Sixty-Four Curves of Degree Six

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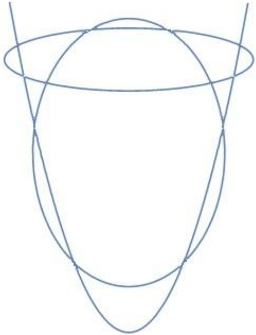
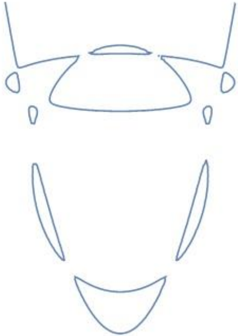
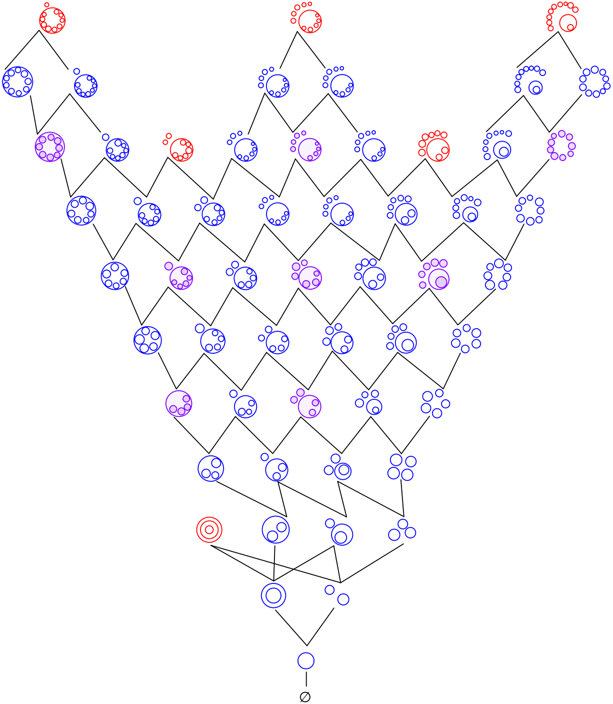




with Nidhi Kaihnsa, Mario Kummer,
Daniel Plaumann and Mahsa Sayyary

Poset



Hilbert's 16th Problem

Classify all real algebraic curves of degree d in the plane $\mathbb{P}^2_{\mathbb{R}}$.
Stratify the parameter space $\mathbb{P}_{\mathbb{R}}^{d(d+3)/2}$.

Assume that the complex curve in $\mathbb{P}^2_{\mathbb{C}}$ is smooth.

Complete answers are known up to $d = 7$, due to *Harnack, Hilbert, Rohn, Petrovsky, Rokhlin, Gudkov, Nikulin, Kharlamov, Viro, ...*

Two curves C_1 and C_2 have same *topological type* if some homeomorphism of $\mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ restricts to a homeo $C_1 \to C_2$.

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Finer notion of equivalence comes from the *discriminant* Δ,

a hypersurface of degree $3(d-1)^2$ in $\mathbb{P}_{\mathbb{R}}^{d(d+3)/2}$.

Points on Δ are singular curves. The *rigid isotopy types* are the connected components of $\mathbb{P}_{\mathbb{R}}^{d(d+3)/2} \backslash \Delta$. Two curves C_1 and C_2 in the same rigid isotopy class have the same topological type

... the converse is not true for $d \geq 5$.

Sextics

Our paper: $d = 6$

Theorem (Rokhlin-Nikulin Classification)

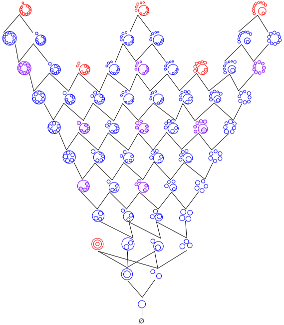
The discriminant of plane sextics is a hypersurface of degree 75 *in* $\mathbb{P}^{27}_{\mathbb{R}}$. *Its complement has* 64 *connected components. The* 64 *rigid isotopy types are grouped into* 56 *topological types, with number of ovals ranging from* 0 *to* 11. *The distribution equals*

# ovals	0	1	2	3	4	5	6	7	8	9	10	11	all
Rigid isotopy	1	1	2	4	4	7	6	10	8	12	6	3	64
Topological	1	1	2	4	4	5	6	7	8	9	6	3	56

The 56 types are seen in our poset.

