On the Cohen-Macauley-property in non-Noetherian rings

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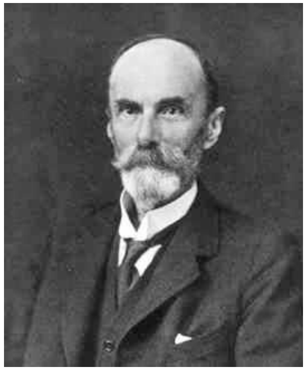
A Noetherian local ring O is called Cohen-Macaulay if the length of a maximal regular sequence $x_1, \ldots x_\ell$ within its maximal ideal m_O equals its Krull dimension:

$$\mathrm{depth}(O) = \dim(O).$$



Francis Macaulay (1862 – 1937)
The algebraic theory of modular systems, Cambridge University Press 1916.

Irvin Cohen (1917 - 1955)
On the structure and ideal theory of complete local rings, Trans. AMS 59 (1946).

Overview

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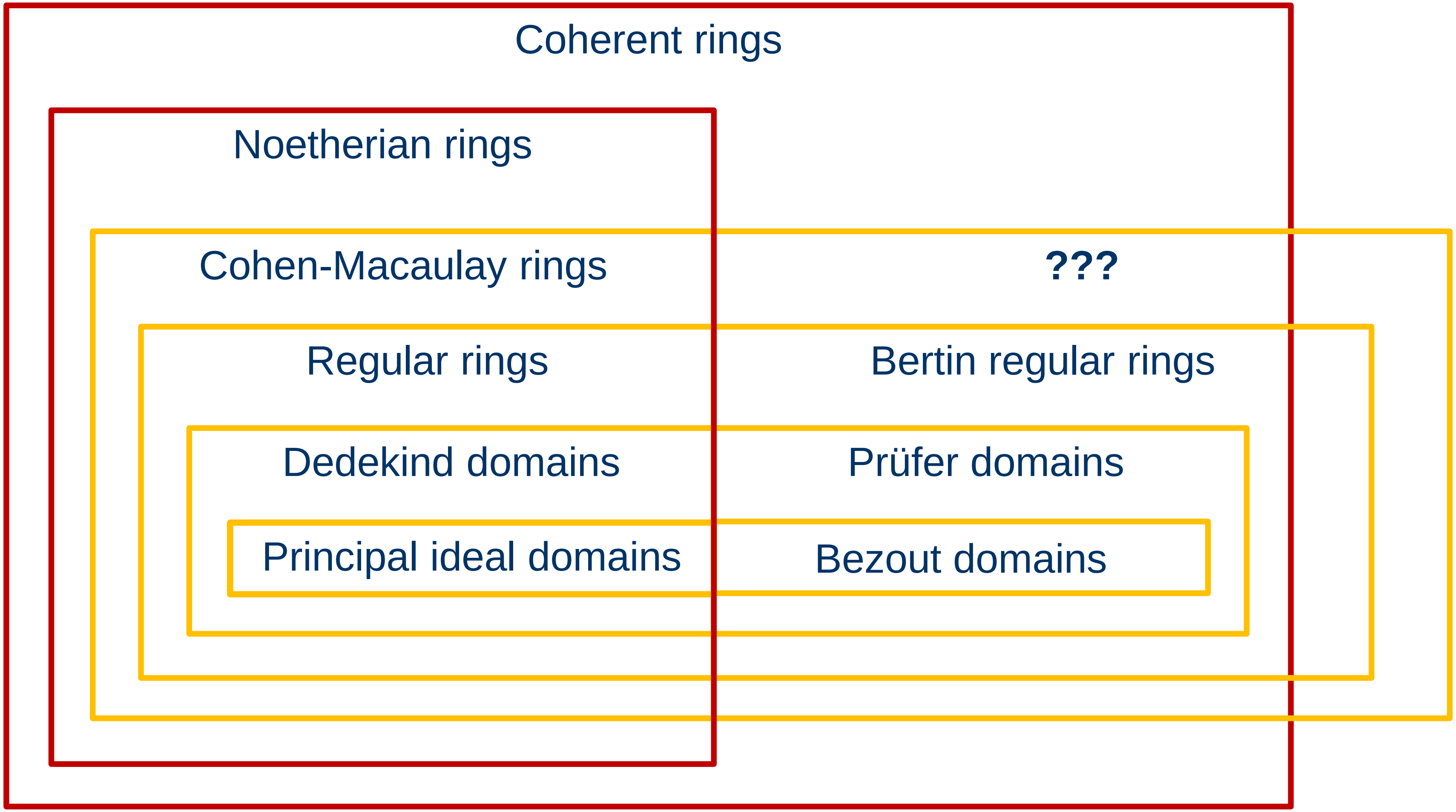
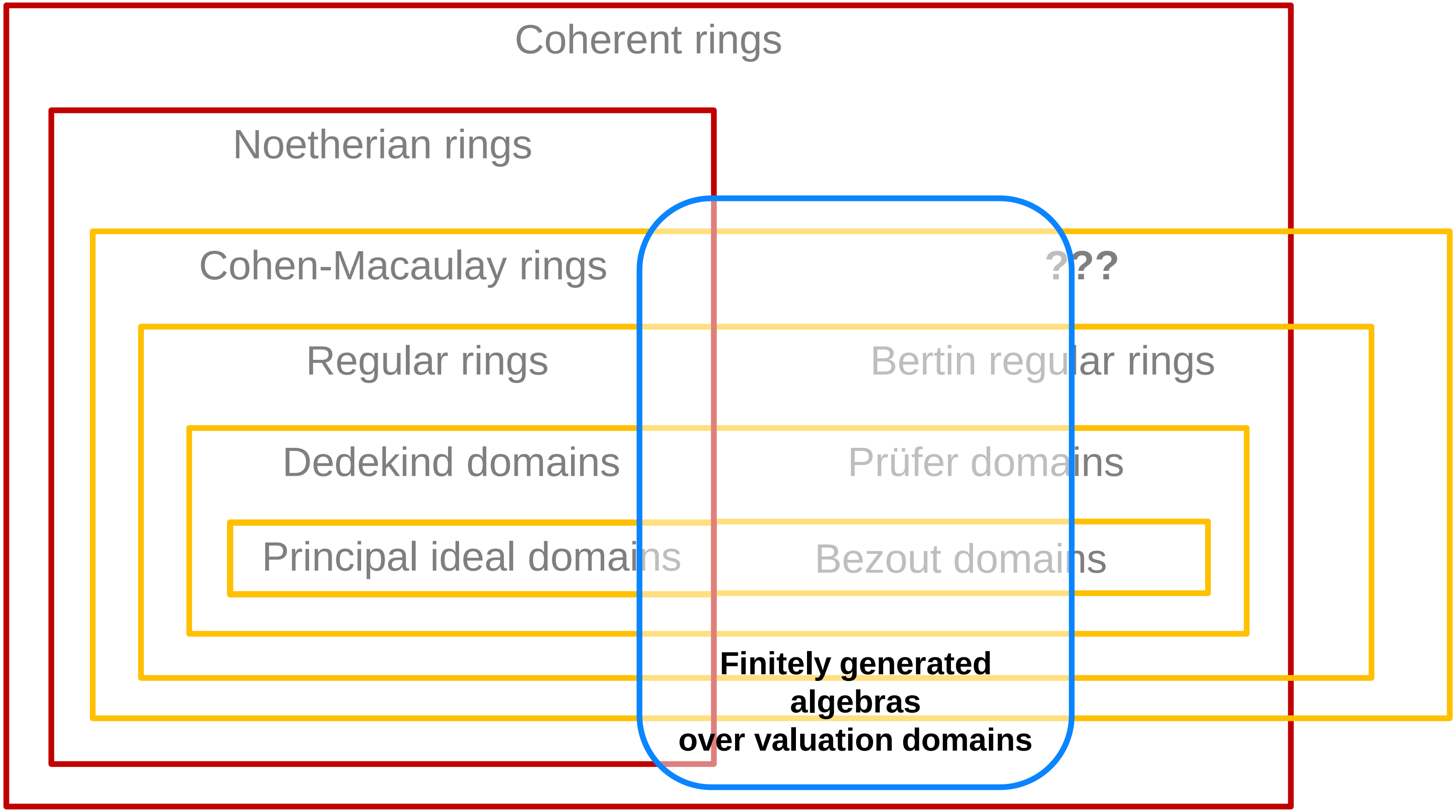
5. Algebras over valuation domains

Introduction and motivation



Introduction and motivation



Introduction and motivation

There exist at least two published attempts to generalize the Cohen-Macaulay-property to non-Noetherian rings:

- Michinori Sakaguchi, Generalized Cohen-Macaulay rings, Hiroshima Mathematical Journal 10 (1980).
- Tracy Hamilton, Thomas Marley, Non-Noetherian Cohen-Macaulay rings, Journal of Algebra 307 (2007).

