the invariance group of Λ/H is $\mathcal{G}(\Lambda)/H$.*

Proof: Take $H \subseteq \mathcal{G}(\Lambda)$. Then $\Lambda^L + H = \Lambda^L$. It follows that for every $g \in G$, we have $\Lambda^L/H + g/H = \Lambda^L/H$ if and only if $\Lambda^L + g = \Lambda^L$. Hence, $\mathcal{G}(\Lambda/H) = \mathcal{G}(\Lambda)/H$. □

In view of Remark 3.4, we obtain from this lemma:

Corollary 3.9 *Assume that the group $G/\mathcal{G}(\Lambda)$ is not trivial and that it is discretely ordered. Then $\Lambda^L/\mathcal{G}(\Lambda)$ has a last and $\Lambda^R/\mathcal{G}(\Lambda)$ has a first element in $G/\mathcal{G}(\Lambda)$.*

3.4 Shifting and additive inversion of cuts

We will now consider shifted cuts $\Lambda + a = (\Lambda^L + a\,,\ \Lambda^R + a)$ obtained from Λ, for arbitrary $a \in G$. Note that Λ is principal if and only if $\Lambda + a$ is.

Since $\Lambda^L + a + g = \Lambda^L + g + a = \Lambda^L + a$ holds if and only if $\Lambda^L + g = \Lambda^L$, we have:

Lemma 3.10 *For every cut Λ and every $a \in G$,*

$$\mathcal{G}(\Lambda + a) \;=\; \mathcal{G}(\Lambda)\;.$$

We will say that the cut Λ **can be shifted into** H if there is some $a \in G$ such that

$$(\Lambda^L + a) \cap H \neq \emptyset \;\text{ and }\; (\Lambda^R + a) \cap H \neq \emptyset\;,$$

or in other words, $((\Lambda^L + a) \cap H\,,\ (\Lambda^R + a) \cap H)$ is a Dedekind cut in H. If this is the case, then the cut $((\Lambda^L + a) \cap H\,,\ (\Lambda^R + a) \cap H)$ in H is realized in H if and only if Λ is realized in G. Note that if Λ is realized in G, then it can be shifted into every non-trivial convex subgroup of G, and its invariance group is trivial. This is a special case of the next lemma.

Theorem 3.11 *The cut Λ can be shifted into H if and only if $\mathcal{G}(\Lambda) \subsetneq H$.*

Proof: First, we show that Λ cannot be shifted into $\mathcal{G}(\Lambda)$ (and hence into no subgroup contained in $\mathcal{G}(\Lambda)$). Take any $a \in G$ and assume that $(\Lambda^L + a) \cap \mathcal{G}(\Lambda) \neq \emptyset$. Since $\mathcal{G}(\Lambda)$ is also the invariance group of $\Lambda + a$, it follows that $(\Lambda^L + a) \cap \mathcal{G}(\Lambda) = ((\Lambda^L + a) \cap \mathcal{G}(\Lambda)) + \mathcal{G}(\Lambda) = \mathcal{G}(\Lambda)$. This implies that $(\Lambda^R + a) \cap \mathcal{G}(\Lambda) = \emptyset$, showing that Λ cannot be shifted into $\mathcal{G}(\Lambda)$.

Now assume that $\mathcal{G}(\Lambda) \subsetneqq H$. By Theorem 3.6, the quasi-cut $(\Lambda^L/H\,,\,\Lambda^R/H)$ induced by Λ in G/H is not a cut. Hence, we can choose some $b \in \Lambda^L$ such that $b/H \in (\Lambda^L/H) \cap (\Lambda^R/H)$. This implies that $b \in \Lambda^L + H$ and $b \in \Lambda^R + H$. This in turn yields that $(\Lambda^L - b) \cap H \neq \emptyset$ and $(\Lambda^R - b) \cap H \neq \emptyset$, i.e., Λ can be shifted into H. □

Since the map $x \mapsto -x$ is order reversing,

$$-\Lambda \;:=\; (-\Lambda^R, -\Lambda^L)$$

is again a cut. We have $-(-\Lambda) = \Lambda$, and $-\Lambda$ is a negative cut if and only if Λ is a positive cut. Note that Λ is principal if and only if $-\Lambda$ is.

Proposition 3.12 *For every cut Λ,*

$$\mathcal{G}(-\Lambda) \;=\; \mathcal{G}(\Lambda)\,, \tag{9}$$

and $-\Lambda$ can be shifted into H if and only if Λ can.

Proof: We have $g \in \mathcal{G}(\Lambda) \;\Leftrightarrow\; \Lambda^R + g = \Lambda^R \;\Leftrightarrow\; -\Lambda^R - g = -\Lambda^R \;\Leftrightarrow\; -g \in \mathcal{G}(-\Lambda) \;\Leftrightarrow\; g \in \mathcal{G}(-\Lambda)$, which proves (9). The second assertion now follows from Theorem 3.11. □

3.5 Cuts induced by edges of convex subgroups

We will say that the cut Λ is a **group $^+$-cut** if it is induced by the upper edge of a convex subgroup H of G, i.e., if $\Lambda = H^+$. We will say that Λ is a **group $^-$-cut** if it is induced by the lower edge of a convex subgroup H of G, i.e., if $\Lambda = H^-$. In both cases, we will call Λ a **group-cut**. Note that 0^+ and 0^- are the only principal group-cuts.

Lemma 3.13 *A cut Λ is a group-cut if and only if $\mathrm{CS}\,(\Lambda)$ is a subgroup of G. If this holds, then*

$$\mathcal{G}(\Lambda) \;=\; \mathrm{CG}\,(\Lambda) \;=\; \mathrm{CS}\,(\Lambda)\;. \tag{10}$$

Proof: If Λ is a group-cut, then $\Lambda = H^+$ or $\Lambda = H^-$ for a convex subgroup H, whence $\mathrm{CS}\,(\Lambda) = H$. Conversely, since $\Lambda = \mathrm{CS}\,(\Lambda)^+$ or $\Lambda = \mathrm{CS}\,(\Lambda)^-$, Λ is a group-cut if the convex set $\mathrm{CS}\,(\Lambda)$ is a subgroup.

Assume that $\mathrm{CS}\,(\Lambda)$ is a convex subgroup. Then $\mathrm{CS}\,(\Lambda) + \mathrm{CS}\,(\Lambda) = \mathrm{CS}\,(\Lambda)$, whence $\mathrm{CS}\,(\Lambda) \subseteq \mathcal{G}(\Lambda)$. In view of assertion (6) of Theorem 3.2, this implies (10). □

Remark 3.14 Paulo Ribenboim ([R], p. 105) defines **distinguished pseudo Cauchy sequences** to be those pseudo Cauchy sequences $(a_\rho)_{\rho<\lambda}$ (where λ is a limit ordinal) in a valued field (K,v) for which the set

$$\{v(a_{\rho+1}-a_\rho)\mid \rho<\lambda\}=\{v(a_\sigma-a_\rho)\mid \rho<\sigma<\lambda\} \tag{11}$$

is cofinal in a convex subgroup H of the value group vK of K. But this just means that the smallest initial segment Λ^L containing the set (11) is at the same time the smallest initial segment containing H. Moreover, H must be non-trivial since the set (11) has no last element. Hence, the cut $\{v(a_{\rho+1}-a_\rho)\mid \rho<\lambda\}^+$ determined by the set (11) is a non-principal group$^+$-cut if and only if the pseudo Cauchy sequence $(a_\rho)_{\rho<\lambda}$ is distinguished.

We call Λ a **ball$^+$-cut** (or a **ball$^-$-cut**) if it is of the form $H^+ + g$ (or $H^- + g$, respectively) for some $g\in G$ and a convex subgroup H. Ball$^+$-cuts and ball$^-$-cuts are called **ball-cuts**. The following is an immediate consequence of the definitions (take $a=-g$):

Lemma 3.15 *The cut Λ is a ball$^+$-cut if and only if there is some $a\in G$ such that $\Lambda+a$ is a group$^+$-cut. The same holds for "$-$" in place of "$+$".*

If H is a convex subgroup of G, then S is the smallest final segment of G containing H if and only if $-S$ is the smallest initial segment of G containing H. Thus,

$$\Lambda = H^+ \iff -\Lambda = H^- .$$

This proves:

Lemma 3.16 *The cut Λ is a group$_0^+$-cut if and only if $-\Lambda$ is a group$_0^-$-cut. The cut Λ is a ball$_0^+$-cut if and only if $-\Lambda$ is a ball$_0^-$-cut. The same holds for group$^+$ and group$^-$.*

We infer from Corollary 3.7 and Lemma 3.13:

Lemma 3.17 *For each cut Λ, the following assertions hold:*

1) The following are equivalent:
   - *a) Λ is a group$^+$-cut,*
   - *b) $\Lambda=\mathcal{G}(\Lambda)^+$,*
   - *c) Λ is a positive cut and $\mathrm{CS}\,(\Lambda)=\mathcal{G}(\Lambda)$.*

2) The following are equivalent:
   - *a) Λ is a group$^-$-cut,*
   - *b) $\Lambda=\mathcal{G}(\Lambda)^-$,*
   - *c) Λ is a negative cut and $\mathrm{CS}\,(\Lambda)=\mathcal{G}(\Lambda)$.*

3) Λ is a ball$^+$-cut if and only if $\Lambda+g=\mathcal{G}(\Lambda)^+$ for some g.

4) Λ is a ball$^-$-cut if and only if $\Lambda+g=\mathcal{G}(\Lambda)^-$ for some g.

Here is another characterization of group $^+$- and group $^-$-cuts:

Lemma 3.18 *The cut Λ is a group$^+$-cut if and only if $0/\mathcal{G}(\Lambda)$ is the last element of $\Lambda^L/\mathcal{G}(\Lambda)$. Symmetrically, Λ is a group$^-$-cut if and only if $0/\mathcal{G}(\Lambda)$ is the first element of $\Lambda^R/\mathcal{G}(\Lambda)$.*

Proof: If Λ is a group$^+$-cut, then by Lemma 3.17, $\Lambda = \mathcal{G}(\Lambda)^+$. Then $0 \in \Lambda^L$ and $g > \mathcal{G}(\Lambda) \Rightarrow g \notin \Lambda^L$. This implies that $0/\mathcal{G}(\Lambda)$ is the last element of $\Lambda^L/\mathcal{G}(\Lambda)$. Conversely, if $0/\mathcal{G}(\Lambda)$ is the last element of $\Lambda^L/\mathcal{G}(\Lambda)$, then there is some $g \in \Lambda^L \cap \mathcal{G}(\Lambda)$. But then, $\mathcal{G}(\Lambda) = g + \mathcal{G}(\Lambda) \subseteq \Lambda^L$. If $a \in G$ with $a > \mathcal{G}(\Lambda)$, then $a/\mathcal{G}(\Lambda) > 0/\mathcal{G}(\Lambda)$, hence $a/\mathcal{G}(\Lambda)$ cannot be in $\Lambda^L/\mathcal{G}(\Lambda)$, and a cannot be in Λ^L. This proves that Λ^L is the least initial segment of G containing $\mathcal{G}(\Lambda)$, that is, $\Lambda = \mathcal{G}(\Lambda)^+$. The proof for the second assertion is similar. □

If $\Lambda^L/\mathcal{G}(\Lambda)$ has last element $a/\mathcal{G}(\Lambda)$, then $(\Lambda^L + g)/\mathcal{G}(\Lambda)$ has last element $(a+g)/\mathcal{G}(\Lambda)$. Thus, we obtain:

Corollary 3.19 *The cut Λ is a ball$^+$-cut if and only if $\Lambda^L/\mathcal{G}(\Lambda)$ admits a largest element. Symmetrically, Λ is a ball$^-$-cut if and only if $\Lambda^R/\mathcal{G}(\Lambda)$ admits a smallest element.*

By means of Lemma 3.18 and Theorem 3.6, the following holds:

Corollary 3.20 *No cut is a group$^+$-cut and a group$^-$-cut at the same time.*

But we have:

Proposition 3.21 *Assume that Λ is a Dedekind cut in G. Then the following are equivalent:*
1) Λ is a ball$^+$-cut and a ball$^-$-cut,
2) $G/\mathcal{G}(\Lambda)$ is discretely ordered.

Proof: If Λ is a ball$^+$-cut and a ball$^-$-cut, then by Corollary 3.19, $\Lambda^L/\mathcal{G}(\Lambda)$ admits a largest element $a/\mathcal{G}(\Lambda)$, and $\Lambda^R/\mathcal{G}(\Lambda)$ admits a smallest element $a'/\mathcal{G}(\Lambda)$. By Theorem 3.6, $(\Lambda^L/\mathcal{G}(\Lambda), \Lambda^R/\mathcal{G}(\Lambda))$ is a cut, so $a/\mathcal{G}(\Lambda) < a'/\mathcal{G}(\Lambda)$. But then $a'/\mathcal{G}(\Lambda)$ is the immediate successor of $a/\mathcal{G}(\Lambda)$ in $G/\mathcal{G}(\Lambda)$. Hence $G/\mathcal{G}(\Lambda)$ is discretely ordered.