Rokhlin (1978) carried out the classification of rigid isotopy types.
Nikulin (1980) completed the proof in his study of real K3 surfaces.

14 Are Dividing



The following eight types consist of two rigid isotopy classes:

$$(41) \quad (21)2 \quad (51)1 \quad (31)3 \quad (11)5 \quad (81) \quad (41)4 \quad 9.$$

The six maximal types necessarily divide their Riemann surface:

$$(91)1 \qquad (51)5 \qquad (11)9 \qquad (61)2 \qquad (21)6 \qquad (\mathrm{hyp}).$$

Corollary

Of the 56 topological types of smooth plane sextics, 42 types are non-dividing, six are dividing, and eight can be dividing or non-dividing. This accounts for all 64 rigid isotopy types in $\mathbb{P}^{27}_{\mathbb{R}} \backslash \Delta$.

Polynomials

Proposition

Each of the 64 rigid isotopy types is realized by a sextic in $\mathbb{Z}[x, y, z]_6$ *whose coefficients have abs. value* $\leq 1.5 \times 10^{38}$.

0	nd	$x^6 + y^6 + z^6$
1	nd	$x^6 + y^6 - z^6$
(11)	nd	$6(x^4 + y^4 - z^4)(x^2 + y^2 - 2z^2) + x^5 y$
2	nd	$(x^4 + y^4 - z^4)((x + 4z)^2 + (y + 4z)^2 - z^2) + z^6$
(21)	nd	$16((x+z)^2+(y+z)^2-z^2)(x^2+y^2-7z^2)((x-z)^2+(y-z)^2-z^2) + x^3 y^3$
(11)1	nd	$((x + 2z)^2 + (y + 2z)^2 - z^2)(x^2+y^2-3z^2)(x^2+y^2-z^2) + x^5 y$
3	nd	$(x^2 + y^2 - z^2)(x^2 + y^2 - 2z^2)(x^2 + y^2 - 3z^2) + x^6$
(hyp)	d	$6(x^2+y^2-z^2)(x^2+y^2-2z^2)(x^2+y^2-3z^2)+x^3 y^3$
(31)	nd	$(10(x^4-x^3 z+2x^2 y^2+3xy^2 z+y^4)+z^4)(x^2+y^2-z^2)+x^5 y$
(21)1	nd	$(10(x^4 - x^3 z+2x^2 y^2+3xy^2 z+y^4)+z^4)((x+z)^2+y^2 - 2z^2)+x^5 y$
(11)2	nd	$(10(x^4-x^3 z+2x^2 y^2+3xy^2 z+y^4)+z^4)(x^2+(y - z)^2 - z^2)+x^5 y$
...	...	... ...

and many more representatives

Robinson Sextic

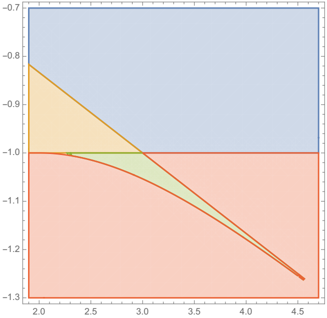
Consider this net of sextics:

$$a(x^6+y^6+z^6)+bx^2y^2z^2+c(x^4y^2+x^4z^2+x^2y^4+x^2z^4+y^4z^2+y^2z^4).$$

For $(a:b:c)=(1:3:-1)$ this a nonnegative sextic that is not SOS.

The discriminant of this net is the following curve of degree 75 in $\mathbb{P}^2_{\mathbb{R}}$:

$$\Delta \;=\; a^3(a+c)^6(3a-c)^{18}(3a+b+6c)^4(3a+b-3c)^8(9a^3-3a^2b+ab^2-3ac^2-bc^2+2c^3)^{12}$$



$(a:b:c)=(19:60:-20)$ gives our sextic for the ten ovals type $10\mathrm{d}$.

Eleven Ovals

Hilbert (1891) argued that type **(51)5** does not exist.
Gudkov (1969) showed that Hilbert had made a mistake.