Both approaches have severe drawbacks.

Introduction and motivation

Attempts to generalise the Cohen-Macaulay-property to non-Noetherian rings have to deal with the rather weak connection between finitely generated ideals and the heights of the minimal prime ideals containing them:

- Krull's principal ideal theorem fails miserably:

  In a valuation domain O of finite dimension every prime ideal is a minimal prime ideal of a principal ideal.

Sakaguchi without any further explanation replaces the Krull by the valuative dimension, although this doesn't improve the situation.

Hamilton and Marley redefine the notion of a regular sequence using Cech cohomology.

In this talk Sakaguchi's approach is discussed.

Valuative dimension

DEFINITION: *The valuative dimension of a domain R is defined to be*

$$\mathrm{Dim}(R) := \sup(\dim(O) : O \text{ is a valuation ring of } \mathrm{Frac}(R)).$$

The valuative dimension of an arbitrary commutative ring R is then defined as

$$\mathrm{Dim}(R) := \sup(\mathrm{Dim}(R/p) : p \in \mathrm{Spec}(R)).$$

- For a Noetherian ring the Krull and the valuative dimension coincide.
- In general $\mathrm{Dim}(R) \geq \dim(R)$ and the difference can be arbitrarily large.

THEOREM: *For a domain R the equality $\mathrm{Dim}(R) = \dim(R)$ is equivalent to the condition*

$$\forall n \in \mathbb{N} \quad \dim(R[X_1, \ldots, X_n]) = \dim(R) + n.$$

Moreover $\mathrm{Dim}(R) = n$ if and only if $\dim(R[X_1, \ldots, X_n]) = 2n$.

Valuative dimension

DEFINITION: *The valuative height of a prime ideal* p *of a ring* R *is defined to be*

$$\mathrm{Ht}(p) := \lim_{k\to\infty} \mathrm{ht}(pR[X_1,\ldots,X_k]).$$

SOME PROPERTIES:

- $\mathrm{Ht}(p) = n \Leftrightarrow \mathrm{ht}(p[X_1,\ldots,X_n]) = n.$
- $\mathrm{Ht}(p/xR) \leq \mathrm{Ht}(p) - 1$ *for every* $x \in p$ *not contained in a minimal prime of* R.
- $\mathrm{Dim}(R) = \sup(\mathrm{Ht}(p) : p \in \mathrm{Spec}(R)).$

Polynomial grade

- For an ideal I of a ring R the length of a maximal regular sequence is denoted by $\text{grade}(I, R)$ – »grade of I«.
- In the Noetherian case the useful equivalences

$$\text{grade}(I, R) > 0 \Leftrightarrow I \text{ contains a non-zerodivisor} \Leftrightarrow \text{ann}(I) = 0$$

  hold.
- This is wrong for non-Noetherian rings: a finitely generated ideal I may consist entirely of zero-divisors although $\text{ann}(I) = 0$.

  Consider the trivial ring extension (»idealisation«)

$$R := K[X, Y] \times \Big(\bigoplus_{p \in K[X,Y] \text{ prime}} K[X, Y]/(p) \Big).$$

  Then $I := ((X, 0), (Y, 0))$ is an example for that phenomenon within the class of coherent rings.