For the converse, assume that $G/\mathcal{G}(\Lambda)$ is discretely ordered. Since Λ is a Dedekind cut in G, we know from Remark 3.5 that $\mathcal{G}(\Lambda) \neq G$. Hence by Corollary 3.9, $\Lambda^L/\mathcal{G}(\Lambda)$ has a last element and $\Lambda^R/\mathcal{G}(\Lambda)$ has a first element in $G/\mathcal{G}(\Lambda)$. Now Corollary 3.19 shows that Λ is a ball$^+$-cut and a ball$^-$-cut. □

Example 3.22 Take $\mathbb{Z}\times\mathbb{Q}$ with the lexicographic ordering, and set

$$\Lambda^L \;:=\; \{(m,q)\mid m\le 0\,,\; q\in\mathbb{Q}\} \quad\text{and}\quad \Lambda^R \;:=\; \{(m,q)\mid m\ge 1\,,\; q\in\mathbb{Q}\}\,.$$

Then Λ is a cut with invariance group $\mathcal{G}(\Lambda)=\{0\}\times\mathbb{Q}$. We have the canonical isomorphism $(\mathbb{Z}\times\mathbb{Q})/\mathcal{G}(\Lambda)\simeq\mathbb{Z}$, which sends the cut induced by Λ to the cut

$$(\{m\in\mathbb{Z}\mid m\le 0\}\,,\,\{m\in\mathbb{Z}\mid m\ge 1\}) \;=\; 0^+$$

in $\mathbb{Z}$. Hence, Λ is a group${}^+$-cut and a ball${}^-$-cut. Indeed, $\{0\}\times\mathbb{Q}=(0,0)\,+\,\{0\}\times\mathbb{Q}$ is a final segment of Λ^L, and $\{1\}\times\mathbb{Q}=(1,0)\,+\,\{0\}\times\mathbb{Q}$ is an initial segment of Λ^R. Further, $\Lambda-(1,0)$ is a ball${}^+$-cut and a group${}^-$-cut. ♢

3.6 Adding cuts

It is not immediately clear how two cuts $\Lambda_1=(\Lambda_1^L,\Lambda_1^R)$ and $\Lambda_2=(\Lambda_2^L,\Lambda_2^R)$ should be added. While it is always true that $\Lambda_1^L+\Lambda_2^L$ is again an initial segment and $\Lambda_1^R+\Lambda_2^R$ is again a final segment with $(\Lambda_1^L+\Lambda_2^L)\cap(\Lambda_1^R+\Lambda_2^R)=\emptyset$, in a general ordered abelian group G it may not be true that $(\Lambda_1^L+\Lambda_2^L)\cup(\Lambda_1^R+\Lambda_2^R)=G$. For instance, consider the cut $\Lambda_1=\Lambda_2=1^+$ in $\mathbb{Z}$. Then $\Lambda_1^L+\Lambda_2^L=\{g\in\mathbb{Z}\mid g\le 2\}$ and $\Lambda_1^R+\Lambda_2^R=\{g\in\mathbb{Z}\mid g\ge 4\}$.

In view of this obstruction, we have two different ways to define $\Lambda_1+\Lambda_2$: either we set $\Lambda_1+\Lambda_2:=(\Lambda_1^L+\Lambda_2^L)^+$, or we set $\Lambda_1+\Lambda_2:=(\Lambda_1^R+\Lambda_2^R)^-$. Both definitions render a monoid structure on the set of all cuts, with neutral element 0^+ in the first case, and 0^- in the second case. In the first case, all group${}_0^+$-cuts turn out to be idempotents, and in the second case, all group${}_0^-$-cuts are idempotents.

In both cases, we can compute $\mathcal{G}(\Lambda_1+\Lambda_2)$ if we are able to compute $\mathcal{G}(\Lambda_1^L+\Lambda_2^L)$ and $\mathcal{G}(\Lambda_1^R+\Lambda_2^R)$ from $\mathcal{G}(\Lambda_1)=\mathcal{G}(\Lambda_1^L)=\mathcal{G}(\Lambda_1^R)$ and $\mathcal{G}(\Lambda_2)=\mathcal{G}(\Lambda_2^L)=\mathcal{G}(\Lambda_2^R)$. This has been done in Lemma 2.4; it yields:

Proposition 3.23 *For any two cuts Λ_1 and Λ_2,*

$$\mathcal{G}(\Lambda_1+\Lambda_2) \;=\; \mathcal{G}(\Lambda_1)\cup\mathcal{G}(\Lambda_2) \;=\; \max\{\mathcal{G}(\Lambda_1),\mathcal{G}(\Lambda_2)\}\,. \tag{12}$$

3.7 A valuation theoretic characterization of the invariance group

Two elements a,b in an ordered abelian group $(G,<)$ are called **archimedean equivalent** if there is some $n\in\mathbb{N}$ such that $n|a|\ge|b|$ and $n|b|\ge|a|$. Denote the equivalence class of a by va, write $vG:=\{vg\mid 0\ne g\in G\}$ and $\infty:=v0$. Order the set $vG\cup\{\infty\}$ as follows: set $va<vb$ if a and b are not archimedean equivalent and $|b|<|a|$. The map

$$v: G\ni a\;\mapsto\; va\in vG\cup\{\infty\}$$

is then a **group valuation**, that is, it satisfies the following two axioms:

(V∞) $vx = \infty$ if and only if $x = 0$,
(VU) $v(x - y) \geq \max\{vx, vy\}$ (ultrametric triangle law).
Note that these axioms imply:
(V$-$) $v(-x) = vx$,
(Vmin) $v(x + y) = \max\{vx, vy\}$ if $vx \neq vy$.
The totally ordered set vG is called the **value set** of the valued abelian group (G, v). The above described valuation induced by an ordering $<$ of an abelian group G is called the **natural valuation** of $(G, <)$. It satisfies:

$$|a| \leq |b| \implies va \geq vb \;, \tag{13}$$

that is, v restricted to $G^{<0}$ preserves $\leq$. It also satisfies

$$0 \neq n \in \mathbb{Z} \implies va \;=\; v(na) \;. \tag{14}$$

From now on, for the remainder of this section, v will always denote the natural valuation.

For any subset $S \subseteq G$ we write $v(S) := \{vg \mid g \in M\}$. This is in contrast to our notation $vH = \{vg \mid 0 \neq g \in H\}$ which excludes the value ∞ when we are dealing with convex subgroups H or with value groups of fields. From (13), we obtain:

Lemma 3.24 *If S is an initial segment of $G^{>0}$ or a final segment of $G^{<0}$, then $v(S)$ is a final segment of vG. If S is a convex subset of G containing 0, then $v(S)$ is a final segment of $vG \cup \{\infty\}$.*

Proof: Take $a \in S$ and $\beta \in vG$ such that $va < \beta$. Choose any $b \in G$ such that $vb = \beta$. Then $b \neq 0$, and $|b| < |a|$ by definition of v. Because of (V$-$), we may choose b positive if and only if a is positive. Then by our assumption on S, it follows that $b \in S$ and therefore, $\beta \in v(S)$.

The proof of the last assertion is left to the reader. $\square$

Lemma 3.25 *The map $v : H \mapsto vH$ establishes an inclusion preserving bijection between the convex subgroups H of G and the final segments S of vG; its inverse map is $S \mapsto v^{-1}(S \cup \{\infty\})$. The map $H \mapsto (vH)^-$ is a bijection between the convex subgroups H of G and the cuts in the value set vG.*

Proof: Take any convex subgroup H of G. By Lemma 3.24, $vH = v(H^{>0})$ is a final segment of vG. Suppose that H' is a convex subgroup of G containing H. Then $vH \subseteq vH'$. Suppose that $vH = vH'$. Then every element $a \in H'$ is archimedean equivalent to an element $b \in H$, whence $|nb| = n|b| \geq |a|$ for some $n \in \mathbb{N}$. Since H is a convex subgroup, it follows that $a \in H$. This proves that $H = H'$. Hence, $v : H \mapsto vH$ is injective.

For every final segment S of vG, we show that $v^{-1}(S \cup \{\infty\})$ is a convex subgroup of G; this implies that $v : H \mapsto vH$ is also surjective, with $S \mapsto v^{-1}(S \cup \{\infty\})$ as its inverse. First, $v^{-1}(S \cup \{\infty\})$ contains 0. Second, if $a, b \in v^{-1}(S \cup \{\infty\})$, then $a - b \in v^{-1}(S \cup \{\infty\})$ since $v(a-b) \geq \min\{va, vb\}$ and S is a final segment. Hence, $v^{-1}(S \cup \{\infty\})$ is a subgroup of G. Finally, if $|a| \leq |b|$ with $b \in v^{-1}(S \cup \{\infty\})$, then $va \geq vb$ by (13), and since S is a final segment of vG we find that $va \in S \cup \{\infty\}$ and thus, $a \in v^{-1}(S \cup \{\infty\})$. This proves that $v^{-1}(S \cup \{\infty\})$ is convex.

The second assertion follows from the first since $S \mapsto S^-$ is a bijection between the final segments of vG and the cuts of vG. □

We will now give a characterization of the invariance group in terms of the natural valuation.

Lemma 3.26 *For every cut Λ in G,*

$$v\mathcal{G}(\Lambda) \;=\; vG \setminus v(\Lambda^R - \Lambda^L)\;.$$

Proof: It follows from (7) that $G^{\geq 0}$ is the disjoint union of $\mathcal{G}(\Lambda)^{\geq 0}$ and $\Lambda^R - \Lambda^L$. Hence $vG = v\mathcal{G}(\Lambda) \cup v(\Lambda^R - \Lambda^L)$. Since $\mathcal{G}(\Lambda)$ is a convex subgroup of G, it follows from the previous lemma that $v\mathcal{G}(\Lambda)$ and $v(\Lambda^R - \Lambda^L)$ are disjoint. Since $v\mathcal{G}(\Lambda) = v\mathcal{G}(\Lambda)^{\geq 0}$ and $vG = vG^{\geq 0}$, this implies our assertion. □

It follows from (13) that $v(\Lambda^{L>0}) \cap v(\Lambda^R)$ contains at most one element if Λ is a positive cut, and that $v(\Lambda^{R<0}) \cap v(\Lambda^L)$ contains at most one element if Λ is a negative cut.

Proposition 3.27 *A positive cut Λ is a group$_0^+$-cut if and only if $v(\Lambda^{L>0}) \cap v(\Lambda^R) = \emptyset$. Similarly, a negative cut Λ is a group$_0^-$-cut if and only if $v(\Lambda^{R<0}) \cap v(\Lambda^L) = \emptyset$.*

Proof: Suppose that Λ is a group$_0^+$-cut and choose a convex subgroup H such that $\Lambda = H^+$. Then $\Lambda^{L>0} = H^{>0} = \mathcal{G}(\Lambda)^{>0}$. From Lemma 3.25 we know that $v(H^{>0}) = vH$ and $v(G \setminus H)$ are disjoint. This proves that $v(\Lambda^{L>0}) \cap v(\Lambda^R) = \emptyset$.

For the converse, assume that the latter holds. Since Λ is a positive cut, $\Lambda^{L^{\geq 0}} \neq \emptyset$. If this set only contains 0, then $\Lambda = 0^+$. Otherwise, $\Lambda^{L>0} \neq \emptyset$. We set $S := v(\Lambda^{L>0})$; then S is a final segment of vG by Lemma 3.24, and $H := v^{-1}(S \cup \{\infty\})$ is a convex subgroup by Lemma 3.25. From $v(\Lambda^{L>0}) \cap v(\Lambda^R) = \emptyset$ we see that $\Lambda^{L>0} = H^{>0}$, that is, $\Lambda = H^+$.

The proof for the case of a negative cut is similar. □

Remark 3.28 Suppose that Λ is a ball$_0^+$-cut, but not a group$_0^+$-cut. Then there is a convex subgroup H and an element $a \notin H$ such that $\Lambda = H^+ + a$. By Lemma 3.25, $a \notin H$ implies that $va < vb$ for all $b \in H$. We observe that $a \in \Lambda^L$. If $a > 0$, then $a > H$ and consequently, $2a = a + a \in \Lambda^R$. If $a < 0$, then $a < H$ and consequently, $-a > H$ and $-a = a + 2(-a) \in \Lambda^R$. Thus, $va \in v(\Lambda^L)$ and $va = v(2a) = v(-a) \in v(\Lambda^R)$. This shows that

$$v(\Lambda^{L>0}) \cap v(\Lambda^R) \;=\; \{va\} \;.$$

A similar assertion holds for ball$_0^-$-cuts which are not group$_0^-$-cuts.

If $(G',<)$ is an ordered group extending $(G,<)$, then we will also write v for the natural valuation of $(G',<)$. Then $(G,v) \subseteq (G',v)$ is an extension of valued groups. For any $x \in G'$ (which we will later assume to realize the cut Λ), we consider the value set

$$v(x-G) \;:=\; \{v(x-b) \mid b \in G\} \;\subseteq\; vG' \cup \{\infty\} \;.$$

Lemma 3.29 *a) If $v(x-G)$ contains an element $\alpha \notin vG$, then α is the largest element of $v(x-G)$.*
b) The set $v(x-G) \cap vG$ is an initial segment of vG.