(91)1 d

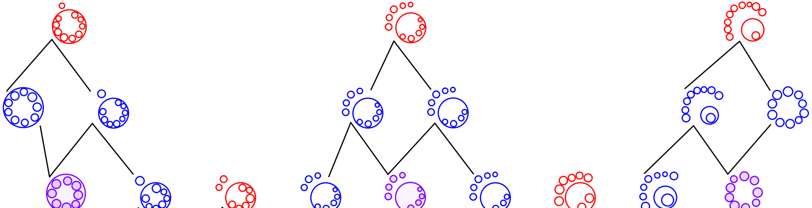
$$(1941536164(yz-x^2)(60(x+z)z-(6x+6z-y)^2)+ \\ 118(10x+8y+3z)(12x+32y+z)(12x-32y-z)(10x-8y-3z))(x^2-yz)-y^6$$

(51)5 d

$$-3401397120x^6 - 3195251840x^5y - 2164525440x^4y^2 - 869728640x^3y^3 + 332217600x^2y^4 \\ +316096000xy^5 + 53760001y^6 + 1597625920x^5z + 368484688000000000x^4yz \\ +798812960000000000x^3y^2z337328640000000000x^2y^3z - 130099200x^2z^4 - 1199390720x^4z^2 \\ -798812960000000000x^3yz^2 - 1275523920000000000000000000x^2y^2z^2 + 764952320x^3z^3 \\ +2342560000000000000000000000y^4z^2 + 365439360000000000x^2yz^3 + 382476160000000000xy^4z \\ +\underline{141724880000000000000000000000000000}y^3z^3 + 161849600000000000y^5z \\ -382476160000000000xyz^4 + 117128000000000000000000000y^2z^4 + 650496000000000000yz^5 - z^6$$

(11)9 d

$$(340291(yz-x^2)((x+2z)z-2(y-2z)^2) \\ +(10x-8y-3z)(12x-27y-z)(12x+28y+z)(10x+7y+3z))(x^2-yz)+y^6$$



SexticClassifier

We wrote fast `Mathematica` code, using built-in method for cylindrical algebraic decomposition. Its input is a sextic $f \in \mathbb{Z}[x, y, z]_6$. Its output is the topological type of $V_{\mathbb{R}}(f)$.

We computed various empirical distributions.
Here is one experiment with $1,500,000$ samples:

1	2	3	(11)	4	(11)1	(21)	5	∅	(11)2	(21)1	6	(31
875109	423099	97834	90316	7594	4360	1180	245	127	118	8	7	2

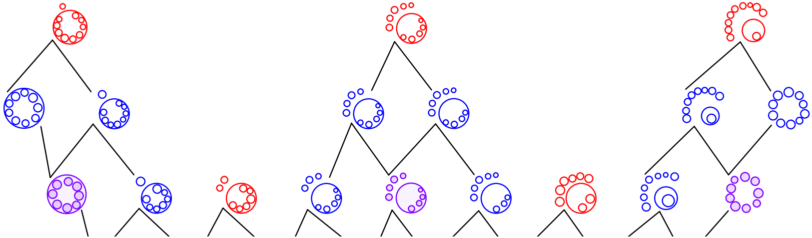
Table: Topological types sampled from the $U(3)$-invariant distribution

For the uniform distribution on $\{-10^{12}, \ldots, 10^{12}\}$ we obtained

1	2	3	(11)	∅	4
77.51%	18.24%	2.09%	1.44%	0.65%	0.06%

Conclusion: *Most types never occur when sampling at random*!!

Transitions



Theorem (Itenberg 1994)

For each edge in our poset, both combinatorial transitions (shrinking or fusing) can be realized by a singular curve with exactly one ordinary node.

Transitions

Theorem

For curves of even degree, every discriminantal transition between rigid isotopy types is one of the following: shrinking an ovals, fusing two ovals, and turning an oval inside out.