Polynomial grade

Definition (M. Hoechster): *For an ideal I of a ring R the polynomial grade of I is defined to be*

$$\operatorname{Grade}(I,R) := \lim_{n\to\infty} \operatorname{grade}(I[X_1,\ldots,X_n], R[X_1,\ldots,X_n])$$

Some properties:

- $\operatorname{Grade}(I,R) > 0 \Leftrightarrow I[X_1,\ldots,X_n]$ *contains a non-zerodivisor for some n.*
- $\operatorname{Grade}(I,R) > 0 \Leftrightarrow \operatorname{ann}(I) = 0.$
- $\operatorname{Grade}((x_1,\ldots,x_r),R) \le r.$
- $\operatorname{Grade}((x_1,\ldots,x_r),R) > 0$ *if and only if* $x_1 + x_2X + \ldots x_rX^{r-1}$ *is a non-zerodivisor in* $(x_1,\ldots,x_r)[X]$.
- $\operatorname{Grade}(\sqrt{I},R) = \operatorname{Grade}(I,R).$
- $\operatorname{Grade}(I,R) = \operatorname{Grade}(IS,S)$ *for a faithfully flat ring extension* $S|R$.

Polynomial grade

PROPOSITION (M. SAKAGUCHI (1978)): *For a local ring O with maximal ideal m_O the inequality*

$$\mathrm{Grade}(m_O, O) \leq \mathrm{Dim}(O)$$

holds.

This result is the main motivation for Sakaguchi's non-Noetherian definition of Cohen-Macaulay-rings.

Polynomial grade

Grade and homological dimension

Definition: *Let R be a ring. The weak dimension* $\mathrm{wdim}(M)$ *of an R-module M is the length of a shortest resolution*

$$0 \to F_\ell \to \ldots \to F_0 \to M \to 0$$

by flat R-modules. The weak dimension of R is defined to be

$$\mathrm{wdim}(R) := \sup(\mathrm{wdim}(M) : M \text{ an } R\text{-module}).$$

Proposition: *A coherent local ring of finite weak dimension is Bertin-regular.*

Theorem (S. Glaz (1989)): *A coherent local ring (O, m_O) satisfies*

$$\mathrm{wdim}(O) = \mathrm{Grade}(m_O, O).$$

Sakaguchi's approach

DEFINITION: *The ring* R *is called polynomially Cohen-Macaulay if the following conditions are satisfied:*

1. $\forall p \in \mathrm{Spec}(R) \quad \mathrm{Dim}(R_p) \neq \infty,$
2. $\forall p, q \in \mathrm{Spec}(R),\ p \subseteq q \quad \mathrm{Grade}(R_p) = \mathrm{Dim}(R_q) - \mathrm{Dim}(R_q/pR_q).$

In particular $\mathrm{Grade}(pR_p, R_p) = \mathrm{Dim}(R_p)$ *for all* $p \in \mathrm{Spec}(R)$.

- Sakaguchi used the nowadays misleading term »generalised Cohen-Macaulay ring«.

Sakaguchi's approach

PROPOSITION: *A noetherian ring R is polynomially Cohen-Macaulay if and only if it is Cohen-Macaulay (in the »classical« sense).*

PROPOSITION:

1. *If R is a polynomially Cohen-Macaulay ring, then $S^{-1}R$ is polynomially Cohen-Macaulay for every multiplicative set $S \subset R$.*
2. *The ring R is polynomially Cohen-Macaulay if and only if R_m is polynomially Cohen-Macaulay for every maximal ideal m of R.*
3. *If O is a local polynomially Cohen-Macaulay ring and $x \in m_O$ is a non-zerodivisor, then O/xO is polynomially Cohen-Macaulay.*

Sakaguchi's approach

THEOREM: *The ring R is polynomially Cohen-Macaulay if and only if it possesses the following properties:*

1. $\forall p \in \mathrm{Spec}(R) \quad \mathrm{Dim}(R_p) \neq \infty,$
2. $\forall p \in \mathrm{Spec}(R) \quad \mathrm{Grade}(R_p) = \mathrm{Dim}(R_p),$
3. $\forall p, q \in \mathrm{Spec}(R),\ p \subseteq q \quad \mathrm{Ht}(q) = \mathrm{Ht}(p) + \mathrm{Ht}(q/p).$

- Property 3 of this theorem can be a motivation to define polynomially catenarian rings and to consequently prove that every polynomially Cohen-Macaulay ring is polynomially catenarian.