Proof: Take $\alpha, \beta \in v(x-G)$ such that $\alpha > \beta$, and $a, b \in G$ such that $\alpha = v(x-a)$, $\beta = v(x-b)$. Then $\beta = \min\{v(x-a), v(x-b)\} = v(a-b) \in vG$. This proves a). Now take $\alpha \in v(x-G)$, $a \in G$ such that $\alpha = v(x-a)$, and $\beta \in vG$ such that $\alpha > \beta$. Choose $c \in G$ such that $vc = \beta$. Then with $b := a + c$, we have that $\beta = \min\{\alpha, \beta\} = \min\{v(x-a), vc\} = v(x-a-c) = v(x-b) \in v(x-G)$. This proves b). □

Lemma 3.30 *If the cut Λ in $(G,<)$ is realized by an element x in an extension $(G',<)$, then*

$$v\mathcal{G}(\Lambda) \;=\; vG \setminus v(x-G) \;=\; \{\alpha \in vG \mid \alpha > v(x-b) \;\textit{ for all }\; b \in G\} \;. \tag{15}$$

Proof: The second equation is an easy consequence of the foregoing lemma. The first equation follows from Lemma 5.9 if we show that

$$v(x-G) \cap vG \;=\; v(\Lambda^R - \Lambda^L) \;.$$

Take any $a \in \Lambda^L$, $b \in \Lambda^R$. Since $x - a \leq b - a$ and $b - x \leq b - a$, we have that $v(x-a) \geq v(b-a)$ and $v(b-x) \geq v(b-a)$. If $v(b-a) = v(x-a)$ or $v(b-a) = v(b-x) = v(x-b)$, then $v(b-a) \in v(x-G)$. If $v(x-a) > v(b-a)$, then $v(b-a) = \min\{v(b-a), v(x-a)\} = v(x-b) \in v(x-G)$. If $v(b-x) > v(b-a)$, then $v(b-a) = \min\{v(b-a), v(b-x)\} = v(x-a) \in v(x-G)$. Hence in all cases, $v(b-a) \in v(x-G)$. This proves that $v(\Lambda^R - \Lambda^L) \subseteq v(x-G) \cap vG$.

To prove the converse inclusion, take $b \in G$ such that $v(x-b) \in vG$. Take $c \in G$ such that $c > 0$ and $vc = v(x-b)$. Then there is some positive integer n such that $nc > |x-b|$. If $b \in \Lambda^L$, then this implies that $b + nc > x$ and thus, $b + nc \in \Lambda^R$. Then $v(x-b) = vc = vnc = v((b+nc)-b) \in v(\Lambda^R - \Lambda^L)$. If $b \in \Lambda^R$, then $b - nc < x$, hence $b - nc \in \Lambda^L$ and again, $v(x-b) = vc = vnc = v(b-(b-nc)) \in v(\Lambda^R - \Lambda^L)$. □

Take an extension $(G,v) \subseteq (G',v)$ of valued abelian groups. If for every $a \in G'$ there is some $b \in G$ such that $v(a-b) > va$, then the extension is called **immediate**. In this case, every element $a \in G' \setminus G$ is the pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (G,v) without a limit in G; this can be shown as in the case of valued fields (cf. [Ka] or [R]). If $a \notin G$, then this pseudo Cauchy sequence can be chosen such that it has no pseudo limit in G. The **breadth** of $(a_\nu)_{\nu<\lambda}$ is defined to be the convex subgroup

$$\{b \in G \mid vb > v(a_{\nu+1} - a_\nu) \text{ for all } \nu < \lambda\} \; .$$

In fact, this is the set of all elements $b \in G$ with the property that a is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ if and only if $a+b$ is.

Note that if the extension $(G,v) \subseteq (G',v)$ is immediate, then $vG' = vG$.

Theorem 3.31 *Suppose that the cut Λ is realized by an element x in an extension $(G',<)$ such that $(G,v) \subseteq (G',v)$ is immediate. If $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (G,v) having pseudo limit x but not having a pseudo limit in G, then its breadth is equal to $\mathcal{G}(\Lambda)$.*

Proof: We denote by B the breadth of $(a_\nu)_{\nu<\lambda}$. Then $vB = \{\alpha \in vG \mid \alpha > v(a_{\nu+1} - a_\nu) \text{ for all } \nu < \lambda\}$. By Lemma 3.25, it suffices to show that $vB = \mathcal{G}(\Lambda)$. By Lemma 3.26, $v\mathcal{G}(\Lambda) = \{\alpha \in vG \mid \alpha > v(x-b) \text{ for all } b \in G\}$. As x is a pseudo limit of $(a_\nu)_{\nu<\lambda}$, we have by definition that $v(a_{\nu+1} - a_\nu) = v(x - a_\nu) \in v(x - G)$ for all $\nu < \lambda$. This shows that

$$\begin{aligned} v\mathcal{G}(\Lambda) &= \{\alpha \in G \mid \alpha > v(x-b) \text{ for all } b \in G\} \\ &\subseteq \{\alpha \in G \mid \alpha > v(a_{\nu+1} - a_\nu) \text{ for all } \nu < \lambda\} \;=\; vB \; . \end{aligned}$$

To prove the converse inclusion, we take any $b \in G$. Suppose that $v(x-b) > v(a_{\nu+1} - a_\nu)$ for all $\nu < \lambda$. But then, since x is a pseudo limit, we have that $v(a_{\nu+1} - a_\nu) = \min\{v(a_{\nu+1} - a_\nu), v(x-b)\} = \min\{v(x-a_\nu), v(x-b)\} = v(b-a_\nu)$. This shows that $b \in G$ is also a pseudo limit of our pseudo Cauchy sequence, in contradiction to our assumption that there be no pseudo limit in G. This proves that the sequence $v(a_{\nu+1} - a_\nu)$, $\nu < \lambda$, is cofinal in $v(x-G)$, which implies that the inclusion "$\supseteq$" holds. □

The **completion** G^c of (G,v) (with respect to the topology induced by v) admits a canonical extension of v and of the ordering from G. With this extension of v, $(G^c|G,v)$ is an immediate extension, and the breadth of every pseudo Cauchy sequence without a pseudo limit in G but with pseudo limit in G^c is $\{0\}$. Thus, we obtain the following

Corollary 3.32 *If the cut Λ is realized in the completion of (G,v), then $\mathcal{G}(\Lambda) = \{0\}$.*

4 Valuation rings and their modules in a field

Take any valued field (K,v). Denote the valuation ring of v by $\mathcal{O}_v$. Then the set $\mathcal{R}$ of all valuation rings which contain $\mathcal{O}_v$ is linearly ordered by inclusion. For every $\mathcal{O}\in\mathcal{R}$, we set

$$H(\mathcal{O}) \;:=\; v\mathcal{O}\cap -v\mathcal{O} \;=\; v\mathcal{O}^\times \;.$$

This is a convex subgroup of the value group vK. In fact, v is finer or equal to the valuation w associated with $\mathcal{O}$, the value group of w is canonically isomorphic to $vK/H(\mathcal{O})$, and the value group of the valuation induced by v on the residue field Kw is canonically isomorphic to $H(\mathcal{O})$ (cf. [Z–S]).

Conversely, for every convex subgroup H of vK, we set

$$\mathcal{O}(H) \;:=\; \{b\in K \mid \exists \alpha\in H:\; \alpha\leq vb\}\;. \tag{16}$$

That is, $v\mathcal{O}(H)$ is the smallest final segment of vK containing H. Note that

$$\mathcal{O}(H)^\times \;=\; v^{-1}(H)\;. \tag{17}$$

We recall the following fact from general valuation theory (cf. [Z–S]):

Lemma 4.1 *The map $H\mapsto\mathcal{O}(H)$ is an order preserving bijection from the set of all convex subgroups of vK onto the set $\mathcal{R}$ of all valuation rings which contain $\mathcal{O}_v$. Its inverse is the order preserving map $\mathcal{R}\ni\mathcal{O}\mapsto H(\mathcal{O})$. Thus,*

$$\mathcal{O}(H(\mathcal{O})) \;=\; \mathcal{O} \quad \textit{and} \quad H(\mathcal{O}(H)) \;=\; H\;. \tag{18}$$

4.1 $\mathcal{O}_v$-modules in K

Now we consider the $\mathcal{O}_v$-modules $M\subseteq K$. For every such module M, we have that

$$vM \;:=\; \{va \mid 0\neq a\in M\}$$

is a final segment of vK, and that

$$M \;=\; \{a\in K \mid va\in vM\}\;. \tag{19}$$

Hence:

Lemma 4.2 *The map $v: M\mapsto vM$ establishes an order preserving bijection between the $\mathcal{O}_v$-modules $M\subseteq K$ and the final segments S of vK; its inverse map is $S\mapsto v^{-1}(S)$. The map $M\mapsto (vM)^-$ is a bijection between the $\mathcal{O}_v$-modules $M\subseteq K$ and the cuts in the value group vK.*

Therefore, we can expect to read off information about M from the invariance group of the associated cut. We start with the case of $M = \mathcal{O} \in \mathcal{R}$. Note that for every $\mathcal{O} \in \mathcal{R}$ with maximal ideal $\mathcal{M}$, we have that $\mathcal{O}$ and $\mathcal{M}$ are $\mathcal{O}_v$-modules.

Lemma 4.3 *For every $\mathcal{O} \in \mathcal{R}$ with maximal ideal $\mathcal{M}$,*

$$H(\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{M}) \quad \textit{and} \quad \mathcal{O} \;=\; \mathcal{O}(\mathcal{G}(v\mathcal{O})) \;=\; \mathcal{O}(\mathcal{G}(v\mathcal{M}))\;. \tag{20}$$

Proof: Since $\mathcal{O}$ is the disjoint union of $\mathcal{O}^\times$ and $\mathcal{M}$, (19) shows that $v\mathcal{O}$ is the disjoint union of $v\mathcal{O}^\times = H(\mathcal{O})$ and $v\mathcal{M}$. Since the latter is a final segment of vK, it follows that $H(\mathcal{O}) < v\mathcal{M}$. Thus, $H(\mathcal{O})$ is an initial segment of $v\mathcal{O}$ and a final segment of $vK \setminus v\mathcal{M}$. Now (20) follows from Corollary 3.7 and Lemma 4.1. □

The next lemma shows that there is at most one $\mathcal{O} \in \mathcal{R}$ such that $cM = \mathcal{O}$ or $cM = \mathcal{M}$ for some non-zero $c \in K$.

Lemma 4.4 *Take any $\mathcal{O}_v$-module $M \subset K$, and $\mathcal{O} \in \mathcal{R}$ with maximal ideal $\mathcal{M}$. If we have $cM = \mathcal{O}$ or $cM = \mathcal{M}$ for some $c \in K$, $c \neq 0$, then $\mathcal{O} = \mathcal{O}(\mathcal{G}(vM))$.*

Proof: By Lemma 4.2, $cM = \mathcal{O}$ is equivalent to $vcM = v\mathcal{O}$, and $cM = \mathcal{M}$ is equivalent to $vcM = v\mathcal{M}$. Both cases imply that

$$\mathcal{G}(vM) \;=\; \mathcal{G}(vc + vM) \;=\; \mathcal{G}(vcM) \;=\; \mathcal{G}(v\mathcal{O}) \;=\; \mathcal{G}(v\mathcal{M}) \;=\; H(\mathcal{O})\;,$$

where we have used Lemma 3.10 for the first and (20) for the last two equalities. By Lemma 4.1, it follows that $\mathcal{O} = \mathcal{O}(H(\mathcal{O})) = \mathcal{O}(\mathcal{G}(vM))$. □

From this lemma together with (20), we deduce:

Corollary 4.5 *For $i = 1, 2$, take $\mathcal{O}_i \in \mathcal{R}$ with maximal ideal $\mathcal{M}_i$. If for some $c \in K$, $c \neq 0$, we have that $c\mathcal{O}_1 = \mathcal{O}_2$ or $c\mathcal{O}_1 = \mathcal{M}_2$ or $c\mathcal{M}_1 = \mathcal{O}_2$ or $c\mathcal{M}_1 = \mathcal{M}_2$, then $\mathcal{O}_1 = \mathcal{O}_2$.*

For the sake of completeness, let us add:

Lemma 4.6 *Take $\mathcal{O} \in \mathcal{R}$ with maximal ideal $\mathcal{M}$. Then there is some nonzero $c \in K$ such that $c\mathcal{O} = \mathcal{M}$ if and only if $vK/H(\mathcal{O})$ is discretely ordered.*

Proof: For an arbitrary valuation ring, it holds that its maximal ideal is generated by one element c if and only if the value of c is the smallest positive element in the associated value group. As $vK/H(\mathcal{O})$ is the value group associated to $\mathcal{O}$, this yields our assertion. □

For later use, we prove:

Lemma 4.7 *Take any $\mathcal{O}_v$-module $M \subset K$. Then $\mathcal{O}(\mathcal{G}(vM))$ is the largest of all rings $\mathcal{O} \in \mathcal{R}$ with the property that M is an $\mathcal{O}$-module.*

Proof: We have that M is an $\mathcal{O}$-module if and only if $\mathcal{O}M = M$, which by Lemma 4.2 is equivalent to $v\mathcal{O} + vM = v\mathcal{O}M = vM$. As vM is a final segment of vK and $H(\mathcal{O}) = v\mathcal{O}^\times$ is an initial segment of $v\mathcal{O}$, the latter is equivalent to $H(\mathcal{O}) + vM = vM$, which in turn holds if and only if $H(\mathcal{O}) \subseteq \mathcal{G}(vM)$. By means of Lemma 4.1, this is equivalent to $\mathcal{O} \subseteq \mathcal{O}(\mathcal{G}(vM))$. □

4.2 The invariance valuation ring of an $\mathcal{O}_v$-module in K

Take an $\mathcal{O}_v$-module $M \subset K$. We set

$$\mathcal{O}(M) \;:=\; \{b \in K \mid bM \subseteq M\} \qquad \text{and} \qquad \mathcal{M}(M) \;=\; \{b \in K \mid bM \subsetneq M\} \cup \{0\}\;. \tag{21}$$

Example 4.8 For every $\mathcal{O} \in \mathcal{R}$ with maximal ideal $\mathcal{M}$,

$$\mathcal{O}(\mathcal{O}) \;=\; \mathcal{O}(\mathcal{M}) \;=\; \mathcal{O} \qquad \text{and} \qquad \mathcal{M}(\mathcal{O}) \;=\; \mathcal{M}(\mathcal{M}) \;=\; \mathcal{M}\;.$$

Further, $\mathcal{O}(K) = \mathcal{O}(\{0\}) = K$ and $\mathcal{M}(K) = \mathcal{M}(\{0\}) = \{0\}$. ♢

Theorem 4.9 *For every $\mathcal{O}_v$-module $M \subset K$, $\mathcal{O}(M)$ is a valuation ring of K with maximal ideal $\mathcal{M}(M)$ and containing $\mathcal{O}_v$. It is the largest of all valuation rings $\mathcal{O}$ of K for which M is an $\mathcal{O}$-module. Further, $\mathcal{O}(M)^\times = \{b \in K \mid bM = M\}$.*

Proof: It is straightforward to prove that $\mathcal{O}(M)$ is a ring. As M is an $\mathcal{O}_v$-module, we have that $\mathcal{O}_vM = M$. Hence, $\mathcal{O}_v \subseteq \mathcal{O}(M)$. By general valuation theory, it follows that $\mathcal{O}(M)$ is a valuation ring.