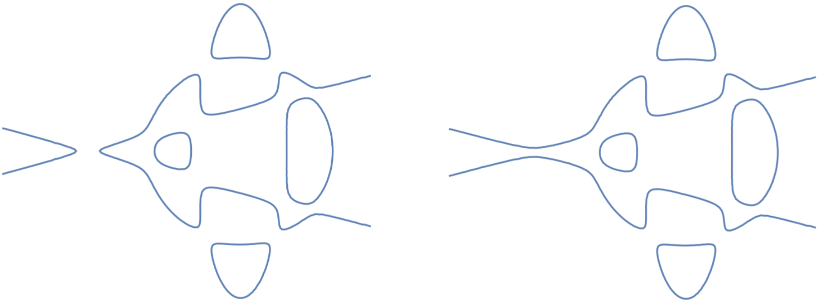


Figure: Type (21)2d transitions into Type (21)2nd by turning inside out.

Sylvester's Formula

We evaluate Δ as the determinant of a 45×45-matrix

...found in [Gelfand-Kapranov-Zelevinsky].

Each entry in the first 30 columns is either 0 or one of the c_{ijk}. The last 15 columns contain cubics in the c_{ijk}. The matrix is

$$(\,\mathbb{R}[x,y,z]_3\,)^3 \oplus \mathbb{R}[x,y,z]_4 \;\longrightarrow\; \mathbb{R}[x,y,z]_8$$

On the first summand, it maps a triple of cubics to an octic:

$$(a,b,c) \;\mapsto\; a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}.$$

On the second summand, it maps a quartic monomial $x^r y^s z^t$ to the octic $\det(M_{rst})$, where M_{rst} is any 3×3-matrix of ternary forms satisfying

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} \;=\; M_{rst} \cdot \begin{pmatrix} x^{r+1} \\ y^{s+1} \\ z^{t+1} \end{pmatrix}.$$

Proposition

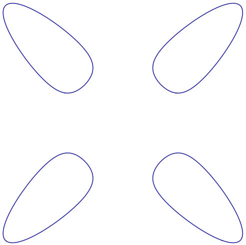
The discriminant Δ is the determinant of this $45{\times}45$-matrix.

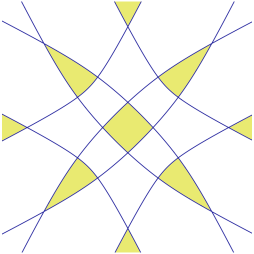
Avoidance Locus

For a real curve C of even degree d, the *avoidance locus* $\mathcal{A}_C$ is the set of lines in $\mathbb{P}^2_{\mathbb{R}}$ that do not intersect $C_{\mathbb{R}}$. This is a semialgebraic subset of the dual projective plane $(\mathbb{P}^2)^{\vee}_{\mathbb{R}}$.

The *dual curve* $C^{\vee}$ has degree $d(d-1)$ in $(\mathbb{P}^2)^{\vee}$.
It divides $(\mathbb{P}^2)^{\vee}_{\mathbb{R}}$ into connected components.

Points on $C^{\vee}$ are lines in $\mathbb{P}^2$ tangent to C.





Proposition

The avoidance locus $\mathcal{A}_C$ is a union of connected components of $(\mathbb{P}^2)^{\vee}_{\mathbb{R}} \setminus C^{\vee}_{\mathbb{R}}$. Each component appearing in $\mathcal{A}_C$ is convex.

Avoiding Sextics

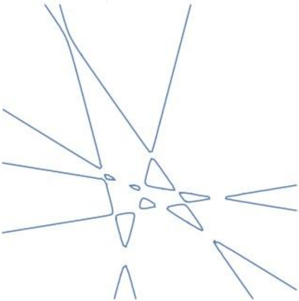


Figure: A smooth sextic of type 10nd whose avoidance locus is empty

Proposition

Let $\deg(C) = d$ *even. The number of convex components of the avoidance locus* $\mathcal{A}_C$ *is bounded above by*

$$\frac{9}{128}d^4 - \frac{9}{32}d^3 + \frac{15}{32}d^2 - \frac{3}{8}d + 1.$$

Corollary

For all $m \in \{0, 1, \ldots, 46\}$ *there exists a smooth sextic* C *in* $\mathbb{P}^2_{\mathbb{R}}$ *whose avoidance locus* $\mathcal{A}_C$ *has exactly* m *convex components.*

