Pushing back the barrier of imperfection

Franz-Viktor Kuhlmann

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Collaborators

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But $v_t K(\vartheta) = \frac{1}{p^\infty}\mathbb{Z} = v_t K$ and $K(\vartheta)v_t = \mathbb{F}_p = Kv_t$, whence $d(K(\vartheta)|K, v_t) = p$.

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But at that time, Abhyankar did not know the notion of the defect.

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where "$\prec_\exists$" means "existentially closed in".

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In the case of model theory, the proof of Ax–Kochen–Ershov Principles is reduced to embedding lemmas for finitely generated extensions of valued fields; these are valued function fields. After having dealt with embeddings of rational function fields, in most cases the only tool available for extending these embeddings to the full function field is Hensel's Lemma.

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Can we do without alteration and can we get from the tame fields to $\mathbb{F}_p((t))$ if we sufficiently push back the barrier of imperfection?

Examples of the barrier

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Theorem (K)

If K is perfect, then the following are equivalent:

a) K is large,
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c) if the function field $F|K$ admits a K-rational place, then $K \prec_\exists F$.

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Theorem

Let K be a large field and $F|K$ an algebraic function field with a K-rational place P. If $(F|K, P)$ admits local uniformization, then $K \prec_\exists F$.

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Note that “extremal” implies “henselian” and “defectless”.

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Completing the characterization would mean pushing the barrier of imperfection.

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Temkin's inseparable uniformization

M. Temkin proved that local uniformization can also be achieved after a finite purely inseparable extension of the function field. But this does not mean that we conquered the defect, we just "stowed it away" in a separable or a purely inseparable extension. However, this tells us something about the type of defect that we have not mastered.

Classification of defect in positive characteristic

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If a field is perfect (such as $\mathbb{F}_p((t))^{1/p^\infty}$), it has no dependent defect extensions.

Can we push the barrier further?

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There is some hope that results on elimination of ramification can be generalized to the case of ground fields that do not admit dependent defect extensions. What would be a good framework for this?

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$(\mathbb{F}_p((t))^{1/p^\infty}, v_t)$ is a henselian deeply ramified field.

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All tame fields are deeply ramified (but not vice versa).

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Can interesting model theoretic results about henselian deeply ramified fields be proven in suitable extensions of the language of valued fields?

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defectless, in general not perfect.

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