The inclusion $\{b \in K \mid bM = M\} \subseteq \mathcal{O}(M)^\times$ holds since $bM = M \Leftrightarrow M = b^{-1}M$ for $b \neq 0$. The converse inclusion holds since if $b, b^{-1} \in \mathcal{O}(M)$, then $M = bb^{-1}M \subseteq bM \subseteq M$ and thus, $bM = M$.

In every valuation ring, the unique maximal ideal consists precisely of all non-units. Hence by what we have already proved, the maximal ideal of $\mathcal{O}(M)$ is $\mathcal{O}(M) \setminus \{b \in K \mid bM = M\} = \mathcal{M}(M)$.

Finally, it follows directly from the definition that if $\mathcal{O}$ is a valuation ring of K such that M is an $\mathcal{O}$-module, then $\mathcal{O} \subseteq \mathcal{O}(M)$. On the other hand, it is also follows from the definition that M is an $\mathcal{O}(M)$-module; therefore, $\mathcal{O}(M)$ is the largest of all valuation rings of K with this property. □

By this theorem and Lemma 4.7 we find that

$$\mathcal{O}(M) \;=\; \mathcal{O}(\mathcal{G}(vM))\;,$$

which shows that definition (21) is coherent with definition (16). Keeping in mind that $H(\mathcal{O}) = \mathcal{G}(v\mathcal{O})$ by (20), we define

$$H(M) \;:=\; \mathcal{G}(vM)\;.$$

Lemma 4.1 shows:

Lemma 4.10 *For every $\mathcal{O}$-module $M \subseteq K$,*

$$H(M) \;=\; H(\mathcal{O}(M)) \;=\; v\mathcal{O}(M)^{\times}\;.$$

Lemma 4.11 *For every $\mathcal{O}$-module $M \subseteq K$ and every nonzero $c \in K$,*

$$\mathcal{O}(cM) \;=\; \mathcal{O}(M) \quad \textit{and} \quad H(cM) \;=\; H(M)\;.$$

Proof: Since c is invertible, we have that $bcM \subseteq cM \;\Leftrightarrow\; cbM \subseteq cM \;\Leftrightarrow\; bM \subseteq M$. □

Lemma 4.12 *1) For every nonzero $c \in K$,*

$$\textit{either } cM \subseteq \mathcal{M}(M)\,, \textit{ or } \mathcal{O}(M) \subseteq cM\;. \tag{22}$$

2) Take $\mathcal{O} \in \mathcal{R}$ and denote its maximal ideal by $\mathcal{M}$. If $\mathcal{O}(M) \subsetneq \mathcal{O}$, then there is some $c \in K$, $c \neq 0$, such that

$$\mathcal{M} \subsetneq cM \subsetneq \mathcal{O}\;. \tag{23}$$

Proof: To start with, we note that for every $c \in K$, $c \neq 0$, and every convex valuation ring $\mathcal{O}$ of K,

$$(vK \setminus vcM) \cap H(\mathcal{O}) \neq \emptyset \;\Leftrightarrow\; cM \subsetneq \mathcal{O} \qquad \text{and} \qquad vcM \cap H(\mathcal{O}) \neq \emptyset \;\Leftrightarrow\; \mathcal{M} \subsetneq cM\;. \tag{24}$$

To see this, recall that $H(\mathcal{O}) = v\mathcal{O}^{\times}$ and observe that cM is an $\mathcal{O}_v$-module in K. Hence, $K \setminus cM \cap \mathcal{O}^{\times} \neq \emptyset \Leftrightarrow cM \subsetneq \mathcal{O}$, and $cM \cap \mathcal{O}^{\times} \neq \emptyset \Leftrightarrow \mathcal{M} \subsetneq cM$. Now (24) follows by means of Lemma 4.1 and (17).

1): We take $\mathcal{O} = \mathcal{O}(M)$. Then $H(\mathcal{O})$ is the invariance group of vcM. Hence by Theorem 3.11, $vcM \cap H(\mathcal{O}) \neq \emptyset$ is equivalent to $H(\mathcal{O}) \subseteq vcM$, which in turn is equivalent to $(vK \setminus vcM) \cap H(\mathcal{O}) = \emptyset$. By means of (24), this yields our assertion.

2): From $\mathcal{O}(M) \subsetneq \mathcal{O}$ it follows from Lemma 4.1 that $H(M) = H(\mathcal{O}(M)) \subsetneq H(\mathcal{O})$. This in turn implies by Lemma 4.10 and Theorem 3.11 that the cut $(vM)^{-}$ can be shifted into $H(\mathcal{O})$. That is, there is some $c \in K$ such that

$$((vK \setminus vM) + vc) \cap H(\mathcal{O}) \neq \emptyset \;\text{ and }\; (vM + vc) \cap H(\mathcal{O}) \neq \emptyset\;.$$

By (24), this implies (23). □

Lemma 4.13 *Take any $\mathcal{O}_v$-module $M \subset K$. Then $cM = \mathcal{M}(M)$ holds for some nonzero $c \in K$ if and only if $(vM)^-$ is a ball$_0^+$-cut. Similarly, $cM = \mathcal{O}(M)$ holds for some nonzero $c \in K$ if and only if $(vM)^-$ is a ball$_0^-$-cut.*

Proof: We observe that $(H(M), v\mathcal{M}(M)) = (v\mathcal{O}(M)^\times, v\mathcal{M}(M))$ is a cut in the ordered set $v\mathcal{O}(M)$. By Lemma 4.2, $cM = \mathcal{M}(M)$ if and only if $vcM = v\mathcal{M}(M)$, which in turn holds if and only if $(vK \setminus vcM) \cap v\mathcal{O}(M) = H(M)$. But this is true if and only if $H(M)$ is a final segment of $vK \setminus vcM$. Now this holds if and only if $(vK \setminus vcM)/H(M)$ admits $0/H(M)$ as its last elements, or in other words, if and only if $(vcM)^-$ is a group$_0^+$-cut (cf. Lemma 3.18). As $(vcM)^- = (vM)^- + vc$, Lemma 3.15 shows that this holds if and only if $(vM)^-$ is a ball$_0^+$-cut. The second half of the proof is similar. □

5 Cuts in ordered fields

Throughout this section, let K be an ordered field and Λ a cut in K. By $(K,+)$ we indicate the ordered additive group of K.

5.1 Shifting and scaling of cuts

For every $S \subseteq K$ and every $c \in K$, we set $cS := \{cb \mid b \in S\}$. If $c > 0$, then the multiplication $b \mapsto cb$ is an order preserving bijection. Hence for every $c > 0$ and every $a \in K$,

$$c\Lambda + a \;:=\; (c\Lambda^L + a\,,\, c\Lambda^R + a)$$

is again a cut in K. For $c < 0$, the multiplication $b \mapsto cb$ is an order reversing bijection. Hence for every $c < 0$ and every $a \in K$,

$$c\Lambda + a \;:=\; (c\Lambda^R + a\,,\, c\Lambda^L + a)$$

is again a cut in K. Note that in both cases, $c\Lambda + a$ is a principal if and only if Λ is; more precisely, if $\Lambda = b^+$ (or $\Lambda = b^-$), then $c\Lambda + a = (cb+a)^+$ if $c > 0$ and $c\Lambda + a = (cb+a)^-$ if $c < 0$ (or respectively, $c\Lambda + a = (cb+a)^-$ if $c > 0$ and $c\Lambda + a = (cb+a)^+$ if $c < 0$).

We have:

Lemma 5.1 *For every nonzero $c \in K$ and all $a \in K$,*

$$\begin{aligned} \mathcal{G}(c\Lambda + a) &= c\mathcal{G}(\Lambda) \\ \mathrm{CS}\,(c\Lambda) &= c\mathrm{CS}\,(\Lambda)\,. \end{aligned}$$

Proof: We know from Lemma 3.10 that $\mathcal{G}(c\Lambda + a) = \mathcal{G}(c\Lambda)$. Since c is invertible, we have that $\Lambda^L + g = \Lambda^L \Leftrightarrow c\Lambda^L + cg = c\Lambda^L$. This proves that $\mathcal{G}(c\Lambda) = c\mathcal{G}(\Lambda)$. The easy proof of the second assertion is left to the reader. $\square$

On the basis of this lemma, we can prove:

Lemma 5.2 *Take $a, c \in K$. If $c > 0$, then Λ is a ball$^+$-cut if and only if $c\Lambda + a$ is, and it is a ball$^-$-cut if and only if $c\Lambda + a$ is. If $c < 0$, then Λ is a ball$_0^+$-cut if and only if $c\Lambda + a$ is a ball$^-$-cut, and it is a ball$^-$-cut if and only if $c\Lambda + a$ is a ball$_0^+$-cut.*

Proof: The reader may verify: $g/\mathcal{G}(\Lambda)$ is the last element of $\Lambda^L/\mathcal{G}(\Lambda)$ if and only if $(cg + a)/\mathcal{G}(\Lambda)$ is the last element of $(c\Lambda^L + a)/c\mathcal{G}(\Lambda)$. Since $c\mathcal{G}(\Lambda) = \mathcal{G}(c\Lambda + a)$ by the foregoing lemma, this gives our assertion. The other parts of the proof are similar. $\square$

5.2 Multiplying cuts

As for addition, it is also not immediately clear how to multiply two cuts Λ_1 and Λ_2. As one can easily find out by experiment, $(\Lambda_1^L \cdot \Lambda_2^L\,,\, \Lambda_1^R \cdot \Lambda_2^R)$ will in general not be a cut. An additional problem here is that multiplying by negative elements switches the order. On the other hand, we have:

Lemma 5.3 *If S_1 and S_2 are symmetric sets, then so is $S_1 \cdot S_2$. If S_1 and S_2 are convex sets, then so is $S_1 \cdot S_2$.*

Proof: The proof for the first assertion is straightforward. For the second assertion, we observe that

$$S_1 \cdot S_2 \;=\; S_1^{\geq 0} \cdot S_2^{\geq 0} \cup S_1^{\leq 0} \cdot S_2^{\geq 0} \cup S_1^{\geq 0} \cdot S_2^{\leq 0} \cup S_1^{\leq 0} \cdot S_2^{\leq 0} \;=\; S_1^{\geq 0} \cdot S_2^{\geq 0} \cup S_1^{\leq 0} \cdot S_2^{\geq 0}\,,$$

where the second inequality holds since $S_1^{\geq 0} \cdot S_2^{\geq 0} = S_1^{\leq 0} \cdot S_2^{\leq 0}$ and $S_1^{\leq 0} \cdot S_2^{\geq 0} = S_1^{\geq 0} \cdot S_2^{\leq 0}$ by the symmetry of S_1 and S_2. For the same reason, $S_1^{\leq 0} \cdot S_2^{\geq 0} = -((-S_1^{\leq 0}) \cdot S_2^{\geq 0}) = -(S_1^{\geq 0} \cdot S_2^{\geq 0})$. Since the operation $S \mapsto -S$ preserves convexity, this set is convex if and only $S_1^{\geq 0} \cdot S_2^{\geq 0}$ is. It therefore suffices to prove that $S_1^{\geq 0} \cdot S_2^{\geq 0}$ is convex, because then the union of the two convex sets $S_1^{\geq 0} \cdot S_2^{\geq 0}$ and $S_1^{\leq 0} \cdot S_2^{\geq 0}$ is convex as they have the element 0 in common.