Avoiding Sextics

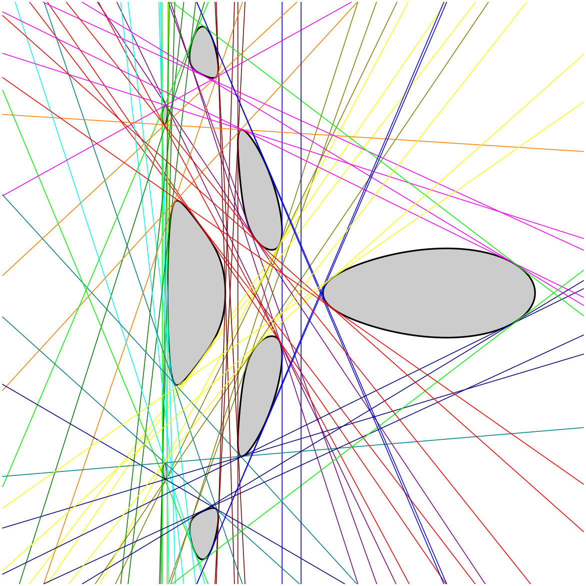


Figure: A sextic C of type 8nd; its 68 *relevant bitangents* represent $\mathcal{A}_C$.

Bitangents and Flexes

A general sextic in $\mathbb{P}^2_{\mathbb{C}}$ has 324 bitangents and 72 inflection points.

Conjecture

The number of real bitangents of a smooth sextic in $\mathbb{P}^2_{\mathbb{R}}$ ranges from 12 to 306. The lower bound is attained by curves of types $0, 1, (11)$ and (hyp). The upper bound is attained by $(51)5$.

Transitions:

(411) C has an *undulation point*.

(222) C has a *tritangent line*.

(321) C has a *flex-bitangent*.

Theorem

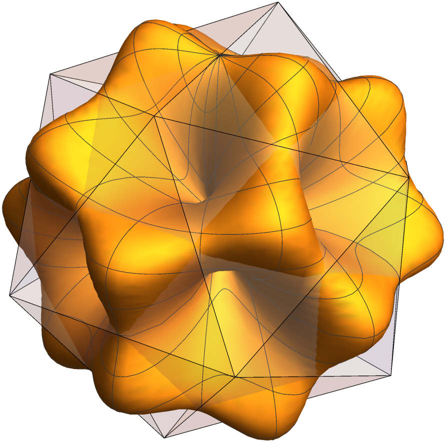
The loci (222) and (321) are irreducible hypersurfaces in $\mathbb{P}^{27}$ of degree 1224 and 306 respectively. Their union is the Zariski closure of the set of smooth sextics having fewer than 324 bitangent lines.

Experiments

Type	Flex	Eigenvec	Bitang	Rank
0	0	3—31	12	3
1	0—12	3—31*	12—56	3
(11)	0—14	11—31*	12—66	10
2	0—8	5—31*	12—52	13
(21)	0—10	7—31*	16—86	14
(11)1	2—6	7—31*	20—66	15
3	0—8	7—31*	24—94	13
(hyp)	0—14	11—31*	12—52	13
(31)	2—10	19—31*	24—90	13
(21)1	0—6	11—31*	28—72	14
(11)2	0—4	11—31*	32—82	13
4	0—2	11—31*	36—54	11
(41)nd	14—16	21—31*	48—90	16
(41)d	12—14	27—31*	98—104	14
(31)1	2—8	15—31*	40—86	14
(21)2nd	10—16	17—31*	54—82	20
(21)2d	8—16	19—31*	60—70	17
(11)3	8—12	19—31*	48—94	14
5	2—10	19—31*	52—112	15
(51)	12—16	21—31*	54—64	14
(41)1	22	27—31*	90—104	14
(31)2	14—18	27—31*	126—130	14
(21)3	16	27—31*	112—116	14
(11)4	6—10	25—31*	76—106	15
6	10—12	23—31*	78—108	14
(61)	16	27—31*	78—88	14
(51)1nd	16	23—25	110—124	15
(51)1d	20—24	29	136	16
(41)2	16—20	29—31	126—128	14
(31)3nd	12	25—31*	124—148	15
(31)3d	20—22	29	132	16
(21)4	14—20	27—31*	138—142	15