Sakaguchi's approach

EXAMPLES:

- *Every ring* R *with* $\mathrm{Dim}(R) = 0$ *is polynomially Cohen-Macaulay.*
- *A local ring* (O, m_O) *with* $\mathrm{Dim}(O) = 1$ *is polynomially Cohen-Macaulay if and only if* $\mathrm{Grade}(m_O, O) > 0$.
- *A local ring* (O, m_O) *with* $\mathrm{Dim}(O) = 2$ *is polynomially Cohen-Macaulay if and only if* $\mathrm{Grade}(m_O, O) = 2$.
- *A Krull domain* R *with* $\mathrm{Dim}(R) \leq 2$ *is polynomially Cohen-Macaulay.*
- *Sakaguchi's article: the polynomial rings* $R[X]$ *and* $R[X,Y]$ *for a one-dimensional valuation ring* R *are polynomially Cohen-Macaulay.*

Algebras over valuation domains

PROPOSITION (BAD NEWS):
A Prüfer domain is polynomially Cohen-Macaulay if and only if $\dim(R) = 1$.

- Consequently the class of polynomially Cohen-Macaulay domains does not contain the class of coherent regular rings, because every Prüfer domain is regular.

- However, using properties of finitely generated algebras over Prüfer domains, that were uncovered only after Sakaguchi's publications one can obtain interesting results for such algebras in the case of a one-dimensional base ring.

Algebras over valuation domains

DEFINITION: *A ring R is called catenarian if for all prime ideals $p \subset q$ of R the lengths of all non-refinable chains of prime ideals starting with p and ending with q are finite and equal.*

A ring R is called universally catenarian if R and all polynomial rings $R[X_1, \ldots, X_n]$, $n \in \mathbb{N}$, are catenarian.

PROPERTIES:

1. *If R is a universally catenarian ring, then $S^{-1}R$ is universally catenarian for every multiplicative set $S \subset R$.*

2. *If R is a universally catenarian ring, then R/I is universally catenarian for every ideal I.*

Algebras over valuation domains

THEOREM: *A universally catenarian ring* R *satisfies* $\dim(R) = \mathrm{Dim}(R)$.

Algebras over valuation domains

Polynomial rings over Prüfer domains

Theorem (G.Sabbagh (1974), B.Alfonsi (1981): *The polynomial ring* $R[X_1, \ldots, X_n]$, $n \in \mathbb{N}$, *over a Prüfer domain is coherent.*

Theorem: *For a Prüfer domain R one has*

$$\mathrm{wdim}(R[X_1, \ldots, X_n]) = n + 1;$$

in particular $R[X_1, \ldots, X_n]$ *is Bertin-regular.*

Theorem (S.Malik, J.Mott (1983), A.Bouvier, M.Fontana (1985)): *A Prüfer domain R with the property* $\dim(R_p) \neq \infty$ *for all* $p \in \mathrm{Spec}(R)$ *is universally catenarian.*

Algebras over valuation domains

Nagata's contribution

Theorem (M. Nagata (1966)): *A finitely generated flat algebra A over a valuation ring R is finitely presented. In particular A is coherent.*

Theorem (M. Nagata (1966)): *Let A be a domain, finitely generated over a valuation ring R. Then every non-refineable chain of prime ideals starting from 0 and ending with q has the length*

$$\ell = \mathrm{ht}(p) + \mathrm{trdeg}(A|R) - \mathrm{trdeg}(A/q|R/p),$$

where $p := q \cap R$.
Moreover: if q is a minimal prime containing pA, then

$$\mathrm{trdeg}(A|R) = \mathrm{trdeg}(A/q|R/p).$$

Algebras over valuation domains

RESULT 1: *Let* (O, m_O) *be a local ring, essentially finitely generated and flat over the one-dimensional valuation ring* R*.*
Then: if O *is Bertin-regular, it is also polynomially Cohen-Macauley and the equations*

$$\text{grade}(m_O, O) = \text{Grade}(m_O, O) = \text{Dim}(O) = \dim(O)$$

hold.

Algebras over valuation domains

RESULT 2: *Let R be a one-dimensional valuation ring.*
Let S be the integral closure of the polynomial ring $R[X]$ in some finite extension of the fraction field of $R[X]$.
Assume that the extension $S|R[X]$ is finite.
Let (O, m_O) be a localisation of S at some prime.
Then O is polynomially Cohen-Macauley and the equations

$$\text{grade}(m_O, O) = \text{Grade}(m_O, O) = \text{Dim}(O) = \dim(O) \leq 2$$

hold.

Thank you for your attention.