Assume that $s_1, s_1' \in S_1^{\geq 0}$ and $s_2, s_2' \in S_2^{\geq 0}$ and that there is some a such that $s_1 s_2 < a < s_1' s_2'$. We may assume that $s_1 \leq s_1'$ and $s_2 \leq s_2'$; otherwise, we replace s_1' by s_1 or s_2' by s_2. We know that $s_1 s_2 \leq s_1 s_2' \leq s_1' s_2'$. If $s_1 s_2 < a \leq s_1 s_2'$, then $s_2 < \frac{a}{s_1} \leq s_2'$, whence $\frac{a}{s_1} \in S_2^{\geq 0}$ and $a = s_1 \cdot \frac{a}{s_1} \in S_1^{\geq 0} \cdot S_2^{\geq 0}$. If $s_1 s_2' \leq a < s_1' s_2'$, then $s_1 \leq \frac{a}{s_2'} < s_1'$, whence $\frac{a}{s_2'} \in S_1^{\geq 0}$ and $a = \frac{a}{s_2'} \cdot s_2' \in S_1^{\geq 0} \cdot S_2^{\geq 0}$. This proves that $S_1^{\geq 0} \cdot S_2^{\geq 0}$ is convex. $\square$

In particular, $\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2)$ is again a convex symmetric subset of K. So among the possibilities for defining multiplication of cuts, we may choose the following natural one:

- if Λ_1 and Λ_2 both are positive cuts or both are negative cuts, then

$$\Lambda_1 \cdot \Lambda_2 \;:=\; (\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2))^+ \,,$$

- if Λ_1 is positive and Λ_2 is negative or vice versa, then

$$\Lambda_1 \cdot \Lambda_2 \;:=\; (\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2))^- \,.$$

We observe that this multiplication is commutative, and that it preserves the order among cuts in the following sense:

Proposition 5.4 *Take two cuts $\Lambda_2 \leq \Lambda_2'$. If Λ_1 is a positive cut, then $\Lambda_1 \cdot \Lambda_2 \leq \Lambda_1 \cdot \Lambda_2'$, and if Λ_1 is a negative cut, then $\Lambda_1 \cdot \Lambda_2 \geq \Lambda_1 \cdot \Lambda_2'$.*

Proof: Assume that $\Lambda_2 \leq \Lambda_2'$ and that Λ_1 is a positive cut. If Λ_2 is a negative cut and Λ_2' is a positive cut, then $\Lambda_1 \cdot \Lambda_2$ is a negative cut and $\Lambda_1 \cdot \Lambda_2'$ is a positive cut, so we have that $\Lambda_1 \cdot \Lambda_2 < \Lambda_1 \cdot \Lambda_2'$. If both Λ_2 and Λ_2' are positive, then $\mathrm{CS}(\Lambda_2) \subseteq \mathrm{CS}(\Lambda_2')$ and therefore, $\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2) \subseteq \mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2')$, whence $\Lambda_1 \cdot \Lambda_2 \leq \Lambda_1 \cdot \Lambda_2'$, as these are positive cuts. If both Λ_2 and Λ_2' are negative, then $\mathrm{CS}(\Lambda_2) \supseteq \mathrm{CS}(\Lambda_2')$ and therefore, $\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2) \supseteq \mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2')$, whence $\Lambda_1 \cdot \Lambda_2 \leq \Lambda_1 \cdot \Lambda_2'$, as these are negative cuts.

The proof for the case of Λ_1 a negative cut is similar. □

We can compute the invariance groups of these new cuts if we can compute $\mathcal{G}(\mathrm{CS}(\Lambda_1) \cdot \mathrm{CS}(\Lambda_2))$ from $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$. This motivates our next proposition. Beforehand, we note:

Lemma 5.5 *Take any convex subgroup H of $(K,+)$. Then for each $s \in K$, sH is again a convex subgroup of $(K,+)$. For every set $S \subseteq K$,*

$$S \cdot H \;=\; \bigcup_{s \in S} sH$$

is again a convex subgroup of $(K,+)$.

Proof: It is clear that sH is a subgroup of $(K,+)$. To show that it is convex, it suffices to show that if $g \in K$ with $0 < g < a \in sH$, then $g \in sH$. We may assume that $s > 0$. We write $a = sb$ with $b \in H$. Then $0 < \frac{g}{s} < b \in H$ and hence $\frac{g}{s} \in H$ by the convexity of H. Hence $g = s\frac{g}{s} \in sH$.

The second assertion follows from the first since the union over any set of convex subgroups is again a convex subgroup. □

Proposition 5.6 *If S_1 and S_2 are convex symmetric subsets of K, then*

$$\begin{aligned} \mathcal{G}(S_1 \cdot S_2) &= S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2) \\ &= S_2^{\geq 0} \cdot \mathcal{G}(S_1) \cup S_1^{\geq 0} \cdot \mathcal{G}(S_2) = \max\{S_2^{\geq 0} \cdot \mathcal{G}(S_1) \,,\, S_1^{\geq 0} \cdot \mathcal{G}(S_2)\} \,. \end{aligned} \tag{25}$$

Proof: The second equality holds because all three sets in (25) are symmetric. The third equality holds because the two sets $S_2 \cdot \mathcal{G}(S_1)$ and $S_1 \cdot \mathcal{G}(S_2)$ are convex symmetric subsets of K and therefore comparable by inclusion. It remains to prove the first equality.

Take any $g \in S_2$ and $a \in \mathcal{G}(S_1)$. To show that $ga \in \mathcal{G}(S_1 \cdot S_2)$, we may assume that $ga \geq 0$, because We have to show that $g_1 \cdot g_2 + ga \in S_1 \cdot S_2$ whenever $g_1 \in S_1$ and $g_2 \in S_2$. Since $S_1 \cdot S_2$ is again convex symmetric by Lemma 5.3, it suffices to consider the case of $g \leq g_2$. We compute: $g_1 \cdot g_2 \leq g_1 \cdot g_2 + ga \leq g_1 \cdot g_2 + g_2 a = (g_1 + a)g_2 \in S_1 \cdot S_2$ since $g_1 + a \in S_1$. By convexity, we find $g_1 \cdot g_2 + ga \in S_1 \cdot S_2$. So we have that $S_2 \cdot \mathcal{G}(S_1) \subseteq \mathcal{G}(S_1 \cdot S_2)$. Symmetrically, one shows that $S_1 \cdot \mathcal{G}(S_2) \subseteq \mathcal{G}(S_1 \cdot S_2)$.

Now it remains to show that $\mathcal{G}(S_1 \cdot S_2) \subseteq S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2)$. We distinguish three cases.

First, assume that S_2 is a convex subgroup. Then by the previous lemma, $S_1 \cdot S_2$ is a convex subgroup and therefore, $\mathcal{G}(S_1 \cdot S_2) = S_1 \cdot S_2$. Further, $\mathcal{G}(S_2) = S_2$ and since $\mathcal{G}(S_1) \subseteq S_1$ by part 4) of Lemma 2.3, we obtain that $S_2 \cdot \mathcal{G}(S_1) \subseteq S_2 \cdot S_1 = S_1 \cdot S_2 = S_1 \cdot \mathcal{G}(S_2)$. Thus,

$$S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2) = S_1 \cdot S_2 = \mathcal{G}(S_1 \cdot S_2) \,.$$

By symmetry, the case of S_1 a convex subgroup is similar.

Now assume that S_1 and S_2 are not subgroups. Hence by Lemma 2.2 there are positive elements $s_1 \in S_1$ and $s_2 \in S_2$ such that $2s_1 > S_1$ and $2s_2 > S_2$. It follows that $s_2 \notin \mathcal{G}(S_2)$. Now take any positive $a \in K$ such that $a \notin S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2)$. Set

$$s_2' := \min\left\{\frac{a}{s_1} \,,\, s_2\right\} \quad \text{and} \quad s_1' := \frac{a}{s_2} \,.$$

By our assumption on a, $\frac{a}{s_1} \notin \mathcal{G}(S_2)$ and $s_1' = \frac{a}{s_2} \notin \mathcal{G}(S_1)$. With $s_2 \notin \mathcal{G}(S_2)$, it also follows that $s_2' \notin \mathcal{G}(S_2)$. So there are positive elements $t_1 \in S_1$ and $t_2 \in S_2$ such that $t_1 + s_1' > S_1$ and $t_2 + s_2' > S_2$. Therefore, $(t_1 + s_1')(t_2 + s_2') > S_1 \cdot S_2$. On the other hand, we have:

$s_1' s_2' \leq s_1' s_2 = a$,
$t_1 s_2' < 2s_1 s_2' \leq 2a$ since $t_1 \in S_1 < 2s_1$,
$s_1' t_2 < s_1' 2s_2 = 2a$ since $t_2 \in S_2 < 2s_2$,

whence

$$S_1 \cdot S_2 < (t_1 + s_1')(t_2 + s_2') = t_1 t_2 + t_1 s_2' + s_1' t_2 + s_1' s_2' \leq t_1 t_2 + 5a \,.$$

This shows that $5a \notin \mathcal{G}(S_1 \cdot S_2)$ and hence, $a \notin \mathcal{G}(S_1 \cdot S_2)$. We have proved that $\mathcal{G}(S_1 \cdot S_2)^{>0} \subseteq S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2)$. Since these sets are symmetric, it again follows that $\mathcal{G}(S_1 \cdot S_2) \subseteq S_2 \cdot \mathcal{G}(S_1) \cup S_1 \cdot \mathcal{G}(S_2)$. $\square$

Since $\mathcal{G}(\Lambda_1 \cdot \Lambda_2) = \mathcal{G}(\mathrm{CS}\,(\Lambda_1 \cdot \Lambda_2))$ by Theorem 3.2 and $\mathrm{CS}\,(\Lambda_1 \cdot \Lambda_2) = \mathrm{CS}\,(\Lambda_1) \cdot \mathrm{CS}\,(\Lambda_2)$ by definition, we obtain:

Corollary 5.7 *For any two cuts Λ_1 and Λ_2,*

$$\begin{aligned}
\mathcal{G}(\Lambda_1 \cdot \Lambda_2) &= \mathrm{CS}\,(\Lambda_2) \cdot \mathcal{G}(\Lambda_1) \,\cup\, \mathrm{CS}\,(\Lambda_1) \cdot \mathcal{G}(\Lambda_2) \\
&= \mathrm{CS}\,(\Lambda_2)^{\geq 0} \cdot \mathcal{G}(\Lambda_1) \,\cup\, \mathrm{CS}\,(\Lambda_1)^{\geq 0} \cdot \mathcal{G}(\Lambda_2) \\
&= \max\{\mathrm{CS}\,(\Lambda_2)^{\geq 0} \cdot \mathcal{G}(\Lambda_1)\,,\, \mathrm{CS}\,(\Lambda_1)^{\geq 0} \cdot \mathcal{G}(\Lambda_2)\}\,.
\end{aligned}$$

5.3 The valuation theory of the invariance group in the field case

For the rest of this paper, we will denote by v the natural valuation of the ordered field K. This is just the natural valuation of the ordered additive group $(K,+)$. We introduce an addition on vK by setting $va + vb = v(ab)$. This turns vK into an ordered abelian group, and v satisfies, in addition to axioms (V∞) and (VU), the following homomorphism axiom for field valuations:

(VH) $v(xy) \;=\; vx + vy\,.$

Among all possible valuations of K, the natural valuation is characterized by the fact that its residue field $\mathcal{O}_v/\mathcal{M}_v$ is an archimedean ordered field (hence embeddable in $\mathbb{R}$).

From Lemma 3.25 and Lemma 4.2 we obtain:

Lemma 5.8 *The convex subgroups of $(K,+)$ are precisely the $\mathcal{O}_v$-submodules of K. Hence for every cut Λ, its invariance group $\mathcal{G}(\Lambda)$ is an $\mathcal{O}_v$-submodule of K.*

This lemma also shows that every subring of K containing $\mathcal{O}_v$ is convex. On the other hand, a convex valuation ring of $(K,<)$, being an $\mathcal{O}_v$-module and containing 1, must contain $\mathcal{O}_v$. Therefore, the set of all convex valuation rings of $(K,<)$ is precisely the set $\mathcal{R}$ of all valuation rings which contain $\mathcal{O}_v$.

For further basic properties of the natural valuation, see [KuS] or [Ku2]. For the convenience of the reader, we will now adapt the results of Section 3.7 to the case of ordered fields, and we will give examples. The proofs for the results in this section can be taken over literally from the corresponding results in Section 3.7.

Lemma 5.9 *For every cut Λ in $(K,<)$,*

$$v\mathcal{G}(\Lambda) \;=\; vK \setminus v(\Lambda^R - \Lambda^L)\,. \tag{26}$$

If $(L,<)$ is an ordered field extending $(K,<)$, then we will also write v for the natural valuation of $(L,<)$. Then $(L|K,v)$ is an extension of valued fields. For any $x \in L$ (which we will later assume to realize the cut Λ), we consider the value set

$$v(x-K) \;:=\; \{v(x-b) \mid b \in K\} \;\subseteq\; vK(x) \cup \{\infty\}\,.$$

Lemma 5.10 *a) If $v(x-K)$ contains an element $\alpha \notin vK$, then α is the largest element of $v(x-K)$.*
b) The set $v(x-K) \cap vK$ is an initial segment of vK.

Lemma 5.11 *If the cut Λ in $(K,<)$ is realized by an element x in an extension $(L,<)$, then*

$$v\mathcal{G}(\Lambda) \;=\; vK \setminus v(x-K) \;=\; \{\alpha \in vK \mid \alpha > v(x-b) \text{ for all } b \in K\}\,. \qquad (27)$$

We give examples to show how to apply this lemma.