Type	Flex	Eigenvec	Bitang	Rank
(11)5nd	6—16	29—31*	116—122	16
(11)5d	8—16	25—31*	120—128	16
7	4—14	25—31*	96—124	14
(71)	20—24	29	108	16
(61)1	20—22	25	104—214	15
(51)2	22	25—31	226—228	15
(41)3	20	23—25	154—214	14
(31)4	22	21	162—214	14
(21)5	16—20	29—31	168	13
(11)6	12—14	27—31*	172—176	14
8	0—12	23—31*	124—142	13
(81)nd	18—22	23	122—196	14
(81)d	18—24	29	124—132	12
(71)1	14—18	21—31	104—240	13
(61)2	18—20	23—31	228—276	13
(51)3	22	25	192—254	13
(41)4nd	14—16	25	188—220	9
(41)4d	18	25	194—230	11
(31)5	20	25—31	198—260	13
(21)6	20	23—31	242—258	15
(11)7	14—16	29—31	216	14
9nd	8—16	25—31*	162—172	15
9d	4—16	29—31*	156	15
(91)	18—22	23	124—236	13
(81)1	16—20	23—31	162—240	14
(51)4	20	27	232—234	10
(41)5	18—20	27—31	232	10
(11)8	14—18	25—31	142—210	13
10	0—24	21—31*	192	12
(91)1	18—22	25—31	200—284	14
(51)5	20—22	25—31	276—306	10
(11)9	16—20	25—31	174—250	14

Critical Points on the Sphere



A sextic f can have as many as 20 local maxima on the unit sphere $\mathbb{S}^2$. The picture shows one with $62 = 2 \cdot 31$ critical points. Its Morse complex is the icosahedron, with f-vector $(12, 30, 20)$.

The critical points are the eigenvectors of f.

Eigenvectors of Tensors

Ternary sextics are symmetric tensors of format $3\times3\times3\times3\times3\times3$.

Such a symmetric tensor f has 28 distinct entries c_{ijk}. A vector $v \in \mathbb{C}^3$ is an *eigenvector* of f if v is parallel to the gradient at v:

$$\mathrm{rank}\begin{pmatrix} x & y & z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \;=\; 1.$$

A general $f \in \mathbb{R}[x,y,z]_d$ has $d^2 - d + 1$ eigenvectors in $\mathbb{P}^2_{\mathbb{C}}$. Among them are pairs of critical points of f on the sphere $\mathbb{S}^2$.

The number of real eigenvectors is $\geq 2\omega + 1$, where ω is the number of ovals (Maccioni 2017). If f is a product of d linear forms, then all complex eigenvectors are real (Abo-Seigal-St 2017).

Conclusion: *The number of real eigenvectors of a general sextic is an odd integer between* 3 *and* 31.

Our table summarizes empirical distribution over the 64 types.

Ranks of Tensors

The *rank* of a symmetric tensor $f \in \mathbb{R}[x, y, z]_d$ is the minimum number of summands in a representation

$$f(x, y, z) \quad = \quad \sum_{i=1}^{r} \lambda_i \, (a_i x + b_i y + c_i z)^d .$$

For a generic sextic f, the complex rank is 10, and the real rank is between 10 and 19 (Michalek-Moon-St-Ventura 2017).

Computing real ranks exactly is very difficult.

We used the numerical software `tensorlab` to determine (our best guess for) the real ranks of the 64 representatives.

Type	Flex	Eigenvec	Bitang	Rank
0	0	3−31	12	3
1	0−12	3−31*	12−56	3
(11)	0−14	11−31*	12−66	10
2	0−8	5−31*	12−52	13
(21)	0−10	7−31*	16−86	14
(11)1	2−6	7−31*	20−66	15
...	...	...	...	...
(41)nd	14−16	21−31*	48−90	16
(41)d	12−14	27−31*	98−104	14
(31)1	2−8	15−31*	40−86	14
(21)2nd	10−16	17−31*	54−82	20
(21)2d	8−16	19−31*	60−70	17
(11)3	8−12	19−31*	48−94	14
5	2−10	19−31*	52−112	15