Example 5.12 Assume that t is algebraically independent over $\mathbb{Q}$. Take v to be the t-adic valuation on $\mathbb{Q}(t)$, that is, v is trivial on $\mathbb{Q}$ and $vt > 0$. The associated place, which sends t to 0, is real since its residue field is $\mathbb{Q}$. Hence there is an ordering $<$ on $\mathbb{Q}(t)$ such that v is the natural valuation of $(\mathbb{Q}(t),<)$. The field $\mathbb{R}(t)$ carries a canonical extension of v, the t-adic valuation on $\mathbb{R}(t)$, and a corresponding extension of $<$. Note that $vr = 0$ for every nonzero r in the subfield $\mathbb{R}$ of $\mathbb{R}(t)$.

Now take any $r \in \mathbb{R} \setminus \mathbb{Q}$. Viewing r as an element of the extension field $(\mathbb{R}(t),<)$ of $(\mathbb{Q}(t),<)$, we take Λ to be the cut induced by r in $(\mathbb{Q}(t),<)$. Take any $b \in \mathbb{Q}(t)$. If $vb < 0 = vr$, then $v(r-b) = vb < 0$. If $vb \geq 0$, then the residue $bv := b/\mathcal{M}_v$ of b lies in $\mathbb{Q}$. On the other hand, the residue of r is r itself, not lying in $\mathbb{Q}$. Hence, $(r-b)v = rv - bv \neq 0$, which implies that $v(r-b) = 0$. We have thus shown that 0 is the maximal element of $v(r-\mathbb{Q}(t))$. Hence by the foregoing lemma, $v\mathcal{G}(\Lambda) = \{\alpha \in vK \mid \alpha > 0\}$. But the latter is the same as $v\mathcal{M}_v$. Hence by Lemma 4.2,

$$\mathcal{G}(\Lambda) \;=\; \mathcal{M}_v\,.$$

In other words: precisely the infinitesimals leave the cut Λ invariant.

We analyze the corresponding cut $(v\mathcal{G}(\Lambda))^-$ in vK. We observe that $vK \setminus v\mathcal{G}(\Lambda) = \{\alpha \in vK \mid \alpha \leq 0\}$ has last element 0. We conclude that the invariance group of $(v\mathcal{G}(\Lambda))^-$ is $\{0\}$ and that $(v\mathcal{G}(\Lambda))^- = 0^+$ is a group$^+_0$-cut in vK. ♢

Example 5.13 We build on the last example. Now we take K to be the real closure of $\mathbb{Q}(t)$. Then $vK = \mathbb{Q}$. Take any $s \in \mathbb{R} \setminus \mathbb{Q}$. We set $\Lambda^L := K^{\leq 0} \cup \{b \in K \mid vb > s\}$ and $\Lambda^R := \{b \in K^{\geq 0} \mid vb < s\}$. Then Λ is a cut in $(K,<)$. The value (with respect to the natural valuation) of any element realizing the cut in some extension field will realize the cut $(\{\alpha \in \mathbb{Q} \mid \alpha < s\}\,,\, \{\alpha \in \mathbb{Q} \mid \alpha > s\})$ in the value group of the extension field. We have $v(\Lambda^R - \Lambda^L) = \{\alpha \in \mathbb{Q} \mid \alpha < s\}$. Hence by (26),

$$v\mathcal{G}(\Lambda) \;=\; vK \setminus \{\alpha \in \mathbb{Q} \mid \alpha < s\} \;=\; \{\alpha \in \mathbb{Q} \mid \alpha > s\}\,.$$

The corresponding cut $(v\mathcal{G}(\Lambda))^-$ in vK is just the above cut in $\mathbb{Q}$ induced by s; its invariance group is $\{0\}$ and it is neither a ball$^+_0$-cut nor a ball$^-_0$-cut. ♢

An extension $(L|K,v)$ of valued fields is called **immediate** if the canonical embeddings of vK in vL and of Kv in Lv are onto. It is easy to show that the extension $(L|K,v)$ is immediate if and only if it is immediate as an extension of valued abelian groups. In this case, every element $x \in L$ is the pseudo limit of a pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in (K,v) indexed by a limit ordinal λ; cf. [Ka] or [R]. If $x \notin K$, then this pseudo Cauchy sequence can be chosen such that it has no pseudo limit in K. The **breadth** of $(a_\nu)_{\nu<\lambda}$ is defined as in the case of valued abelian groups (see above).

Theorem 5.14 *Suppose that the cut Λ is not realized in $(K,<)$, but is realized by an element x in an extension $(L,<)$ such that $(L|K,v)$ is immediate. If $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) having pseudo limit x but not having a pseudo limit in K, then its breadth is equal to $\mathcal{G}(\Lambda)$.*

The **completion** K^c of (K,v) (with respect to the topology induced by v) admits a canonical extension of v and of the ordering from K. With this extension of v, $(K^c|K,v)$ is an immediate extension of valued fields, and the breadth of every pseudo Cauchy sequence without a pseudo limit in K but with pseudo limit in K^c is $\{0\}$. Thus, we obtain the following corollary:

Corollary 5.15 *If the cut Λ is realized in the completion of (K,v), then $\mathcal{G}(\Lambda) = \{0\}$.*

Theorem 5.14 can be used to construct interesting examples. Here is one.

Example 5.16 Assume that the elements t_i, $i \in \mathbb{N}$, are algebraically independent over $\mathbb{Q}$. Define a valuation on $K := \mathbb{Q}(t_i \mid i \in \mathbb{N})$ by setting $vt_1 \gg vt_2 \gg \ldots \gg vt_i \gg \ldots \gg 0$. That is,

$$vK \;=\; \mathbb{Z}vt_1 \oplus \mathbb{Z}vt_2 \oplus \ldots \oplus \mathbb{Z}vt_i \oplus \ldots\;,$$

lexicographically ordered. Now define a pseudo Cauchy sequence in (K,v) by setting $a_\nu := \sum_{i=1}^{\nu} t_i^{-1}$ for all $\nu \in \mathbb{N}$. This is a pseudo Cauchy sequence since $v(a_{\nu+1} - a_\nu) = vt_{\nu+1}^{-1} = -vt_{\nu+1}$ is monotonically increasing with ν. Observe that this sequence of values is cofinal in the negative part of the value group vK. Hence, the breadth of our pseudo Cauchy sequence is just $\mathcal{O}_v$. The pseudo Cauchy sequence has no pseudo limit in K (it can actually be shown using Kaplansky's theory that every such pseudo limit must be transcendental over K).

The place assiociated with v is real. Indeed, it is easy to show that $\mathcal{O}_v/\mathcal{M}_v = \mathbb{Q}$. Therefore, there exists an ordering $<$ on K such that v is the natural valuation of $(K,<)$. Now we consider the cut Λ induced in $(K,<)$ by any pseudo limit of our pseudo Cauchy sequence in any extension field. Then by Theorem 5.14,

$$\mathcal{G}(\Lambda) \;=\; \mathcal{O}_v\;.$$

The reader should note that vK has no smallest nonzero convex subgroup. By Lemma 4.1, this means that there is no smallest valuation ring properly containing $\mathcal{O}_v$.

In contrast to Example 5.12, we now have that $v\mathcal{G}(\Lambda) = v\mathcal{O}_v = \{\alpha \in vK \mid \alpha \geq 0\}$ has first element 0. We conclude that the invariance group of $(v\mathcal{G}(\Lambda))^-$ is again $\{0\}$ and that this time, $(v\mathcal{G}(\Lambda))^-$ is a group$_0^-$-cut. As vK obviously has no smallest positive element, it is not discretely ordered. Hence by Proposition 3.21, $(v\mathcal{G}(\Lambda))^- = 0^-$ cannot be a ball$_0^+$-cut in vK. ♢

5.4 The invariance valuation ring of a cut

We set

$$\mathcal{O}(\Lambda) \;:=\; \mathcal{O}(\mathcal{G}(\Lambda)) \;=\; \{b \in K \mid b\mathcal{G}(\Lambda) \subseteq \mathcal{G}(\Lambda)\}$$

and denote the maximal ideal $\mathcal{M}(\mathcal{G}(\Lambda))$ of $\mathcal{O}(\Lambda)$ by $\mathcal{M}(\Lambda)$. From Theorem 4.9 and Lemma 4.10, we obtain:

Proposition 5.17 *The ring $\mathcal{O}(\Lambda)$ is a convex valuation ring of $(K,<)$. The group*

$$H(\Lambda) \;:=\; H(\mathcal{O}(\Lambda)) \;=\; v\mathcal{O}(\Lambda)^\times$$

is the invariance group of $(v\mathcal{G}(\Lambda))^-$ in vK.

The convex valuation ring $\mathcal{O}(\Lambda)$ of $(K,<)$ will be called the **invariance valuation ring** of the cut Λ in $(K,<)$. From Lemma 5.1 and Lemma 4.11, we obtain:

Proposition 5.18 *For all $a,c \in K$, $c \neq 0$, we have that*

$$\mathcal{O}(c\Lambda + a) \;=\; \mathcal{O}(\Lambda) \qquad \textit{and} \qquad H(c\Lambda + a) \;=\; H(\Lambda)\;.$$

5.5 The multiplicative invariance group of a cut

For every $S \subseteq K$, we define

$$\mathcal{G}^\times(S) \;:=\; \{b \in K^{>0} \mid bS = S\}\;.$$

Let Λ be any cut in the ordered field $(K,<)$. We are now interested in the set of all $b \in K^\times$ such that $b\Lambda = \Lambda$.

Proposition 5.19 *The set*

$$\mathcal{G}^\times(\Lambda) \;:=\; \{b \in K^\times \mid b\Lambda = \Lambda\} \;=\; \mathcal{G}^\times(\Lambda^L) \;=\; \mathcal{G}^\times(\Lambda^R)$$

is a convex subgroup of the ordered multiplicative group $(K^{>0},\,\cdot\,,\,>\,)$ of positive elements of K. It is equal to $\mathcal{G}^\times(\mathrm{CS}\,(\Lambda))$.

Proof: Note that if $b \in K^\times$ is negative, then we will always have $b\Lambda \neq \Lambda$. Hence, $\mathcal{G}^\times(\Lambda) \subseteq K^{>0}$. It is straightforward to show that $\mathcal{G}^\times(\Lambda)$ is a multiplicative subgroup of $(K^{>0}, \cdot)$. Take $a, b \in \mathcal{G}^\times(\Lambda)$ and $c \in K$ such that $a < c < b$. If Λ^L contains positive elements, then $\Lambda^L = a\Lambda^L \subseteq c\Lambda^L \subseteq b\Lambda^L = \Lambda^L$; otherwise, $\Lambda^L = b\Lambda^L \subseteq c\Lambda^L \subseteq a\Lambda^L = \Lambda^L$. In both cases, we find that $c\Lambda = \Lambda$, that is, $c \in \mathcal{G}^\times(\Lambda)$. This proves that $\mathcal{G}^\times(\Lambda)$ is convex.

The last assertion follows from the fact that $\mathrm{CS}\,(b\Lambda) = b\mathrm{CS}\,(\Lambda)$ (cf. Lemma 5.1). □

We will call $\mathcal{G}^\times(\Lambda)$ the **multiplicative invariance group of the cut** Λ.

Lemma 5.20 *If $c \in K^{>0}$, then $\mathcal{G}^\times(c\Lambda) = \mathcal{G}^\times(\Lambda)$.*

Proof: Since c is invertible, we have $bc\Lambda^L = c\Lambda^L \Leftrightarrow cb\Lambda^L = c\Lambda^L \Leftrightarrow b\Lambda^L = \Lambda^L$. □

Lemma 5.21 *For every cut Λ,*

$$\Big(\mathcal{G}^\times(\Lambda) - 1\Big) \cdot \mathrm{CS}\,(\Lambda) \;\subseteq\; \mathcal{G}(\Lambda)\;. \tag{28}$$

Proof: Take $c \in \mathcal{G}^\times(\Lambda) - 1$. Then $c + 1 \in \mathcal{G}^\times(\Lambda)$ and therefore, $(c + 1)\Lambda = \Lambda$. This implies that $(c + 1)\Lambda^{L^{\geq 0}} = \Lambda^{L^{\geq 0}}$ and $(c + 1)\Lambda^{R^{\leq 0}} = \Lambda^{R^{\leq 0}}$.

Assume first that $\Lambda^{L^{\geq 0}} \neq \emptyset$, and take an element g in this set. We wish to show that $cg \in \mathcal{G}(\Lambda)$. By the symmetry of $\mathrm{CS}\,(\Lambda)$ it will then follow that $c\mathrm{CS}\,(\Lambda) \subseteq \mathcal{G}(\Lambda)$. Suppose that $c \geq 0$. Then it suffices to show that $cg + \Lambda^{L^{\geq 0}} \subseteq \Lambda^{L^{\geq 0}}$. Take any $g' \in \Lambda^{L^{\geq 0}}$; we have to show that $cg + g' \in \Lambda^L$, so we may actually assume that $g' \geq g$. Then $cg + g' \leq cg' + g' = (c + 1)g' \in \Lambda^L$, so we obtain that $cg + g' \in \Lambda^L$. Now suppose that $c \leq 0$. Then it suffices to show that $cg + \Lambda^R \subseteq \Lambda^R$. Take any $g' \in \Lambda^R$; we have to show that $cg + g' \in \Lambda^R$. Since $g < g'$ and $c \leq 0$, we have that $cg + g' \geq cg' + g' = (c+1)g' \in \Lambda^R$, so we obtain that $cg + g' \in \Lambda^R$.

Now assume that $\Lambda^{R^{\leq 0}} \neq \emptyset$, and take an element g in this set. Again, we wish to show that $cg \in \mathcal{G}(\Lambda)$. Suppose that $c \geq 0$. Then it suffices to show that $cg + \Lambda^{R^{\leq 0}} \subseteq \Lambda^{R^{\leq 0}}$. Take any $g' \in \Lambda^{R^{\leq 0}}$; we have to show that $cg + g' \in \Lambda^R$, so we may actually assume that $g' \leq g$. Then $cg + g' \geq cg' + g' = (c+1)g' \in \Lambda^R$, so we obtain that $cg + g' \in \Lambda^R$. Now suppose that $c \leq 0$. Then it suffices to show that $cg + \Lambda^L \subseteq \Lambda^L$. Take any $g' \in \Lambda^L$; we have to show that $cg + g' \in \Lambda^L$. Since $g > g'$ and $c \leq 0$, we have that $cg + g' \leq cg' + g' = (c+1)g' \in \Lambda^L$, so we obtain that $cg + g' \in \Lambda^L$. □

Proposition 5.22 *The following are equivalent:*

a) *Λ is not a group$_0$-cut,*

b) $2 \notin \mathcal{G}^\times(\Lambda)$,

c) $\mathcal{G}^\times(\Lambda) - 1 \subseteq \mathcal{M}_v\,$,

d) *$\mathcal{G}^\times(\Lambda) - 1$ is a convex subgroup of the ordered additive group of K.*

Proof: a)⇔b): By Proposition 5.19, $\mathcal{G}^\times(\Lambda) = \mathcal{G}^\times(\mathrm{CS}\,(\Lambda))$. By Lemma 3.13, Λ is a group${}_0$-cut if and only if $\mathrm{CS}\,(\Lambda)$ is a group. Hence if Λ is a group${}_0$-cut, then $2\mathrm{CS}\,(\Lambda) = \mathrm{CS}\,(\Lambda)$ and therefore, $2 \in \mathcal{G}^\times(\mathrm{CS}\,(\Lambda)) = \mathcal{G}^\times(\Lambda)$. Conversely, if Λ is not a group${}_0$-cut and hence $\mathrm{CS}\,(\Lambda)$ is not a group, then by Lemma 2.2, there is a positive element $g \in \mathrm{CS}\,(\Lambda)$ such that $2g > \mathrm{CS}\,(\Lambda)$, showing that $2 \notin \mathcal{G}^\times(\mathrm{CS}\,(\Lambda)) = \mathcal{G}^\times(\Lambda)$.

b)⇒c): Assume that $\mathcal{G}^\times(\Lambda) - 1$ contains an element $d \notin \mathcal{M}_v$. There is some $n \in \mathbb{N}$ such that $n|d| > 1$. If d is positive, we get that $(1+d)^n \geq 1 + nd > 2$, and if d is negative, we get that $(1+d)^{-n} \geq 1 + n|d| > 2$. But 1, $(1+d)^n$ and $(1+d)^{-n}$ are elements of the convex group $\mathcal{G}^\times(\Lambda)$. Hence it follows that $2 \in \mathcal{G}^\times(\Lambda)$.

c)⇒d): Assume that $\mathcal{G}^\times(\Lambda) - 1 \subseteq \mathcal{M}_v$. Since $\mathcal{G}^\times(\Lambda)$ is convex, the same holds for $\mathcal{G}^\times(\Lambda) - 1$. Take any $a, b \in \mathcal{G}^\times(\Lambda) - 1$; we have to show that $a - b \in \mathcal{G}^\times(\Lambda) - 1$. Setting $c := \max\{|a|, |b|\}$, we have that $-2c \leq a - b \leq 2c$. Because of convexity, we only have to show that $-2c,\ 2c \in \mathcal{G}^\times(\Lambda) - 1$. But $c \in \mathcal{G}^\times(\Lambda) - 1$ or $-c \in \mathcal{G}^\times(\Lambda) - 1$ by our choice of c. That is, we have to show: if $c \in \mathcal{G}^\times(\Lambda) - 1$, then $-2c,\ 2c \in \mathcal{G}^\times(\Lambda) - 1$. As $(1+c) \in \mathcal{G}^\times(\Lambda)$, we have that $(1+c)^n \in \mathcal{G}^\times(\Lambda)$ for all $n \in \mathbb{Z}$. Note that by our general assumption, $c \in \mathcal{M}_v$. Thus, $|1 \pm 3c - (1+c)^{\pm 3}| \ll |c|$ and therefore,

$$\begin{array}{lll} 1 \leq 1 + 2c \leq (1+c)^3 & \text{and} \quad (1+c)^{-3} \leq 1 - 2c \leq 1 & \text{if } c \geq 0 \\ (1+c)^3 \leq 1 + 2c \leq 1 & \text{and} \quad 1 \leq 1 - 2c \leq (1+c)^{-3} & \text{if } c \leq 0 \end{array}$$

By convexity, this implies that $-2c,\ 2c \in \mathcal{G}^\times(\Lambda) - 1$. Now we have proved that $\mathcal{G}^\times(\Lambda) - 1$ is a convex subgroup of $(K, +, <)$.

d)⇒b): Since $\mathcal{G}^\times(\Lambda) \subseteq K^{>0}$, $-1 \notin \mathcal{G}^\times(\Lambda) - 1$. Hence if $\mathcal{G}^\times(\Lambda) - 1$ is a convex additive subgroup of K, then $1 \notin \mathcal{G}^\times(\Lambda) - 1$, that is, $2 \notin \mathcal{G}^\times(\Lambda)$. □

Remark 5.23 In this proposition, the element 2 can be replaced by any other rational number > 1, or more generally, any element $1+d$ where $d \in K$ is positive and archimedean comparable to 1.

Theorem 5.24 *1) If Λ is a group${}_0$-cut, then*

$$\mathcal{G}^\times(\Lambda) \;=\; (\mathcal{O}(\Lambda)^\times)^{>0}\,. \tag{29}$$

2) Assume that Λ is not a group${}_0$-cut. Then

$$\mathcal{G}^\times(\Lambda) \;=\; 1 + \frac{1}{g}\,\mathcal{G}(\Lambda)\,, \tag{30}$$

for every $g \in \mathrm{CS}\,(\Lambda) \setminus \mathrm{CG}\,(\Lambda)$ (such an element exists by Lemma 3.13). Furthermore,

$$\mathcal{O}(\mathcal{G}^\times(\Lambda) - 1) \;=\; \mathcal{O}(\Lambda)\,. \tag{31}$$

3) For every cut Λ,

$$\mathcal{G}(\Lambda) \;=\; \Big(\mathcal{G}^\times(\Lambda) - 1\Big) \cdot \mathrm{CS}\,(\Lambda)\,. \tag{32}$$

Proof: 1): If Λ is a group $_0$-cut, then $\mathcal{G}(\Lambda) = \mathrm{CS}(\Lambda)$ by Lemma 3.13. On the other hand, using Proposition 5.19, $\mathcal{G}^\times(\Lambda) = \mathcal{G}^\times(\mathrm{CS}(\Lambda)) = (\mathcal{O}(\mathrm{CS}(\Lambda)) \setminus \mathcal{M}(\mathrm{CS}(\Lambda)))^{>0} = (\mathcal{O}(\Lambda)^\times)^{>0}$. This proves (29).

2): First, let us assume that $g > 0$. Since $g \notin \mathrm{CG}(\Lambda)$, there is some $n \in \mathbb{N}$ such that $ng > \mathrm{CS}(\Lambda)$. Take any $a \in \mathcal{G}(\Lambda)^{\geq 0}$. Set $c = 1 + \frac{a}{g}$. Then $c \geq 1$. To show that $c \in \mathcal{G}^\times(\Lambda)$, it suffices to show that $cg' \in \mathrm{CS}(\Lambda)$ whenever $g \leq g' \in \mathrm{CS}(\Lambda)$. We know that $g' < ng$ and compute: $cg' = g' + g'\frac{a}{g} = g' + \frac{g'}{g}a < g' + na \in \Lambda^L$ since $na \in \mathcal{G}(\Lambda)^{\geq 0}$. By convexity, $cg' \in \Lambda^L$. So we have shown that $c \in \mathcal{G}^\times(\Lambda)$, whence $a = g(c-1) \in g(\mathcal{G}^\times(\Lambda) - 1)^{\geq 0}$. Hence, $\mathcal{G}(\Lambda)^{\geq 0} \subseteq g(\mathcal{G}^\times(\Lambda) - 1)^{\geq 0}$. Since $\mathcal{G}(\Lambda)$ is symmetric, and the same holds for $\mathcal{G}^\times(\Lambda) - 1$ by Proposition 5.22, we have proved that $\mathcal{G}(\Lambda) \subseteq g(\mathcal{G}^\times(\Lambda) - 1)$, and together with the assertion of Lemma 5.21, this gives

$$\mathcal{G}(\Lambda) \;=\; g(\mathcal{G}^\times(\Lambda) - 1)\;, \tag{33}$$

which is equivalent to (30).

Now suppose that $g < 0$. Then by symmetry, $-g$ is a positive element of $\mathrm{CS}(\Lambda) \setminus \mathrm{CG}(\Lambda)$, so we obtain $\mathcal{G}(\Lambda)^{\geq 0} = -g(\mathcal{G}^\times(\Lambda) - 1)^{\geq 0}$. By symmetry, this again yields (33).

From (33), we obtain

$$\mathcal{O}(\Lambda) \;=\; \mathcal{O}(\mathcal{G}(\Lambda)) \;=\; \mathcal{O}(g(\mathcal{G}^\times(\Lambda) - 1)) \;=\; \mathcal{O}(\mathcal{G}^\times(\Lambda) - 1)$$

by Proposition 5.18.

3): In view of Lemma 5.21, it suffices to show that $\mathcal{G}(\Lambda) \subseteq (\mathcal{G}^\times(\Lambda) - 1) \cdot \mathrm{CS}(\Lambda)$. If Λ is a group $_0$-cut, then by Proposition 5.22, $1 \in \mathcal{G}^\times(\Lambda) - 1$, which implies our assertion. If Λ is not a group $_0$-cut, then by (33),

$$\mathcal{G}(\Lambda) \;=\; g(\mathcal{G}^\times(\Lambda) - 1) \;\subseteq\; (\mathcal{G}^\times(\Lambda) - 1) \cdot \mathrm{CS}(\Lambda)\;.$$

□

5.6 Projecting cuts into residue fields

Take a convex valuation ring $\mathcal{O}$ of $(K, <)$ with maximal ideal $\mathcal{M}$. Its residue field $\mathcal{O}/\mathcal{M}$ is again an ordered field, with the ordering induced through the residue map. We will say that the cut Λ **can be projected into the residue field** $\mathcal{O}/\mathcal{M}$ if there are elements $a, c \in K$ such that $c > 0$ and that $c\Lambda + a$ induces a Dedekind cut in $\mathcal{O}/\mathcal{M}$ via the residue map. This means that

$$\Big(((c\Lambda^L + a) \cap \mathcal{O})/\mathcal{M}\,,\; ((c\Lambda^R + a) \cap \mathcal{O})/\mathcal{M} \Big) \tag{34}$$

is a cut in $\mathcal{O}/\mathcal{M}$, with both sets nonempty.