Type	Flex	Eigenvec	Bitang	Rank
(11)5nd	6−16	29−31*	116−122	16
(11)5d	8−16	25−31*	120−128	16
7	4−14	25−31*	96−124	14
(71)	20−24	29	108	16
(61)1	20−22	25	104−214	15
(51)2	22	25−31	226−228	15
(41)3	20	23−25	154−214	14
(31)4	22	21	162−214	14
(21)5	16−20	29−31	168	13
(11)6	12−14	27−31*	172−176	14
8	0−12	23−31*	124−142	13
(81)nd	18−22	23	122−196	14
(81)d	18−24	29	124−132	12
(71)1	14−18	21−31	104−240	13

Quartic Surfaces

Our 64 sextics represent K3 surfaces over $\mathbb{Q}$.

The two basic models for algebraic K3 surfaces are quartic surfaces in $\mathbb{P}^3$ and double-covers of $\mathbb{P}^2$ branched at a sextic curve. A real K3 surface is orientable and has ≤ 10 connected components. Its Euler characteristic is between -18 and 20. (Silhol 1989)

Can construct quartic surfaces with desired topology from our curves:

Example

Let F be the quartic

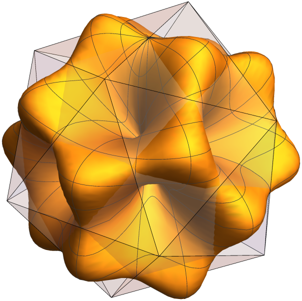
$$100w^4-12500w^2x^2+104x^4-12500w^2y^2+1640x^2y^2+1550y^4+12500w^2yz$$
$$-75x^2yz - 1552y^3z + 9375w^2z^2 - 487x^2z^2 - 1533y^2z^2 + 354yz^3 + 314z^4.$$

The surface $V_{\mathbb{R}}(F)$ is connected of genus 10, so $\chi = 20$.

Example (Rohn 1913)

Let $G = \tau(s_1^2 - 6s_2)^2 + (s_1^2 - 4s_2)^2 - 64s_4$, where s_i is the ith elementary symmetric polynomial in x, y, z, w and $\tau = \frac{16\sqrt{10}-20}{135}$.
Then $V_{\mathbb{R}}(G)$ consists of 10 spheres, so $\chi = -18$.

Conclusion



The geometry and topology of real algebraic varieties is a beautiful subject, with many great results, especially from the Russian school.

We seek to connect this to current problems and developments in **Applied Algebraic Geometry**. This requires *computational and experimental work* with polynomials. We studied explicit sextics like

$$(1941536164(yz-x^2)(60(x+z)z-(6x+6z-y)^2)+$$
$$118(10x+8y+3z)(12x+32y+z)(12x-32y-z)(10x-8y-3z))(x^2-yz)-y^6$$

Quiz: What does the real picture look like for this curve?

Eleven Ovals

(91)1 d

$$(1941536164(yz-x^2)(60(x+z)z-(6x+6z-y)^2)+ \\ 118(10x+8y+3z)(12x+32y+z)(12x-32y-z)(10x-8y-3z))(x^2-yz)-y^6$$

(51)5 d

$$-3401397120x^6-3195251840x^5y-2164525440x^4y^2-869728640x^3y^3+332217600x^2y^4 \\ +316096000xy^5+53760001y^6+1597625920x^5z+36848468800000000000x^4yz \\ +7988129600000000000x^3y^2z3373286400000000000x^2y^3z-130099200x^2z^4-1199390720x^4z^2 \\ -7988129600000000000x^3yz^2-12755239200000000000000000000000x^2y^2z^2+764952320x^3z^3 \\ +2342560000000000000000000000000y^4z^2+36543936000000000000x^2yz^3+38247616000000000000xy^4z \\ +\underline{141724880000000000000000000000000000000000}y^3z^3+16184960000000000000y^5z \\ -38247616000000000000xyz^4+117128000000000000000000000000y^2z^4+6504960000000000000yz^5-z^6$$

(11)9 d

$$(340291(yz-x^2)((x+2z)z-2(y-2z)^2) \\ +(10x-8y-3z)(12x-27y-z)(12x+28y+z)(10x+7y+3z))(x^2-yz)+y^6$$