Lemma 5.25 *The cut Λ can be projected into the residue field $\mathcal{O}/\mathcal{M}$ if and only if there is some $c \in K$ such that*

$$\mathcal{M} \subseteq c\mathcal{G}(\Lambda) \subsetneqq \mathcal{O} . \tag{35}$$

Proof: Take $0 \neq c \in K$. By Theorem 3.11, the cut $c\Lambda$ can be shifted into $\mathcal{O}$ if and only if $c\mathcal{G}(\Lambda) = \mathcal{G}(c\Lambda) \subsetneqq \mathcal{O}$. From Theorem 3.6 we know that (34) is a cut (and hence a Dedekind cut) in $\mathcal{O}/\mathcal{M}$ if and only if $\mathcal{M} \subseteq \mathcal{G}(c\Lambda) = c\mathcal{G}(\Lambda)$. □

From this lemma together with part 2) of Lemma 4.12, we obtain that Λ can be projected into $\mathcal{O}/\mathcal{M}$ if $\mathcal{O}(\Lambda) \subsetneqq \mathcal{O}$. On the other hand, we have:

Lemma 5.26 *If $\mathcal{O} \subsetneqq \mathcal{O}(\Lambda)$, then Λ cannot be projected into $\mathcal{O}/\mathcal{M}$.*

Proof: Assume that $\mathcal{O} \subsetneqq \mathcal{O}(\Lambda)$. Since the maximal ideal of a valuation ring consists exactly of its non-units and a unit in $\mathcal{O}$ is also a unit in $\mathcal{O}(\Lambda)$, we find that $\mathcal{M}(\Lambda) \subsetneqq \mathcal{M}$. If $c\mathcal{G}(\Lambda) \subsetneqq \mathcal{O}$ for some $c \in K$, then $c\mathcal{G}(\Lambda) \subsetneqq \mathcal{O}(\Lambda)$. Since $\mathcal{O}(\Lambda) = \mathcal{O}(\mathcal{G}(\Lambda))$ and $\mathcal{M}(\Lambda) = \mathcal{M}(\mathcal{G}(\Lambda))$, this implies by (22) that $c\mathcal{G}(\Lambda) \subseteq \mathcal{M}(\Lambda) \subsetneqq \mathcal{M}$. Hence by Lemma 5.25, Λ cannot be projected into $\mathcal{O}/\mathcal{M}$. □

Now it remains to determine under which conditions Λ can be projected into the residue field $\mathcal{O}(\Lambda)/\mathcal{M}(\Lambda)$. By Lemma 5.25, this is the case if and only if there is $c \in K$ such that $\mathcal{M}(\Lambda) \subseteq c\mathcal{G}(\Lambda) \subsetneqq \mathcal{O}(\Lambda)$. Again, (22) shows that this holds if and only if $\mathcal{M}(\Lambda) = c\mathcal{G}(\Lambda)$. By Lemma 4.13, this in turn holds if and only if $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^+$-cut. We summarize what we have proved:

Theorem 5.27 *1) Take any convex valuation ring $\mathcal{O}$ of $(K, <)$. If $\mathcal{O}(\Lambda) \subsetneqq \mathcal{O}$, then the cut Λ can be projected into the residue field $\mathcal{O}/\mathcal{M}$. If $\mathcal{O} \subsetneqq \mathcal{O}(\Lambda)$, then it cannot be projected into $\mathcal{O}/\mathcal{M}$.*
2) The cut Λ can be projected into $\mathcal{O}(\Lambda)/\mathcal{M}(\Lambda)$ if and only if $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^+$-cut.

Example 5.28 Consider our Example 5.12. We have that $\mathcal{G}(\Lambda) = \mathcal{M}_v$ and therefore, $\mathcal{O}(\Lambda) = \mathcal{O}(\mathcal{M}_v) = \mathcal{O}_v$. We have already seen that $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^+$-cut. Hence by our theorem, the cut Λ can be projected into $\mathcal{O}_v/\mathcal{M}_v = \mathbb{Q}$. This is what we expected since by construction, the cut comes from the subfield $\mathbb{Q}$. In fact, for the projection we can set $c = 1$ and $a = 0$.

Now consider our Example 5.13. We have noted that the invariance group of $(v\mathcal{G}(\Lambda))^-$ is $\{0\}$. Hence by Proposition 5.17, $H(\Lambda) = \{0\}$ and consequently, $\mathcal{O}(\Lambda) = \mathcal{O}(H(\Lambda)) = \mathcal{O}(\{0\}) = \mathcal{O}_v$, as before. But now, $(v\mathcal{G}(\Lambda))^-$ is neither a ball$_0^+$-cut nor a ball$_0^-$-cut. Hence by our theorem, the cut Λ cannot be projected into $\mathcal{O}_v/\mathcal{M}_v$ (which in this case is the real closure of $\mathbb{Q}$). This is because the cut comes from a cut in the value group.

Finally, consider our Example 5.16. We have that $\mathcal{G}(\Lambda) = \mathcal{O}_v$ and therefore again, $\mathcal{O}(\Lambda) = \mathcal{O}(\mathcal{O}_v) = \mathcal{O}_v$. We noted already that $(v\mathcal{G}(\Lambda))^-$ is a group$_0^-$-cut but not a ball$_0^+$-cut. Hence by our theorem, the cut Λ cannot be projected into $\mathcal{O}_v/\mathcal{M}_v = \mathbb{Q}$. Even worse: by our construction, there is no smallest convex valuation ring $\mathcal{O}$ of $(K,<)$ such that Λ can be projected into $\mathcal{O}/\mathcal{M}$.

The convex valuation rings of $(K,<)$ correspond bijectively to the convex subgroups of $vK = \mathbb{Z}vt_1 \oplus \mathbb{Z}vt_2 \oplus \ldots \oplus \mathbb{Z}vt_i \oplus \ldots$. These convex subgroups are precisely $\{0\}$ and the subgroups of the form $H_k := \mathbb{Z}vt_k \oplus \mathbb{Z}vt_{k+1} \oplus \ldots$, for all $k \in \mathbb{N}$. We set $\mathcal{O}_k := \mathcal{O}(H_k)$ and denote its maximal ideal by $\mathcal{M}_k$. Then the residue field $\mathcal{O}_k/\mathcal{M}_k$ can be identified with the field $\mathbb{Q}(t_i \mid i \geq k)$. Looking at the pseudo Cauchy sequence defined in Example 5.16, we see that precisely the summands $t_1^{-1}, \ldots, t_{k-1}^{-1}$ of our a_ν's do not lie in $\mathcal{O}_k$. So if we define a new pseudo Cauchy sequence by setting $a'_\nu := a_{\nu+k-1} - a_{k-1}$, then we obtain a sequence consisting of elements in $\mathcal{O}_k$, which consequently also defines a cut in $\mathcal{O}_k$. In fact, this new cut is $\Lambda - a_{k-1}$. The projection of the new pseudo Cauchy sequence $(a'_\nu)_{\nu\in\mathbb{N}}$ via the residue map into the residue field $\mathcal{O}_k/\mathcal{M}_k = \mathbb{Q}(t_i \mid i \geq k)$ renders a pseudo Cauchy sequence without a limit in that field. Actually, since we regard the residue field $\mathbb{Q}(t_i \mid i \geq k)$ as a subfield of K, the projection is the identity on the elements of the pseudo Cauchy sequence, and the cut in the residue field is contained by cutting down $\Lambda - a_{k-1}$. This cut is again a Dedekind cut, not realized in $\mathcal{O}_k/\mathcal{M}_k$. Observe that there is no $a \in K$ such that for *every* $k \in \mathbb{N}$, $\Lambda - a$ would induce a Dedekind cut via the residue map in $\mathcal{O}_k/\mathcal{M}_k$. $\diamondsuit$

Finally, we determine the cuts which originate from the upper or lower edge of convex valuation rings or of their maximal ideals:

Theorem 5.29 *1) $c\Lambda + a = \mathcal{O}^+$ for some $a, c \in K$, $c > 0$, if and only if $\mathcal{O} = \mathcal{O}(\Lambda)$, the cut $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^-$-cut, and Λ is a ball$^+$-cut.*
2) $c\Lambda + a = \mathcal{O}^-$ for some $a, c \in K$, $c > 0$, if and only if $\mathcal{O} = \mathcal{O}(\Lambda)$, the cut $(v\mathcal{G}(\Lambda))$ is a ball$_0^-$-cut, and Λ is a ball$^-$-cut.
3) $c\Lambda + a = \mathcal{M}^+$ for some $a, c \in K$, $c > 0$, if and only if $\mathcal{O} = \mathcal{O}(\Lambda)$, the cut $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^+$-cut, and Λ is a ball$^+$-cut.
4) $c\Lambda + a = \mathcal{M}^-$ for some $a, c \in K$, $c > 0$, if and only if $\mathcal{O} = \mathcal{O}(\Lambda)$, the cut $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^+$-cut, and Λ is a ball$^-$-cut.
In case 1) and 2), Λ cannot be projected into $\mathcal{O}(\Lambda)/\mathcal{M}(\Lambda)$. In case 3), we obtain as possible projections all cuts whose initial segment has a last element. In case 4), we obtain as possible projections all cuts whose final segment has a first element.

Proof: We prove part 1). Assume first that $c\Lambda + a = \mathcal{O}^+$ for some $a, c \in K$, $c > 0$. Then by Corollary 3.7, $\mathcal{O} = \mathcal{G}(c\Lambda + a)$. Thus by Proposition 5.18, $\mathcal{O}(\Lambda) = \mathcal{O}(c\Lambda + a) = \mathcal{O}(\mathcal{G}(c\Lambda + a)) = \mathcal{O}(\mathcal{O}) = \mathcal{O}$. As $\mathcal{O} = \mathcal{G}(c\Lambda + a)$ is a final segment of $c\Lambda^L + a$, we know from Lemma 3.18 that $c\Lambda + a$ is a group$^+$-cut. Hence by Lemma 5.2, the cut Λ is a ball$^+$-cut. Using Lemma 5.1, we compute: $c\mathcal{G}(\Lambda) = \mathcal{G}(c\Lambda + a) = \mathcal{O} = \mathcal{O}(\Lambda) = \mathcal{O}(\mathcal{G}(\Lambda))$. By Lemma 4.13 it follows that $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^-$-cut.

To prove the converse, assume that $\mathcal{O} = \mathcal{O}(\Lambda)$, that $(v\mathcal{G}(\Lambda))^-$ is a ball$_0^-$-cut, and that Λ is a ball$^+$-cut. Then by Lemma 4.13, there is some nonzero $c \in K$ such that $c\mathcal{G}(\Lambda) = \mathcal{O}(\mathcal{G}(\Lambda)) = \mathcal{O}(\Lambda)$. Since $\mathcal{G}(\Lambda) = -\mathcal{G}(\Lambda)$, we can choose c positive. Since Λ is a ball$^+$-cut, the same holds for $c\Lambda$ by Lemma 5.2. Now Corollary 3.19 shows that there is some $a \in K$ such that $\mathcal{G}(c\Lambda)$ is a final segment of $c\Lambda^L + a$. But by Lemma 5.1, by what we have already proved, and by our assumption on $\mathcal{O}$, we have that $\mathcal{G}(c\Lambda) = c\mathcal{G}(\Lambda) = \mathcal{O}(\Lambda) = \mathcal{O}$. This proves that $c\Lambda + a = \mathcal{O}^+$ for some $a, c \in K$, $c > 0$.

The proofs of parts 2), 3) and 4) are similar. The further assertions follow from part 2) of Theorem 5.27, and from Corollary 3.19 together with the remark preceding that corollary. □

Remark 5.30 Let us consider the four cases of the foregoing theorem, for a fixed $\mathcal{O}$. Then we find that the four cuts cannot be transformed into each other through transformations of the form $(\Lambda^L, \Lambda^R) \mapsto (c\Lambda^L + a\,,\, c\Lambda^R + a)$, with $c > 0$. Nevertheless, each of the four cuts can be transformed into the other. We show this for two pairs, the others are similar or combinations of these transformations.
1)$\mapsto$2): If Λ is a cut satisfying 1), then $-\Lambda := (-\Lambda^R, -\Lambda^L)$ is a cut satisfying 2). Indeed, if $\mathcal{O}$ is a final segment of $c\Lambda^L + a$, then it is an initial segment of $-(c\Lambda^L + a) = c(-\Lambda^L) - a$.
1)$\mapsto$3): Assume that Λ is a cut satisfying 1). Then Λ^L contains positive elements. For this case, we define a cut $\Lambda^{-1} = (\Lambda_1^L, \Lambda_1^R)$ as follows. We set $\Lambda_1^L := \{d \in K \mid d \leq 0\} \cup \{b^{-1} \mid b \in \Lambda^R\}$ and $\Lambda_1^R := \{b^{-1} \mid 0 < b \in \Lambda^L\}$. Since $b^{-1} \in \mathcal{M}$ if and only if $b \notin \mathcal{O}$, it follows that $\{b^{-1} \mid b \in \Lambda^R\} = \{d \in \mathcal{M} \mid d > 0\}$. Therefore, $\mathcal{M}$ is a final segment of Λ_1^L.

The connection between the four cuts can also be understood as follows. Suppose that an element x in some extension field realizes the cut of 1). Then $-x$ realizes the cut of 2), x^{-1} realizes the cut of 3), and $-x^{-1}$ realizes the cut of 4). If we extend the natural valuation to that extension field, we will find that the value γ of x and $-x$ is not archimedean comparable to any value in vK. It realizes the cut $H(\mathcal{O})^-$; so γ itself is negative. Further, $-\gamma$ is the value of x^{-1} and of $-x^{-1}$. It realizes the cut $H(\mathcal{O})^+$ and is positive.

Exercise. Take any cut Λ such that Λ^L contains positive elements. Compute the invariance group and the invariance valuation ring of Λ^{-1}.

References

[Ka] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

[Ku1] Kuhlmann, F.-V.: *Henselian function fields*, Ph.D. thesis, Heidelberg (1989)

[Ku2] Kuhlmann, F.-V.: *Valuation theory*, book in preparation. Preliminary versions of several chapters are available on the web site http://math.usask.ca/˜fvk/Fvkbook.htm

[Ku3] Kuhlmann, F.-V.: *Invariance valuation ring of cuts in ordered fields*, manuscript, Baton Rouge, (1996)

[KuS] Kuhlmann, S.: *On the structure of nonarchimedean exponential fields I*, Archive for Math. Logic **34** (1995), 145–182

[M] Madden, J.: *Two methods in the study of k-vector lattices*, dissertation, Wesleyan University (1993)

[R] Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal (1964)

[T] Tressl, M.: *Dedekind cuts in polynomially bounded, o-minimal expansions of real closed fields*, doctoral thesis, Regensburg (1996)

[W] Wehrung, F.: *Monoids of intervals of ordered abelian groups*, J. Algebra **182** (1996), 287–328

[Z–S] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin (1960)

Department of Mathematics and Statistics, University of Saskatchewan,
106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6
email: fvk@math.usask.ca